The best constant approximant operators in Lorentz spaces $\Gamma_{p,w}$ and their applications

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Abstract

In the present article we extend the best constant approximant operator from Lorentz spaces $\Gamma_{p,w}$ to $\Gamma_{p-1,w}$ for any $1 < p < \infty$ and $w \geq 0$ a locally integrable weight function, and from $\Gamma_{1,w}$ to the space of all measurable functions $L^0$. Then we establish several properties of the extended best constant approximant operator and finally, we prove a generalized version of the Lebesgue Differentiation Theorem in $L^0$.

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0. Introduction

In 2001, Mazzone and Cuenya [14] introduced a new method of extension of the best constant approximant operator from the space $L^p(R^n) + L^\infty(R^n)$, $1 < p < \infty$, to $L^{p-1}(R^n) + L^\infty(R^n)$, and from $L^1(R^n) + L^\infty(R^n)$ to $L^0$. They also demonstrated several properties of the expanded operator, in particular its monotonicity in the sense of Landers and Rogge [9], which they further used to prove a new type of Lebesgue’s Differentiation Theorem (LDT) for local approximation in $L^p(R^n) + L^\infty(R^n)$, $0 \leq p < \infty$. Recall that the classical LDT has been proved by Henri Lebesgue and states that any locally integrable function $f$ can be recovered a.e. from an integral average $\frac{1}{\mu(B(v,\epsilon))} \int_{B(v,\epsilon)} f$ for $\epsilon$ approaching zero, where $\epsilon$ is the radius of the ball $B(v,\epsilon)$ with center at $v$. This integral average can be interpreted as the best constant approximant of $f$ on $B(v,\epsilon)$. In the same spirit as in [14], Favier and Zo [6] constructed the extended best constant

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approximant operator over Orlicz space $L^{ϕ'}(\mathbb{R}^n)$, where $ϕ'$ is a derivative of an Orlicz function $ϕ$, and established the analogous generalization of LDT in $L^{ϕ'}(\mathbb{R}^n)$. Very recently similar problems concerning the extensions of the best constant approximants and LDT in Orlicz–Lorentz spaces $L_{p,ϕ}$ have been investigated for the Orlicz function $ϕ$ and the weight function $w ≥ 0$, by Levis, Cuenya and Priori in the papers [13] and [10].

The purpose of this paper is to study the existence and the properties of the extension of the best constant approximant operator from Lorentz spaces $Γ_{p, w}$, $1 < p < ∞$, to $Γ_{p−1, w}$, and from $Γ_{1, w}$ to the space of all measurable functions $L^0$. Recall that $Γ_{p, w}$, for $0 < p < ∞$ and $w$ a nonnegative weight function, is a set of all Lebesgue measurable functions such that $\int_0^∞ (f^{∗∗})^pw < ∞$, where $f^{∗∗}(t) = H^1(f)(t) = \frac{1}{t} \int_0^t f^∗$ is the Hardy operator and $0 < α ≤ ∞$. The extension under consideration is constructed in the spirit of [14] on the basis of the results obtained in [4], where the Gâteaux derivative of the norm in $Γ_{p, w}$ has been thoroughly investigated. Finally, maximal inequalities for the extended best constant approximant operators and the generalization of LDT in $L^0$ are investigated.

The present article is organized as follows. The preliminaries contain all necessary notions, definitions and auxiliary results, which will be used later. Here we also recall some earlier results, which play a crucial role in further research, especially in the investigation of the extension of the best constant approximant operator in $Γ_{p, w}$.

The main result of Section 2 is a thorough presentation of the extension procedure of the best constant approximant operator in $Γ_{p, w}$ in the spirit of the expansion given first in [14] and then developed in [6,13]. The simple but important fact that $Γ_{p, w}(A) ⊂ Γ_{r, w}(A)$ for $p > r$ and $A ⊂ (0, α)$ of positive and finite measure allows us to expand the best constant approximant operator from normed Lorentz space $Γ_{p, w}$ to quasi-normed Lorentz space $Γ_{p−1, w}$ for $p > 1$ and $w$ a nonnegative weight function. We also present, in this section, the result about the extension of the best constant approximant operator from Lorentz spaces $L_{p, w}$ to $L^0$. The main theorems are preceded by several technical lemmas.

In Section 3, first we characterize some basic properties of the extended best constant approximant operators. Next we prove that the right-hand Gâteaux derivative of the norm in $Γ_{1, w}$ at $fχ_A$ in the direction $χ_A$ is monotone with respect to $f ∈ L^0$, where $A ⊂ (0, α)$ with $μ(A) < ∞$. Then we apply this result to show that the extended best constant approximant operator over $L^0$ is monotone in the sense of the definition introduced by Landers and Rogge [9].

In the last Section 4, we establish the weak inequality for the maximal functions that correspond to the best constant approximation. Finally, we prove convergence of the extended best constant approximant of any $f ∈ L^0$ to this function f a.e., which is a new type of Lebesgue’s Differentiation Theorem.

Further studies of different convergence theorems of the extended best constant approximants in $Γ_{p, w}$ for $0 < p < ∞$ are conducted in the paper [5], which is currently in a preliminary version.

1. Preliminaries

Let $\mathbb{R}$ and $\mathbb{N}$ be the set of all real and natural numbers, respectively. For any $A ⊂ (0, α)$ define $A^c = [0, α) \setminus A$. Let us have $0 < α ≤ ∞$ and let $μ$ be the Lebesgue measure on $\mathbb{R}$. We denote by $L^0$ the space of all extended real valued $μ$-measurable and finite functions a.e. on $(0, α)$. Denote the outer measure on $\mathbb{R}$ by $μ^*$, the support of $f ∈ L^0$ by $S(f) = \text{supp}(f)$ and the restriction of $f$ to the set $A ⊂ [0, α)$ by $f|_A$. By a simple function (resp., step function) we mean any measurable function which attains only a finite number of values (resp., a finite number of values
on a finite number of disjoint intervals). The distribution function \( d_f \) of a function \( f \in L^0 \) is given by \( d_f(\lambda) = \mu(s \in [0, \alpha) : |f|(s) > \lambda) \) for all \( \lambda \geq 0 \). Two functions \( f, g \in L^0 \) are called equimeasurable if \( d_f(\lambda) = d_g(\lambda) \) for all \( \lambda \geq 0 \). We define the decreasing rearrangement for any \( f \in L^0 \) by \( f^*(t) = \inf(s > 0 : d_f(s) \leq t) \), \( t > 0 \). For given \( f \in L^0 \) we denote the maximal function of \( f^* \) by \( f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)ds \). It is well known that \( f^* \leq f^{**} \) and \( f^{**} \) is decreasing and subadditive, i.e. \( (f + g)^{**} \leq f^{**} + g^{**} \) for any \( f, g \in L^0 \). For the properties of \( d_f, f^* \) and \( f^{**} \) see [1,8].

A quasi-normed lattice \((E, \| \cdot \|_E)\) is called a quasi-normed function space if it is a sublattice of \( L^0 \) satisfying the following conditions:

1. If \( f \in L^0, g \in E \) and \( |f| \leq |g| \) a.e., then \( f \in E \) and \( \|f\|_E \leq \|g\|_E \).
2. There exists a strictly positive \( f \in E \).

If \((E, \| \cdot \|_E)\) is complete then it is said to be a quasi-Banach function space. The quasi-norm \( \| \cdot \|_E \) or the space \((E, \| \cdot \|_E)\) is called order continuous if for any \( f \in E \) and \( |f_n| \leq |f| \) with \( |f_n| \to 0 \) a.e. we have \( \|f_n\|_E \to 0 \). We also mention that \( E \) has the Fatou property if whenever \( 0 \leq f_n \in E \) for all \( n \in \mathbb{N}, f \in L^0, f_n \uparrow f \) a.e. and \( \sup_n \|f_n\|_E < \infty \) then \( f \in E \) and \( \|f_n\|_E \uparrow \|f\|_E \).

We say that a quasi-Banach function space \((E, \| \cdot \|_E)\) is rearrangement invariant (r.i. for short) if whenever \( f \in L^0 \) and \( g \in E \) with \( d_f = d_g \) then \( f \in E \) and \( \|f\|_E = \|g\|_E \). Given a r.i. quasi-Banach function space \( E \) let \( \phi_E \) denote its fundamental function, that is \( \phi_E(0) = 0 \) and \( \phi_E(t) = \|\chi_{(0,t)}\|_E \) for any \( t \in (0, \alpha) \) [1].

Let \( 0 < p < \infty \) and \( w \in L^0 \) be a nonnegative weight function. Lorentz space \( \Gamma_{p,w} \) is a subspace of all \( f \in L^0 \) for which

\[
\|f\| = \|f\|_{\Gamma_{p,w}} := \left( \int_0^\alpha f^{** p} w \right)^{1/p} = \left( \int_0^\alpha f^{** p} (t) w(t) dt \right)^{1/p} < \infty.
\]

Given a measurable set \( A \subset [0, \alpha) \), by \( \Gamma_{p,w}(A) \) we denote the set of \( f \in L^0 \) restricted to \( A \) and satisfying the above inequality. Throughout the paper we assume that \( w \) belongs to the class \( D_p \), which means that it satisfies the following conditions:

\[
W(s) := \int_0^s w < \infty \quad \text{and} \quad W_p(s) := s^p \int_s^\alpha t^{-p} w(t) dt < \infty
\]

for all \( 0 < s \leq \alpha \) if \( \alpha < \infty \), and for all \( 0 < s < \infty \) otherwise. These two conditions guarantee that \( \Gamma_{p,w} \neq (0) \). Unless we state otherwise, we also assume that

\[
W(\infty) = \int_0^\infty w = \infty, \quad \text{whenever} \quad \alpha = \infty.
\]

It follows from this assumption that \( \| \cdot \|_{\Gamma_{p,w}} \) is order continuous. Therefore for all \( g \in \Gamma_{p,w}, f_n, f \in L^0 \) and \( |f_n| \leq |g| \) a.e. for any \( n \in \mathbb{N} \) if \( f_n \) converges to \( f \) a.e., then we have \( \lim_{n \to \infty} (f_n - f)^{**}(t) = 0 \) for all \( t \in (0, \alpha) \) and \( \lim_{n \to \infty} \|f_n - f\|_{\Gamma_{p,w}} = 0 \). We also have that \( \lim_{t \to \infty} f^*(t) = 0 \) for \( f \in \Gamma_{p,w} \) if \( \alpha = \infty \).

The space \((\Gamma_{p,w}, \| \cdot \|)\) is a r.i. quasi-Banach function space with the Fatou property and order continuous norm. In the case where \( 1 \leq p < \infty \), then it is a Banach space. For more details about the properties of \( \Gamma_{p,w} \), the reader is referred to [7,4].

Let \((\Omega_1, \mu_1)\) and \((\Omega_2, \mu_2)\) be \( \sigma \)-finite measure spaces. A map \( \gamma \) from \( \Omega_1 \) into \( \Omega_2 \) is said to be a measure preserving transformation if whenever \( E \) is a \( \mu_2 \)-measurable subset of \( \Omega_2 \), the set \( \gamma^{-1}[E] = \{ u \in \Omega_1 : \gamma(u) \in E \} \) is a \( \mu_1 \)-measurable subset of \( \Omega_1 \) and \( \mu_1(\gamma^{-1}[E]) = \mu_2(E) \).
Given \( A, B \subset [0, \infty) \) such that \( \mu(A) = \mu(B) \), there exists a measure preserving transformation \( \delta : A \to B \) (see [16, Theorem 17, p. 410]).

**Definition 1.1 ([2])**. Let \( f, h \in L^0 \) and let \( A \subset [0, \alpha) \). Define

\[
\tau_{(f,h)}(t) = d_f([f](t)) + \mu(u : |f|(u) = |f|(t) \text{ and } h(u) \text{ sign}(f(u)) > h(t) \text{ sign}(f(t)))) + \mu(u : |f|(u) = |f|(t) \text{ and } h(u) \text{ sign}(f(u)) = h(t) \text{ sign}(f(t))) \text{ and } u \leq t,
\]

and

\[
\psi_{(f,A)}(t) = \mu(u \in A : f(u) > f(t)) + \mu(u \in A : f(u) = f(t), u \leq t),
\]

for all \( t \in [0, \alpha) \).

In 1970, Ryff proved in [18] that \( \psi_{(f,[0,1])} : [0, 1] \to [0, 1] \) is a measure preserving transformation for any \( f \in L^0 \) and \( |f| = f^* \circ \psi_{(f,[0,1])} \) a.e. on \([0, 1] \). In 1993, Carothers, Haydon and Lin established in [2] that \( \tau_{(f,h)} \) is a measure preserving transformation from \( S(f) \) onto \( S(f^*) \) such that \(|f| = f^* \circ \psi_{(f,h)} \) a.e. on \( S(f) \) for any \( f \in L^0 \) with \( d_f(\lambda) < \infty \) for all \( \lambda > 0 \) and any \( h \in L^0 \). Notice that for any \( f \in L^0 \) with \( d_f(\lambda) < \infty \) for any \( \lambda > 0 \) and \( h \in L^0 \) if \( \mu(u : |f|(u) = v) = 0 \) for every \( v > 0 \), then we have that \( \tau_{(f,h)}(t) = d_f([f](t)) \) and it is the unique measure preserving transformation up to measure zero satisfying \(|f| = f^* \circ \tau_{(f,h)} \) a.e. on \( S(f) \). In 2007, Levis and Cuenya showed in [12] that \( \psi_{(|f|,S(f))} : S(f) \rightarrow S(f^*) \) is a measure preserving transformation for any \( f \in L^0 \) such that \( d_f(\lambda) < \infty, \lambda > 0 \). We have that \( \psi_{(f,A)} \) is a measure preserving transformation from \( A \subset [0, \mu(A)] \) for any \( f \in L^0 \) and \( 0 < \mu(A) < \infty \). Notice also that \( \psi_{(|f|,A)}(s) = \tau_{(|f|,A),0}(s) \) for any \( s \in A \).

**Definition 1.2 ([4,12])**. Let \( f, g \in \Gamma_{p,w} \) and let \( \tau_{(f,g)}, \tau_{(g|S(g)|S(f),0)} \) be measure preserving transformations given by **Definition 1.1**. Define

\[
\rho_{(f,g)}(s) = \begin{cases} 
\tau_{(f,g)}(s), & \text{if } s \in S(f), \\
\tau_{(g|S(g)|S(f),0)}(0), & \text{if } s \in S(g) \setminus S(f).
\end{cases}
\]

**Definition 1.3 ([4])**. Let \( f \in \Gamma_{p,w} \) and \( A \subset [0, \alpha) \). Define

\[
K_{(f,A)}(u,t) = \frac{1}{t} \int_A (1 - 2\bar{\chi}_{[f < u]}(s)) \chi_{(0,t)}(\rho_{((f-u)\chi_A,\chi_A)}(s))ds,
\]

and

\[
S_{(f,A)}^p(u) = \int_0^\alpha K_{(f,A)}(u,t) ((f - u)\chi_A)^{**(p-1)}(t)w(t)dt
\]

for any \( u \in \mathbb{R} \) and \( t \in (0, \alpha) \).

**Theorem 1.4 ([4, Theorem 4.3])**. Let \( 1 \leq p < \infty \), \( u \in \mathbb{R} \) and \( f \in \Gamma_{p,w} \) and let \( A \subset [0, \alpha) \), \( 0 < \mu(A) < \infty \). Then the right-hand Gâteaux derivative of the norm at \((f - u)\chi_A\) in the direction \( \chi_A \) is given by

\[
G_+((f - u)\chi_A, \chi_A) = S_{(f,A)}^p(u).
\]

Let \((X, \| \cdot \|)\) be a real normed space and let \( Y \) be a subset of \( X \) and \( x \in X \). Denote the closed unit ball and the unit sphere of \( X \) by \( B_X \) and \( S_X \) respectively. An element \( \tilde{x} \in Y \) is called best
approximant to \( x \) from \( Y \) if
\[
\| x - \tilde{x} \| = \inf \{ \| x - y \| : y \in Y \}.
\]
A nonempty subset \( Y \) of \( X \) is called a set of existence if for every \( x \in X \) there exists at least one element \( \tilde{x} \in Y \) for which the above equation is satisfied. Let \( K \) be a linear subspace (resp., a convex subset) of a normed space \( X \), and \( x \in X \setminus \overline{K} \). Then \( \tilde{x} \in K \) is a best approximant to \( x \) from \( K \) if and only if \( G_+(x - \tilde{x}, y) \geq 0 \) (resp., \( G_+(x - \tilde{x}, \tilde{x} - y) \geq 0 \)) for all \( y \in K \) (see [15, 19]).

**Definition 1.5** ([3, 15]). Let \( Y \) be a subset of a normed space \( X \). For any \( x \in X \) define
\[
\mathcal{C}_Y(x) = \{ y \in Y : \| x - y \| \leq \| x - z \| \text{ for all } z \in Y \} = \{ y \in Y : \| x - y \| = \inf_{z \in Y} \| x - z \| \}.
\]
Throughout the rest of the paper let \( A \subset [0, \alpha) \) be a set of positive and finite measure. Define \( \mathbb{K}(A) = \{ c \chi_A : c \in \mathbb{R} \} \). It is well known that \( C_{\mathbb{K}(A)}(f) \) is convex, compact and a set of existence for all \( f \in \Gamma_{p,w} \) (see [3, 15]).

**Theorem 1.6** ([4, Theorem 7.5]). Let \( 1 \leq p < \infty \) and let \( f \in \Gamma_{p,w} \setminus \mathbb{K}(A) \). Then \( u \in \mathbb{K}(A) \) is the best constant approximant of \( f \) if and only if
\[
S_{(f,A)}^p(u) \geq 0 \quad \text{and} \quad S_{(-f,A)}^p(-u) \geq 0.
\]

**Definition 1.7.** Let \( 1 \leq p < \infty \) and \( f \in \Gamma_{p,w} \). Then we define
\[
\underline{f}_{(p,A)}(f) = \min \left\{ u : u \chi_A \in C_{\mathbb{K}(A)}(f \chi_A) \right\}, \quad \overline{f}_{(p,A)}(f) = \max \left\{ u : u \chi_A \in C_{\mathbb{K}(A)}(f \chi_A) \right\},
\]
and
\[
T_{(p,A)}(f) := \left[ \underline{f}_{(p,A)} \chi_A, \overline{f}_{(p,A)} \chi_A \right].
\]
Since \( C_{\mathbb{K}(A)}(f) \) is compact and convex for any \( f \in \Gamma_{p,w} \), we notice that \( -\infty < \underline{f}_{(p,A)} \leq \overline{f}_{(p,A)} < \infty \) and \( T_{(p,A)}(f) = C_{\mathbb{K}(A)}(f \chi_A) \). In further investigation, we call the map \( T_{(p,A)} \) the best constant approximant operator on \( \Gamma_{p,w} \) and every element \( u \in T_{(p,A)}(f) \) the best constant approximant of \( f \in \Gamma_{p,w}(A) \).

**Lemma 1.8** ([13]). Let \( f \in L^0 \) and \( c < d \). Then for all \( s \in A \) we have:

(i) if \( f(s) \geq d \), then
\[
d_{(f-c)\chi_A}(f(s) - c) \leq d_{(f-d)\chi_A}(f(s) - d)
\]
and
\[
\rho((f-c)\chi_A,s) \leq \rho((f-d)\chi_A,s),
\]
(ii) if \( f(s) < c \), then
\[
d_{(f-d)\chi_A}(d - f(s)) \leq d_{(f-c)\chi_A}(c - f(s))
\]
and
\[
\rho((f-d)\chi_A,s) \leq \rho((f-c)\chi_A,s).
\]

**Lemma 1.9** ([13]). Let \( f \in L^0 \) and \( u_0 \in \mathbb{R} \). Then for all \( s \in A \) we have
\[
\lim_{u \to u_0} \rho((f-u)\chi_A,s) = \rho((f-u_0)\chi_A,s).
\]
2. Extension of the best constant approximant operators

In this section we extend the best constant approximant operator \( T_{(1,A)} \) from \( \Gamma_{1,w} \) to \( L^0 \) and \( T_{(p,A)} \) from \( \Gamma_{p,w} \) to \( \Gamma_{p-1,w} \) for any \( 1 < p < \infty \). This is a gradual process, which is done with the support of several technical lemmas. The extension method is based on the construction developed in [6,13,14]. While extending the operator from \( L^p \) to \( L^{p-1} \) for \( 1 < p < \infty \) (see [14]) is a quite simple process, here the construction is much more complicated. The first step in the extension process is to show an inclusion of \( \Gamma_{p,w}(A) \) in \( \Gamma_{r,w}(A) \) for \( p > r \) and \( A \subset (0,\alpha) \) with \( 0 < \mu(A) < \infty \). This result enables us to investigate an expansion of the best constant approximant operator from \( \Gamma_{p,w}(A) \) to \( \Gamma_{p-1,w}(A) \) if \( p > 1 \). Next we prove that for all \( f \in \Gamma_{p-1,w} \) if \( p > 1 \), and \( f \in \Gamma_{1,w} \) if \( p = 1 \), the function \( S_{(f,A)}^p(u) \) is well-defined, decreasing and left-continuous with respect to \( u \in \mathbb{R} \). In view of Theorem 1.6 and the properties of \( S_{(f,A)}^p(u) \) we establish the existence of a set value operator \( T_{(p,A)} \) in \( \Gamma_{p-1,w}(A) \) if \( p > 1 \), and in \( L^0 \) if \( p = 1 \), which we call an extended best constant approximant operator.

**Lemma 2.1.** Let \( f \in L^0 \) and \( t \in (0,\alpha) \). Then \( K_{(f,A)}(u,t) \) is a decreasing function with respect to \( u \in \mathbb{R} \).

**Proof.** Define

\[
P(u) = \frac{1}{t} \int_{A \cap [f \geq u]} X_{(0,t)}(\rho(\rho_{(f-u)\chi_{A},\chi_{A}})),
\]

\[
Q(u) = \frac{1}{t} \int_{A \cap [f < u]} X_{(0,t)}(\rho(\rho_{(f-u)\chi_{A},\chi_{A}}))
\]

for any \( u \in \mathbb{R} \). Then we get

\[
K_{(f,A)}(u,t) = \frac{1}{t} \int_{A} \left( X_{[f \geq u]} - X_{[f < u]} \right) X_{(0,t)}(\rho(\rho_{(f-u)\chi_{A},\chi_{A}})) = P(u) - Q(u)
\]

for every \( u \in \mathbb{R} \). It is sufficient to prove that \( P \) is decreasing and \( Q \) is increasing. Let \( c, d \in \mathbb{R} \) be such that \( c < d \). By Lemma 1.8 we have

\[
P(d) = \frac{1}{t} \int_{A \cap [f \geq d]} X_{(0,t)}(\rho(\rho_{(f-d)\chi_{A},\chi_{A}})) \leq \frac{1}{t} \int_{A \cap [f \geq c]} X_{(0,t)}(\rho(\rho_{(f-c)\chi_{A},\chi_{A}}))
\]

\[
\leq \frac{1}{t} \int_{A \cap [f \geq c]} X_{(0,t)}(\rho(\rho_{(f-c)\chi_{A},\chi_{A}})) = P(c),
\]

and

\[
Q(c) = \frac{1}{t} \int_{A \cap [f < c]} X_{(0,t)}(\rho(\rho_{(f-c)\chi_{A},\chi_{A}})) \leq \frac{1}{t} \int_{A \cap [f < d]} X_{(0,t)}(\rho(\rho_{(f-d)\chi_{A},\chi_{A}}))
\]

\[
\leq \frac{1}{t} \int_{A \cap [f < d]} X_{(0,t)}(\rho(\rho_{(f-d)\chi_{A},\chi_{A}})) = Q(d).
\]

**Lemma 2.2.** Let \( 0 < r < p < \infty \) and \( w \in D_r \). Then \( w \in D_p \) and

\[\Gamma_{p,w}(A) \subset \Gamma_{r,w}(A)\].

**Proof.** Since \( w \in D_r \), we have

\[
W(s) = \int_{0}^{s} w(t)dt < \infty \quad \text{and} \quad W_r(s) = s^{r} \int_{s}^{\alpha} t^{-r} w(t)dt < \infty
\]
for every \( s \in (0, \alpha) \). Hence
\[
W_p(s) = s^p \int_s^\alpha t^{r-p} r - t w(t) dt \leq s^p \int_s^\alpha t^{r-p} t^{r} w(t) dt = W_r(s),
\]
which implies that \( w \in D_p \). Let \( f \in \Gamma_{p,w}(A) \). Since \( r < p \), there exists \( p' > r \) such that \( \frac{1}{p} + \frac{1}{p'} = \frac{1}{r} \) and by the Hölder inequality we get
\[
\int_0^{\mu(A)} (f \chi_A)^{*r} (t) w(t) dt \leq \left( \int_0^{\mu(A)} (f \chi_A)^{**p} (t) w(t) dt \right)^{r/p} W(\mu(A))^{r/p} < \infty.
\]
Notice that for any \( t \geq \mu(A) \) we have
\[
(f \chi_A)^{*r}(t) = \frac{\mu(A)}{t} (f \chi_A)^{*r}(\mu(A)),
\]
whence
\[
\int_{\mu(A)}^\alpha (f \chi_A)^{*r} (t) w(t) dt \leq \mu(A) \int_{\mu(A)}^\alpha t^{r} w(t) dt < \infty.
\]
Therefore \( f \in \Gamma_{r,w}(A) \), which finishes the proof. \( \square \)

**Remark 2.3.** Notice that, by Lemma 2.2 any nonnegative weight function \( w \in D_{p-1} \) for \( 1 < p < \infty \) satisfies condition \( D_p \) and \( \Gamma_{p,w}(A) \subset \Gamma_{p-1,w}(A) \) for any \( A \subset (0, \alpha) \) with \( 0 < \mu(A) < \infty \). In fact, to expand a best constant approximant operator we assume that a nonnegative weight function \( w \in D_{p-1} \) if \( 1 < p < \infty \), and \( w \in D_1 \) if \( p = 1 \).

**Proposition 2.4.** Let \( 1 \leq p < \infty \). If \( f \in \Gamma_{p-1,w} \) for \( p > 1 \), and \( f \in L^0 \) for \( p = 1 \), then \( S^p_{(f,A)}(u) \) is decreasing and left-continuous with respect to \( u \in \mathbb{R} \).

**Proof.** Let \( f \in L^0 \) and \( u_n \uparrow u_0 \). We claim that
\[
\lim_{n \to \infty} 1 - 2 \chi_{\{f < u_n\}}(s) = 1 - 2 \chi_{\{f < u_0\}}(s) \quad (1)
\]
for a.a. \( s \in A \). Let \( s \in A \) and \(|f(s)| < \infty \). If \( s \in \{f < u_0\} \), then there exists \( N_0 \in \mathbb{N} \) such that for all \( n \geq N_0 \) we have \( f(s) < u_n \) and so
\[
\chi_{\{f < u_n\}}(s) = 1 = \chi_{\{f < u_0\}}(s).
\]
Now suppose that \( s \in A \setminus \{f < u_0\} \). Then for all \( n \in \mathbb{N} \) we get \( f(s) \geq u_n \). Thus,
\[
\chi_{\{f < u_n\}}(s) = 0 = \chi_{\{f < u_0\}}(s)
\]
for any \( n \in \mathbb{N} \), which proves the claim (1). By Lemma 1.9 we have
\[
\lim_{n \to \infty} \rho((f-u_n)\chi_A,\chi_A)(s) = \rho((f-u_0)\chi_A,\chi_A)(s)
\]
for all \( s \in A \) and consequently,
\[
\lim_{n \to \infty} \chi_{(0,t)}(\rho((f-u_n)\chi_A,\chi_A)(s)) = \chi_{(0,t)}(\rho((f-u_0)\chi_A,\chi_A)(s))
\]
for a.a. \( s \in A \). Hence, by condition (1) we get
\[
\lim_{n \to \infty} \left( 1 - 2 \chi_{\{f < u_n\}}(s) \right) \chi_{(0,t)}(\rho((f-u_n)\chi_A,\chi_A)(s))
\]
\[
= \left( 1 - 2 \chi_{\{f < u_0\}}(s) \right) \chi_{(0,t)}(\rho((f-u_0)\chi_A,\chi_A)(s))
\]
Lemma 2.1: we have that $u^p$ for any $t \leq \frac{1}{t}$ and by the Lebesgue Dominated Convergence Theorem we get

$$\lim_{n \to \infty} K_{(f, A)}(u_n, t) = K_{(f, A)}(u_0, t)$$

for any $t \in (0, \alpha)$. Since $\rho_{((-f - u)X, A)}: A \to [0, \mu(A)]$ for any $u \in \mathbb{R}$ is a measure preserving transformation, we obtain

$$|K_{(f, A)}(u, t)| \leq \frac{1}{t} \int_A |1 - 2\chi_{(f < u)}|\chi_0(t)\rho_{((-f - u)X, A)} = \frac{1}{t} \int_A \chi\rho_{((-f - u)X, A)}^{(0, t]}$$

$$= \frac{1}{t} \mu\left(\rho_{((-f - u)X, A)}^{[[0, t]}} \right) = \chi_{[0, \mu(A)]}(t) + \frac{\mu(A)}{t} \chi_{(\mu(A), \alpha)}(t) = (\chi_A)^*(t)$$

for any $u \in \mathbb{R}$ and all $t \in (0, \alpha)$. Now we will consider two cases.

(Case 1). Let $\rho = 1$. Immediately by Lemma 2.1 we have that $S_{(f, A)}^1$ is decreasing. By conditions (2), (3) and by the Lebesgue Dominated Convergence Theorem we get

$$\lim_{n \to \infty} S_{(f, A)}^1(u_n) = \lim_{n \to \infty} \int_0^\alpha K_{(f, A)}(u_n, t)w(t)dt$$

$$= \int_0^\alpha K_{(f, A)}(u_0, t)w(t)dt = S_{(f, A)}^1(u_0),$$

which finishes the first case.

(Case 2). Let $1 < p < \infty$ and $f \in \Gamma_{p-1, w}$. Pick out $t \in (0, \alpha)$. Define a mapping

$$\phi_t(u) = ((f + u)\chi_A)^*(t)$$

for every $u \in \mathbb{R}$. Let $\lambda \in (0, 1)$ and $u, v \in \mathbb{R}$, $u < v$. By the property of the maximal function we get

$$((f + \lambda u + (1 - \lambda)v)\chi_A)^*(t) \leq \lambda ((f + u)\chi_A)^*(t) + (1 - \lambda) ((f + v)\chi_A)^*(t),$$

whence and by convexity of a power function $u^p$ we obtain that $\phi_t$ is convex. By Proposition 4.2 in [4] we have

$$\lim_{\epsilon \to 0^+} \frac{(f + u)\chi_A + \epsilon \chi_A)^*(t) - ((f + u)\chi_A)^*(t)}{\epsilon}$$

$$= P_{((f + u)\chi_A, A)}(t) + T_{((f + u)\chi_A, A)}(t)$$

$$= \frac{1}{t} \int_{A \setminus \{f \neq -u\}} \chi_0(t)(\rho_{((f + u)\chi_A, A)} - \frac{1}{t} \int_{A \setminus \{f \neq -u\}} \chi_0(t)(\rho_{((f + u)\chi_A, A)})$$

$$+ \frac{1}{t} \int_{A \setminus \{f \neq -u\}} \chi_0(t)(\rho_{((f + u)\chi_A, A)}) = K_{(f, A)}(-u, t)$$

for all $t \in (0, \alpha)$. Then the derivative

$$\frac{d}{du}^+ (\phi_t(u)) = pK_{(f, A)}(-u, t)((f + u)\chi_A)^{(p-1)}(t)$$

is increasing and so

$$K_{(f, A)}(-u, t)((f + u)\chi_A)^{(p-1)}(t) \leq K_{(f, A)}(-v, t)((f + v)\chi_A)^{(p-1)}(t)$$
for any $u, v \in \mathbb{R}$, $u < v$ and for all $t \in (0, \alpha)$. Hence
\[
S_{(f,A)}^p(-u) \leq S_{(f,A)}^p(-v).
\]
Defining $a = -u$ and $b = -v$ we have
\[
S_{(f,A)}^p(a) \leq S_{(f,A)}^p(b)
\]
for $b < a$, which implies that $S_{(f,A)}^p$ is decreasing. By continuity of $((f-u)\chi_A)^{**}(t)$ with respect to $u$ we get
\[
\lim_{n \to \infty} ((f - u_n)\chi_A)^{**}(t) = ((f - u_0)\chi_A)^{**}(t)
\]
for all $t \in (0, \alpha)$. Thus, by condition (2) we obtain
\[
\lim_{n \to \infty} K_{(f,A)}(u_n, t) \left( ((f - u_n)\chi_A)^{**}(t) \right)^{p-1} = K_{(f,A)}(u_0, t) \left( ((f - u_0)\chi_A)^{**}(t) \right)^{p-1}
\]
for any $t \in (0, \alpha)$. Clearly, there exists $M > 0$ such that $|u_n| \leq M$ for every $n \in \mathbb{N}$. Consequently,
\[
((f - u_n)\chi_A)^{**}(t) \leq M(\chi_A)^{**}(t) + (f\chi_A)^{**}(t)
\]
for any $n \in \mathbb{N}$ and $t \in (0, \alpha)$. Combining this with condition (3) we get
\[
|K_{(f,A)}(u_n, t)| \left( ((f - u_n)\chi_A)^{**}(t) \right)^{p-1} \leq (M(\chi_A)^{**}(t) + (f\chi_A)^{**}(t))^{p-1}
\]
for all $n \in \mathbb{N}$ and $t \in (0, \alpha)$. It is well known that for any $0 < p < \infty$ there exists $C > 0$ such that
\[
(M(\chi_A)^{**}(t) + (f\chi_A)^{**}(t))^{p-1} \leq C \left( (M(\chi_A)^{**}(t))^{p-1} + C \left( (f\chi_A)^{**}(t) \right)^{p-1} \right),
\]
whence
\[
\int_0^\alpha \left( (M(\chi_A)^{**}(t) + (f\chi_A)^{**}(t))^{p-1} \right) w(t) dt
\]
\[
\leq M^{p-1}C\|\chi_A\|_{p^{-1},w}^{p-1} + C\|f\chi_A\|_{p^{-1},w}^{p-1} < \infty,
\]
by the assumption that $f \in L_{p^{-1},w}$. Finally, by conditions (5) and (6) we complete the proof of case 2. \[\square]\[\]

**Lemma 2.5.** Let $f \in L^0$. Then for a.a. $t \in A$ we get
\[
\lim_{u \to -\infty} \rho((f-u)\chi_{A},\chi_{A})(t) = \psi(-f,A)(t) \quad \text{and} \quad \lim_{u \to -\infty} \rho((f-u)\chi_{A},\chi_{A})(t) = \psi(f,A)(t).
\]

**Proof.** Let $(u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ and $u_n \to \infty, v_n \to -\infty$ as $n \to \infty$. Then for a.a. $t \in A$ there exists $N_t \in \mathbb{N}$ such that for any $n \geq N_t$ we get $v_n < f(t) < u_n$. So, for any $n \geq N_t$ we have
\[
\rho((f-u_n)\chi_{A},\chi_{A})(t) = \tau((f-u_n)\chi_{A},\chi_{A})(t)
\]
\[
= d((f-u_n)\chi_{A})(u_n - f(t)) + \mu(s \in A : |f(s) - u_n| = |f(t) - u_n|, \text{sign}(f(s) - u_n) > \text{sign}(f(t) - u_n))
\]
\[
+ \mu(s \in A : |f(s) - u_n| = |f(t) - u_n|, \text{sign}(f(s) - u_n) = \text{sign}(f(t) - u_n), s < t)
\]
\[
= \mu(s \in A : f(s) \geq 2u_n - f(t))
\]
\[
+ \mu(s \in A : f(s) < f(t)) + \mu(s \in A : f(s) = f(t), s \leq t),
\]
as well as
\[
\rho((f-u_n)_{\mathcal{A}, \mathcal{A}})(t) = \mu(s \in A : f(s) < 2v_n - f(t)) + \mu(s \in A : f(s) > f(t)) + \mu(s \in A : f(s) = f(t), s \leq t).
\]

Since \( f \) is finite a.e. on \([0, \alpha]\) and \( \mu(A) < \infty \), we obtain
\[
\lim_{n \to \infty} \mu(s \in A : f(s) \geq 2u_n - f(t)) = 0 \quad \text{and} \quad \lim_{n \to \infty} \mu(s \in A : f(s) < 2v_n - f(t)) = 0
\]
for a.a. \( t \in A \), from which we conclude that
\[
\lim_{n \to \infty} \rho((f-u_n)_{\mathcal{A}, \mathcal{A}})(t) = \psi(-f_A)(t) \quad \text{and} \quad \lim_{n \to \infty} \rho((f-v_n)_{\mathcal{A}, \mathcal{A}})(t) = \psi(f_A)(t)
\]
for a.a. \( t \in A \). \( \Box \)

**Lemma 2.6.** Let \( f \in L^0 \). Then for any \( t \in (0, \alpha) \) we have
\[
\lim_{u \to -\infty} K(f_A)(u, t) = - (\chi_A)^{**}(t) \quad \text{and} \quad \lim_{u \to \infty} K(f_A)(u, t) = (\chi_A)^{**}(t).
\]

**Proof.** Let \((u_n)_{n \in \mathbb{N}}\), \((v_n)_{n \in \mathbb{N}} \subset \mathbb{R}\) and \( u_n \to \infty, v_n \to -\infty \) as \( n \to \infty \). Let \( s \in A \) and \( |f(s)| < \infty \). Then there is \( N_s \in \mathbb{N} \) such that for all \( n \geq N_s \) we have \( v_n < f(s) < u_n \).

Consequently,
\[
\lim_{n \to \infty} 1 - 2\chi_{\{f < u_n\}}(s) = -1 \quad \text{and} \quad \lim_{n \to \infty} 1 - 2\chi_{\{f < v_n\}}(s) = 1
\]
for a.a. \( s \in A \). Since \( \psi(\pm f_A) \) are measure preserving transformations, by Lemma 2.5 we obtain
\[
\lim_{n \to \infty} \chi_{(0,t)} \left( \rho((f-u_n)_{\mathcal{A}, \mathcal{A}})(s) \right) = \chi_{(0,t)} \left( \psi(-f_A)(s) \right),
\]
and
\[
\lim_{n \to \infty} \chi_{(0,t)} \left( \rho((f-v_n)_{\mathcal{A}, \mathcal{A}})(s) \right) = \chi_{(0,t)} \left( \psi(f_A)(s) \right)
\]
for a.a. \( s \in A \) and for any \( t \in (0, \alpha) \). Hence, by condition (7) we get
\[
\lim_{n \to \infty} \left( 1 - 2\chi_{\{f < u_n\}}(s) \right) \chi_{(0,t)} \left( \rho((f-u_n)_{\mathcal{A}, \mathcal{A}})(s) \right) = -\chi_{(0,t)} \left( \psi(-f_A)(s) \right),
\]
and
\[
\lim_{n \to \infty} \left( 1 - 2\chi_{\{f < v_n\}}(s) \right) \chi_{(0,t)} \left( \rho((f-v_n)_{\mathcal{A}, \mathcal{A}})(s) \right) = \chi_{(0,t)} \left( \psi(f_A)(s) \right)
\]
for a.a. \( s \in A \) and for any \( t \in (0, \alpha) \). Therefore, we conclude that
\[
\lim_{n \to \infty} K(f_A)(u_n, t) = -\frac{1}{t} \int_A \chi_{(0,t)} \left( \psi(-f_A) \right),
\]
and
\[
\lim_{n \to \infty} K(f_A)(v_n, t) = \frac{1}{t} \int_A \chi_{(0,t)} \left( \psi(f_A) \right)
\]
for any \( t \in (0, \alpha) \). Finally, by the fact that \( \psi(\pm f_A) \) are measure preserving transformations we obtain
\[
\frac{1}{t} \int_A \chi_{(0,t)} \left( \psi(-f_A) \right) = \frac{1}{t} \mu \left( \psi^{-1}(-f_A)((0, t]) \right) = (\chi_A)^{**}(t),
\]
and
\[ \frac{1}{t} \int_A \chi(0,t) \left( \Psi(f,A) \right) = (\chi_A)^{**}(t) \]
for any \( t \in (0, \alpha) \), which finishes the proof. \( \square \)

**Proposition 2.7.** Let \( 1 < p < \infty \). Then for any \( f \in \Gamma_{p-1,w} \) we have
\[ \lim_{u \to \infty} \int_0^{\mu(A)} K(f,A)(u, t) ((f - u)\chi_A)^{**}(p-1) (t) w(t) dt = -\infty, \]
and
\[ \lim_{u \to -\infty} \int_0^{\mu(A)} K(f,A)(u, t) ((f - u)\chi_A)^{**}(p-1) (t) w(t) dt = \infty. \]

**Proof.** Define
\[ U(f,A)(u, t) = -1 + \frac{2}{t} \int_{A \cap \{ f \geq u \}} \chi(0,t)(\Psi(f,A)), \]
and
\[ L(f,A)(u, t) = 1 - \frac{2}{t} \int_{A \cap \{ f < u \}} \chi(0,t)(\Psi(-f,A)) \]
for any \( u \in \mathbb{R} \) and \( t \in (0, \alpha) \). Let \( s \in A \) and \( |f(s)| < \infty \). Notice that
\[ \rho((f-u)\chi_A, \chi_A)(s) \geq \psi(-f,A)(s) \quad \text{and} \quad \rho((f-v)\chi_A, \chi_A)(s) \geq \psi(f,A)(s) \]
for every \( u, v \in \mathbb{R} \) such that \( u > f(s) \geq v \). Hence we get
\[ K(f,A)(u, t) = 1 - \frac{2}{t} \int_{A \cap \{ f < u \}} \chi(0,t)(\rho((f-u)\chi_A, \chi_A)) \geq L(f,A)(u, t), \quad (8) \]
and
\[ K(f,A)(v, t) = -1 + \frac{2}{t} \int_{A \cap \{ f \geq v \}} \chi(0,t)(\rho((f-v)\chi_A, \chi_A)) \leq U(f,A)(v, t) \quad (9) \]
for all \( t \in (0, \mu(A)) \). Define
\[ t_u = \mu(A \cap \{ f \geq u \}) \quad \text{and} \quad s_u = \mu(A \cap \{ f < u \}) \]
for all \( u \in \mathbb{R} \). We claim that
\[ A \cap \{ f \geq u \} = \psi_{(f,A)}^{-1} ([0, t_u]) \quad \text{and} \quad A \cap \{ f < u \} = \psi_{(-f,A)}^{-1} ([0, s_u]) \quad (10) \]
a.e. for any \( u \in \mathbb{R} \). Let \( s \in A \cap \{ f \geq u \} \). Then we have
\[ \{ y \in A : f(y) > f(s) \} \cup \{ y \in A : f(y) = f(s), y \leq s \} \subset \{ y \in A : f(y) \geq u \}, \]
and so
\[ \psi_{(f,A)}(s) \leq \mu(A \cap \{ f \geq u \}) = t_u. \]
Therefore,
\[ \psi_{(f,A)}(A \cap \{ f \geq u \}) \subset [0, t_u], \]
and consequently,
\[ A \cap \{ f \geq u \} \subset \psi_{(f,A)}^{-1} \left( \psi_{(f,A)}(A \cap \{ f \geq u \}) \right) \subset \psi_{(f,A)}^{-1} ([0, t_u]). \]
Hence, by definition of $t_u$, since $\psi_{(f,A)}$ is a measure preserving transformation from $A$ onto $[0, \mu(A)]$, we obtain the first equation of our claim. Analogously, we can show the second equation of condition (10). Now we will prove that

$$U_{(f,A)}(u, t) = \chi_{[0,t_u]}(t) + \left(\frac{2t_u}{t} - 1\right) \chi_{[t_u, \alpha]}(t), \quad (11)$$

and

$$L_{(f,A)}(u, t) = -\chi_{[0,s_u]}(t) - \left(\frac{2s_u}{t} - 1\right) \chi_{[s_u, \alpha]}(t) \quad (12)$$

for any $u \in \mathbb{R}$ and $t \in (0, \alpha)$. According to condition (10) for all $u \in \mathbb{R}$ and $t \in (0, \alpha)$ we have

$$U_{(f,A)}(u, t) = -1 + \frac{2}{t} \int_{\mathcal{A}[f \geq u]} \chi_{\psi_{(f,A)}^{-1}([0,t])}$$

$$= -1 + \frac{2}{t} \mu \left(\psi_{(f,A)}^{-1}([0,t_u]) \cap \psi_{(f,A)}^{-1}([0,t])\right)$$

$$= -1 + \frac{2}{t} \mu \left([0,t_u] \cap (0,t)\right) = \chi_{[0,t_u]}(t) + \left(\frac{2t_u}{t} - 1\right) \chi_{[t_u, \alpha]}(t),$$

which implies condition (11). Similarly, we can show (12). Extending $U_{(f,A)}(u, t)$ and $L_{(f,A)}(u, t)$ by continuity to the entire interval $[0, \alpha)$, the functions are continuous with respect $t \in [0, \alpha)$. Since $\lim_{u \to -\infty} t_u = \lim_{u \to -\infty} s_u = 0$, we have

$$\lim_{u \to -\infty} U_{(f,A)}(u, t) = -1 \quad \text{and} \quad \lim_{u \to -\infty} L_{(f,A)}(-u, t) = 1$$

for every $t \in [0, \mu(A)]$. Clearly $(U_{(f,A)}(n,t))_{n \in \mathbb{N}}$ and $(L_{(f,A)}(-n,t))_{n \in \mathbb{N}}$ are decreasing and increasing sequences of continuous functions on a compact interval $[0, \mu(A)]$ respectively. Hence, by Theorem 7.13 [17] we obtain that $(U_{(f,A)}(n,t))_{n \in \mathbb{N}}$ and $(L_{(f,A)}(-n,t))_{n \in \mathbb{N}}$ converge uniformly for $t \in [0, \mu(A)]$ to $-1$ and $1$. So, for any $0 < \epsilon < 1$ there exists $N_\epsilon \in \mathbb{N}$ such that for all $n \geq N_\epsilon$ and $t \in [0, \mu(A)]$ we get

$$|U_{(f,A)}(n,t) + 1| < \epsilon \quad \text{and} \quad |L_{(f,A)}(-n,t) - 1| < \epsilon.$$ 

Therefore,

$$U_{(f,A)}(n,t) < \epsilon - 1 \quad \text{and} \quad 1 - \epsilon < L_{(f,A)}(-n,t)$$

for all $n \geq N_\epsilon$ and for any $t \in [0, \mu(A)]$. Consequently,

$$\int_0^{\mu(A)} U_{(f,A)}(n,t) ((f - n) \chi_A)^{**(p-1)}(t) w(t) dt$$

$$\leq (\epsilon - 1) \int_0^{\mu(A)} ((f - n) \chi_A)^{**(p-1)}(t) w(t) dt,$$

and

$$\int_0^{\mu(A)} L_{(f,A)}(-n,t) ((f + n) \chi_A)^{**(p-1)}(t) w(t) dt$$

$$\geq (1 - \epsilon) \int_0^{\mu(A)} ((f + n) \chi_A)^{**(p-1)}(t) w(t) dt$$

for all $n \geq N_\epsilon$. Thus, by inequalities (8) and (9) we finish the proof. \qed
Proposition 2.8. Let \( f \in L^0 \). Then
\[
\lim_{u \to -\infty} S^1_{(f,A)}(u) = -\|\chi_A\|_{\Gamma_{1,w}} \quad \text{and} \quad \lim_{u \to -\infty} S^1_{(f,A)}(u) = \|\chi_A\|_{\Gamma_{1,w}}.
\]
If \( p > 1 \), then for any \( f \in \Gamma_{p-1,w} \) we have
\[
\lim_{u \to -\infty} S^p_{(f,A)}(u) = -\infty \quad \text{and} \quad \lim_{u \to -\infty} S^p_{(f,A)}(u) = \infty.
\]

Proof. Let \( p = 1 \). Immediately, by Lemma 2.6 and by condition (3) we get
\[
\lim_{u \to -\infty} S^1_{(f,A)}(u) = -\|\chi_A\|_{\Gamma_{1,w}} \quad \text{and} \quad \lim_{u \to -\infty} S^1_{(f,A)}(u) = \|\chi_A\|_{\Gamma_{1,w}}.
\]
Now assume that \( 1 < p < \infty \). Let \( t \in [\mu(A), \alpha) \). Since \( \rho_{((f-u)\chi_A,\chi_A)} : A \to [0, \mu(A)] \) is a measure preserving transformation, we have
\[
K_{(f,A)}(u, t) = \frac{1}{t} \int_{A \cap \{f \geq u\}} \chi_{\rho_{((f-u)\chi_A,\chi_A)}^{-1}([0,t])} - \frac{1}{t} \int_{A \cap \{f < u\}} \chi_{\rho_{((f-u)\chi_A,\chi_A)}^{-1}([0,t])} = \frac{1}{t} \mu(A \cap \{f \geq u\}) - \frac{1}{t} \mu(A \cap \{f < u\})
\]
for every \( u \in \mathbb{R} \). There exists \( N \in \mathbb{N} \) such that for all \( n \geq N \) we obtain
\[
\mu(A \cap \{f \geq n\}) \leq \frac{\mu(A)}{4}, \quad \mu(A \cap \{f < n\}) \geq \frac{3\mu(A)}{4},
\]
and
\[
\mu(A \cap \{f \geq -n\}) \geq \frac{3\mu(A)}{4}, \quad \mu(A \cap \{f < -n\}) \leq \frac{\mu(A)}{4}.
\]
Thus, by condition (13) we get
\[
K_{(f,A)}(n, t) = \frac{1}{t} \mu(A \cap \{f \geq n\}) - \frac{1}{t} \mu(A \cap \{f < n\}) \leq -\frac{\mu(A)}{2t},
\]
and
\[
K_{(f,A)}(-n, t) = \frac{1}{t} \mu(A \cap \{f \geq -n\}) - \frac{1}{t} \mu(A \cap \{f < -n\}) \geq \frac{\mu(A)}{2t}
\]
for any \( n \geq N \) and \( t \in [\mu(A), \alpha) \). Therefore,
\[
\int_{\mu(A)}^{\alpha} K_{(f,A)}(n, t) ((f - n)\chi_A)^{**(p-1)}(t) w(t) dt
\leq -\frac{\mu(A)}{2} \int_{\mu(A)}^{\alpha} ((f - n)\chi_A)^{**(p-1)}(t) \frac{w(t)}{t} dt,
\]
and
\[
\int_{\mu(A)}^{\alpha} K_{(f,A)}(-n, t) ((f + n)\chi_A)^{**(p-1)}(t) w(t) dt
\geq \frac{\mu(A)}{2} \int_{\mu(A)}^{\alpha} ((f + n)\chi_A)^{**(p-1)}(t) \frac{w(t)}{t} dt.
\]
Furthermore

\[(f \pm n)\chi_A \]  
\[(f \pm n)\chi_A \]  
\[(f \pm n)\chi_A \]  
\[= \pm \left( (f \pm n)\chi_A \right)\chi_{[0, \mu(A)]}(t) \]

whence

\[
\int_\mu(A) \left( (f \pm n)\chi_A \right)^{(p-1)}(t) \frac{w(t)}{t} \, dt = \left( (f \pm n)\chi_A \right)^{(p-1)}(\mu(A)) \frac{W_p(\mu(A))}{\mu(A)}.
\]

Consequently,

\[
\lim_{n \to \infty} \int_\mu(A) \left( (f \pm n)\chi_A \right)^{(p-1)}(t) \frac{w(t)}{t} \, dt = \infty,
\]

from which we conclude

\[
\lim_{n \to \infty} \int_\mu(A) K_{(f,A)}(n, t) \left( (f - n)\chi_A \right)^{(p-1)}(t) w(t) \, dt = -\infty,
\]

and

\[
\lim_{n \to \infty} \int_\mu(A) K_{(f,A)}(-n, t) \left( (f + n)\chi_A \right)^{(p-1)}(t) w(t) \, dt = \infty.
\]

Thus, by Proposition 2.7 we complete the proof. \( \square \)

**Theorem 2.9.** Let \( 1 \leq p < \infty \). Assume \( f \in \Gamma_{p-1, w} \) if \( p > 1 \) and \( f \in L^0 \) if \( p = 1 \); then there are constants \( \underline{f}_{(p,A)} \), \( \overline{f}_{(p,A)} \) \( \in \mathbb{R} \) such that

\[
\underline{f}_{(p,A)} = \min \left\{ u : S_{(f,A)}^p(-u) \geq 0 \right\}, \quad \overline{f}_{(p,A)} = \max \left\{ u : S_{(f,A)}^p(u) \geq 0 \right\}.
\]

In addition \(-\overline{f}_{(p,A)} = -f_{(p,A)} \).

**Proof.** By Propositions 2.4 and 2.8 we get that \( S_{(f,A)}^p \) is decreasing and left-continuous for any \( 1 \leq p < \infty \), as well as for \( p = 1 \),

\[
\lim_{u \to -\infty} S_{(f,A)}^1(u) = -\|\chi_A\| \Gamma_{1, w}, \quad \lim_{u \to -\infty} S_{(f,A)}^1(u) = \|\chi_A\| \Gamma_{1, w},
\]

and for \( p > 1 \),

\[
\lim_{u \to -\infty} S_{(f,A)}^p(u) = -\infty, \quad \lim_{u \to -\infty} S_{(f,A)}^p(u) = \infty.
\]

Hence there exist constants given by condition (14). Clearly

\[
-\overline{f}_{(p,A)} = \max\{u : S_{(f,A)}^p(u) \geq 0\} = -\min\{v : S_{(f,A)}^{p-1}(-v) \geq 0\} = -\underline{f}_{(p,A)}
\]

for \( p \geq 1 \). \( \square \)

**Definition 2.10.** Let \( T_{(p,A)} \) be a best constant approximant operator given by Definition 1.7. Theorem 2.9 allows us to extend the operator \( T_{(1,A)} \) from \( \Gamma_{1, w} \) to \( L^0 \) and, in the case where \( 1 < p < \infty \), \( T_{(p,A)} \) from \( \Gamma_{p, w} \) to \( \Gamma_{p-1, w} \) through

\[
T_{(p,A)}(f) = [\underline{f}_{(p,A)}\chi_A, \overline{f}_{(p,A)}\chi_A]
\]

for all \( f \in L^0 \) if \( p = 1 \), and \( f \in \Gamma_{p-1, w} \) if \( p > 1 \).
Corollary 2.11. Let \( f \in \Gamma_{p-1,w} \) if \( p > 1 \), and \( f \in L^0 \) if \( p = 1 \). Then the following conditions are equivalent:

(i) \( u \in T_{(p,A)}^p(f) = [f_{(p,A)} X_A, \overline{F}_{(p,A)} X_A] \),

(ii) \( S_{(f,A)}^p(u) \geq 0 \) and \( S_{(-f,A)}^p(-u) \geq 0 \),

(iii) \[
\int_0^\alpha \frac{1}{t} \int_{A \cap \{f < u\}} \chi_{0,t} \left( \rho\left( (f-u) X_A, X_A \right) \right) ((f-u) X_A)^{(p-1)}(t) w(t) \, dt \\
\leq \int_0^\alpha \frac{1}{t} \int_{A \cap \{f \geq u\}} \chi_{0,t} \left( \rho\left( (f-u) X_A, X_A \right) \right) ((f-u) X_A)^{(p-1)}(t) w(t) \, dt,
\]

and

(iv) \[
\int_0^\alpha \frac{1}{t} \int_{A \cap \{f > u\}} \chi_{0,t} \left( \rho\left( (f-u) X_A, X_A \right) \right) ((f-u) X_A)^{(p-1)}(t) w(t) \, dt \\
\leq \int_0^\alpha \frac{1}{t} \int_{A \cap \{f \leq u\}} \chi_{0,t} \left( \rho\left( (f-u) X_A, X_A \right) \right) ((f-u) X_A)^{(p-1)}(t) w(t) \, dt,
\]

Proof. (i) \( \iff \) (ii). In fact, the equivalence of the given conditions is a consequence of Proposition 2.4, Theorem 2.9 and Definition 2.10.

(ii) \( \iff \) (iii) \( \iff \) (iv). The proof follows immediately from the definition of \( S_{(f,A)}^p(u) \) and by the fact that

\[
(\chi_A)^{(p)}(t) = \frac{1}{t} \int_{A \cap \{\pm f \geq u\}} \chi_{0,t} \left( \rho\left( \pm (f-u) X_A, X_A \right) \right) + \frac{1}{t} \int_{A \cap \{\pm f < u\}} \chi_{0,t} \left( \rho\left( \pm (f-u) X_A, X_A \right) \right)
\]

for any \( t \in (0, \alpha) \). \( \square \)

3. Properties of the extended best constant approximant operators in \( \Gamma_{p,w} \)

Landers and Rogge in [9] defined a new concept of monotonicity for a set valued operator \( M \) defined on \( L^0 \) with \( M(f) \subset L^0 \) for \( f \in L^0 \). Let \( f, g \in L^0 \) and \( u \in M(f), v \in M(g) \). Then the operator \( M \) is monotone if \( f \leq g \) implies \( \min\{u, v\} \in M(f) \) and \( \max\{u, v\} \in M(g) \). Using the extended best constant approximant operator \( T_{(p,A)} \) instead of \( M \), the new monotone property can be written in the following way.
Let $f, g \in \Gamma_{p-1,w}$ if $p > 1$, and $f, g \in L^0$ if $p = 1$. We say that the extended best constant approximant operator $T_{(p,A)}$ is monotone if

$$f \leq g \Rightarrow f_{(p,A)} \leq g_{(p,A)} \quad \text{and} \quad \overline{f}_{(p,A)} \leq \overline{g}_{(p,A)}.$$ 

In the next theorem we list basic properties of the extended best constant approximant operators.

**Theorem 3.2.** Let $1 \leq p < \infty$. Assume that $f \in \Gamma_{p-1,w}$ if $p > 1$, and $f \in L^0$ if $p = 1$. Then the extended best constant approximant operator $T_{(p,A)}$ satisfies the following conditions:

(i) If $g \in L^0$ is constant on $A$, then $T_{(p,A)}(f + g) = T_{(p,A)}(f) + g\chi_A$.

(ii) For any $a \in \mathbb{R}$ we get $T_{(p,A)}(af) = aT_{(p,A)}(f)$.

**Proof.** Define $h = f + g$. Then

$$S_{p(h,A)}(u) = S_{p(f+g,A)}(u) = S_{p(f,A)}(u - g) \quad \text{and} \quad S_{p(h,-A)}(-u) = S_{p(-f,A)}(-u + g).$$

Hence, by Corollary 2.11 we obtain that $u\chi_A \in T_{(p,A)}(h)$ if and only if $(u - g)\chi_A \in T_{(p,A)}(f)$, which is (i). Clearly, the condition (ii) holds for $a = 0$. If $a > 0$, then $\rho[(af - au)\chi_A,\chi_A](s) = \rho((f - u)\chi_A,\chi_A)(s)$ and $\rho((af + au)\chi_A,\chi_A)(s) = \rho((f + u)\chi_A,\chi_A)(s)$ for any $s \in A$ and $u \in \mathbb{R}$. Thus,

$$S_{p(af,A)}(au) = a^{p-1}S_{p(f,A)}(u) \quad \text{and} \quad S_{p(-af,A)}(-au) = a^{p-1}S_{p(-f,A)}(-u).$$

In the case where $a < 0$ we get $\rho[(af - au)\chi_A,\chi_A](s) = \rho((-f + u)\chi_A,\chi_A)(s)$ and $\rho((-af + au)\chi_A,\chi_A)(s) = \rho((-f - u)\chi_A,\chi_A)(s)$ for any $s \in A$ and $u \in \mathbb{R}$. Hence,

$$S_{p(af,A)}(au) = |a|^{p-1}S_{p(-f,A)}(-u) \quad \text{and} \quad S_{p(-af,A)}(-au) = |a|^{p-1}S_{p(f,A)}(u).$$

Therefore, by Corollary 2.11 we have that $au \in T_{(p,A)}(af)$ if and only if $u \in T_{(p,A)}(f)$, which implies condition (ii) for $a \neq 0$. 

Now we will show that the extended best constant approximant operator $T_{(1,A)}$ is monotone in the space $L^0$. In order to attain this goal we need to show several technical results. Monotonicity of the operator is the key property in the proof of the LDT shown in the next section. Recall now the standard simple functions employed often to approximate measurable functions.

**Definition 3.3.** Let $f \in L^0$. Define

$$\tilde{f}_n(s) = \begin{cases} \text{sign}(f(s)) \frac{k}{2^n} & \text{if} \quad \frac{k}{2^n} < |f(s)| \leq \frac{k + 1}{2^n}, k \in \mathbb{N} \cap [1, n2^{n-1}], \\ \text{sign}(f(s))n & \text{if} \quad |f(s)| > n, \\ 0 & \text{if} \quad |f(s)| \leq \frac{1}{2^n}, \end{cases}$$

for any $n \in \mathbb{N}$ and $s \in [0, \alpha)$.

**Remark 3.4.** Let $f, g \in L^0$ and $f \leq g$ a.e. It is easy to see that $|\tilde{f}_n| \leq |f|$ and $\tilde{f}_n \leq \tilde{g}_n$ a.e. for all $n \in \mathbb{N}$. Furthermore, by Lemma 3.6 in [13] we have

$$\lim_{n \to \infty} \rho_{(f\chi_A,\chi_A)}(s) = \rho_{(f\chi_A,\chi_A)}(s)$$

for any $s \in A$. 

The next definition is exclusively technical and serves to prove the monotone property of the extended best constant approximant operator for simple functions.

**Definition 3.5.** Let \( h \) be a simple function given by \( h = \sum_{k=1}^{n} h_k \chi_{A_k} \) such that \( A_i \cap A_j = \emptyset \) for any \( i \neq j \). Define
\[
\delta_h = \min\{|h_i| - |h_j| : h_i \neq h_j\}, \quad \tilde{\delta}_h = \min\{|h_i| - |h_j| : |h_i| \neq |h_j|\},
\]
and
\[
\gamma_h = \min\{|h_i| : |h_i| > 0\}, \quad \beta_h = \begin{cases} \min\{\delta_h, \gamma_h\} & \text{if } \delta_h > 0, \\ \frac{\gamma_h}{4} & \text{if } \delta_h = 0, \end{cases} \quad \tilde{\beta}_h = \frac{\min\{\tilde{\delta}_h, \gamma_h\}}{4}.
\]

Notice that if \( \delta_h > 0 \), then \( \beta_h = \tilde{\beta}_h \). In the case where \( \delta_h = 0 \) we have \( \tilde{\beta}_h \leq \beta_h \). Let us now recall a result which was established by Levis and Cuenya in [11].

**Lemma 3.6.** Let \( f \) be a simple function, \( 0 < \epsilon < \beta_f \) and let \( h \) be a measurable function such that \( 0 \leq h < \epsilon \). Then there exists a measure preserving transformation \( \sigma : [0, 1] \to [0, 1] \) such that
\[
(f \chi_{[0,1]} + sh \chi_{[0,1]} + t \chi_{[0,1]})^* \circ \sigma = |f + sh + t| \chi_{[0,1]}
\]
for all \( s \in \{0, 1\} \) and \( t \in [0, \epsilon] \) a.e. on \( [0, 1] \).

The following example shows that Lemma 3.6 is not valid for \( s \in \{0, -1\} \). It explains why the proof of the next Lemma 3.8 consists of two different parts, of which the first one relies on Lemma 3.6, while the second one is totally different.

**Example 3.7.** Let \( f = \chi_{[0, \frac{1}{2}]} - \chi_{[\frac{1}{2}, 1]} \). Then we have that \( f = f \chi_{[0,1]} \) and \( \beta_f = \tilde{\beta}_f = \frac{1}{4} \). Let \( 0 < \epsilon < \frac{1}{4} \) and \( h = \frac{\epsilon}{2} \chi_{[0, \frac{1}{2}]} \). It follows that there is no measure preserving transformation \( \sigma : [0, 1] \to [0, 1] \) such that for any \( t \in [0, \epsilon] \) and for \( s \in \{0, -1\} \),
\[
(f - h + t \chi_{[0,1]})^* \circ \sigma = |f - h + t| \chi_{[0,1]}
\]
a.e. on \( [0, 1] \). Indeed if \( t = 0 \), then
\[
(f - h \chi_{[0,1]})^* \circ \sigma = |f - h| \chi_{[0,1]} \quad \text{and} \quad (f \chi_{[0,1]})^* \circ \sigma = |f| \chi_{[0,1]}
\]
a.e. on \( [0, 1] \), which implies that \( \sigma([0, \frac{1}{2}]) = [\frac{1}{2}, 1] \) and \( \sigma([\frac{1}{2}, 1]) = [0, \frac{1}{2}] \) a.e. On the other hand in the case where \( t = \epsilon \) we have
\[
(f - h + \epsilon \chi_{[0,1]})^* \circ \sigma = |f - h + \epsilon| \chi_{[0,1]} \quad \text{and} \quad (f + \epsilon \chi_{[0,1]})^* \circ \sigma = |f + \epsilon| \chi_{[0,1]}
\]
a.e. \( [0, 1] \). Hence, we conclude that \( \sigma([0, \frac{1}{2}]) = [0, \frac{1}{2}] \) and \( \sigma([\frac{1}{2}, 1]) = [\frac{1}{2}, 1] \) a.e. Therefore, we get a contradiction.

**Lemma 3.8.** Let \( f = \sum_{k=1}^{n} a_k \chi_{A_k} \) be a simple function with \( A_i \cap A_k = \emptyset \) for \( i \neq k \) and let \( \epsilon > 0 \), \( a \geq 0 \) and \( g = \chi_{A_j} \) for some \( 1 \leq j \leq n \). If \( 0 < |a - \epsilon| < \tilde{\beta}_f(f + eg) \chi_A \), then
\[
\text{sign}(a - \epsilon) S_{(f + eg, A)}^1(0) \leq \text{sign}(a - \epsilon) S_{(f + ag, A)}^1(0).
\]

**Proof.** Define for any \( u \in \mathbb{R}, \)
\[
f_u = (f + ug) \chi_A.
\]
Without loss of generality we assume that \( j = 1 \) and \( A_k = A_k \cap A \) for all \( 1 \leq k \leq n \).
Suppose first that $0 < \epsilon - a < \tilde{\beta}_f$. By Lemma 3.6 there is a measure preserving transformation $\sigma : A \rightarrow [0, \mu(A)]$ such that for any $s \in [0, 1]$ and $t \in [0, \epsilon - a]$ we have

$$(f(1-s)a + se + t\chi_A)^* \circ \sigma = |f(1-s)a + se + t\chi_A|$$

a.e. on $A$. By convexity of the absolute value function we get $|f_\epsilon| - |f_a| \leq |f_\epsilon + t\chi_A| - |f_a + t\chi_A|$ a.e. on $A$ and for all $t \in [0, \epsilon - a]$. Hence

$$\int_{A} |f_\epsilon| (\chi_{(0,v)}(\sigma)) - \int_{A} |f_a| (\chi_{(0,v)}(\sigma)) \leq \int_{A} |f_\epsilon + t\chi_A| (\chi_{(0,v)}(\sigma)) - \int_{A} |f_a + t\chi_A| (\chi_{(0,v)}(\sigma))$$

for any $v \in (0, \alpha)$. By Lemma 2.2 [4] we conclude that

$$\int_{0}^{v} (f_\epsilon)^* - \int_{0}^{v} (f_a)^* \leq \int_{0}^{v} (f_\epsilon + t\chi_A)^* - \int_{0}^{v} (f_a + t\chi_A)^*$$

for every $v \in (0, \alpha)$ and consequently,

$$\|f_\epsilon\| - \|f_a\| = \int_{0}^{\alpha} (f_\epsilon)^* w - \int_{0}^{\alpha} (f_a)^* w$$

$$\leq \int_{0}^{\alpha} (f_\epsilon + t\chi_A)^* w - \int_{0}^{\alpha} (f_a + t\chi_A)^* w = \|f_\epsilon + t\chi_A\| - \|f_a + t\chi_A\|$$

for any $t \in [0, \epsilon - a]$. Therefore,

$$\frac{\|f_a + t\chi_A\| - \|f_a\|}{t} \leq \frac{\|f_\epsilon + t\chi_A\| - \|f_\epsilon\|}{t}$$

for any $t \in (0, \epsilon - a)$. Thus, by Theorem 4.3 [4] we get

$$S_{(f+\epsilon A)}^{(s)}(0) = \lim_{t \downarrow 0} \frac{\|f_a + t\chi_A\| - \|f_a\|}{t} \leq \lim_{t \downarrow 0} \frac{\|f_\epsilon + t\chi_A\| - \|f_\epsilon\|}{t} = S_{(f+\epsilon A)}^{(s)}(0).$$

Now suppose that $0 < a - \epsilon < \tilde{\beta}_f$. We will consider two cases. Define

$$R_a = \{|a_1 + a|, |a_2|, \ldots, |a_n|\} \quad \text{and} \quad E_a = R_a \setminus [0, |a_1 + a|].$$

Case 1. Let $a_1 + a \neq 0$. If $\lambda \in R_a$, by the assumption that $0 < a - \epsilon < \tilde{\beta}_f$ we have

$$|a_1 + \epsilon| \neq \lambda \quad \text{and} \quad \text{sign}(a_1 + a) = \text{sign}(a_1 + \epsilon).$$

(15)

Notice that if $\lambda \in E_a$, then

$$\lambda < \min\{|a_1 + a|, |a_1 + \epsilon|\} \quad \text{or} \quad \max\{|a_1 + a|, |a_1 + \epsilon|\} < \lambda.$$

Therefore,

$$\mu(v : f_a(v) = \pm \lambda) = \mu(v : f_\epsilon(v) = \pm \lambda)$$

for any $\lambda \in E_a$ and also

$$d_{f_a}(\lambda) = d_{f_\epsilon}(\lambda)$$

(17)

for every $\lambda \in E_a \cup \{0\}$. Observe that in the case where $|a_k| \neq |a_1 + a|$ for all $2 \leq k$ we have

$$\mu(v : |f_\epsilon(v)| = |a_1 + a|) = 0.$$
Now assume that $a_1 + a < 0$. Then we get that $|a_1 + a| < |a_1 + \varepsilon|$ and

$$ s := \mu(v : f_a(v) = |a_1 + a|) = \mu(v : f_\varepsilon(v) = |a_1 + a|), $$

$$ r := d_{f_a}(|a_1 + a|) = d_{f_\varepsilon}(|a_1 + a|) - \mu(A_1) = d_{f_a}(|a_1 + \varepsilon|) = d_{f_\varepsilon}(|a_1 + \varepsilon|), $$

$$ t := \mu(v : f_a(v) = a_1 + a) = \mu(v : f_\varepsilon(v) = a_1 + a) + \mu(A_1). $$

Moreover, by condition (15) we obtain

$$ \mu(v : f_a(v) = |a_1 + \varepsilon|) = \mu(v : f_\varepsilon(v) = |a_1 + \varepsilon|) = 0, $$

and also

$$ \mu(v : f_a(v) = a_1 + \varepsilon) = \mu(v : f_\varepsilon(v) = a_1 + \varepsilon) - \mu(A_1) = 0. $$

Now according to conditions (16), (17) and (18) we have

$$ S_{(f_\varepsilon, A)}(0) = \int_0^\alpha \left( \int_{A \cap \{ f_\varepsilon \neq 0 \}} \text{sign}(f_\varepsilon) \chi(0, v) \circ \rho(f_\varepsilon, \chi_A) \right) \frac{w(v)}{v} dv $$

$$ + \int_{A \cap \{ f_\varepsilon = 0 \}} \chi(0, v) \circ \rho(f_\varepsilon, \chi_A) \frac{w(v)}{v} dv $$

$$ = \int_0^\alpha \left( \sum_{\lambda \in \mathcal{E}_a} \int d_{f_\varepsilon}(\lambda) + \mu(f_\varepsilon = \lambda) \chi(0, v) \right) \frac{w(v)}{v} dv $$

$$ - \int d_{f_\varepsilon}(\lambda) + \mu(f_\varepsilon = \lambda) \chi(0, v) + \int \mu(A) \chi(0, v) \frac{w(v)}{v} dv $$

$$ + \int_0^\alpha \left( \int_{r + \mu(A_1) + s} \chi(0, v) - \int_{r + \mu(A_1) + s} \chi(0, v) \right) \frac{w(v)}{v} dv, $$

and

$$ S_{(f_a, A)}(0) = \int_0^\alpha \left( \sum_{\lambda \in \mathcal{E}_a} \int d_{f_a}(\lambda) + \mu(f_a = \lambda) \chi(0, v) \right) \frac{w(v)}{v} dv $$

$$ - \int d_{f_a}(\lambda) + \mu(f_a = \lambda) \chi(0, v) + \int \mu(A) \chi(0, v) \frac{w(v)}{v} dv $$

$$ + \int_0^\alpha \left( \int_{r} \chi(0, v) - \int_{r + s} \chi(0, v) \right) \frac{w(v)}{v} dv. $$

Consequently,

$$ S_{(f_a, A)}(0) - S_{(f_\varepsilon, A)}(0) = \int_0^\alpha \left( \int_{r} \chi(0, v) - \int_{r + \mu(A_1) + s} \chi(0, v) \right) \frac{w(v)}{v} dv $$

$$ + \int_0^\alpha \left( \int_{r + \mu(A_1)} \chi(0, v) - \int_{r + s + \mu(A_1)} \chi(0, v) \right) \frac{w(v)}{v} dv \geq 0. $$

Now suppose that $a_1 + a > 0$. Then

$$ s := \mu(v : f_a(v) = a_1 + a) = \mu(v : f_\varepsilon(v) = a_1 + a) + \mu(A_1), $$

$$ r := d_{f_a}(a_1 + a) = d_{f_\varepsilon}(a_1 + a), $$
Let $f$ be a constant function and $a$. Clearly, we can consider $\mu(v : f_a(v) = -(a_1 + a)) = \mu(v : f_\epsilon(v) = -(a_1 + a))$.

Since $0 < a_1 + \epsilon < a_1 + a$ and $0 < a - \epsilon < \bar{\beta}_{f_a}$, we conclude that

$$d_{f_a}(a_1 + \epsilon) = d_{f_\epsilon}(a_1 + \epsilon) + \mu(A_1) = d_{f_a}(a_1 + a) + \mu(v : |f_a(v)| = a_1 + a) = r + s + t,$$

and

$$\mu(v : f_a(v) = a_1 + \epsilon) = \mu(v : f_\epsilon(v) = a_1 + \epsilon) - \mu(A_1) = 0,$$

as well as

$$\mu(v : f_a(v) = -(a_1 + \epsilon)) = \mu(v : f_\epsilon(v) = -(a_1 + \epsilon)) = 0.$$

Therefore, by conditions (16), (17) and (18) we get

$$S^1_{(f_a, A)}(0) - S^1_{(f_\epsilon, A)}(0) = \int_0^\alpha \left( \int_{r+s-\mu(A_1)}^{r+s+t} \chi(0,v) - \int_{r+s+t-\mu(A_1)}^{r+s+t} \chi(0,v) \right) \frac{w(v)}{v} dv + \int_0^\alpha \left( \int_{r+s-\mu(A_1)}^{r+s+t} \chi(0,v) - \int_{r+s}^{r+s+t} \chi(0,v) \right) \frac{w(v)}{v} dv \geq 0.$$

Case 2. Let $a_1 + a = 0$. Since $0 < a - \epsilon < \bar{\beta}_{f_a}$, we have that $a_1 + \epsilon < 0$. Notice that if $\lambda \in E_a$, then $\lambda > |a_1 + \epsilon|$. Consequently, for all $\lambda \in E_a$ we have

$$d_{f_a}(\lambda) = d_{f_\epsilon}(\lambda) \quad \text{and} \quad \mu(v : f_a(v) = \pm \lambda) = \mu(v : f_\epsilon(v) = \pm \lambda),$$

and

$$d_{f_\epsilon}(0) = d_{f_a}(0) + \mu(A_1) \quad \text{and} \quad \mu(v : f_\epsilon(v) = a_1 + \epsilon) = \mu(A_1).$$

Define $u := d_{f_\epsilon}([a_1 + \epsilon])$. Then, by conditions (19) and (20) we obtain

$$S^1_{(f_a, A)}(0) - S^1_{(f_\epsilon, A)}(0) = \int_0^\alpha \left( \int_{d_{f_a}(0)}^{d_{f_a}(0) + \mu(A_1)} \chi(0,v) + \int_u^\alpha \chi(0,v) \right) \frac{w(v)}{v} dv \geq 0,$$

which finishes the proof. \hfill \Box

**Lemma 3.9.** Let $f$ be a constant function and $a \in \mathbb{R}$ and let $B \subset A$, $h = \chi_B$. Then

$$\text{sign}(a) K_{(f, A)}(0, t) \leq \text{sign}(a) K_{(f+ah, A)}(0, t)$$

for all $t \in (0, \alpha)$ and consequently,

$$\text{sign}(a) S^1_{(f, A)}(0) \leq \text{sign}(a) S^1_{(f+ah, A)}(0).$$

**Proof.** Clearly, we can consider $a \neq 0$. Notice that the second inequality is an immediate consequence of (21). Let $a > 0$. If $f \geq 0$, then we have

$$1 - 2\chi_{[f < 0]}(s) = 1 - 2\chi_{[f+ah < 0]}(s) = 1$$

for a.a. $s \in A$. Hence, by the fact that $\rho_{(f, A), X_A}$ and $\rho_{((f+ah)X_A, X_A)}$ are measure preserving transformations from $A$ onto $[0, \mu(A)]$ we obtain

$$K_{(f, A)}(0, t) = \frac{1}{t} \int_{[0, t]} \chi_{\rho_{(f, A), X_A}^{-1}[(0,t)]} = \frac{1}{t} \mu \left( \rho_{(f, A), X_A}^{-1}[(0, t)] \right) = (\chi_A)^*(t) = K_{(f+ah, A)}(0, t).$$
for any $t \in (0, \alpha)$, which implies (21). Now consider $f < 0$. If $a < |f|$, then for a.a. $s \in A$

$$1 - 2\chi_{\{f < 0\}}(s) = 1 - 2\chi_{\{f + ah < 0\}}(s) = -1,$$

and similarly to before

$$K_{(f, A)}(0, t) = -\frac{1}{t} \int_A \chi(0, t)(\rho_{(f \chi_A, \chi_A)})$$

$$= -\frac{1}{t} \int_A \chi(0, t)(\rho_{((f + ah)\chi_A, \chi_A)}) = K_{(f + ah, A)}(0, t)$$

for any $t \in (0, \alpha)$. In the case where $a \geq |f|$ we have

$$1 - 2\chi_{\{f < 0\}}(s) = -1 \leq \chi_B(s) - \chi_{A \setminus B}(s) = 1 - 2\chi_{\{f + ah < 0\}}(s)$$

for a.a. $s \in A$, which allows us to conclude, for any $t \in (0, \alpha)$,

$$K_{(f + ah, A)}(0, t) = \frac{1}{t} \int_B \chi(0, t)(\rho_{((f + ah)\chi_A, \chi_A)}) - \frac{1}{t} \int_{A \setminus B} \chi(0, t)(\rho_{((f + ah)\chi_A, \chi_A)})$$

$$\geq -\frac{1}{t} \int_A \chi(0, t)(\rho_{(f\chi_A, \chi_A)})$$

$$= -\frac{1}{t} \int_A \chi(0, t)(\rho_{f\chi_A, \chi_A}) = K_{(f, A)}(0, t).$$

We proceed analogously in the remaining cases. □

**Proposition 3.10.** Let $f, g$ be simple functions such that $f \leq g$ a.e. on $A$. Then

$$S_{(f, A)}^1(0) \leq S_{(g, A)}^1(0).$$

**Proof.** Let $f = \sum_{k=1}^n a_k \chi_{E_k}$ and $g = \sum_{k=1}^n b_k \chi_{E_k}$, where $E_i \cap E_j = \emptyset$ for $i \neq j$ and $E_k \subset A$ for any $1 \leq k \leq n$. Since $f \leq g$ a.e. on $A$, we get $a_k \leq b_k$ for any $1 \leq k \leq n$. Define

$$f_0 = f, \quad f_m = \sum_{k=1}^m b_k \chi_{E_k} + \sum_{k=m+1}^n a_k \chi_{E_k} \quad \text{and} \quad h_m = \chi_{E_m}$$

for any $1 \leq m \leq n$. We can easily see that

$$f_n = g \quad \text{and} \quad f_{m+1} = f_m + (b_{m+1} - a_{m+1})h_{m+1}$$

for any $0 \leq m \leq n - 1$. Now we will prove that for all $0 \leq m \leq n - 1$,

$$S_{(f_m, A)}^1(0) \leq S_{(f_{m+1}, A)}^1(0). \quad (22)$$

In the case where $b_{m+1} = a_{m+1}$ for some $0 \leq m \leq n - 1$ we obtain condition (22). Assume that $a = b_{m+1} - a_{m+1} > 0$ for some $0 \leq m \leq n - 1$. Define

$$C = \{ u \in [0, \alpha] : S_{(f_m, A)}^1(0) \leq S_{(f_m + uh_{m+1}, A)}^1(0) \}.$$ 

We finish the proof if we show that $a \in C$. If $f_m$ is a constant function on $A$, then by Lemma 3.9 we have that $a \in C$. Otherwise if $f_m$ is not constant, then by Lemma 3.8 we get

$$S_{(f_m, A)}^1(0) \leq S_{(f_m + \epsilon h_{m+1}, A)}^1(0).$$
for all $0 < \epsilon < \tilde{\rho}_{f,\mu}$ and consequently there exists $c = \sup\{C\} > 0$. Assume that $f_m + ch_{m+1}$ is constant on $A$. Since $a - c \geq 0$, by Lemma 3.9 we get

$$S^1(f_m,A)(0) \leq S^1(f_m+ch_{m+1},A)(0) \leq S^1(f_m+ah_{m+1},A)(0),$$

which concludes the claim. In the case where $f_m + ch_{m+1}$ is not constant, pick out $t \in C$ such that $t > 0$ and $0 < c - t < \tilde{\rho}(f_m+ch_{m+1})_A$. Thus, by Lemma 3.8 we have

$$S^1(f_m,A)(0) \leq S^1(f_m+ch_{m+1},A)(0) \leq S^1(f_m+ah_{m+1},A)(0).$$

Therefore $c \in C$. Now suppose that $c < a$. Then we can choose $0 < r < a$ such that $0 < r - c < \tilde{\rho}(f_m+ch_{m+1})_A$. Applying Lemma 3.8 we obtain

$$S^1(f_m,A)(0) \leq S^1(f_m+ch_{m+1},A)(0) \leq S^1(f_m+rh_{m+1},A)(0),$$

whence $r \in C$ and by assumption $r > c$ we get a contradiction. So $a = c$ and $a \in C$. $\square$

**Proposition 3.11.** Let $f \in \Gamma_{p-1,w}$ if $1 < p < \infty$, and $f \in L^0$ if $p = 1$. Then

$$\lim_{n \to \infty} S^p_{(f_n,A)}(0) = S^p_{(f,A)}(0).$$

**Proof.** Let $f \in L^0$. First we show that for $t \in (0,\alpha)$,

$$\lim_{n \to \infty} K_{(f_n,A)}(0,t) = K_{(f,A)}(0,t).$$

It is easy to check that if $f(s) < 0$, then

$$1 - 2\chi_{(f_n < 0)}(s) = 1 - 2\chi_{(f < 0)}(s) = -1,$$

and if $f(s) \geq 0$, then

$$1 - 2\chi_{(f_n < 0)}(s) = 1 - 2\chi_{(f < 0)}(s) = 1$$

for sufficiently large $n \in \mathbb{N}$. Hence for a.a. $s \in A$,

$$\lim_{n \to \infty} 1 - 2\chi_{(f_n < 0)}(s) = 1 - 2\chi_{(f < 0)}(s).$$

Since $\rho(f_{A,A}) : A \to [0,\mu(A)]$ is a measure preserving transformation, we get

$$\mu(A \setminus \rho^{-1}_{(f,A,A)}[[t,0]]) = \mu(A)$$

for any $t \in (0,\alpha)$. Consequently, by Remark 3.4 we obtain

$$\lim_{n \to \infty} \chi(0,t) \left( \rho_{(f_{A,A})}(s) \right) = \chi(0,t) \left( \rho_{(f_{A,A})}(s) \right)$$

for a.a. $s \in A$ and for any $t \in (0,\alpha)$. Hence, by condition (23) we have

$$\lim_{n \to \infty} \left( 1 - 2\chi_{(f_n < 0)}(s) \right) \chi(0,t) \left( \rho_{(f_{A,A})}(s) \right) \chi(0,t) \left( \rho_{(f_{A,A})}(s) \right)$$

for a.a. $s \in A$, and for all $t \in (0,\alpha)$. Therefore, $\lim_{n \to \infty} K_{(f_n,A)}(0,t) = K_{(f,A)}(0,t)$ for all $t \in (0,\alpha)$. Hence, by condition (3) we get

$$\lim_{n \to \infty} S^1_{(f_n,A)}(0) = S^1_{(f,A)}(0).$$

Let now $f \in \Gamma_{p-1,w}$. Since $|f_n(s)| \uparrow |f(s)|$ for a.a. $s \in A$, we get $(\tilde{f}_nA)^{**}(t) \uparrow (fA)^{**}(t)$ for any $t \in (0,\alpha)$. Thus for any $t \in (0,\alpha)$,

$$\lim_{n \to \infty} K_{(f_n,A)}(0,t) \left( \tilde{f}_nA \right)^{**}(t) = K_{(f,A)}(0,t) \left( fA \right)^{**}(t).$$
By condition (3) we have
\[ |K(\bar{f}_n, A)(0, t)| (\bar{f}_n \chi_A)^{**} \leq (f \chi_A)^{**}, \]
which implies that \( \lim_{n \to \infty} S^{p(\bar{f}_n, A)}(0) = S^{p(f, A)}(0). \)

**Proposition 3.12.** Let \( f, g \in L^0, f \leq g \) a.e. on \( A. \) Then for any \( u \in \mathbb{R} \) we have
\[ S^{1}_{(f, A)}(u) \leq S^{1}_{(g, A)}(u). \]

**Proof.** Define \( h = f - u \) and \( k = g - u. \) By Remark 3.4 we get that \( \bar{h}_n \leq \bar{k}_n \) for all \( n \in \mathbb{N}, \) a.e. on \( A \) and consequently, by Proposition 3.10 we conclude that
\[ S^{1}_{(\bar{h}_n, A)}(0) \leq S^{1}_{(\bar{k}_n, A)}(0) \]
for every \( n \in \mathbb{N}. \) Finally, by Proposition 3.11 we obtain
\[ S^{1}_{(h, A)}(0) = \lim_{n \to \infty} S^{1}_{(\bar{h}_n, A)}(0) \leq \lim_{n \to \infty} S^{1}_{(\bar{k}_n, A)}(0) = S^{1}_{(k, A)}(0), \]
which implies that \( S^{1}_{(f, A)}(u) \leq S^{1}_{(g, A)}(u). \)

**Theorem 3.13.** The extended best constant approximant operator \( T_{(1, A)} \) is monotone on \( L^0. \)

**Proof.** Let \( f, g \in L^0 \) and \( f \leq g \) a.e. on \( A. \) According to Proposition 3.12 we get that
\[ 0 \leq S^{1}_{(f, A)}(\bar{f}_A) \leq S^{1}_{(g, A)}(\bar{f}_A). \]
Hence, by definition of \( \bar{g}_A \) we have \( \bar{f}_A \leq \bar{g}_A. \) It is easy to see that \( (\bar{g}_A) \leq (\bar{f}_A). \) Now applying Theorem 2.9 we obtain
\[ f_A = (\bar{f}_A) \leq (\bar{g}_A) = g_A, \]
which completes the proof.

**4. Convergence of extended best constant approximants on \( L^0 \)**

In this section, we establish the convergence of extended best constant approximants on \( L^0, \) a generalized version of LDT.

**Lemma 4.1.** For any nonnegative function \( f \in L^0 \) we get \( f^{(1, A)} \geq 0. \)

**Proof.** Let \( u < 0. \) Then \( f - u \geq -u > 0. \) Since \( \rho((f+u)\chi_A, \chi_A) \) is a measure preserving transformation, we obtain
\[ \frac{1}{t} \int f A \chi_{(0,t)}(\rho(u-f)\chi_A, \chi_A) = \frac{1}{t} \int A \chi_{\rho(u-f)\chi_A}[0, t)] = (\chi_A)^{**}(t) \]
for any \( t \in (0, \alpha). \) Consequently,
\[ S^{1}_{(-f, A)}(-u) = \int_0^\alpha \frac{1}{t} \int A \chi_{(0,t)}(\rho(u-f)\chi_A, \chi_A) w(t)dt \]
\[ - \int_0^\alpha \frac{1}{t} \int A \chi_{(0,t)}(\rho(u-f)\chi_A, \chi_A) w(t)dt \]
\[ = \int_0^\alpha (\chi_A)^{**}(t) w(t)dt. \]
Since $(\chi_A)^{**}(t) > 0$ for any $t \in (0, \alpha)$, by the fact that $w$ is nontrivial, i.e. there is $B \subset (0, \alpha)$ of positive measure such that $w(t) > 0$ for any $t \in B$, we obtain $S^1_{(-f, A)}(-u) < 0$. By Proposition 2.4 the function $S^1_{(-f, A)}(v)$ is left-continuous with respect to $v$ and by definition of $f_{(-1, A)}$ we get that $S^1_{(-f, A)}(-f_{(-1, A)}) \geq 0$. Now applying monotonicity of $S^1_{(-f, A)}(v)$ for $v$ we have $f_{(-1, A)} > u$. Finally, for arbitrarily chosen $u < 0$ we obtain that $f_{(-1, A)} \geq 0$. □

**Definition 4.2.** Let $f \in L^0$ and let $T_{(1, B(v, \epsilon))}$ be an extended best constant approximant operator on $L^0$. Let the maximal function $M_L f : (0, \alpha) \to \overline{R}$ be defined as

$$M_L f(v) = \sup \{|m| : m(\chi_{B(v, \epsilon)}(f)) \epsilon > 0, B(v, \epsilon) \subset (0, \alpha)|.$$ 

**Theorem 4.3.** Let $f \in L^0$ and let $\phi$ be the fundamental function of $\Gamma_{1, w}$. Then there exists $C > 0$ such that

$$\phi(\mu_s(M_L f > s)) \leq C \phi(d_f(s))$$

for any $s > 0$.

**Proof.** Let $f \in L^0$. Define $Hf : (0, \alpha) \to \overline{R}$ as a new maximal function by

$$Hf(v) = \sup \{|f|_{B(v, \epsilon)} : \epsilon > 0, B(v, \epsilon) \subset (0, \alpha)|.$$ 

By Theorem 3.13 the extended best constant approximant operator $T_{(1, B(v, \epsilon))}$ is monotone, which yields that

$$|f|_{B(v, \epsilon)} \leq f_{B(v, \epsilon)} \leq \sup \{|f|_{B(v, \epsilon)} : \epsilon > 0, B(v, \epsilon) \subset (0, \alpha)|.$$ 

Therefore, by Theorem 2.9 and Lemma 4.1 we get

$$\max \{|f|_{B(v, \epsilon)}, |f|_{B(v, \epsilon)}| \leq \sup \{|f|_{B(v, \epsilon)} : \epsilon > 0, B(v, \epsilon) \subset (0, \alpha)|.$$ 

Consequently, $M_L f(v) \leq Hf(v)$ for all $v \in (0, \alpha)$. Let $s > 0$. Define

$$\Omega_s = \{v \in (0, \alpha) : Hf(v) > s\}.$$ 

Notice that for any $v \in \Omega_s$ there exists $\epsilon_v > 0$ such that

$$B(v, \epsilon_v) \subset (0, \alpha) \quad \text{and} \quad |f|_{B(v, \epsilon_v)} > s,$$

(24)

Let $c < \mu_s(\Omega_s)$ and let $B = \bigcup_{v \in \Omega_s} B(v, \epsilon_v)$. Since $\Omega_s \subset B$, we obtain that $c < \mu(B)$. By regularity of the Lebesgue measure $\mu$ there exists a compact set $K \subset B$ such that $c < \mu(K)$. Since a collection $V = \{B(v, \epsilon_v) : v \in \Omega_s\}$ is a covering of the set $K$, by Vitali’s Lemma 3.2 in [1] there is a finite collection of pairwise disjoint sets $\{B(v_k, \epsilon_k)\}_{k=1}^n \subset V$ such that

$$c < 4 \sum_{k=1}^n \mu(B(v_k, \epsilon_k)).$$

Define $B_s = \bigcup_{k=1}^n B(v_k, \epsilon_k)$. Since $\phi$ satisfies the triangle inequality, we get

$$\phi(c) \leq \phi\left(4 \sum_{k=1}^n \mu(B(v_k, \epsilon_k))\right) \leq 4\phi\left(\sum_{k=1}^n \mu(B(v_k, \epsilon_k))\right) = 4\phi(\mu(B_s)) = 4 \int_0^\alpha \left(\frac{1}{t} \int_0^\mu(B_s) \chi_{(0, t)}\right) w(t) dt.$$ 

(25)
Let $\rho = \rho((|f|^{-1}B_{B_{\alpha}})\chi_{B_{\alpha}})$ be a measure preserving transformation given by Definition 1.2. Then

$$\int_0^{\mu(B_{\alpha})} \chi_{(0,t)} = \int_{B_{\alpha}} \chi_{(0,t)} = \int_{B_{\alpha} \cap ||f|<|f|B_{\alpha})} \chi_{(0,t)} + \int_{B_{\alpha} \cap ||f|>|f|B_{\alpha})} \chi_{(0,t)},$$

whence and by condition (25) we conclude that

$$\phi(c) \leq 4 \int_0^\alpha \left( \frac{1}{t} \int_{B_{\alpha} \cap ||f|<|f|B_{\alpha})} \chi_{(0,t)} \right) w(t) dt + 4 \int_0^\alpha \left( \frac{1}{t} \int_{B_{\alpha} \cap ||f|>|f|B_{\alpha})} \chi_{(0,t)} \right) w(t) dt.$$

Now applying Corollary 2.11 (iii) we obtain

$$\phi(c) \leq 8 \int_0^\alpha \left( \frac{1}{t} \int_{B_{\alpha} \cap ||f|>|f|B_{\alpha})} \chi_{(0,t)} \right) w(t) dt. \quad (26)$$

Since $|f|B_{B(v,\epsilon_k)} \leq |f|B_{B_{\alpha}}$ for all $1 \leq k \leq n$, by Theorem 3.13 we get

$$|f|B_{B(v,\epsilon_k)} \leq |f|B_{B_{\alpha}}.$$

Hence, by the Hardy–Littlewood inequality and by condition (24),

$$\frac{1}{t} \int_{B_{\alpha} \cap ||f|>|f|B_{\alpha})} \chi_{(0,t)} \leq \frac{1}{t} \int_0^t \chi_{(0,\mu(B_{\alpha} \cap ||f|>|f|B_{\alpha})}) \leq \frac{1}{t} \int_0^t \chi_{(0,\mu(B_{\alpha} \cap ||f|>|f|B_{\alpha})}) \leq \frac{1}{t} \int_0^t \chi_{(0,d_f(s))} \geq \chi_{(0,d_f(s))}^{**}(t).$$

Consequently, by condition (26) we have

$$\phi(c) \leq 8 \int_0^\alpha \left( \chi_{(0,d_f(s))}^{**}(t) w(t) dt = 8\phi(d_f(s)).$$

Finally, since $c$ is arbitrary and $c < \mu_*(\Omega_s)$, we get

$$\phi(\mu_*(\Omega_s)) = \phi(\mu_*(\Omega_s)) \leq 8\phi(d_f(s))$$

for all $s > 0$. Therefore for any $s > 0$,

$$\phi(\mu_*(\Omega_s)) \leq 8\phi(d_f(s)). \quad \square$$

We finish with a new version of LDT in $L^0$, namely we prove the convergence of $f_\epsilon$, an extended best constant approximant of $f \in L^0$, to $f$.

**Theorem 4.4.** Let $f \in L^0$ and $f_\epsilon(v)\chi_{B(v,\epsilon)} \in T_{(1,B(v,\epsilon))}(f)$ be an extended best constant approximant of $f$. Then for a.a. $v \in (0,\alpha)$,

$$\lim_{\epsilon \to 0} f_\epsilon(v) = f(v).$$
Proof. Let $0 < \beta < \alpha$ and let $\phi$ be the fundamental function of $I_{1,w}^1$. Define for any $v \in (0, \beta)$,

$$L_f(v) = \limsup_{\epsilon \to 0} |f_\epsilon(v) - f(v)|.$$

Let $g$ be a step function. Define $h = f - g$. Then, by Theorem 3.2, we get that for a.a. $v \in (0, \beta)$ there exists $\epsilon_v > 0$ such that

$$T_{1,B(v,\epsilon)}(h) = T_{1,B(v,\epsilon)}(f) - g \chi_{B(v,\epsilon)}$$

for all $\epsilon \in (0, \epsilon_v)$. Consequently, for a.a. $v \in (0, \beta)$ there is a net $\{h_\epsilon(v)\}_{\epsilon \in (0, \epsilon_v)} \subset \mathbb{R}$ such that

$$(f_\epsilon(v) - g(v)) \chi_{B(v,\epsilon)} = h_\epsilon(v) \chi_{B(v,\epsilon)} \in T_{1,B(v,\epsilon)}(h)$$

for any $\epsilon \in (0, \epsilon_v)$ and so

$$L_f(v) = \limsup_{\epsilon \to 0} |f_\epsilon(v) - f(v)| = \limsup_{\epsilon \to 0} |h_\epsilon(v) - h(v)|.$$

Thus,

$$L_f(v) \leq \limsup_{\epsilon \to 0} |h_\epsilon(v)| + \limsup_{\epsilon \to 0} |h|(v) \leq M_L h(v) + |h|(v)$$

for a.a. $v \in (0, \beta)$ and consequently for $s > 0$,

$$\mu_*(v \in (0, \beta) : L_f(v) > 2s) \leq \mu_*(v \in (0, \beta) : M_L h(v) + |h|(v) > 2s).$$

Clearly

$$\{v \in (0, \beta) : M_L h(v) + |h|(v) > 2s\} \subset \{v \in (0, \beta) : M_L h(v) > s\} \cup \{v \in (0, \beta) : |h|(v) > s\},$$

whence for any $s > 0$,

$$\mu_*(v \in (0, \beta) : L_f(v) > 2s) \leq \mu_*(v \in (0, \beta) : M_L h(v) > s) + \mu_*(v \in (0, \beta) : |h|(v) > s) = \mu_*(v \in (0, \beta) : M_L h(v) > s) + d_h\chi_{(0,\beta)}(s).$$

Since $\phi$ is subadditive, by Theorem 4.3 there exists $C_{\beta} > 0$ such that

$$\phi(\mu_*(v \in (0, \beta) : L_f(v) > 2s)) \leq \phi(\mu_*(v \in (0, \beta) : M_L h(v) > 2)) + \phi\left(d_h\chi_{(0,\beta)}(s)\right) \leq (C_{\beta} + 1)\phi\left(d(f - g)\chi_{(0,\beta)}(s)\right)$$

for all $s > 0$. Since $g$ is an arbitrary step function, we can replace $g$ by a sequence of step functions $(g_n)_{n \in \mathbb{N}}$ such that $g_n \to f$ as $n \to \infty$ a.e. on $(0, \alpha)$, which implies that

$$\lim_{n \to \infty} d(f - g_n)\chi_{(0,\beta)}(s) = 0$$

for any $s > 0$. Furthermore, by order continuity of the norm in $I_{1,w}^1$ we have $\phi(t) = 0$ if and only if $t = 0$. Therefore for any $s > 0$,

$$\mu_*(v \in (0, \beta) : L_f(v) > s) = 0.$$

So $L_f(v) = 0$, for a.a. $v \in (0, \beta)$ and by the fact that $\beta \in (0, \alpha)$ is arbitrary the proof is finished. □
References