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Constructions of partially balanced bipartite block designs

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Abstract

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Methods of construction of group divisible partially balanced bipartite block designs are presented. These designs have unequal replications of treatments.

1. Introduction

Rao [12] defined and constructed partially balanced block designs with different numbers of replications as discussed in Preece [7]. Rao also discussed their statistical analysis. A design of this type has two groups of treatments (treatments in each group having a constant replication) such that there is partial balance within the groups and balance between the groups. The balance structure between the groups allows to estimate treatment differences (about treatments belonging to the groups) with the same variance.

An incomplete block design with a set of v_1 treatments occurring r_1 times and another set of v_2 treatments occurring r_2 ($r_1 \neq r_2$) times arranged into b blocks of size k, is called a group divisible (GD) partially balanced bipartite (PBB) design if:

(i) the treatments in the ith set can be divided into m_i groups each of size n_i ($i = 1, 2$) and any two treatments in the same group are called first associates, otherwise they are called second associates;

(ii) any two treatments in the ith set which are jth associates occur together in $\lambda_{i(j)}$ blocks for $j = 1, 2$;

(iii) any two treatments from different sets occur together in $\lambda_{12} = \lambda_{21}$ (>0) blocks.

Note that when $\lambda_{12} = 0$, the design is disconnected and hence we give the restriction $\lambda_{12} > 0$. Furthermore, when $\lambda_{2(1)} = \lambda_{2(2)}$, it is denoted by $\lambda_{2(\ )}$. The following parametric relations hold for a GD PBB design:

$$v_1r_1 + v_2r_2 = bk, \qquad r_1(k-1) = (n_1-1)\lambda_{1(1)} + n_1(m_1-1)\lambda_{1(2)} + \lambda_{12}v_2,$$
$$r_2(k-1) = \lambda_{12}v_1 + (n_2-1)\lambda_{2(1)} + n_2(m_2-1)\lambda_{2(2)}.$$

These designs may be considered as an extension of balanced bipartite block designs, studied by Corsten [2], Nair and Rao [6], and Kageyama and Sinha [5]. They have some statistical advantage (cf. Kageyama and Sinha [5]). GD PBB designs with $m_i = m$ and $n_i = n$ $(i = 1, 2)$ may be useful for $2 \times m \times n$ factorial experiments (cf. Puri and Kageyama [8], Puri, Mehta and Kageyama [9]). In general, the construction problem on GD PBB designs may be more complicated than that of balanced bipartite block designs. Here, a method of constructing GD PBB designs with two different replications, from a regular GD partially balanced incomplete block (PBIB) design, is provided first. A table of these designs in the range of parameters $k \leq 10$ is also given. Furthermore, other methods are presented in a line similar to Kageyama and Sinha [5].

The definitions of other terms discussed here are from Raghavarao [11].

2. Construction I (composition method)

Rao [12] presented several methods of construction of PBB designs. We first give a construction method of PBB designs from a regular GD PBB design of mn treatments with m groups of n treatments each.

Theorem 2.1. *The existence of a regular GD design with parameters*

$$v' = mn,\ b' = 2nr,\ r' = r,\ k' = m/2,\ m,\ n,\ \lambda_1' = 0,\ \lambda_2' \tag{2.1}$$

for even m, implies the existence of a GD PBB design with parameters

$$\begin{gathered} v_1 = mn,\ v_2 = m,\ b = 4nr,\ r_1 = 2r,\ r_2 = 2nr,\ k = m,\ \lambda_{1(1)} = 0, \\ \lambda_{1(2)} = 2\lambda_2',\ \lambda_{2(\ )} = 2n^2\lambda_2',\ \lambda_{12} = r. \end{gathered} \tag{2.2}$$

Proof. Let the mn treatments in the original regular GD design be arranged in an $n \times m$ array (i.e., m groups of n treatments each).

$$\begin{array}{cccc} 1 & 2 & \cdots & m \\ m+1 & m+2 & \cdots & 2m \\ \vdots & \vdots & \ddots & \vdots \\ (n-1)m+1 & (n-1)m+2 & \cdots & nm \end{array}$$

Any two treatments in the same group $\{i, m+i, \ldots, (n-1)m+i\}$ $(i = 1, 2, \ldots, m)$ are first associates, otherwise they are second associates. Now to each block of the regular GD design with parameters (2.1), we add $m-k'$ treatments $nm+i$, if there are no treatments from the ith group for $i = 1, 2, \ldots, m$. Thus we get a set of b' blocks each of size $k = m$. Next, in order to get another set of b' blocks each of size m, to each of the b' blocks in the original regular GD design, we add treatments $nm+i$ corresponding to any treatment from the ith group $jm+i$ tor $j = 0, 1, \ldots, n-1$, that is, this addition part is just a complement of the addition part in the first set. Then by taking a union of these two sets of $2b'$ blocks, we can obtain a GD PBB design with the required parameters (2.2). Note that $k' = k/2$ is necessary for the constancy of block sizes in the resulting design. The proof is completed. □

The mn treatments in the first group are partially balanced with a GD association scheme, and the second group of treatments, $mn+1, \ldots, m(n+1)$, are balanced and the treatments' comparisons between the two groups are also balanced.

Given below is a table of GD PBB designs through Theorem 2.1 in the range of $k \leq 10$. The references to design numbers are from Clatworthy [1].

No.	v_1	v_2	b	r_1	r_2	k	$\lambda_{1(1)}$	$\lambda_{1(2)}$	$\lambda_{2(\ )}$	λ_{12}	Source
1	8	4	48	12	24	4	0	2	8	6	R 29
2	12	4	108	18	54	4	0	2	18	9	R 39
3	12	6	40	10	20	6	0	2	8	5	R 70
4	12	6	80	20	40	6	0	4	16	10	R 77
5	24	6	160	20	80	6	0	2	32	10	R 93
6	24	8	84	14	42	8	0	2	18	7	R 125
7	40	10	144	18	72	10	0	2	32	9	R 161

Example 2.1. A plan of a GD PBB design of No. 1 in the table can be constructed from the design R 29 ($v' = 8$, $b' = 24$, $r' = 6$, $k' = 2$, $m = 4$, $n = 2$, $\lambda_1' = 0$, $\lambda_2' = 1$) in [1] as follows:

First set

$(1, 2, C, D)$ $(3, 5, B, D)$ $(4, 7, A, B)$
$(6, 8, A, C)$ $(1, 3, B, D)$ $(2, 7, A, D)$
$(4, 6, A, C)$ $(5, 8, B, C)$ $(1, 4, B, C)$
$(2, 3, A, D)$ $(5, 6, C, D)$ $(7, 8, A, B)$
$(1, 6, C, D)$ $(2, 4, A, C)$ $(3, 8, A, B)$
$(5, 7, B, D)$ $(1, 7, B, D)$ $(2, 8, A, C)$
$(3, 6, A, D)$ $(4, 5, B, C)$ $(1, 8, B, C)$
$(2, 5, C, D)$ $(3, 4, A, B)$ $(6, 7, A, D)$

Second set

$(1, 2, A, B)$ $(3, 5, A, C)$ $(4, 7, C, D)$
$(6, 8, B, D)$ $(1, 3, A, C)$ $(2, 7, B, C)$
$(4, 6, B, D)$ $(5, 8, A, D)$ $(1, 4, A, D)$
$(2, 3, B, C)$ $(5, 6, A, B)$ $(7, 8, C, D)$
$(1, 6, A, B)$ $(2, 4, B, D)$ $(3, 8, C, D)$
$(5, 7, A, C)$ $(1, 7, A, C)$ $(2, 8, B, D)$
$(3, 6, B, C)$ $(4, 5, A, D)$ $(1, 8, A, D)$
$(2, 5, A, B)$ $(3, 4, C, D)$ $(6, 7, B, C)$

Note that A, B, C and D are the symbols being added to the original design R 29.

3. Construction II (pattern method)

Next, further methods of construction of PBB designs, different from those of Rao [12], are presented. The idea of construction is from Kageyama and Sinha [5]. The proofs are straightforward.

A GD design with one additional block itself yields the following: Without loss of generality, the additional block can be expressed as $\{1, 2, \ldots, n(m-1)\}$ (for (i)) and $\{n(m-1)+1, \ldots, nm\}$ (for (ii)).

Theorem 3.1. *When $m \geq 3$, the existence of a GD design with parameters $v' = mn$, b', r', k', λ_1', λ_2', implies the existence of a GD PBB design with parameters*

(i) *if $k' = n(m-1)$,*

$$v_1 = n(m-1),\ v_2 = n,\ b = b' + 1,\ r_1 = r' + 1,\ r_2 = r',\ k = k',$$
$$\lambda_{1(1)} = \lambda_1' + 1,\ \lambda_{1(2)} = \lambda_2' + 1,\ \lambda_{2(\ )} = \lambda_1',\ \lambda_{12} = \lambda_2';$$

(ii) *if $k' = n$,*

$$v_1 = n(m-1),\ v_2 = n,\ b = b' + 1,\ r_1 = r',\ r_2 = r' + 1,\ k = k',$$
$$\lambda_{1(1)} = \lambda_1',\ \lambda_{1(2)} = \lambda_2',\ \lambda_{2(\ )} = \lambda_1' + 1,\ \lambda_{12} = \lambda_2'.$$

Example 3.1. A semi-regular GD design, SR 65 in [1], with parameters $v' = b' = 9$, $r' = k' = 6$, $\lambda_1' = 3$, $\lambda_2' = 4$, $m = n = 3$, yields a GD PBB design with parameters $v_1 = 6$, $v_2 = 3$, $b = 10$, $r_1 = 7$, $r_2 = 6$, $k = 6$, $\lambda_{1(1)} = 4$, $\lambda_{1(2)} = 5$, $\lambda_{2(\ )} = 3$, $\lambda_{12} = 4$.

Remark 3.1. Theorem 3.1 can be generalized in a routine way as follows. In Case (i) of the theorem, we have a group divisible PBB design with parameters $v_1 = n(m-1)$, $v_2 = n$, $b = b' + p$, $r_1 = r' + p$, $r_2 = r'$, $k = k'$, $\lambda_{1(1)} = \lambda_1' + p$, $\lambda_{1(2)} = \lambda_2' + p$, $\lambda_{2(\ )} = \lambda_1'$, $\lambda_{12} = \lambda_2'$ for a positive integer p. In Case (ii) of the theorem, we have a group divisible PBB design with parameters $v_1 = n(m-1)$, $v_2 = n$, $b = b' + q$, $r_1 = r'$, $r_2 = r' + q$, $k = k'$, $\lambda_{1(1)} = \lambda_1'$, $\lambda_{1(2)} = \lambda_2'$, $\lambda_{2(\ )} = \lambda_1' + q$, $\lambda_{12} = \lambda_2'$ for a positive integer q.

Remark 3.2. As another generalization of Theorem 3.1, we can state the following. When $m \geq 3$, the existence of a GD design with parameters $v' = mn$, b', r', $k' = nl$, λ_1', λ_2' for a positive integer l such that $1 \leq l \leq m-1$, implies the existence of a GD PBB design with parameters

$$v_1 = nl,\ v_2 = n(m-l),\ b = b' + 1,\ r_1 = r' + 1,\ r_2 = r',\ k = k',$$
$$\lambda_{1(1)} = \lambda_1' + 1,\ \lambda_{1(2)} = \lambda_2' + 1,\ \lambda_{2(1)} = \lambda_1',\ \lambda_{2(2)} = \lambda_2',\ \lambda_{12} = \lambda_2'.$$

For example, if we take an SR 36 in Clatworthy [1] as a starting design, this generalization yields a GD PBB design with parameters $v_1 = v_2 = 4$, $b = 9$, $r_1 = 5$, $r_2 = 4$, $k = 4$, $\lambda_{1(1)} = 1$, $\lambda_{1(2)} = 3$, $\lambda_{2(1)} = 0$, $\lambda_{2(2)} = 2$, $\lambda_{12} = 2$.

Theorem 3.2. *The existence of a GD design with parameters v', b', r', k', λ_i', $i = 1, 2$, and a balanced incomplete block (BIB) design with parameters v'', b'', r'', k', λ'', where $v' < v''$, implies the existence of a GD PBB design with parameters*

$$v_1 = v',\ v_2 = v'' - v',\ b = b' + b'',\ r_1 = r' + r'',\ r_2 = r'',\ k = k',$$

$$\lambda_{1(i)} = \lambda_i' + \lambda'',\ i = 1, 2,\ \lambda_{2(\ )} = \lambda'',\ \lambda_{12} = \lambda''.$$

This follows from the pattern as

$$\left[\begin{array}{c|c} \text{GD design} & \\ O & \text{BIB design} \end{array}\right].$$

Example 3.2. Consider a GD design, SR 35, with parameters $v = 6$, $b = 9$, $r = 6$, $k = 4$, $\lambda_1 = 3$, $\lambda_2 = 4$, $m = 2$, $n = 3$, and a BIB design with parameters $v = 9$, $b = 18$, $r = 8$, $k = 4$, $\lambda = 3$ (Takeuchi [13]). Then we have a GD PBB design with parameters $v_1 = 6$, $v_2 = 3$, $b = 27$, $r_1 = 14$, $r_2 = 8$, $k = 4$, $\lambda_{1(1)} = 6$, $\lambda_{1(2)} = 7$, $\lambda_{2(\ )} = 3$, $\lambda_{12} = 3$.

Remark 3.3. Theorem 3.2 can be generalized, for example, to the following pattern.

$$\left[\begin{array}{c|cc} \text{BIB design} & \text{PBIB design} & O \\ & O & \text{PBIB design} \end{array}\right].$$

Theorem 3.3. *The existence of a GD design with parameters $v' = mn$, b', r', k', λ_1', λ_2', $m = 2$, and two GD designs with respective parameters $v'' = n$, b'', r'', k', λ_i'' $(i = 1, 2)$, and $v''' = n$, b''', r''', k', $\lambda_i'''(i = 1, 2)$, implies the existence of a GD PBB design with parameters*

$$v_1 = n = v_2,\ b = b' + b'' + b''',\ r_1 = r' + r'',\ r_2 = r' + r''',\ k = k',$$

$$\lambda_{1(i)} = \lambda_1' + \lambda_i'',\ i = 1, 2,\ \lambda_{2(i)} = \lambda_i' + \lambda_i''',\ i = 1, 2,\ \lambda_{12} = \lambda_2'.$$

This follows from the pattern as

$$\left[\begin{array}{c|cc} \text{GD design } (m = 2) & \text{GD design} & \text{O} \\ & \text{O} & \text{GD design} \end{array}\right].$$

Remark 3.4. The similar patterns can be considered as

$$\left[\begin{array}{c|cc} \text{GD design (any } m) & \text{BIB design} & O \\ & O & \text{BIB design} \end{array}\right]$$

or

$$\left[\begin{array}{c|cc} \text{GD design (any } m) & \text{BIB design} & J \\ & J & \text{BIB design} \end{array}\right].$$

Remark 3.5. In Theorem 3.3, if the last two GD designs are the same and have the parameters $v''=n$, b'', r'', k', λ_i'' $(i=1,2)$, then the resulting design can also be regarded as a nested 3-associate GD design (Homel and Robinson [3], Raghavarao [10]), with parameters

$$v=2n \text{ (i.e., } v_1=n=v_2), \ b=b'+2b'', \ r=r'+r'', \ k=k',$$
$$\lambda_1=\lambda_1'+\lambda_1'', \ \lambda_2=\lambda_1'+\lambda_2'', \ \lambda_3=\lambda_2'.$$

As another pattern which is called a reinforcement of the GD design, we have the following.

Theorem 3.4. *The existence of a GD design with parameters v', b', r', k', λ_1', λ_2', implies the existence of a GD PBB design with parameters*

$$v_1=v', \ v_2=v'-k'+1, \ b=b'+v'-k'+1, \ r_1=r'+v'-k'+1,$$
$$r_2=b'+1, \ \lambda_{1(1)}=\lambda_1'+v'-k'+1, \ \lambda_{1(2)}=\lambda_2'+v'-k'+1, \ \lambda_{2(\ )}=b',$$
$$\lambda_{12}=r'+1.$$

This follows from the pattern as

$$\begin{bmatrix} \text{GD design} & J_{v'\times p} \\ J_{p\times b'} & I_{v'-k'+1} \end{bmatrix} \quad \text{with } p=v'-k'+1.$$

Example 3.3. Consider a design SR 18 with parameters $v=6$, $b=4$, $r=2$, $k=3$, $\lambda_1=0$, $\lambda_2=1$, $m=3$, $n=2$. Then Theorem 3.4 yields a GD PBB design with parameters $v_1=6$, $v_2=4$, $b=8$, $r_1=6$, $r_2=5$, $\lambda_{1(1)}=4$, $\lambda_{1(2)}=5$, $\lambda_{2(\ )}=4$, $\lambda_{12}=3$.

Remark 3.6. A similar pattern can be taken as

$$\begin{bmatrix} \text{GD design} & J_{v'\times p} \\ J_{p\times b'} & O \end{bmatrix} \quad \text{with } p=v'-k'.$$

As a pattern of another modification of Theorem 3.4,

$$\begin{bmatrix} \text{GD design} & \text{GD design} \\ J_{p\times b'} & O \end{bmatrix} \quad \text{with } p=k''-k'$$

yields the following.

Theorem 3.5. *When $k''>k'$, the existence of two GD designs with respective parameters v', b', r', k', λ_1', λ_2', and v', b'', r'', k'', λ_1'', λ_2'', having the same GD association scheme, implies the existence of a GD PBB design with parameters*

$$v_1=v', \ v_2=k''-k', \ b=b'+b'', \ r_1=r'+r'', \ r_2=b', \ k=k'',$$
$$\lambda_{1(i)}=\lambda_i'+\lambda_i'', \ i=1,2, \ \lambda_{2(\ )}=b', \ \lambda_{12}=r'.$$

Example 3.4. Consider a GD design, SR 26, with parameters $v=12$, $b=16$, $r=4$, $k=3$, $\lambda_1=0$, $\lambda_2=1$, $m=3$, $n=4$, and a GD design, SR 68, with parameters $v=b=12$, $r=k=6$, $\lambda_1=2$, $\lambda_2=3$, $m=3$, $n=4$. Then we get through Theorem 3.5 a GD PBB design with parameters $v_1=12$, $v_2=3$, $b=28$, $r_1=10$, $r_2=16$, $k=6$, $\lambda_{1(1)}=2$, $\lambda_{1(2)}=4$, $\lambda_{2(\ )}=16$, $\lambda_{12}=4$.

As an augmentation technique, a pattern as

$$\begin{bmatrix} \text{GD design} \\ J_{p\times b'} \end{bmatrix}$$

shows that the existence of a GD design with parameters v', b', r', k', λ_1', λ_2', implies the existence of a GD PBB design with parameters

$$v_1=v',\ v_2=p,\ b=b',\ r_1=r',\ r_2=b',\ k=k'+p,$$

$$\lambda_{1(i)}=\lambda_i',\ i=1,2,\ \lambda_{2(\ )}=b',\ \lambda_{12}=r'$$

for any positive integer p. This observation is essentially the same as Construction 4.1 of Rao [12].

A similar augmentation can be applied to an α-resolvable PBIB design for $\alpha\geqslant 1$.

Theorem 3.6. *The existence of an α-resolvable GD design with parameters v', $b'=\beta t$, $r'=\alpha t$, k', λ_i', $i=1,2$, implies the existence of a GD PBB design with parameters*

$$v_1=v',\ v_2=t,\ b=b',\ r_1=r',\ r_2=\beta,\ k=k'+1,$$

$$\lambda_{1(i)}=\lambda_i',\ i=1,2,\ \lambda_{2(\ )}=0,\ \lambda_{12}=\alpha.$$

The proof is similar to that of Theorem 2.3 of Kageyama and Sinha [5]. For α-resolvable GD designs refer to Clatworthy [1], and Kageyama and Mohan [4].

It is known (cf. Rao [12]) that the complement of a PBB design is again a PBB design. Since the complementation does not change a GD association scheme, the PBB designs constructed here yield other GD PBB designs, through complementation. Furthermore, note that taking copies of a PBB design yields another PBB design.

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