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REALIZATION THEORY FOR AUTOMATA IN CATEGORIES

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This paper solves a problem of Arbib and Manes from the categorial automata theory: for which input processes $F : K \to K$ do minimal realizations always exist? A generalization is exhibited, giving a model of finite automata and treating all functors, not only input processes. Here is a rough formulation of the obtained results:

1. Minimal realizations exist iff F preserves cointersections.
2. If so, F must be an input process.
3. For sets and vector spaces
   a) finite minimal realizations exist for every functor F;
   b) a descriptive characterization of functors with all minimal realizations is exhibited.

These results show, inter alia, that the model of Arbib and Manes is not "too wide". Among the basic examples of machines are namely 1) tree machines, where K = sets and $F = F_\Omega$ is a sum of finite hom-functors; 2) bilinear machines, where K = vector spaces and $F(-) = - \otimes \Sigma$. And behold!, each functor with minimal realizations is proved to be 1) for K = sets a quotient of some F_Ω, 2) for K = vector spaces, and for F additive or normal, just $- \otimes \Sigma$

The present paper generalizes the results of the author's paper [1]; this generalization has been announced in [2]. I am much indebted to Věra Trnková, Václav Koubek and Jan Reiterman, with whom I had valuable discussions on these problems. In particular, Koubek's ideas on set functors were a big help. Trnková has characterized set functors with minimal realizations in [7]. We present an enriched characterization, having an analogy for vector spaces. The presented proof, independent of the original one, is a common work of Koubek and myself. A generalization of the presented results to triple machines is going to appear in [3].

1. Basic notions

1.1. Following [1] we denote, for a functor $F : K \to K$, by $K(F)$ the category of F-algebras and F-homomorphisms. An F-algebra is a pair (X, ω), where

$\omega : FX \to X$ is a morphism in K. An F-homomorphism $p : (X, \omega) \to (X', \omega')$ is such a K-morphism $p : X \to X'$ that $p \cdot \omega = \omega' \cdot Fp$. Given an object A in K, by a free F-algebra over A is meant an F-algebra (A^*, φ), with a morphism $s : A \to A^*$ ("the injection of generators"), having the following property: for every F-algebra (X, ω) and every morphism $f : A \to X$ there exists a unique homomorphism $f^* : (A^*, \varphi) \to (X, \omega)$ such that $f = f^* \cdot s$.

Arbib and Manes [4] call F an input process provided that for every object A the free algebra (A^*, φ) exists. In the terminology of Barr [5], this means that F generates a free triple. In [1] a transfinite construction for free F-algebras is exhibited: its effectivity is verified in [10]. Let us mention a special (finitary) case, which we shall need later on.

1.2. Assume that K has finite sums and colimits of sequences. For a fixed object A define a sequence $s_n : W_n \to W_{n+1}$ $(n = 0, 1, 2, \ldots)$ in K as follows: $W_0 = A$, $W_1 = A \vee FA, \ldots, W_{n+1} = A \vee FW_n, \ldots$; $s_0 : W_0 \to W_1$ is the sum-injection and $s_{n+1} = 1_A \vee Fs_n$ $(: A \vee FW_n \to A \vee FW_{n+1})$. Then we have sum-injections $FW_n \to W_{n+1}$:

$$\begin{array}{ccccccccccc} FA = FW_0 & \xrightarrow{Fs_0} & FW_1 & \xrightarrow{Fs_1} & FW_2 & \xrightarrow{Fs_2} & \cdots & \longrightarrow & FA^* & & \\ & \searrow & & \searrow & & \searrow & & & \downarrow \varphi & & \\ A = W_0 & \xrightarrow[s_0]{} & W_1 & \xrightarrow[s_1]{} & W_2 & \xrightarrow[s_2]{} & W_3 & \xrightarrow[s_3]{} \cdots & \longrightarrow & A^*. & \end{array}$$

Denote (optimistically) by A^* the colimit of the sequence (W_n, s_n). Assuming that F preserves this colimit, we get a unique $\varphi : FA^* \to A^*$, for which the above diagram commutes. In this case we say that the free-algebra algorithm converges for A.

Theorem [1]. *If the free-algebra algorithm converges then (A^*, φ), together with the colimit-injection $s : A \to A^*$, is a free F-algebra over A.*

1.3. For fixed objects I and Y in K, Arbib and Manes define the following category of machines. The objects, machines, are tuples $M = (Q, \delta, \beta, \tau)$ where (Q, δ) is an F-algebra and $\tau : I \to Q$, $\beta : Q \to Y$ are morphisms of K. The morphisms, simulations, from M to M' are such F-homomorphisms $p : (Q, \delta) \to (Q', \delta')$ for which $p \cdot \tau = \tau'$ and $\beta' \cdot p = \beta$. If F is an input process, the morphism $\tau : I \to Q$ can be uniquely extended to a homomorphism $r = \tau^* : (I^*, \varphi) \to (Q, \delta)$, called the run map of M.

$$\begin{array}{ccccc} & & I & & \\ & & \downarrow \tau & & \\ FI^* & \xrightarrow{\varphi} & I^* & \xrightarrow{f} & Y \\ \downarrow Fr & & \downarrow r & \nearrow \beta & \\ FQ & \xrightarrow{\delta} & Q & & \end{array}$$

Then $f_M = \beta \cdot r : I^* \to Y$ denotes the behavior of M. If r is epi, M is said to be a reachable machine.

Given a behavior, i.e. a morphism $f : I^* \to Y$, its minimal realization is a reachable machine M with $f_M = f$ and with the following property: for every other reachable machine M' with $f_{M'} = f$ there exists a unique simulation $p : M' \to M$.

Definition. An input process F is said to admit minimal realization if every behavior $f : I^* \to Y$ (with I, Y arbitrary) has a minimal realization.

1.4. The above notions are generalized by V. Trnková [8] for functors, which are not necessarily input processes. Let us call a simulation $p : M \to M_0$ a reduction if p is epi. Then the minimal reduction of a machine M is its reduction $p : M \to M_0$ such that for every other reduction $p' : M \to M_0'$ there exists a unique reduction $q : M_0' \to M_0$ with $p = q \cdot p'$.

Definition. A functor F (not necessarily an input process) is said to admit minimal reduction if every machine has a minimal reduction.

The following easy proposition is mentioned in [8].

Proposition. *If F is an input process then it admits minimal reduction iff it admits minimal realization.*

(The idea of sufficiency is: given a behavior $f : I^* \to Y$, reduce the "free" machine (I^*, φ, s, f); of necessity: the minimal reduction of M is obtained as the minimal realization of f_M.)

1.5. Finite machines play an important role in the automata theory. To include finiteness in the above categorial model, we would either have to define finite objects or to consider them axiomatically. The latter approach has the big advantage that one theory serves for finite machines as well as for arbitrary machines. To do that, we consider a class $\mathscr{E}$ of epimorphisms of K as an additional parameter. (No assumptions on $\mathscr{E}$ are made a priori — in particular, no factorizations are expected!). Then, considering as $\mathscr{E}$ all epis $A \to B$ with B "finite", we get a theory of finite machines; while for $\mathscr{E}$ = all epis we get the above theory.

Assume that K, F and $\mathscr{E}$ are given. A machine M is said to be regular if it has a reduction $p : M \to M_0$ with $p \in \mathscr{E}$. And F is said to admit minimal (more precisely, $\mathscr{E}$-minimal) reduction iff for every regular machine M there exists a minimal reduction $p : M \to M_0$ with $p \in \mathscr{E}$. If, moreover, F is an input process then a machine M is reachable (more precisely, $\mathscr{E}$-reachable) if $r \in \mathscr{E}$ where $r = \tau^*$ denotes the run map. A behavior is regular if it has a reachable realization. And F is said to admit $\mathscr{E}$-minimal realization iff every regular behavior has a minimal realization.

It should be noted that if $\mathscr{E}$ contains all isomorphisms (or, at least, all identities) then every behavior is regular and every machine is regular. On the other hand, choosing K = sets, $\mathscr{E}$ = all epis $A \to B$ with B finite and $F(-) = -x\Sigma$, $I = \{1\}$, $Y = \{0, 1\}$, a regular behavior is a mapping $f : \Sigma^* \to \{0, 1\}$ such that $f^{-1}(1)$ is a regular language in the usual sense. Hence the terminology.

1.6. Quotients of an object A, i.e. epis $e : A \to X$ (up to isomorphism), are naturally ordered: $e \leqslant e'$ iff $k \cdot e = e'$ for some $k : X \to X'$. The least upper bounds are called cointersections (dually to intersections of subobjects). Since no image factorizations are assumed, we must give a little broader definition; briefly, cointersections are multiple pushouts of epis:

The cointersection of a class of epis $e_i : A \to X_i$, $i \in I$ ($I \neq \emptyset$), is an epi $e : A \to X$ such that

1) $e_i \leqslant e$ for all $i \in I$

2) given a morphism $g : A \to Y$ such that $g = f_i \cdot e_i$ for some $f_i : X_i \to Y$ ($i \in I$), then there exists $f : X \to Y$ with $g = f \cdot e$. We put $e = \bigcap^* e_i$.

A class $\mathscr{E}$ of epis is said to be closed to cointersections if $\bigcap^* e_i$ always exists and belongs to $\mathscr{E}$, whenever $e_i \in \mathscr{E}$ for all $i \in I$. A functor F is said to preserve $\mathscr{E}$-cointersections if for $e = \bigcap^* e_i$, $e_i \in \mathscr{E}$, always Fe is a multiple pushout of Fe_i (Fe need not be an epi). Shortly: $F \bigcap^* e_i = \bigcap^* Fe_i$, whenever $e_i \in \mathscr{E}$.

We shall need a weaker condition on F:

Definition. A functor F is said to preserve $\mathscr{E}$-cointersections of congruences if $F \bigcap^* e_i = \bigcap^* Fe_i$ holds for any collection of F-homomorphisms $e_i : (A, \omega) \to (X_i, \omega_i)$ with $e_i \in \mathscr{E}$.

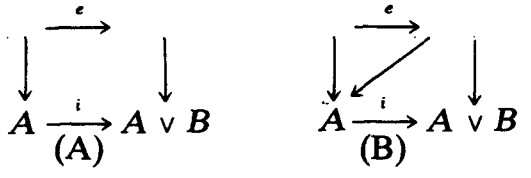
1.7. To make the following statements shorter we use an abbreviation for five very natural conditions on the class $\mathscr{E}$. Notice that 1) none of them contradicts the intuition of "finite quotients" as elements of $\mathscr{E}$, 2) if $\mathscr{E}$ is a part of a factorization system $(\mathscr{E}, \mathscr{M})$ and if $\mathscr{M}$ contains all sum-injections $i : A \to A \vee B$, then these conditions (but the last) are satisfied, whenever K is cocomplete and $\mathscr{E}$-cowell powered (cf. [6]).

Definition. We say that a class $\mathscr{E}$ of epimorphisms is standard if

S1. $\mathscr{E}$ is closed to cointersections.

S2. $\mathscr{E}$ is closed to finite sums, i.e. given $e_i = A_i \to B_i$ in $\mathscr{E}$, $i = 1, 2$, then $e_1 \vee e_2 : A_1 \vee A_2 \to B_1 \vee B_2$ is in $\mathscr{E}$, too.

S3. Any commuting square (A), with $e \in \mathscr{E}$ and $i : A \to A \vee B$ canonical, can be completed to a commuting diagram (B):



S4. Given $e \in \mathscr{E}$ and a split epi r then also $e \cdot r \in \mathscr{E}$.

S5. Given $e : A \to B$ in $\mathscr{E}$ where A is an initial object, then e is an isomorphism.

2. Minimal realization and reduction

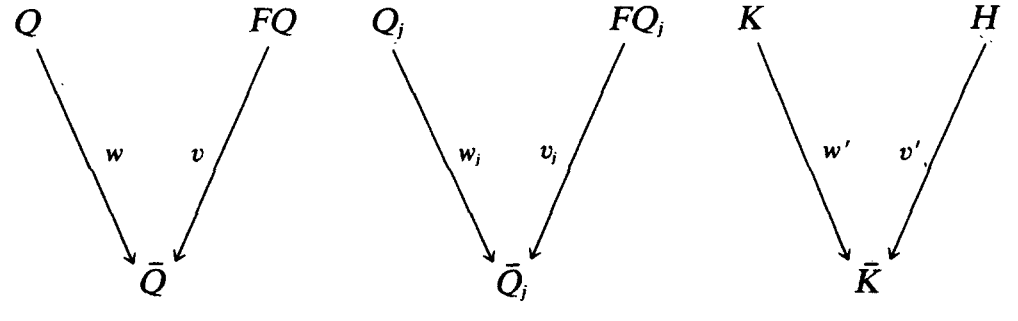
2.1. Theorem. *Let $\mathscr{E}$ be a class of epimorphisms of a category K, closed to cointersections. Then every functor $F : K \to K$, which preserves $\mathscr{E}$-cointersections of congruences, admits $\mathscr{E}$-minimal reduction.*

Proof. Given a regular machine M, consider the (non-void!) class $e_i : M \to M_i$, $i \in I$, of all of its reductions with $e_i \in \mathscr{E}$. Since e_i are F-homomorphisms, their cointersection $e = \bigcap^* e_i$ is preserved by F. Via $Fe = \bigcap^* Fe_i$ we see that also e is a homomorphism — and there is a unique machine M_0 such that $e : M \to M_0$ is a simulation. Then this is the $\mathscr{E}$-minimal reduction of M.

2.2. Reduction Theorem I. *Let K be a finitely complete category with a standard class $\mathscr{E}$ of epis, let $F : K \to K$ be a functor such that $e \in \mathscr{E}$ implies $Fe \in \mathscr{E}$. Then F admits $\mathscr{E}$-minimal reduction iff it preserves $\mathscr{E}$-cointersections of congruences.*

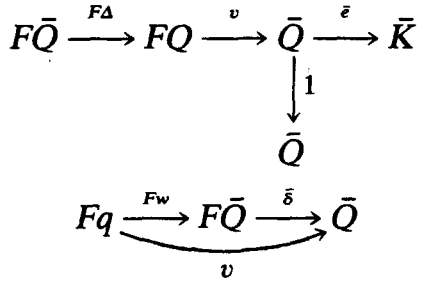
Proof. A) We want to prove the necessity. Let $e_j : Q \to Q_j$, $j \in J$ ($J \neq \emptyset$) be $\mathscr{E}$-epis, which are congruences of an F-algebra. Thus we have $\delta : FQ \to Q$ and $\delta_j : FQ_j \to Q_j$ with $e_j : (Q, \delta) \to (Q_j, \delta_j)$. Denote $k : Q \to K$ their cointersection, $k = \bigcap^* e_j$ with $k = k_j \cdot e_j$ for $j \in J$. It is our task to show that $Fk = \bigcap^* Fe_j$. Notice that, since $k \in \mathscr{E}$ and $F(\mathscr{E}) \subset \mathscr{E}$, we have $Fk \in \mathscr{E}$. Given morphisms $h : FQ \to H$ and $h_j : FQ_j \to H$ with $h = h_j \cdot Fe_j$, we must find a morphism $d : FK \to H$ with $h = d \cdot Fk$. (Then also $h_j = d \cdot Fk_j$, because each Fe_j is epi and $h_j \cdot Fe_j = (d \cdot Fk_j) \cdot Fe_j$.) That will conclude the proof.

B) We shall work with F-automata where $I = \bar{Q}$ and $Y = \bar{K}$; here $\bar{Q}$ denotes the sum $Q \vee FQ$ with injections w, v; analogously $\bar{K} = K \vee H$ and $\bar{Q}_j = Q_j \vee FQ_j$:



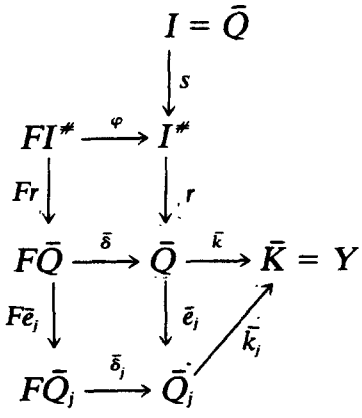
Furthermore, we put $\bar{e}_j = e_j \vee Fe_j : \bar{Q} \to \bar{Q}_j$ and $\bar{k}_j = k_j \vee h_j : \bar{Q}_j \to \bar{K}$ (also $\bar{k} = k \vee h$).

Now we define an F-automaton $M = (\bar{Q}, \bar{\delta}, \beta, \tau)$ where $\tau = 1_{\bar{Q}}$, $\beta = \bar{k}\,(: \bar{Q} \to \bar{K} = Y)$ and $\delta = v \cdot F\Delta$ with $\Delta : \bar{Q} \to Q$



defined as follows: $\Delta \cdot w = 1_Q$ and $\Delta \cdot v = \delta$. Analogously define machines $M_j = (\bar{Q}_j, \bar{\delta}_j, \beta_j, \tau_j)$, $j \in J$, with $\tau_j = \bar{e}_j$; $\beta_j = \bar{k}_j$ and $\bar{\delta}_j = v_j \cdot F\Delta_i$ with $\Delta_j \cdot w_j = 1$ and $\Delta_j \cdot v_j = \delta_j$. Notice that $\bar{\delta} \cdot Fw = v$!

It can be easily verified that all $\bar{e}_j$ present machine morphisms $\bar{e}_j : M \to M_j$:

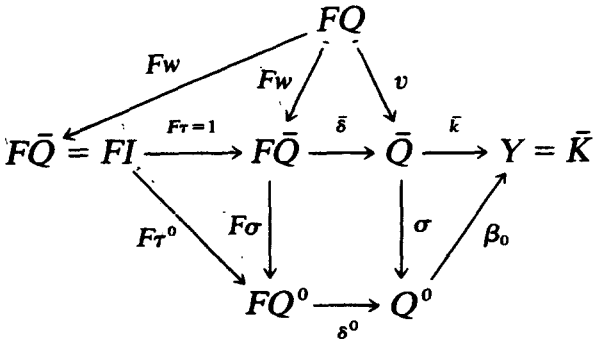


As usual, we denote $r = \tau^{\#}$ and $r_j = \tau_j^{\#}$. Notice that $\bar{e}_j \cdot r$ is a homomorphism; moreover, since $\tau = 1$ and $\tau_j = \bar{e}_j$, we have $(\bar{e}_j \cdot r) \cdot s = \bar{e}_j \cdot \tau = \tau_j$—necessarily $\bar{e}_j \cdot r = (\tau_j)^{\#} = r_j$. This implies that M_j are $\mathscr{E}$-reachable machines via Conditions S2, S3 (see 1.6): r is a retraction ($r \cdot s = \tau = 1$) and $\bar{e}_j \in \mathscr{E}$ (we have $e_j \in \mathscr{E}$, thus $Fe_j \in \mathscr{E}$ and so $e_j \vee Fe_j \in \mathscr{E}$) and so $r_j = \bar{e}_j \cdot r \in \mathscr{E}$. Moreover, M and M_j have the same behavior $f = \bar{k} \cdot r = \bar{k}_j \cdot (\bar{e}_j \cdot \bar{r}_j)$. Thus, f is $\mathscr{E}$-regular.

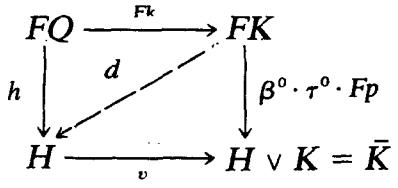
C) Since f is $\mathscr{E}$-regular, it has an $\mathscr{E}$-minimal realization, say $M^0 = (Q^0, \delta^0, \beta^0, \tau^0)$. Since each M_j is an $\mathscr{E}$-reachable realization of f, there leads a machine morphism $\sigma_j : M_j \to M^0$. Moreover, $\sigma_j \cdot \bar{e}_j$ is clearly independent of j; put $\sigma = \sigma_j \cdot \bar{e}_j : M \to M^0$.

Since $\tau^0 = \sigma_j \cdot \tau_j$, and since $\tau_j \cdot w_j = \bar{e}_j \cdot w = w_j \cdot e_j$, we obtain $\tau^0 \cdot w = (\sigma_j \cdot w_j) \cdot e_j$ for all $j \in J$. Now, $k = \bigcap^* e_j$ and so there exists $p : K \to Q^0$ with $\tau^0 \cdot w = p \cdot k$.

D) It is clear that the following diagram commutes:



In particular, $k \cdot v = \beta^0 \cdot \delta^0 \cdot F\tau^0 \cdot Fw$. We have $\tau^0 \cdot w = p \cdot k$ and, since $\bar{k} = k \vee h$, also $\bar{k} \cdot v = v' \cdot h$. Therefore, $v' \cdot h = \beta^0 \cdot \delta^0 \cdot Fp \cdot Fk$. This means that the square



commutes. Via Condition S4 (see 1.6) we obtain $d = FK \to H$ with $h = d \cdot Fk$. That is what we wanted to prove.

2.3. The condition "F preserves cointersections of $\mathscr{E}$-congruences" is unpleasant, because the concept of a congruence engages in itself F-algebras. The natural question is if it cannot be strengthend to "F preserves all $\mathscr{E}$-cointersections". Generally, the answer is negative: there exist input processes, not preserving all $\mathscr{E}$-cointersections, which admit $\mathscr{E}$-minimal realization. To strengthen the above theorem, we shall have to assume a bit more about K.

Recall that an object D is initial, if to any other object X there leads exactly one morphism $0 \to X$ (examples: $\emptyset$ in SET; the trivial space $\{\mathbf{0}\}$ in VECT). Dually, an object T is terminal if there always leads just one morphism $X \to T$ (examples: any singleton set in SET; $\{\mathbf{0}\}$, again, in VECT). We say that a category K is *connected* if for arbitrary non-initial objects X, Y there exists some morphism $X \to Y$. Examples: sets, vector spaces, groups, topological spaces etc. are connected categories; graphs and unary algebras form typical non-connected categories.

2.4. Reduction Theorem II. *Let K be a finitely cocomplete, connected category with a terminal object and a standard class $\mathscr{E}$ of epis. Let $F : K \to K$ be a functor such that $e \in \mathscr{E}$ implies $Fe \in \mathscr{E}$. Then F admits $\mathscr{E}$-minimal reduction iff it preserves $\mathscr{E}$-cointersections.*

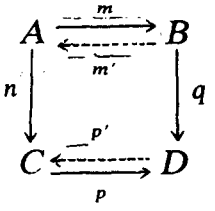
Proof. All we have to show is that for any class $e_j : Q \to Q_j$ $(j \in J)$ of $\mathscr{E}$-epis there exist $\delta : FQ \to Q$ and $\delta_j : FQ_j \to Q_j$ such that $e_j : (Q, \delta) \to (Q_j, \delta_j)$ are homomorphisms. This is easy in case Q is initial (then, by S5 in 1.6, all e_j are isomorphisms). If contrary, there exists a morphism, say d, from the terminal object T to Q. We have canonical morphisms $c : FQ \to T$ and $c_j : FQ_j \to T$. It suffices to put $\delta = d \cdot c$ and $\delta_j = e_j \cdot d \cdot c_j$ $(j \in J)$. Then $c = c_j \cdot Fe_j$ implies $e_j \cdot \delta = \delta_j \cdot Fe_j$.

2.5. In the rest of this section we assume that K is a category with colimits of sequences and with pushouts and pullbacks. We assume further that colimits of sequences commute with pullbacks. To avoid confusion, we say that a morphism $c : A \to B$ is a coequalizer (rather than a regular epi) if it is the coequalizer of some pair $f_1, f_2 : X \to A$.

Lemma. *Let $t_n : X_n \to X_{n+1}$, $n = 0, 1, 2, \ldots$, be a sequence of split monics with a colimit $v_n : X_n \to A$. Then each v_n is a split monic, too.*

Proof. For each n choose $r_n : X_{n+1} \to X_n$ with $r_n \cdot t_n = 1$. Fix n_0 and define a bound $w_n : X_n \to X_{n_0} : w_n = t_{n_0-1} \cdot \ldots \cdot t_n$ if $n < n_0$; $w_n = r_{n_0} \cdot \ldots \cdot r_n$ if $n > n_0$ and $w_{n_0} = 1$. Then there exists $u : A \to X_{n_0}$ with $u \cdot v_n = w_n$. In particular, $u \cdot v_{n_0} = 1$.

Proposition. *A commuting square*



is an absolute pullback if there exist $p' = D \to C$, $m' : B \to A$ with $p' \cdot p = 1$, $m' \cdot m = 1$ and $p' \cdot q = n \cdot m'$, and if q is a split monic. This is so e.g. whenever the square is a pushout and $m = n$ is a split monic.

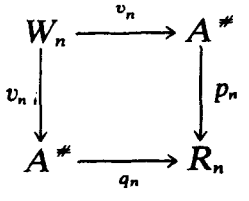
Proof. Let m', p' exist. Given a functor H and $f : X \to HB$, $g : X \to HC$ *with* $Hq \cdot f = Hp \cdot g$, put $k = Hm' \cdot f : X \to HA$. Then $Hn \cdot k = H(n \cdot m') \cdot f = H(p' \cdot q) \cdot f = Hp' \cdot Hp \cdot g = g$. Further $Hm \cdot k = f$ since Hq is a (split) monic and $Hq \cdot Hm \cdot k = Hp \cdot Hn \cdot k = Hp \cdot g = Hq \cdot f$.

If the square is a pushout and if $m = n$, $m' \cdot m = 1$, then use the fact that $(n \cdot m') \cdot m = 1_A \cdot n$: there exists a unique p' with $p' \cdot q = n \cdot m'$ and $p' \cdot p = 1_A$.

Theorem. *Let F preserve colimits of sequences of coequalizers. Then the free-algebra algorithm* (1.2) *converges for every A with* $\hom(FA, A) \neq \emptyset$.

Proof. A) All $s_n : W_n \to W_{n+1}$ are split monics. Indeed, choose a morphism $d_0' : FA \to A$ and define $d_0 : A \vee FA \to A$ by $d_0 = 1_A$ on A, $d_0 = d_0'$ on FA. Then $d_0 \cdot s_0 = 1_A$, i.e. s_0 is a split monic. If s_n is a split monic, i.e. $d_n \cdot s_n = 1$, then so is $s_{n+1} = 1_A \vee Fs_n$: put $d_{n+1} = 1_A \vee Fd_n$, then $d_{n+1} \cdot s_{n+1} = 1$.

B) We are to prove that F preserves the colimit $v_n : W_n \to A^*$ of (W_n, s_n). By the above lemmas, v_n are split monics and so the pushouts



are absolute pullbacks. Here R_n is just the amalgam sum $R_n = A^* \vee_{W_n} A^*$ and for $m > n$ we have a natural factorization $r_{n,m} : R_n \to R_m$. It is easy to verify that 1) $r_{n,m}$ is a coequalizer of $p_n \cdot v_m$ and $q_n \cdot v_m$, 2) the colimit of the sequence $(R_n, r_{n,n+1})$ is just

A^* with the natural factorizations $R_n \to A^*$. Now, F preserves this colimit, thus FR_n "converge" to FA^*.

C) For each n we have a pullback

$$\begin{array}{ccc} FW_n & \longrightarrow & FA^* \\ \downarrow & & \downarrow \\ FA^* & \longrightarrow & FR_n \end{array}$$

and, since pullbacks commute with colimits of sequences, also

$$\begin{array}{ccc} B & \xrightarrow{z} & FA^* \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle 1} \\ FA^* & \xrightarrow[1]{} & FA^* \end{array}$$

is a pullback, where $B = \text{colim}\,(FW_n, Fs_n)$ and z is the canonical morphism. This proves that z is an isomorphism, thus F preserves the colimit of (W_n, s_n).

2.6. For the next theorem we assume that

1) K is a connected, cocomplete category with a terminal object,
2) K has pullbacks, which commute with colimits of sequences,
3) $\mathscr{E}$ is a standard class of epimorphisms, containing all coequalizers.

Realization Theorem. *The following conditions are equivalent for every functor $F: K \to K$ such that $e \in \mathscr{E}$ implies $Fe \in \mathscr{E}$:*

(i) *F admits $\mathscr{E}$-minimal reduction;*
(ii) *F is an input process and admits $\mathscr{E}$-minimal realization;*
(iii) *F preserves $\mathscr{E}$-cointersections.*

Proof. This is a corollary of Realization Theorem II, only we have to show that if F preserves $\mathscr{E}$-cointersections, then it is an input process. But this follows from the above theorem: since F preserves $\mathscr{E}$-cointersections, it preserves colimits of sequences of $\mathscr{E}$-morphisms (given $e_{n,m} : A_n \to A_m$ in $\mathscr{E}$, $n \leqq m$, then the colimit of this sequence is just the cointersection $\bigcap^*_{m=0}{}^{\infty} e_{0,m}$). Thus, by the above theorem, the free-algebra algorithm converges for every A (recall that K is connected!).

In [1] it is proved that if F is an algorithmic input process (i.e., if the free-algebra algorithm always stops) and F admits minimal realization then Fe is an epimorphism, whenever e is an epimorphism with a non-initial domain. Thus, in case $\mathscr{E} = \{e : A \to B;\ A \text{ is non-initial, } e \text{ is epi}\}$, the above assumption $Fe \in \mathscr{E}$ for $e \in \mathscr{E}$ can be omitted.

3. Sets and vector spaces: finite machines

3.1. Denote by SET the category of sets and mappings and by VECT the category of vector spaces (over a fixed but arbitrary field) and linear transformations. These categories are connected, cocomplete, have a terminal object and pullbacks commute with colimits of sequences; thus, the preceding results apply here.

In the present section we consider cointersections of finite quotients, i.e. $\mathscr{E}$-quotients, where $\mathscr{E}$ is the class of all epis with finite range (finite-dimensional quotients in VECT, where $\mathscr{E}$ are epis with a finite-dimensional range).

3.2. *Note.* In all current concrete categories, particularly in SET and VECT, cointersections can be characterized as follows. Given $k = \bigcap^*_{i\in I} k_i$ $(k : X \to Y)$, then for $x, y \in X$ we have $k(x) = k(y)$ iff there exist $x = a_0, a_1, \ldots, a_n = y$ in X and $i_1, i_2, \ldots, i_n$ in I with $k_{i_t}(a_{t-1}) = k_{i_t}(a_t)$ for $t = 1, 2, \ldots, n$.

Theorem. *Cointersections of finite quotients are absolute colimits in* SET. *In particular, the finite minimal realization problem is solvable for an arbitrary functor* $F : \mathrm{SET} \to \mathrm{SET}$.

Proof. Let $k_i : X \to X_i$, $i \in I$, be epis with X_i finite, let $k = \bigcap^* k_i$, $k : X \to Y$. Given a sequence $s = (i_1, \ldots, i_n)$ in I, by an s-path from x to y in X is meant a sequence $x = a_0, a_1, \ldots, a_n = y$ with $k_{i_t}(a_{t-1}) = k_{i_t}(a_t)$ for $t = 1, \ldots, n$.

A) There exists a sequence $s = (i_1, i_2, \ldots, i_n)$ such that $k(x) = k(y)$ for $x, y \in X$ iff an s-path leads form x to y.

Proof. There clearly exists a finite set $J \subset I$ with $k = \bigcap^*_{j\in J} k_j$. Order J into a sequence, then s is obtained by repeating this sequence m-times, where $m = \sum_{j\in J} \operatorname{card} X_j$.

B) For each i_t $(t = 1, 2, \ldots, n)$ as in A) there exists $p_t : X_{i_t} \to X$ with $k_{i_t} \cdot p_t = 1$, such that for the mapping $d = (p_n \cdot k_{i_n}) \cdot \ldots \cdot (p_2 \cdot k_{i_2}) \cdot (p_1 \cdot k_{i_1}) : X \to X$ there is $\bar{k} : Y \to X$ with $d^n = \bar{k} \cdot k$ (here $d^n = d \cdot d \cdot \ldots \cdot d$, as usual).

Proof. Choose $p_0 : Y \to X$ with $k \cdot p_0 = 1$, put $A_0 = p_0(Y)$. Define p_t and $A_t \subset X$ by induction: $A_t = k_{i_t}^{-1}(k_{i_t}(A_{t-1}))$; p_t fulfills $k_{i_t} \cdot p_t = 1$ and $p_t(k_{i_t}(A_{t-1})) \subset A_{t-1}$. Clearly $d(A_t) \subset A_{t-1}$ and, for any s-path $a_0, \ldots, a_n$, $a_0 \in A_0$ implies $a_t \in A_t$ — hence $d^n(a_0) = d^n(a_n)$. This implies that, whenever $k(x) = k(y)$ then $d^n(x) = d^n(y)$; thus we can define $\bar{k}$ by $\bar{k}(k(x)) = d^n(x)$.

C) The cointersection is absolute.

Proof. Let $F : \mathrm{SET} \to Q$ be a functor, let $f_i : FX_i \to Z$ and $f : FX \to Z$ be given in Q with $f = f_i \cdot Fk_i$. We shall find $g : FY \to Z$ with $g \cdot Fk = f$ (since epis in SET split, Fk_i and Fk are epis).

Put $g = f \cdot F\bar{k}$, we shall show that $f \cdot F(\bar{k} \cdot k) = f$. To this it suffices to verify $f \cdot Fd = f$ or only $f \cdot F(p_t \cdot k_{i_t}) = f$ for each t. We have $f = f_{i_t} \cdot Fk_{i_t} = f_{i_t} \cdot F(k_{i_t} \cdot p_t \cdot k_{i_t}) = (f_{i_t} \cdot Fk_{i_t}) \cdot (Fp_t \cdot Fk_{i_t}) = f \cdot F(p_t \cdot k_{i_t})$.

3.3. Theorem. *Cointersections of finite-dimensional quotients are absolute colimits in* VECT. *In particular, the finite minimal realization problem is solvable for an arbitrary functor* $F: \mathrm{VECT} \to \mathrm{VECT}$.

The proof for vector spaces is analogous to that for sets. This result cannot be extended to modules. E.g., in the category of Abelian groups the intrinsic hom-functors of finite cyclic groups do not preserve cointersections of finitely generated quotients.

In a subsequent paper I shall show that all endofunctors of the category of sets and relations also preserve all existing cointersections of finite quotients. See [2].

4. Sets and vector cases: infinite machines

4.1. This section is devoted to a characterization of set functors (i.e., endofunctors of SET) and vector functors (endofunctors of VECT) which admit minimal realization. Since all epis in SET and VECT split and are thus preserved by all functors, we shall actually characterize functors which preserve cointersections. See Realization Theorem, all assumptions of which are clearly fulfilled by SET and VECT.

Set functors which admit minimal realizations were characterized by Trnková [7] as quotient functors of sums of finite hom-functors. In the present paper an independent proof is exhibited, which is a common work of Koubek and myself.

4.2. A diagram over a well-ordered set is called a chain (more in detail, an α-chain, if α is the corresponding ordinal). A sequence is then an ω_0-chain. A strictly increasing chain of monics is such that none of the monics is an isomorphism.

Definition. An object is said to be finite if it is not a union of a strictly increasing, infinite chain of its subobjets.

Examples. A universal algebra is a finite object iff it is finitely generated; a set, graph, topological space is a finite object iff it is finite.

Definition. A functor $F: K \to L$ is finitary iff for arbitrary A in K and $f, g: FA \to X$ in L with $f \neq g$ there exists a finite subobject $m: B \to A$ such that $f \cdot Fm \neq g \cdot Fm$.

Example. A set functor is finitary iff it is a quotient functor of a sum of finite hom-functors.

Proof. It is easy to see that each $\hom(A, -)$, with A finite, is a finitary functor. Also, a factor functor of a sum of finitary functors is easily seen to be finitary. The converse follows from the Yoneda lemma: for each $a \in F\dot{n}$, where $\dot{n} =$

$\{0, 1, \ldots, n-1\}$, we have a transformation $\hom(\dot{n}, -) \to F$. This defines a transformation $\tau : \coprod_{n=0}^{\infty} \coprod_{a \in F\dot{n}} \hom(\dot{n}, -) \to F$. The finitarity of F makes it clear that τ is an epi-transformation.

4.3. Theorem. *The following conditions are equivalent for each set functor:*

(COIN) *F preserves cointersections.*

(E) *F preserves colimits of chains of epis.*

(M) *F preserves colimits of chains of monics.*

(FIL) *F preserves filtered colimits.*

(FIN) *F is finitary.*

(Q) *F is a quotient functor of a sum of finite hom-functors.*

Proof. (COIN) $\to$ (E) Given a chain $e_{i,j} : A_i \to A_j$ $(i \leqslant j < \alpha)$, its colimit is just the cointersection $\bigcap^*_{j<\alpha} e_{0,j}$.

(E) $\to$ (M) Let $m_{ij} : X_i \to X_j$ $(i \leqslant j < \alpha)$ be a chain of subsets $(X_i \subset X_j)$. The colimit is $X = \bigcup_{i<\alpha} X_i$ and we are to verify that $FX = \bigcup_{i<\alpha} Fv_i(FX_i)$ where $v_i : X_i \to X$ denote the inclusion maps. To this end, denote by Y_i the amalgam sum $Y_i = X \vee_{X_i} X$. Then we clearly get a natural chain of epis $e_{ij} : Y_i \to Y_j$ $(i \leqslant j < \alpha)$ with a colimit, presented by the natural factorizations $Y_i \to X$. Since F preserves this colimit, we obtain: FX is the amalgam sum $FX \vee_W FX$ where $W = \bigcup_{i<\alpha} Fv_i(FX_i)$. This proves that $W = FX$.

(M) $\to$ (FIN) Assume that F is not finitary and choose a set X with the least power such that $FX \neq \bigcup_{Y \subset X, Y \text{ finite}} Fv(FY)$ ($v : Y \to X$ denotes the inclusion map). This leads to a contradiction: X must be infinite, thus it is a colimit of a chain $Y_i \to Y_j$ of subsets where card $Y_i <$ card X. But then F cannot preserve this colimit.

(FIN) $\to$ (FIL) Let $D : \mathcal{D} \to \mathrm{SET}$ be a filtered diagram with a colimit $v_d : Dd \to X$; let $w_d : FDd \to T$ be a bound of FD. We want to define $s : FX \to T$ with $w_d = s \cdot Fv_d$. Choose $a \in FX$. Since F is finitary, we can find a finite subset $H \subset X$ with $a \in Fv(FH)$ ($v : H \to X$ is the inclusion). Since every set functor preserves finite non-void intersections (see [9]), H can be chosen as the least set with this property.

Since H is a finite set and the diagram D is filtered, there clearly exists $d \in \mathcal{D}$ such that $H \subset v_d(Dd)$ (indeed, $X = \bigcup_{d \in \mathcal{D}} v_d(Dd)$). Then $a \in Fv_d(FDd)$ and we can choose $b \in FDd$ with $Fv_d(b) = a$. Put $s(a) = w_d(b)$. This is a correct way of defining $s : FX \to T$: if $Fv_{d'}(b') = a$, then $w_d(b) = w_{d'}(b')$, because the diagram is filtered. This kind of definition guarantees $s \cdot Fv_d = w_d$.

The uniqueness of s follows from the finitarity of F: since D is filtered, clearly $FX = \bigcup_{d \in \mathcal{D}} Fv_d(FD_d)$.

(FIL) $\to$ (COIN) Let $k = \bigcap^* k_i$ and $p = \bigcap^* Fk_i$, $(k_i = X \to X_i,\ k : X \to T)$. We are to verify that $Fk \leqslant p$, i.e. that whenever $Fk(a) = Fk(b)$, then also $p(a) = p(b)$. Since every set is a filtered colimit of its finite subsets, there exists a finite subset $Y \subset X$ with $a, b \in Fv(FY)$ (where $v : Y \to X$ is the inclusion).

There exists a finite set Z, $Y \subset Z \subset X$, such that the restrictions $k_i' : Z \to k_i(Z)$ of

k_i have the following property: given $k' = \bigcap^* k'_i$, then $k'(y_1) = k'(y_2)$ iff $k(y_1) = k(y_2)$ for arbitrary $y_1, y_2 \in Y$. (*Proof*: For each y_1, y_2 with $k(y_1) = k(y_2)$ there exist $a_0, a_1, \ldots, a_n$ in X, $a_0 = y_1$ and $a_n = y_2$, such that for some $i_1, \ldots, i_n$ we have $k_{i_t}(a_{t-1}) = k_{i_t}(a_t)$, $t = 1, \ldots, n$ (that is the description of cointersections in SET); now, constitute Z of all the points $a_0, a_1, \ldots, a_n$, one n-tuple for each y_1, y_2.) Moreover, denoting by $u : Y \to Z$ the inclusion, clearly $F(k \cdot v)(a_1) = F(k \cdot v)(b_1)$ implies $F(k' \cdot u)(a_1) = F(k' \cdot u)(b_1)$ for arbitrary $a_1, b_1 \in FY$.

Now we use the fact that F preserves cointersections of finite quotients (see section 3): $Fk' = \bigcap^* Fk'_i$. Since k'_i are restrictions of k_i, also Fk'_i are restrictions of Fk_i. Thus, given $a_1, b_1 \in FY$ with $F(k' \cdot u)(a_1) = F(k' \cdot u)(b_1)$, then necessarily $p \cdot Fw(a_1) = p \cdot Fw(b_1)$. Therefore, given $a, b \in Fv(FY)$ with $Fk(a) = Fk(b)$, we have $p(a) = p(b)$.

4.4. There is a number of similarities between the categories SET and VECT. E.g., monics and epis split, finite intersections are absolute pullbacks (this follows immediately from Proposition 2.5). Indeed, the whole proof of the above theorem can be applied to VECT as well, after the last condition has been excluded.

Theorem. *The following conditions are equivalent for each vector functor F:*

(COIN) *F preserves cointersections.*
(E) *F preserves colimits of chains of epis.*
(M) *F preserves colimits of chains of monics.*
(FIL) *F preserves filtered colimits.*
(FIN) *F is finitary.*

4.5. The analogy between SET and VECT stops as soon as we investigate linear functors. A vector functor F is linear if $F(f + g) = Ff + Fg$ (i.e., F is additive) and $F(\lambda f) = \lambda(Ff)$ (F is homogeneous). E.g., each intrinsic hom-functor is linear. We denote by $\oplus$ the direct product (sum) and by $\otimes$ the tensor product.

Proposition. *A vector functor F is linear iff it is a colimit of a (big) diagram of intrinsic hom-functors. Equivalently, vectors in FX and transformations $\tau : \hom(X, -) \to F$ are in a one-to-one correspondence via*

$$\tau \mapsto \tau_X(\mathrm{id}_X).$$

Proof. This is just the standard Yoneda lemma.

Proposition. *Every linear functor preserves all finite limits and colimits.*

Proof. Being additive, a linear functor preserves biproducts. Let us prove that every linear functor F preserves cokernels (analogously kernels), then the proposition is clear.

Given a subspace $m : A \to B$ (m is the inclusion map), use the above proposition to obtain a functor G, which is a sum of intrinsic hom-functors, and a transformation $\tau : G \to F$ such that τ_B is onto. (Put $G = \coprod_{x \in FB} \hom(B_x, -)$ with $B_x = B$.) It is clear that G preserves co-kernels, thus $Gk = \operatorname{coker} Gm$ where $k : B \to B/A$ denotes the co-kernel of m. We shall verify that also $Fk = \operatorname{coker} Fm$.

Since Fk is onto and $Fk \cdot Fm = 0$, it suffices to find, for each $p : FB \to X$ with $p \cdot Fm = 0$, such $q : F(B/A) \to X$ that $p = q \cdot Fk$. There exists $h : B/A \to B$ with $kh = 1$. Put $q = p \cdot Fh$. Then $p \cdot \tau_B \cdot Gm = 0$ implies that there exists $q_0 : G(B/A) \to X$ with $p \cdot \tau_B = q_0 \cdot Gk$. Therefore, $p = q \cdot Fk$, because τ_B is onto and $p \cdot \tau_B = q_0 \cdot G(khk) = p \cdot \tau_B \cdot G(hk) = p \cdot F(hk) \cdot \tau_B = q \cdot Fk \cdot \tau_B$.

Proposition. *A linear functor F preserves sums iff there exists a space Σ such that F is naturally equivalent to the functor*

$$X \mapsto X \otimes \Sigma, \quad f \mapsto f \otimes \mathrm{id}_\Sigma.$$

Proof. The functors $X \mapsto X \otimes \Sigma$ are left adjoints to $\hom(\Sigma, -)$ and therefore they preserve colimits.

Let F be a functor, preserving sums. Then $F0 = 0$ (where $0 = \{\mathbf{0}\}$ denotes the trivial space), for 0 is the sum of zero summands. Since every vector functor preserves finite intersections, we see that $f \cap g = 0$ implies $Ff \cap Fg = 0$ for subobjects f, g.

Every space X is a sum of copies of the one-dimensional space S of scalars — therefore, FX is a sum of copies of FS. In particular, for each $\boldsymbol{x} \in FX$ there exists $f : S \to X$ and $\boldsymbol{y} \in FS$ with $Ff(\boldsymbol{y}) = \boldsymbol{x}$.

Choose a base $\{\boldsymbol{b}_i ; i \in I\}$ for FS and for each i define a subfunctor F_i of F by $F_iX = \{\boldsymbol{x} \in FX ; \boldsymbol{x} = Ff(\boldsymbol{b}_i)$ for some $f : S \to X\}$. Then we have a transformation $\tau : \bigoplus_{i \in I} F_i \to F$, defined by $\tau_X(\sum \lambda_i x_i) = \sum \lambda_i x_i$ for $x_i \in F_iX$. This is indeed an epi-transformation. Let us verify that it is a natural equivalence. To this end, it suffices to show that if $i \neq j$ then $F_iX \cap F_jX = 0$. Indeed, if $Ff(\boldsymbol{b}_i) = Fg(\boldsymbol{b}_j)$, then f, $g : S \to X$ are subobjects of X with $f \cap g = 0$ (else $f = \lambda \cdot g$, but then $Fg(\boldsymbol{b}_j) \neq Fg(\lambda \boldsymbol{b}_i) = Ff(\boldsymbol{b}_i)$, for Fg is one-to-one). This implies $Ff \cap Fg = 0$, i.e., $Ff(\boldsymbol{b}_i) = \mathbf{0}$. We see that τ is a natural equivalence. Note that $\dim F_iS = 1$ for each i and F_i preserves colimits.

It remains to show that if $F_iS = S$ and F_i preserves colimits, then F_i is naturally equivalent to the identity functor Id: then F is equivalent to a sum of copies of Id and this sum is clearly equivalent to $X \mapsto X \otimes \Sigma$ (where $\dim \Sigma = \operatorname{card} I$). Define a transformation $\mu : \mathrm{Id} \to F_i$ as follows: given $\boldsymbol{x} \in X$ we have a linear mapping $f : S \to X$ with $f(\lambda) = \lambda \boldsymbol{x}$; then $\mu_X(\boldsymbol{x}) = F_if(1)$. This is clearly a transformation, which is both monic (for F_i is faithful) and epi. Thus, it is a natural equivalence.

4.6. Theorem. *The following conditions are equivalent for every vector functor:*

(A-COIN) *F preserves cointersections and is additive or normal or co-normal.*

(L-FIN) *F is linear and finitary.*
(COL) *F preserves colimits.*
(TENS) *F is naturally equivalent to a functor* $X \mapsto X \otimes \Sigma$, $f \mapsto f \otimes \mathrm{id}_\Sigma$.
(Q) *F is a quotient functor of a sum of finite-dimensional intrinsic hom-functors.*

Proof. (A–COIN)→(TENS) By the above theorem, it suffices to verify that F preserves sums. Let F be co-normal (if F is normal, we proceed analogously; if F is additive then it preserves finite sums which, together with cointersections, yield all finite colimits — hence, F is co-normal). Let $A = \bigoplus_{i \in I} A_i$, with injections $v_i : A_i \to A$. To prove that $FA \cong \bigoplus_{i \in I} Fv_i\,(FA_i)$ it suffices to verify

a) $Fv_i \cap Fv_j = 0$ if $i \neq j$,

b) $FA = \bigcup Fv_i$.

Since F is co-normal, $F0 = 0$ ($=$ coker 1) and so $v_i \cap v_j = 0$ implies $Fv_i \cap Fv_j = 0$. Further, since $A = \bigcup v_i$, we have $0 = \text{coker } 1_A = \bigcap^* \text{coker } v_i$ and so $F0 = \bigcap^* \text{coker } Fv_i$, i.e. $FA = \bigcup Fv_i$.

(TENS)→(L-FIN) Obvious.

(L-FIN)→(Q) This is the same as for SET (Example in 4.2).

(Q)→(COL) The quotient of any linear functor is obviously a linear functor, again; thus, it remains to show that F preserves sums (if G does and $\varepsilon : G \to F$ is an epi-transformation). By the above proposition, $GX \cong X \otimes \Sigma$. Then $FX \cong X \otimes \Sigma'$ where $\Sigma' = \varepsilon_S(S \otimes \Sigma)$, as is easily seen.

(COL)→(A–COIN) Obvious.

Example. The functors $FX = X \otimes X$, $FX = X \oplus \Sigma$ for a fixed Σ, and the constant functors are examples of non-linear finitary functors.

4.7. The considerations about non-linear vector functors can be hardly expected to have an analogy for modules over general rings. On the other hand, the last theorem has an immediate generalization to R-Mod, the category of left R-modules over a ring R.

Theorem. *Let* $F : R\text{-Mod} \to S\text{-Mod}$ *be a linear* (!) *functor. Then the following are equivalent:*

(L–COIN) *F preserves cointersections.*
(LC-FIL) *F preserves filtered colimits and is co-normal.*
(L–COL) *F preserves colimits.*
(LC–FIN) *F is finitary and conormal.*

In case $R = S$, *further equivalent conditions are*

(LC–TENS) *F is a conormal quotient functor of a functor* $X \mapsto X \otimes \Sigma$, $f \mapsto f \otimes \mathrm{id}_\Sigma$.
(LC–Q) *F is a conormal quotient functor of a sum of finitely-generated intrinsic hom-functors.*

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