New sufficient conditions for the extendability of quaternary linear codes

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Abstract

For an \([n, k, d]_4\) code \(C\) with \(d\) odd, we define the diversity of \(C\) as the 3-tuple \((\Phi_0, \Phi_1, \Phi_2)\) with

\[
\Phi_0 = \frac{1}{3} \sum_{4|i, i \geq 0} A_i,
\]

\[
\Phi_j = \frac{1}{3} \sum_{i \equiv j \pmod{4}} A_i \quad \text{for } j = 1, 2 \text{ when } d \equiv 1 \pmod{4},
\]

\[
\Phi_j = \frac{1}{3} \sum_{i \equiv j \pmod{4}} A_i \quad \text{for } j = 1, 2 \text{ when } d \equiv 3 \pmod{4},
\]

where \(A_i\) stands for the number of codewords with weight \(i\). We prove that an \([n, k, d]_4\) code with \(d\) odd, \(k \geq 3\), is extendable if \(\Phi_0 + \Phi_2 = \theta_{k-2} + 2 \times 4^{k-2}\) or if \(\Phi_0 = \theta_{k-4}\), where \(\theta_j = (4^{j+1} - 1)/3\). For the case when \(k = 3\), we determine all possible diversities and the corresponding spectra, which yield that \(C\) is extendable if \((\Phi_0, \Phi_1, \Phi_2) \notin \{(6, 1, 3), (6, 3, 3), (2, 3, 7)\}\). Geometric necessary and sufficient conditions for the non-extendability of \(C\) when \((\Phi_0, \Phi_1, \Phi_2) \in \{(6, 1, 3), (6, 3, 3), (2, 3, 7)\}\) are also given.

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1. Introduction

Let $C$ be an $[n,k,d]_q$ code, that is a linear code over $GF(q)$ of length $n$ with dimension $k$ whose minimum Hamming distance is $d$, where $GF(q)$ stands for the finite field of order $q$. The weight distribution of $C$ is the list of numbers $A_i$ which is the number of codewords of $C$ with weight $i$. The weight distribution with $(A_0, A_d, \ldots) = (1, \alpha, \ldots)$ is also expressed as $0^1d^\alpha \ldots$.

We only consider non-degenerate codes having no coordinate which is identically zero. The code obtained by deleting the same coordinate from each codeword of $C$ is called a punctured code of $C$. If there exists an $[n+1,k,d+1]_q$ code $C'$ which gives $C$ as a punctured code, $C$ is called extendable (to $C'$) and $C'$ is an extension of $C$. It is well known that $[n,k,d]_2$ codes with $d$ odd are always extendable [6]. The condition ‘$d$ is odd’ for $q = 2$ would be replaced by ‘$q$ and $d$ are relatively prime’ for general $q$. But this is not enough for $q > 2$. Our aim is to find easily checkable conditions to see if a given $[n,k,d]_q$ code with $\gcd(d,q) = 1$ is extendable or not (Extension Problem). See [3,19–21] for $q = 3$ and [7,12,13,15–17,22] for general $q$.

Let $C$ be an $[n,k,d]_4$ code with $k \geq 3$, $d$ odd. We define the diversity of $C$ as the 3-tuple $(\Phi_0, \Phi_1, \Phi_2)$ with

$$\Phi_0 = \frac{1}{3} \sum_{4|i, i > 0} A_i,$$

$$\Phi_j = \frac{1}{3} \sum_{i \equiv -j \pmod{4}} A_i \quad \text{for } j = 1, 2 \text{ when } d \equiv 1 \pmod{4},$$

$$\Phi_j = \frac{1}{3} \sum_{i \equiv j \pmod{4}} A_i \quad \text{for } j = 1, 2 \text{ when } d \equiv 3 \pmod{4}.$$

For quaternary linear codes, the following is known.

**Theorem 1.1.** (See [16,18,22].) An $[n,k,d]_4$ code $C$ with diversity $(\Phi_0, \Phi_1, \Phi_2), d$ odd, is extendable if one of the following conditions holds:

1. $\Phi_1 = 0$,  
2. $\Phi_2 = 0$,  
3. $\Phi_0 + \Phi_2 < \theta_{k-2} + 4^{k-2}$,

where $\theta_{k-2} = (4^{k-1} - 1)/3$.

There are many more extendable $[n,k,d]_4$ codes satisfying none of the conditions of Theorem 1.1. Our purpose in this paper is to show the following new extension theorems for quaternary linear codes.

**Theorem 1.2.** Let $C$ be an $[n,k,d]_4$ code with diversity $(\Phi_0, \Phi_1, \Phi_2), k \geq 3$, $d$ odd. Then $C$ is extendable if $\Phi_0 + \Phi_2 = \theta_{k-2} + 2 \times 4^{k-2}$. 
Theorem 1.3. Let $C$ be an $[n, k, d]_4$ code with diversity $(\Phi_0, \Phi_1, \Phi_2)$, $k \geq 3$, $d$ odd. Then

1. $\Phi_0 \geq \theta_{k-4}$.
2. $(\Phi_1, \Phi_2) = (3 \times 4^{k-3}, 9 \times 4^{k-3})$ or $(9 \times 4^{k-3}, 9 \times 4^{k-3})$ when $\Phi_0 = \theta_{k-4}$.
3. $C$ is extendable when $\Phi_0 = \theta_{k-4}$.

Since $\Phi_0 + \Phi_2 = \theta_{k-2} + 4^{k-2}$ for $\Phi_0$ and $\Phi_2$ in Theorem 1.3(2), an $[n, k, d]_4$ code with $\Phi_0 = \theta_{k-4}$ satisfies none of the conditions in Theorem 1.1.

As for the case when $k = 3$, we solve the Extension Problem as follows.

Theorem 1.4. Let $C$ be an $[n, 3, d]_4$ code with diversity $(\Phi_0, \Phi_1, \Phi_2)$, $d$ odd. Then

1. $\Phi_0 + \Phi_2 \in \{5, 9, 13\}$.
2. $C$ is extendable if $\Phi_0 + \Phi_2 \neq 9$.
3. $C$ is extendable if $(\Phi_0, \Phi_1, \Phi_2) \notin \{(6, 1, 3), (6, 3, 3), (2, 3, 7)\}$.
4. $C$ is extendable if $A_d < 24$ when $(\Phi_0, \Phi_1, \Phi_2) \in \{(6, 1, 3), (6, 3, 3), (2, 3, 7)\}$.
5. $C$ is not extendable if $A_d > 30$ when $(\Phi_0, \Phi_1, \Phi_2) = (6, 1, 3)$.
6. $C$ is not extendable if $A_d > 24$ when $(\Phi_0, \Phi_1, \Phi_2) = (6, 3, 3)$ or $(2, 3, 7)$.

Furthermore, we give necessary and sufficient conditions for the non-extendability of $C$ when

(i) $24 \leq A_d \leq 30$, $(\Phi_0, \Phi_1, \Phi_2) = (6, 1, 3)$,
(ii) $A_d = 24$, $(\Phi_0, \Phi_1, \Phi_2) = (6, 3, 3)$ or $(2, 3, 7)$

by means of geometrical terms in Section 3 (Theorems 3.1–3.3).

The following theorem gives an algorithm for finding a column to be added to a generator matrix of $C$ to get an extension of $C$ when $C$ satisfies one of the sufficient conditions in Theorems 1.2, 1.3.

Theorem 1.5. Let $C$ be an $[n, k, d]_4$ code with diversity $(\Phi_0, \Phi_1, \Phi_2)$, $k \geq 3$, $d$ odd, satisfying $\Phi_0 + \Phi_2 = \theta_{k-2} + 2 \times 4^{k-2}$ or $\Phi_0 = \theta_{k-4}$. Let $G = [g_1, g_2, \ldots, g_k]^T$ be a generator matrix for $C$ with rows $g_1, \ldots, g_k$ in $V(n, 4)$. Then, one can select $k - 1$ codewords $c_1, \ldots, c_{k-1}$ with

$$ c_i = a_1g_1 + \cdots + a_ig_k \quad (1 \leq i \leq k - 1), $$

so that $a_1, \ldots, a_{k-1}$ are linearly independent, where $a_i = (a_{i1}, \ldots, a_{ik}) \in V(k, 4)$, and that

(i) $\text{wt}(c_i) \equiv 0 \pmod{4}$ for $1 \leq i \leq k - 3$ and $\text{wt}((\mu a_{k-1} + v a_{k-2}) \cdot G) \not\equiv 0, d \pmod{4}$ for all $\mu, v \in \text{GF}(4)$, $(\mu, v) \neq (0, 0)$ when $\Phi_0 = \theta_{k-4}$.
(ii) $\text{wt}(\sum_{i=1}^{k-1} \mu_i a_i) \equiv 0 \pmod{2}$ for all $(\mu_1, \ldots, \mu_{k-1}) \in V(k - 1, 4)$ when $\Phi_0 + \Phi_2 = \theta_{k-2} + 2 \times 4^{k-2}$.

Taking a non-zero vector $h^T \in V(k, 4)$ orthogonal to $a_1, \ldots, a_{k-1}$, $[G, h]$ generates an extension of $C$, where $M^T$ denotes the transpose of a matrix $M$.

An important application of the extension theorems is in proving new results about optimal linear codes by showing that a code with certain parameters cannot exist, see [7,10,14]. An
application in finite geometry is given by Hill [7]. Another application is finding new codes from known ones, e.g. (1) of the next example.

Example 1. Let \( \omega \) and \( \bar{\omega} \) be the roots of \( x^2 + x + 1 \) over GF(2) so that GF(4) = \{0, 1, \omega, \bar{\omega} = \omega^2\}.

(1) Let \( C_1 \) be the [138, 6, 99]_4 code with the weight distribution

\[
\begin{align*}
0^{199} & 1^{45} 0^{100} 3^{78} 1^{102} 1^{216} 1^{102} 1^{029} 1^{103} 1^{342} 1^{104} 1^{234} 1^{105} 1^{288} 1^{106} 1^{135} 1^{107} 1^{18} 1^{106} 1^{69} 1^{109} 1^{36} 1^{110} 1^{63} 1^{111} 1^{42} 1^{112} 1^{18} 1^{114} 1^{108} 1^{116} 1^{108} 1^{117} 1^{36} 1^{118} 1^{108} 1^{120} 3
\end{align*}
\]

(division (405, 192, 448))

found by Gulliver and Östergård [4]. Then, by Theorem 1.2, \( C_1 \) is extendable to a [139, 6, 100]_4 code, which is new [2]. Note that all of a [96, 6, 67]_4 code, a [190, 6, 137]_4 code and a [198, 6, 143]_4 code found in [4] are also extendable by Theorem 1.1.

(2) Let \( C_2 \) be the [17, 4, 7]_4 code with generator matrix

\[
G_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & \omega & \bar{\omega} & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & \bar{\omega} & 1 & 1 & \bar{\omega} & 1 & \omega & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & \omega & \bar{\omega} & 1 & 1 & \bar{\omega} & 0 & \bar{\omega} & 1 & \omega & \bar{\omega}
\end{bmatrix}.
\]

The weight distribution of \( C_2 \) is \( 0^{17} 3^{9} 10^{33} 11^{9} 12^{3} 13^{99} 14^{75} 15^{24} \) (division (1, 36, 36)), and \( C_2 \) is extendable by Theorem 1.3. By Theorem 1.5 one can find \( h^T = (1, \omega, \bar{\omega}, 1) \) so that \([G_2, h]\) gives an extension of \( C_2 \) whose weight distribution is \( 0^{18} 3^{93} 10^{12} 11^{27} 12^{12} 13^{21} 14^{108} 15^{45} 16^{24} \).

2. Geometric preliminaries

We denote by PG\((r, q)\) the projective geometry of dimension \( r \) over GF\((q)\). A \( j \)-flat is a projective subspace of dimension \( j \) in PG\((r, q)\). 0-flats, 1-flats, 2-flats, 3-flats and \((r - 1)\)-flats are called points, lines, planes, solids and hyperplanes respectively as usual. We denote by \( F_j \) the set of \( j \)-flats of PG\((r, q)\) and denote by \( \theta_j \) the number of points in a \( j \)-flat, i.e. \( \theta_j = (q^{j + 1} - 1)/(q - 1) \). We set \( \theta_{-1} = 0 \).

Let \( C \) be an \([n, k, d]_q\) code with a generator matrix \( G \). Then the columns of \( G \) can be considered as a multiset of \( n \) points in \( \Sigma = \text{PG}(k - 1, q) \) denoted also by \( C \). An \( i \)-point is a point of \( \Sigma \) which has multiplicity \( i \) in \( C \). Denote by \( \gamma_0 \) the maximum multiplicity of a point from \( \Sigma \) in \( C \) and let \( C_i \) be the set of \( i \)-points in \( \Sigma \) for \( 0 \leq i \leq \gamma_0 \). For any subset \( S \) of \( \Sigma \) we define the multiplicity of \( S \) with respect to \( C \), denoted by \( m_C(S) \), as

\[
m_C(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap C_i|,
\]

where \(|T|\) denotes the number of points in \( T \) for a subset \( T \) of \( \Sigma \). Then we obtain the partition \( \Sigma = C_0 \cup C_1 \cup \cdots \cup C_{\gamma_0} \) such that

\[
n = m_C(\Sigma),
\]

\[
n - d = \max \{ m_C(\pi) \mid \pi \in \mathcal{F}_{k-2} \}.
\]
Conversely such a partition of $\Sigma$ as above gives an $[n, k, d]_q$ code in the natural manner. Denote by $a_i$ the number of hyperplanes $\pi$ with $m_C(\pi) = i$. Note that

$$a_i = A_{n-i}/(q-1) \quad \text{for } 0 \leq i \leq n-d. \quad \text{(2.1)}$$

The list of $a_i$’s is called the spectrum of $C$. Simple counting arguments yield the following.

**Lemma 2.1.**

1. $$\sum_{i=0}^{n-d} a_i = \theta_{k-1}.$$
2. $$\sum_{i=1}^{n-d} ia_i = n\theta_{k-2}.$$
3. $$\sum_{i=2}^{n-d} \binom{i}{2} a_i = \binom{n}{2} \theta_{k-3} + q^{k-2} \sum_{s=2}^{\gamma_0} \binom{s}{2} \lambda_s, \quad \text{where } \lambda_s = |C_s|.$$

Since (length) − (minimum distance) = $n - d$ holds also for an extension of $C$, we get the following.

**Lemma 2.2.** $C$ is extendable if and only if there exists a point $P \in \Sigma$ such that $m_C(\pi) < n - d$ for all hyperplanes $\pi$ through $P$.

Let $\Sigma^*$ be the dual space of $\Sigma$ (considering $F_{k-2}$ as the set of points of $\Sigma^*$). Then Lemma 2.2 is equivalent to the following:

**Lemma 2.3.** $C$ is extendable if and only if there exists a hyperplane $\Pi$ of $\Sigma^*$ such that $\Pi \subset \{\pi \in F_{k-2} \mid m_C(\pi) < n - d\}$.

From now on, we assume that $C$ is an $[n, k, d]_4$ code with $d$ odd, $k \geq 3$. We define $F_i$, $0 \leq i \leq 3$, $F_d$, $F_e$, $F_{02}$ and $F$ as follows:

$$F_i = \{\pi \in F_{k-2} \mid m_C(\pi) \equiv n+i \pmod{4}\} \quad \text{for } 0 \leq i \leq 3 \text{ when } d \equiv 1 \pmod{4},$$

$$F_i = \{\pi \in F_{k-2} \mid m_C(\pi) \equiv n-i \pmod{4}\} \quad \text{for } 0 \leq i \leq 3 \text{ when } d \equiv 3 \pmod{4},$$

$$F_d = \{\pi \in F_{k-2} \mid m_C(\pi) = n-d\}, \quad F_e = F_3 \setminus F_d,$$

$$F_{02} = F_0 \cup F_2, \quad F = F_0 \cup F_1 \cup F_2.$$

Then, by (2.1), we have

$$\Sigma^* = F \cup F_3,$$

$$|F_d| = a_{n-d} = A_d/3,$$

$$\Phi_i = |F_i| \quad \text{for } 0 \leq i \leq 2.$$
Lemma 2.4. $C$ is extendable if and only if $F \cup F_c$ contains a hyperplane of $\Sigma^*$.

A subset $B$ of $\text{PG}(r, q)$ is called a blocking set with respect to $j$-flats if every $j$-flat of $\text{PG}(r, q)$ meets $B$ in at least one point. $B$ is called non-trivial if $B$ contains no $(r - j)$-flat of $\text{PG}(r, q)$, see [9]. Since $F$ forms a blocking set with respect to lines in $\Sigma^*$ [16], $F_d$ contains no line of $\Sigma^*$. Hence Lemma 2.4 is equivalent to the following:

Lemma 2.5. $C$ is not extendable if and only if $F_d$ forms a non-trivial blocking set of $\Sigma^*$ with respect to hyperplanes.

A $t$-flat $\Pi$ of $\Sigma^*$ with $|\Pi \cap F_0| = h$, $|\Pi \cap F_1| = i$, $|\Pi \cap F_2| = j$ is called an $(h, i, j)_t$-flat. An $(h, i, j)_1$-flat is called an $(h, i, j)$-line. An $(h, i, j)$-plane, an $(h, i, j)$-solid and an $(h, i, j)$-hyperplane are defined similarly.

Lemma 2.6. Let $l$ be a line of $\Sigma^*$ with $F \cap l = \{\pi_1, \ldots, \pi_u\}$, $m_C(\pi_i) = s_i$, $1 \leq i \leq u$. Then $\sum_{i=1}^u s_i \equiv n + (u - 1)(n - d) \pmod{4}$.

Proof. Put $l \setminus F = \{\pi_{u+1}, \ldots, \pi_{\theta_1}\}$. Then $m_C(\pi_j) \equiv n - d \pmod{4}$ for $u + 1 \leq j \leq \theta_1$. Since $\sum_{i=1}^{\theta_1} (m_C(\pi_i) - m_C(l)) + m_C(l) = n$, we get $\sum_{i=1}^u s_i + (\theta_1 - u)(n - d) \equiv n \pmod{4}$. □

We denote by $F^j_*$ the set of $j$-flats of $\Sigma^*$, so $F^j_* = F_{k-2-j}$, $0 \leq j \leq k - 2$. From Lemma 2.6, all possible $|l \cap F|$ and $|l \cap F_i|$ $(i = 0, 1, 2)$ for $l \in F^*_1$ are as in Table 2.1 for $d \equiv \pm 1 \pmod{4}$.

From Table 2.1 we get the following:

Lemma 2.7. Every $(h, i, j)$-line satisfies $h + j \in \{1, 3, 5\}$.

Corollary 2.1. For any $(h, i, j)_t$-flat in $\Sigma^*$, $t \geq 1$, it holds that $h + j$ is odd.

Remark 1. It follows from Lemma 2.7 that $F_{02} = F_0 \cup F_2$ forms a blocking set with respect to lines meeting every line in one, three or five points in $\Sigma^*$. In general, the smallest non-trivial minimal blocking set with respect to lines in $\text{PG}(r, 4)$ are cones with an $(r - 3)$-flat as the vertex and a Fano plane as a base [1]. Hence $C$ is extendable if $|F_{02}| = \Phi_0 + \Phi_2 < \theta_{k-2} + 2 \times 4^{k-3}$, where $\theta_{k-2} + 2 \times 4^{k-3}$ is the cardinality of such a cone. But a stronger fact is known for $F_{02}$. Actually, $F_{02}$ is a trivial blocking set if $|F_{02}| < \theta_{k-2} + 4^{k-2}$, giving Theorem 1.1(3) [16].

\begin{table}[h]
\centering
\begin{tabular}{c|cccccccccccc}
\hline
$|l \cap F|$ & 1 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 5 \\
\hline
$|l \cap F_0|$ & 1 & 0 & 2 & 1 & 0 & 3 & 0 & 1 & 5 & 3 & 1 & 1 & 0 & 2 \\
$|l \cap F_1|$ & 0 & 1 & 0 & 2 & 0 & 1 & 3 & 1 & 0 & 0 & 0 & 4 & 2 & 2 \\
$|l \cap F_2|$ & 0 & 1 & 1 & 0 & 3 & 0 & 1 & 2 & 0 & 2 & 4 & 0 & 3 & 1 \\
\hline
\end{tabular}
\caption{Table 2.1}
\end{table}
Now, assume $2 \leq t \leq k - 1$ and let $\Pi \in \mathcal{F}_t^s$. Denote by $c_{h,i,j}^{(t)}$ the number of $(h, i, j)_{t-1}$ flats in $\Pi$ and let $\varphi_s^{(t)} = |\Pi \cap F_s|$, $s = 0, 1, 2$. $(\varphi_0^{(t)}, \varphi_1^{(t)}, \varphi_2^{(t)})$ is called the diversity of $\Pi$ and the list of $c_{h,i,j}^{(t)}$’s is called its spectrum. Counting arguments similar to Lemma 2.1 yield the following:

\[ \sum_{(h,i,j) \in A_{t-1}} c_{h,i,j}^{(t)} = \theta_1, \] (2.2)

\[ \sum_{(i_0,i_1,i_2) \in A_{t-1}} i_j c_{i_0,i_1,i_2}^{(t)} = \theta_{t-1} \varphi_j^{(t)} \quad \text{for } j = 0, 1, 2, \] (2.3)

\[ \sum_{(i_0,i_1,i_2) \in A_{t-1}} \left( \frac{i_j}{2} \right) c_{i_0,i_1,i_2}^{(t)} = \theta_{t-2} \left( \varphi_h^{(t)} + \varphi_j^{(t)} \right) \quad \text{for } j = 0, 1, 2, \] (2.4)

\[ \sum_{(i_0,i_1,i_2) \in A_{t-1}} \left( i_h + \frac{i_j}{2} \right) c_{i_0,i_1,i_2}^{(t)} = \theta_{t-2} \left( \varphi_h^{(t)} + \varphi_j^{(t)} \right) \quad \text{for } (h, j) = (0, 1), (0, 2), (1, 2), \] (2.5)

where $A_{t-1}$ is the set of all possible $(\varphi_0^{(t-1)}, \varphi_1^{(t-1)}, \varphi_2^{(t-1)})$. Note that (2.5) can be replaced by

\[ \sum_{(i_0,i_1,i_2) \in A_{t-1}} i_h i_j c_{i_0,i_1,i_2}^{(t)} = \theta_{t-2} \varphi_h^{(t)} \varphi_j^{(t)} \quad \text{for } (h, j) = (0, 1), (0, 2), (1, 2). \] (2.6)

We have already seen that

\[ A_1 = \{(1,0,0), (0,1,1), (2,0,1), (1,2,0), (0,0,3), (3,1,0), (0,3,1), \]

\[ (1,1,2), (5,0,0), (3,0,2), (1,0,4), (1,4,0), (0,2,3), (2,2,1)\}. \]

3. Proof of Theorem 1.4

We first determine all possible $(\varphi_0^{(2)}, \varphi_1^{(2)}, \varphi_2^{(2)}) \in A_2$ and the corresponding spectra. Let $\delta$ be a $(\varphi_0, \varphi_1, \varphi_2)$-plane of $\Sigma^*$ whose spectrum is the list of $c_{h,i,j}$’s. Since every line meets $F_{02}$ in at least one point by Lemma 2.7, $\varphi_0 + \varphi_2 \geq 5$. Corollary 2.1 implies that $\varphi_0 + \varphi_2$ is odd. Suppose $\varphi_0 + \varphi_2 = 19$. Then we can take a line on $\delta$ which meets $F_{02}$ in exactly four points, contradicting Lemma 2.7. We can also show $\varphi_0 + \varphi_2 \neq 17$ similarly. Hence

\[ \varphi_0 + \varphi_2 \in \{5, 7, 9, 11, 13, 15, 21\}. \]

It is obvious that $c_{h,i,j} > 0$ implies $h + j = 5$ if and only if $\varphi_0 + \varphi_2 = 21 (= \theta_2)$. Solving (2.2)–(2.5) under the condition $\varphi_0 + \varphi_2 = 21$ with the aid of a computer, we get all the solutions as Table 3.1.

It can be confirmed that (2.2)–(2.5) have no solution when $\varphi_0 + \varphi_2 \in \{7, 11, 15\}$ with the aid of a computer, but we give a computer-free proof in Lemma 3.4 below. From now on in this section, let $C$ be an $[n, 3, d]$ code with odd $d$ whose diversity is $(\varphi_0, \varphi_1, \varphi_2)$. Then $\delta = \Sigma^*$ and $\varphi_0 + \varphi_2 < 21$ since $F_{02} \neq \Sigma^*$. 


Table 3.1
$(\phi_0 + \phi_2 = 21)$

<table>
<thead>
<tr>
<th>$\phi_0$</th>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$c_{5,0,0}$</th>
<th>$c_{3,0,2}$</th>
<th>$c_{1,0,4}$</th>
</tr>
</thead>
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<td>16</td>
<td>1</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
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<td>14</td>
<td>0</td>
<td>7</td>
<td>14</td>
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<td>0</td>
<td>21</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Lemma 3.1.

1. $c_{h,i,j} > 0$ implies $h + j = 3$ or $5$ if and only if $\phi_0 + \phi_2 = 15$.
2. $c_{h,i,j} > 0$ implies $h + j = 1$ or $5$ if and only if $\phi_0 + \phi_2 = 5$.
3. $c_{h,i,j} > 0$ implies $h + j = 1$ or $3$ if and only if $\phi_0 + \phi_2 = 7$ or $9$.
4. $\delta$ contains an $(h_m, i_m, j_m)$-line with $h_m + j_m = 2m - 1$ for $m = 1, 2, 3$ if and only if $\phi_0 + \phi_2 = 11$ or $13$.

Proof. (4) is obtained from (1)–(3). Using (2.2)–(2.6), it can be calculated that

$$\sum_{(h,i,j) \in \Lambda_1} (h + j - s)(h + j - t)c_{h,i,j} = X(X - 1) + 5X(1 - s - t) + 21st$$

for integers $s$ and $t$, where $X = \phi_0 + \phi_2$. So, we have

$$\sum_{(h,i,j) \in \Lambda_1} (h + j - 3)(h + j - 5)c_{h,i,j} = (X - 15)(X - 21),$$

$$\sum_{(h,i,j) \in \Lambda_1} (h + j - 1)(h + j - 5)c_{h,i,j} = (X - 5)(X - 21),$$

$$\sum_{(h,i,j) \in \Lambda_1} (h + j - 1)(h + j - 3)c_{h,i,j} = (X - 7)(X - 9).$$

Hence our assertion follows. □

A $\kappa$-set $K$ in PG$(2, q)$ no $\nu + 1$ points of which are collinear is called a $(\kappa, \nu)$-arc. Recall that a blocking set consisting of 7 points in PG$(2, 4)$ is a Fano plane. See [8] for arcs and blocking sets in PG$(2, q)$.

Lemma 3.2.

1. $F_{02}$ forms a line if $\phi_0 + \phi_2 = 5$.
2. $F_{02}$ forms a Fano plane if $\phi_0 + \phi_2 = 7$.
3. $F_{02}$ consists of a line and a $(6, 2)$-arc if $\phi_0 + \phi_2 = 11$.
4. $\delta \setminus F_{02}$ forms a $(6, 2)$-arc if $\phi_0 + \phi_2 = 15$. 

Proof. (1), (2), (4) are straightforward from Lemma 3.1. Assume \( \varphi_0 + \varphi_2 = 11 \). By Lemma 3.1(4), \( F_{02} \) contains a line, say \( l \). Suppose that \( \delta \) contains another line \( l' \) and let \( l \cap l' = \{ P \} \). Then the other two points of \( F_{02} \) not in \( l \cup l' \) and \( P \) must be collinear by Lemma 2.7. But a line containing none of these three points meets \( F_{02} \) in exactly two points, a contradiction. Hence \( F_{02} \) contains only one line and \( F_{02} \setminus l \) forms a \((6, 2)\)-arc. □

Lemma 3.3. \( C \) satisfies

\[ \varphi_0 + \varphi_2 = \sum_{i \equiv n \pmod{2}} a_i \not\equiv 3 \pmod{4} \quad \text{if} \quad \sum_{i \equiv n \pmod{2}} i a_i \equiv n \pmod{2}. \]

Proof. Suppose

\[ \sum_{i \equiv n \pmod{2}} a_i \equiv 3 \pmod{4}. \] (3.1)

Assume that \( n \) is odd and let

\[ \sum_{i \equiv n \pmod{2}} i a_i = n + 2s \] (3.2)

for some integer \( s \). By Lemma 2.1(2) we have

\[ \sum_{i \equiv n-d \pmod{2}} i a_i = 4n - 2s. \] (3.3)

Substituting (3.1) and (3.2) in the equality of Lemma 2.1(3), we get

\[ \sum_i \binom{i}{2} a_i \equiv \frac{n(n-1)}{2} \equiv \frac{n-1}{2} \equiv \sum_{i \equiv n \pmod{2}} \frac{i-1}{2} a_i - s + 1 \pmod{2} \]

whence, by (3.3), we obtain

\[ \sum_{i \equiv n \pmod{2}} \frac{(i-1)^2}{2} a_i + \sum_{i \equiv n-d \pmod{2}} \frac{i^2 - 2i}{2} a_i \equiv \sum_{i \equiv n-d \pmod{2}} \frac{i}{2} a_i - s + 1 \equiv 1 \pmod{2} \]

giving a contradiction (the left-hand side is even).

A similar argument for the case when \( n \) is even also yields a contradiction. □

Lemma 3.4. It holds that \( \varphi_0 + \varphi_2 \notin \{7, 11, 15\} \).

Proof. Recall that \( \delta = \Sigma^* \) is the set of lines in \( \Sigma = \text{PG}(2, 4) \).

(1) Suppose \( \varphi_0 + \varphi_2 = 7 \). Let \( F_{02} = \{l_1, \ldots, l_7\} \) and let \( H \) be the set of points of \( \Sigma \) such that three lines of \( l_1, \ldots, l_7 \) are passing through. Then \( H \) forms a Fano plane by Lemma 3.2(2), and there is only one of \( l_1, \ldots, l_7 \) through \( Q \) for any point \( Q \in \Sigma \setminus H \). Hence

\[ \sum_{i \equiv n \pmod{2}} i a_i = \sum_{i=1}^{7} m_C(l_i) = n + 2m_C(H) \]

which is contradictory to Lemma 3.3.
(2) Suppose \( \varphi_0 + \varphi_2 = 11 \). \( F_{02} \) consists of a line and a \((6, 2)\)-arc by Lemma 3.2(3). Let \( P \) be the point of \( \Sigma \) corresponding to the line of \( \delta \) contained in \( F_{02} \) and let \( F_{02} \setminus P = \{l_1, \ldots, l_6\} \). Setting \( H = l_1 \cup \cdots \cup l_6 \) as a subset of \( \Sigma \), we get

\[
\sum_{i \equiv n \pmod{2}} ia_i = n + 2mc_c(H) + 4mc_c(P)
\]

giving a contradiction by Lemma 3.3.

(3) Suppose \( \varphi_0 + \varphi_2 = 15 \). Let \( \delta \setminus F_{02} = \{l_1, \ldots, l_6\} \) and let \( H = \Sigma \setminus (l_1 \cup \cdots \cup l_6) \) as the set of points of \( \Sigma \). Then, by Lemma 3.2(4), we obtain

\[
\sum_{i \equiv n \pmod{2}} ia_i = 3n + 2mc_c(H)
\]

giving a contradiction again. \( \square \)

Thus we have proved Theorem 1.4(1). Theorem 1.4(2) follows by Lemma 2.4 since \( F_{02} \) contains a line of \( \delta \) when \( \varphi_0 + \varphi_2 \neq 9 \) by Lemmas 3.1, 3.2, 3.4. When \( \varphi_0 + \varphi_2 = 9 \), \( F_{02} \) forms a minimal blocking 9-set (a Hermitian curve [8, Theorem 13.13]) by Lemma 3.1(3).

Solving (2.2)–(2.5) with the aid of a computer, one can get Tables 3.2–3.4 under the conditions \( \varphi_0 + \varphi_2 = 5, 9, 13 \), respectively. We have confirmed that quaternary linear codes of dimension three corresponding to each spectrum of Tables 3.2–3.4 do exist [11]. Tables 3.1–3.4 are used to investigate the extendability of quaternary linear codes of dimension \( k \geq 4 \) in Section 4.

From Table 3.3, the diversities satisfying \( c_{h,i,j} = 0 \) for all \((h, i, j) \in \Lambda_1 \) with \( h + i + j = 5 \) are

\[
(\varphi_0, \varphi_1, \varphi_2) = (2, 3, 7), (6, 1, 3), (6, 3, 3).
\]

Hence Theorem 1.4(3) follows. We give the necessary and sufficient conditions for the non-extendability of \([n, 3, d]_4\) codes with such diversities.
Table 3.3

\[(\varphi_0 + \varphi_2 = 9)\]

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We denote by \((\chi_1, \chi_2, \ldots)\) the smallest flat containing subsets \(\chi_1, \chi_2, \ldots\) of \(\Sigma^*\).

**Theorem 3.1.** Assume that \((\varphi_0, \varphi_1, \varphi_2) = (2, 3, 7)\). Let \(l\) be the \((2, 0, 1)\)-line in \(\delta\) and put \(l \cap F_3 = \{B_1, B_2\}\). Then \(C\) is not extendable if and only if either

1. \(|F_d| = 9\) or
2. \(|F_d| = 8, F_e = \{B_i\}, i = 1 \ or \ 2\).

**Proof.** From Table 3.3, the spectrum of \(\delta\) is

\[(c_{1,0,0}, c_{0,1,1}, c_{2,0,1}, c_{0,0,3}, c_{0,3,1}, c_{1,1,2}) = (2, 6, 1, 5, 1, 6)\].

Let \(l = \{B_1, B_2, P_1, P_2, R\}, P_1, P_2 \in F_0, R \in F_2\), and let \(m\) be the \((0, 3, 1)\)-line, \(m \cap F_1 = \{Q_1, Q_2, Q_3\}\). If \(l \cap m = \{B_1\}\), then the other three lines through \(B_1\) must be \((0, 0, 3)\)-lines, so \(\varphi_2 \geq 1 + 1 + 3 \times 3 = 11\), a contradiction. Hence \(l \cap m = \{R\}\). Let \(B_0\) be the point of \(F_3\) on \(m\). Then \(\langle B_0, B_i \rangle\) is a \((0, 0, 3)\)-line for \(i = 1, 2\), and \(\langle B_j, Q_j \rangle\) is a \((0, 1, 1)\)-line for \(i = 1, 2, j = 1, 2, 3\). This implies that every point of \(F_3\) other than \(B_0, B_1, B_2\) lies on a \((1, 1, 2)\)-line, so the points of \(F_3\) other than \(B_1, B_2\) must belong to \(F_d\) if \(C\) is not extendable by Lemma 2.5. Since there is no \((h, i, j)\)-line through \(B_i\) with \(h + j = 4\) for \(i = 1, 2\), at most one of \(B_1, B_2\) may remain in \(F_e\). Hence our assertion follows. \(\square\)
Table 3.4
($\phi_0 + \phi_2 = 13$)

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Theorem 3.2. Assume that $(\phi_0, \phi_1, \phi_2) = (6, 1, 3)$. Let $l$ be the $(0, 0, 3)$-line in $\delta$. Then $C$ is not extendable if and only if one of the following conditions (1)–(5) holds:

1. $|F_d| = 11$,
2. $|F_d| = 10$, $F_e \cap l = \emptyset$,
3. $|F_d| = 9$, $F_e$ is contained in a $(1, 0, 0)$-line or a $(0, 1, 1)$-line, $F_e \cap l = \emptyset$,
4. $|F_d| = 8$, $F_e$ is contained in a $(1, 0, 0)$-line, $F_e \cap l = \emptyset$,
5. $|F_d| = 8$, $F_e$ forms a $(3, 2)$-arc, $F_e \cap l = \emptyset$, and any two of $F_e$ are contained in a $(1, 0, 0)$-line or a $(0, 1, 1)$-line.

Proof. From Table 3.3, the spectrum of $\delta$ is

$$(c_{1,0,0}, c_{0,1,1}, c_{2,0,1}, c_{0,0,3}, c_{3,1,0}) = (6, 3, 9, 1, 2).$$
Assume that $C$ is not extendable. Let $l_1, l_2$ be the $(3, 1, 0)$-lines. Then $l_i \cap l$ must be in $F_d$, so in $F_d$ by Lemma 2.5 for $i = 1, 2$, whence $F_3 \cap l$ is contained in $F_d$. If $|F_d| = 7$, $F_d$ forms a Fano plane, which is impossible since $|F_d \cap l| = 2$. Hence $|F_d| \geq 8$. The conditions (1)–(5) are obtained from Lemma 2.5 and the spectrum of $\delta$. The “if” part is straightforward. □

**Theorem 3.3.** Assume that $(\varphi_0, \varphi_1, \varphi_2) = (6, 3, 3).$ Let $l_1, l_2, l_3$ be $(3, 1, 0)$-lines and $l_4, l_5, l_6$ be $(1, 1, 2)$-lines. Put $K = F_3 \setminus (l_1 \cup \cdots \cup l_6)$. Then $|K| = 3$, and $C$ is not extendable if and only if either

1. $|F_d| = 9$ or
2. $|F_d| = 8$, $F_e = \{ B \}$ for some $B \in K$.

**Proof.** The spectrum of $\delta$ is

$$(c_{1,0,0}, c_{0,1,1}, c_{2,0,1}, c_{1,2,0}, c_{3,1,0}, c_{1,1,2}) = (3, 6, 3, 3, 3).$$

Let $l_i \cap F_3 = \{ B_i \}$, $1 \leq i \leq 6$. $B_1, B_2, B_3$ are distinct since any two of $l_1, l_2, l_3$ meets in a point of $F_0, B_4, B_5, B_6$ are also distinct as well. If $B_1 = B_4$, then all of the three $(1, 0, 0)$-lines must pass through the point, so $\varphi_0 = 7$, a contradiction. Hence $B_1, \ldots, B_6$ are distinct and $|K| = 3$.

Assume that $C$ is not extendable. It holds that $B_1, \ldots, B_6 \in F_d$ by Lemma 2.5, so $F_e \subset K$. Suppose $|F_d| = 7$ so that $F_d$ forms a Fano plane. Let $S_0$ be a point of $F_d$ on a $(1, 1, 2)$-line. The lines through $S_0$ are one $(1, 1, 2)$-line, two $(1, 0, 0)$-lines $m_1, m_2$, one $(1, 2, 0)$-line $m_3$ containing another point $S_1$ of $F_3$ and one $(2, 0, 1)$-line $m_4$ containing another point $S_2$ of $F_3$. Since $m_3$ and $m_4$ have only two points of $F_3$ and since $S_0 \in F_d$, we have $S_1, S_2 \notin F_d$. Thus $F_d$ is contained in $m_1 \cup m_2$, which is impossible if $F_d$ is a Fano plane. Hence $|F_d| \geq 8$, so $|F_e| = 0$ or 1. This completes the proof of “only if” part. When $F_e = \{ B \}$ for some $B \in K$, the lines through $B$ are one $(1, 0, 0)$-line, one $(0, 1, 1)$-line, two $(2, 0, 1)$-lines and one $(1, 2, 0)$-line. Hence “if” part follows. □

Since $|F_d| = a_{n-d} = A_d/3$ by (2.1), we obtain Theorem 1.4(4)–(6) from Theorems 3.1–3.3. Geometrical characterizations of $(\varphi_0, \varphi_1, \varphi_2)$-planes are given by Kawakami [11] for all $(\varphi_0, \varphi_1, \varphi_2) \in A_2$. Determining $A_3$ and the corresponding spectra will be difficult even using a computer.

**Example 2.** Note that a line of $\Sigma$ given by the equation

$$a_0X_0 + a_1X_1 + a_2X_2 = 0, \quad (a_0, a_1, a_2) \neq (0, 0, 0), \quad a_0, a_1, a_2 \in GF(4),$$

can be seen as a point of $\Sigma^*$ with the coordinate vector $(a_0, a_1, a_2)$ and vice versa [8]. We give two examples with the same weight distribution such that one is extendable but the other is not.

1. Let $C_1$ be the $[17, 3, 11]_4$ code with generator matrix

$$G_1 = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & \omega & \omega & \omega & \omega & \omega & \omega & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & \omega & 0 & \omega & \omega & \omega & \omega & \omega & 1
\end{bmatrix}.$$

The weight distribution of $C_1$ is $0^{11}1^{24}2^{9}3^{14}4^{9}5^{9}6^{9}$ (diversity $(6, 1, 3)$), so $|F_d| = A_{11}/3 = 8$. Two points $P_1(1, \omega, 0)$ and $P_2(0, 1, \omega)$ are in $F_0$, for $(1, \omega, 0)G_1$ and $(0, 1, \omega)G_1$
are codewords with weight 12. It can be checked that the line \( l = \langle P_1, P_2 \rangle \) is a \((3, 1, 0)\)-line which has the point \((1, 1, \omega)\) of \( \mathcal{F}_e \) giving a codeword of weight 15. Since \( l \) is given by the equation \( X_0 + \bar{\omega}X_1 + X_2 = 0 \), adding the column vector \([1, \bar{\omega}, 1]^T\) to \( G_1 \) we get an extension of \( C_1 \) with the weight distribution \( 0^{12}1^{30}13^615^12^616^917^6 \).

(2) Let \( C_2 \) be the \([17, 3, 11]_4\) code with generator matrix
\[
G_2 = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & \bar{\omega} & 1 & 0 & \omega & 0 & 1 & \omega \\
0 & 1 & 0 & 1 & 0 & \omega & \omega & 1 & 1 & 1 & \bar{\omega} & 1 & 0 & \omega & 0 & 1 & \omega & 1 & 1 & 1 & \omega & \omega & \bar{\omega} & 1 & \omega & 0
\end{bmatrix}.
\]
The weight distribution of \( C_2 \) is \( 0^{11}1^{24}12^913^314^915^616^9 \), just the same as for \( C_1 \)’s. We have \( F_e = \{B_1(1, 0, \omega), B_2(0, 1, \omega), B_3(1, \omega, 0)\} \) since \((1, 0, \omega)G_2, (0, 1, \omega)G_2, (1, \omega, 0)G_2\) are codewords with weight 15. It can be shown that \( \langle B_1, B_2 \rangle, \langle B_1, B_3 \rangle \) are \((1, 0, 0)\)-lines and that \( \langle B_2, B_3 \rangle \) is a \((0, 1, 1)\)-line. Hence, satisfying the condition (5) of Theorem 3.2, \( C_2 \) is not extendable.

We define \( \Psi_t \) from \( \Lambda_t \) as:
\[
\Psi_t = \left\{ \varphi_0^{(t)} + \varphi_2^{(t)} \mid (\varphi_0^{(t)}, \varphi_1^{(t)}, \varphi_2^{(t)}) \in \Lambda_t \right\}.
\]
We have seen the following:

**Lemma 3.5.**

(1) \( \Psi_1 = \{1, 3, 5\} \).
(2) \( \Psi_2 = \{5, 9, 13, 21\} \).

It is known that the smallest value in \( \Psi_t \) is \( \theta_t - 1 \) and that \( C \) is extendable when \( \Psi_{t-1} = \theta_{t-2} \) [16]. Let \( \psi_{2t} = 2\theta_{2t-1} - \theta_{t-1} \) and \( \psi_{2t+1} = (\theta_{2t+1} + \theta_t)/2 \) for \( t \geq 1 \). We pose the following conjecture, which is true for \( t = 2 \).

**Conjecture.**

(1) \( \Psi_t = \{4a + 1 \mid a \in \Psi_{t-1}\} \cup \{\psi_t\} \) for \( t \geq 2 \).
(2) \( F_{2t} \cap \Pi \) forms a non-singular Hermitian variety (see [8]) in a \((\varphi_0^{(t)}, \varphi_1^{(t)}, \varphi_2^{(t)})\) flat \( \Pi \) of \( \Sigma^* \) if \( \psi_0^{(t)} + \psi_2^{(t)} = \psi_t \) for \( t \geq 2 \).

**Remark 2.** We have solved the simultaneous equations (2.2)–(2.5) for \( t = 2 \) to get the necessary and sufficient conditions for the extendability of quaternary linear codes of dimension three. But it looks infeasible to solve (2.2)–(2.5) completely for \( t \geq 3 \) even with the aid of a computer.

### 4. Proof of Theorems 1.2, 1.3, 1.5

In this section we prove Theorems 1.2, 1.3, 1.5 using the determination of \((\varphi_0^{(2)}, \varphi_1^{(2)}, \varphi_2^{(2)})\)-planes in the previous section (Tables 3.1–3.4).
Lemma 4.1. Let $\Pi$ be a $t$-flat of $\Sigma^*$ with diversity $(\varphi_0, \varphi_1, \varphi_2)$, $t \geq 2$.

1. There exists an $(h, i, j)_{t-1}$ flat in $\Pi$ with $h + j = \theta_{t-1}$ if $\varphi_0 + \varphi_2 = \theta_{t-1} + 2 \times 4^{t-1}$.
2. It holds that $\varphi_0 + \varphi_2 = \theta_t$ if $\varphi_0 + \varphi_2 > \theta_{t-1} + 2 \times 4^{t-1}$.

Proof. We prove (1) and (2) simultaneously by induction on $t$. For $t = 2$, (1) and (2) hold by Lemmas 3.1(4), 3.5(2). Assume $t \geq 3$ and that $\varphi_0 + \varphi_2 = \theta_{t-1} + 2 \times 4^{t-1}$. Suppose $c_{h, i, j} = 0$ for all $(h, i, j) \in \Lambda_{t-1}$ with $h + j = \theta_{t-1}$. It follows from the induction hypothesis for (2) that $h + j \leq \theta_{t-2} + 2 \times 4^{t-2}$ for all $(h, i, j) \in \Lambda_{t-1}$ with $c_{h, i, j} > 0$. So, by (2.2) and (2.3), we have

$$\theta_{t-1}(\varphi_0 + \varphi_2) = \sum_{(h, i, j) \in \Lambda_{t-1}} (h + j)c_{h, i, j} \leq (\theta_{t-2} + 2 \times 4^{t-2}) \sum_{(h, i, j) \in \Lambda_{t-1}} c_{h, i, j} = (\theta_{t-2} + 2 \times 4^{t-2})(4\theta_{t-1} + 1),$$

whence

$$\varphi_0 + \varphi_2 \leq (\theta_{t-2} + 2 \times 4^{t-2}) \left(4 + \frac{1}{\theta_{t-1}}\right) = \theta_{t-1} - 1 + 2 \times 4^{t-1} + \frac{\theta_{t-2} + 2 \times 4^{t-2}}{\theta_{t-1}}.$$

Since $(\theta_{t-2} + 2 \times 4^{t-2})/\theta_{t-1} < 1$, we get $\varphi_0 + \varphi_2 \leq \theta_{t-1} + 2 \times 4^{t-1} - 1$, which contradicts our assumption. Hence (1) holds.

Next, we assume $\varphi_0 + \varphi_2 < \theta_t$ to prove (2). From the induction hypothesis for (2), we have

$$\sum_{(h, i, j) \in \Lambda_{t-1}} (h + j - \theta_{t-1})(h + j - \theta_{t-2} - 2 \times 4^{t-2})c_{h, i, j} \geq 0. \quad (4.1)$$

Using (2.2)–(2.5), the left-hand side of (4.1) can be calculated as

$$\sum_{(h, i, j) \in \Lambda_{t-1}} (h + j)(h + j - 1)c_{h, i, j} - (\theta_{t-1} + \theta_{t-2} + 2 \times 4^{t-2} - 1) \sum_{(h, i, j) \in \Lambda_{t-1}} (h + j)c_{h, i, j}$$

$$+ \theta_{t-1}(\theta_{t-2} + 2 \times 4^{t-2}) \sum_{(h, i, j) \in \Lambda_{t-1}} c_{h, i, j}$$

$$= 2\theta_{t-1} \left(\frac{\varphi_0 + \varphi_2}{2}\right) - (\theta_{t-1} + \theta_{t-2} + 2 \times 4^{t-2} - 1)\theta_{t-1}(\varphi_0 + \varphi_2) + \theta_{t-1}(\theta_{t-2} + 2 \times 4^{t-2})$$

$$= (\varphi_0 + \varphi_2 - \theta_t)\left\{\theta_{t-2}(\varphi_0 + \varphi_2) - \theta_{t-1}(\theta_{t-2} + 2 \times 4^{t-2})\right\}.$$

Since $\varphi_0 + \varphi_2 - \theta_t < 0$, we obtain

$$\theta_{t-2}(\varphi_0 + \varphi_2) - \theta_{t-1}(\theta_{t-2} + 2 \times 4^{t-2}) \leq 0,$$

whence

$$\varphi_0 + \varphi_2 \leq \theta_{t-1}\left(1 + \frac{\theta_{t-2} + 2 \times 4^{t-2}}{\theta_{t-2}}\right) = \theta_{t-1} + 2 \times 4^{t-1} + 1 + \frac{2\theta_{t-3} + 1}{\theta_{t-2}}.$$
Since \((2\theta_{t-3} + 1)/\theta_{t-2} < 1\) for \(t \geq 3\), we get \(\varphi_0 + \varphi_2 \leq \theta_{t-1} + 2 \times 4^{t-1} + 1\). Hence, by Corollary 2.1, we obtain \(\varphi_0 + \varphi_2 \leq \theta_{t-1} + 2 \times 4^{t-1}\). This completes the proof.  

Lemma 4.1(2) implies that \(F_{02}\) must be a trivial blocking set when \(\varphi_0 + \varphi_2 = \theta_{t-1} + 2 \times 4^{t-1}\). Hence Theorem 1.2 follows by Lemma 2.4.

Setting \(t = k - 1\), the following lemma yields Theorem 1.3(1).

**Lemma 4.2.** It holds that \(\varphi_{0}^{(t)} \geq \theta_{t-3}\) for all \((\varphi_{0}^{(t)}, \varphi_{1}^{(t)}, \varphi_{2}^{(t)}) \in \Lambda_{t}\).

**Proof.** The lemma is obvious for \(t = 2\). Assume \(t \geq 3\). From the induction hypothesis, we obtain

\[
\sum_{(h,i,j) \in \Lambda_{t-1}} (h - \theta_{t-4})c_{h,i,j} \geq 0.
\]

(4.2)

Using (2.2) and (2.3), the left-hand side of (4.2) can be calculated as

\[
\sum_{(h,i,j) \in \Lambda_{t-1}}hc_{h,i,j} - \theta_{t-4} \sum_{(h,i,j) \in \Lambda_{t-1}}c_{h,i,j} = \theta_{t-1} \varphi_{0}^{(t)} - \theta_{t-4} \theta_{t},
\]

whence

\[
\varphi_{0}^{(t)} \geq \theta_{t-4}\theta_{t} = \theta_{t-4}\left(4 + \frac{1}{\theta_{t-1}}\right) = \theta_{t-3} - 1 + \frac{\theta_{t-4}}{\theta_{t-1}}.
\]

This implies that \(\varphi_{0}^{(t)} \geq \theta_{t-3}\).

An \(f\)-subset \(F\) of \(PG(r, q)\) is called an \(\{f, m; r, q\}\)-minihyper if

\[
m = \min\{m \mid |F \cap \Pi| \geq m \text{ for all hyperplane } \Pi\}.
\]

**Lemma 4.3.** (See [5].) For \(1 \leq s \leq r\), a subset \(F\) of \(PG(r, q)\) is a \(\{\theta_{s}, \theta_{s-1}; r, q\}\)-minihyper if and only if \(F\) is an \(s\)-flat.

An \(s\)-flat \(S\) is called the axis of \(\Pi\) of type \((a, b, c)\) if every hyperplane of \(\Pi\) not containing \(S\) has the same diversity \((a, b, c)\) and if there is no hyperplane of \(\Pi\) through \(S\) whose diversity is \((a, b, c)\). Then the spectrum of \(\Pi\) satisfies \(c_{a,b,c}^{(t)} = \theta_{t} - \theta_{t-1-s}\) and the axis is unique if it exists. The axis is helpful to characterize the geometrical structure of \(\Pi\), see [20].

**Lemma 4.4.** Let \(\Pi\) be a \(t\)-flat of \(\Sigma^{*}\) with diversity \((\theta_{t-3}, \varphi_{1}^{(t)}, \varphi_{2}^{(t)})\), \(t \geq 3\).

(1) \((\varphi_{1}^{(t)}, \varphi_{2}^{(t)}) = (3 \times 4^{t-2}, 9 \times 4^{t-2})\) or \((9 \times 4^{t-2}, 9 \times 4^{t-2})\).

(2) When \((\varphi_{1}^{(t)}, \varphi_{2}^{(t)}) = (3 \times 4^{t-2}, 9 \times 4^{t-2})\), the spectrum of \(\Pi\) is

\[
(c_{0}^{(t)}, c_{1}^{(t)}, c_{2}^{(t)}, c_{3}^{(t)}, c_{4}^{(t)}, c_{5}^{(t)} = (9, 3, \theta_{t} - \theta_{2})
\]

and \(\Pi\) has a \((\theta_{t-3}, 0, 0)_{t-3}\) flat which is the axis of \(\Pi\) of type \((\theta_{t-4}, 3 \times 4^{t-3}, 9 \times 4^{t-3})\).
(3) When $(\varphi_1^{(t)}, \varphi_2^{(t)}) = (9 \times 4t^{-2}, 9 \times 4t^{-2})$, the spectrum of $\Pi$ is

$$(c^{(t)}_{\theta_1-3,3 \times 4t^{-2}, 4t^{-2}}, c^{(t)}_{\theta_1-3,0,3 \times 4t^{-2}}, c^{(t)}_{\theta_1-3,2 \times 4t^{-2}, 3 \times 4t^{-2}}, c^{(t)}_{\theta_1-4,9 \times 4t^{-3}, 9 \times 4t^{-3}}) = (9, 3, 9, \theta_t - \theta_2)$$

and $\Pi$ has a $(\theta_{t-3}, 0, 0)_{t-3}$ flat which is the axis of $\Pi$ of type $(\theta_{t-4}, 9 \times 4t^{-3}, 9 \times 4t^{-3})$.

**Proof.** We prove (1)–(3) simultaneously by induction on $t$. $L = F_0 \cap \Pi$ is a $(\theta_{t-3}, \theta_{t-4}, t, 4)$-minihyper by Lemma 4.2. Hence $L$ is a $(t - 3)$-flat by Lemma 4.3.

We first consider the case $t = 3$. Let $P$ be the point of $F_0$ in the $(1, \varphi_1^{(3)}, \varphi_2^{(3)})$-solid $\Pi$ and let $\delta_0$ be a plane in $\Pi$ not containing $P$. Then, from Table 3.3, $\delta_0$ is a $(0, 3, 9)$-plane or a $(0, 9, 9)$-plane.

Assume that $\delta_0$ is a $(0, 3, 9)$-plane. Since the spectrum of $\delta_0$ is $(c_0^2, c_{0,3}^2, c_{0,2,3}^2) = (9, 9, 3)$, it is easy to see that every point of $\delta_0$ is on a $(0, 0, 3)$-line or a $(0, 2, 3)$-line, say $l$. From Tables 3.1–3.4, the plane $\langle P, l \rangle$ must be a $(1, 0, 12)$-plane consisting of two $(1, 0, 0)$-lines and three $(1, 0, 4)$-lines or a $(1, 8, 12)$-plane consisting of two $(1, 4, 0)$-lines and three $(1, 0, 4)$-lines. Hence $F$ forms a cone with $P$ as the vertex and $(F_1 \cup F_2) \cap \delta_0$ as a base, giving the assertion (2) with $t = 3$ and $P$ as the axis of $\Pi$ of type $(0, 3, 9)$. Similarly, we obtain (3) with $t = 3$ when $\delta_0$ is a $(0, 9, 9)$-plane.

Assume $t \geq 4$. Let $\pi_0$ be a hyperplane not containing $L$. From $|\pi_0 \cap L| = \theta_{t-4}$ and the induction hypothesis, $\pi_0$ is a $(\theta_{t-4}, 3 \times 4t^{-3}, 9 \times 4t^{-3})_{t-1}$ flat or a $(\theta_{t-4}, 9 \times 4t^{-3}, 9 \times 4t^{-3})_{t-1}$ flat.

Assume that $\pi_0$ is a $(\theta_{t-4}, 3 \times 4t^{-3}, 9 \times 4t^{-3})_{t-1}$ flat. The hyperplanes of $\Pi$ through a fixed $(\theta_{t-5}, 3 \times 4t^{-4}, 9 \times 4t^{-4})_{t-2}$ flat in $\pi_0$ are $(\theta_{t-4}, 3 \times 4t^{-3}, 9 \times 4t^{-3})_{t-1}$ flats from the induction hypothesis for (1)–(3) and Lemma 4.2. Counting the numbers of points of $F_1$ and $F_2$, we get

$$\varphi_1^{(t)} = (3 \times 4t^{-3} - 3 \times 4t^{-4}) \times 5 + 3 \times 4t^{-4} = 3 \times 4t^{-2},$$

$$\varphi_2^{(t)} = (9 \times 4t^{-3} - 9 \times 4t^{-4}) \times 5 + 9 \times 4t^{-4} = 9 \times 4t^{-2}.$$ 

Similarly, we get $(\varphi_1^{(t)}, \varphi_2^{(t)}) = (9 \times 4t^{-2}, 9 \times 4t^{-2})$ when $\pi_0$ is a $(\theta_{t-4}, 9 \times 4t^{-3}, 9 \times 4t^{-3})_{t-1}$ flat. Hence, $\varphi_1^{(t)}$ is determined according to the diversity of $\pi_0$. It follows that every hyperplane of $\Pi$ not containing $L$ is a $(\theta_{t-4}, 3 \times 4t^{-3}, 9 \times 4t^{-3})_{t-1}$ flat when $\pi_0$ is a $(\theta_{t-4}, 3 \times 4t^{-3}, 9 \times 4t^{-3})_{t-1}$ flat and that every hyperplane of $\Pi$ not containing $L$ is a $(\theta_{t-4}, 9 \times 4t^{-3}, 9 \times 4t^{-3})_{t-1}$ flat when $\pi_0$ is a $(\theta_{t-4}, 9 \times 4t^{-3}, 9 \times 4t^{-3})_{t-1}$ flat. Hence a $(\theta_{t-4}, 3 \times 4t^{-3}, 9 \times 4t^{-3})_{t-1}$ flat and a $(\theta_{t-4}, 3 \times 4t^{-3}, 9 \times 4t^{-3})_{t-1}$ flat cannot exist together in $\Pi$. Next, we determine the spectrum of $\Pi$.

We prove (2) by induction on $t$. (3) can be proved similarly. From the induction hypothesis for $t - 1$, the spectrum of a $(\theta_{t-4}, 3 \times 4t^{-3}, 9 \times 4t^{-3})_{t-1}$ flat $\pi_0$ is

$$(c^{(t-1)}_{\theta_1-4,4t^{-3}, 4t^{-3}}, c^{(t-1)}_{\theta_1-4,0,3 \times 4t^{-3}}, c^{(t-1)}_{\theta_1-4,2 \times 4t^{-3}, 3 \times 4t^{-3}}, c^{(t-1)}_{\theta_1-5,3 \times 4t^{-4}, 9 \times 4t^{-4}}) = (9, 3, \theta_{t-1} - \theta_2).$$

Let $\delta_1$ be a $(\theta_{t-4}, 4t^{-3}, 4t^{-3})_{t-2}$ flat in $\pi_0$ and let $\pi_1 = (L, \delta_1)$ be a $(\theta_{t-3}, a, b)_{t-1}$ flat. Then the $(t-1)$-flats in $\Pi$ through $\delta_1$ are $(\theta_{t-4}, 3 \times 4t^{-3}, 9 \times 4t^{-3})_{t-1}$ flats except $\pi_1$, so

$$\varphi_1^{(t)} = (3 \times 4t^{-3} - 3 \times 4t^{-3}) \times 4 + a = 3 \times 4t^{-2},$$

$$\varphi_2^{(t)} = (9 \times 4t^{-3} - 4t^{-3}) \times 4 + b = 9 \times 4t^{-2}.$$
Hence \( \pi_1 \) is a \((\theta_{t-3}, 4^t-2, 4^t-2)_{t-1}\) flat and
\[
c_{\theta_{t-3},2\times4^t-2,3\times4^t-2}^{(t)} \geq c_{\theta_{t-4},4^t-3,4^t-3}^{(t-1)} = 9. \tag{4.3}
\]

Similarly, we get the following relations (4.4) and (4.5) when \( \delta_1 \) is a \((\theta_{t-4}, 0, 3 \times 4^t-3)_{t-2}\) flat and a \((\theta_{t-4}, 2 \times 4^t-3, 3 \times 4^t-3)_{t-2}\) flat respectively:
\[
c_{\theta_{t-3},0,3\times4^t-3}^{(t)} \geq c_{\theta_{t-4},0,3\times4^t-3}^{(t-1)} = 9, \tag{4.4}
\]
\[
c_{\theta_{t-3},2\times4^t-3,3\times4^t-3}^{(t)} \geq c_{\theta_{t-4},2\times4^t-3,3\times4^t-3}^{(t-1)} = 3. \tag{4.5}
\]

Since the number of \((t-1)\)-flats in \( \Pi \) through a fixed \((t-3)\)-flat is \( \theta_2 \), the equalities of (4.3)–(4.5) hold. All the \((t-1)\)-flats in \( \Pi \) except \((\theta_{t-3}, 4^t-2, 4^t-2)_{t-1}\) flats, \((\theta_{t-3}, 0, 3 \times 4^t-2)_{t-1}\) flats and \((\theta_{t-3}, 2 \times 4^t-2, 3 \times 4^t-2)_{t-1}\) flats are \((\theta_{t-4}, 3 \times 4^t-3, 9 \times 4^t-3)_{t-1}\) flats. Hence \( L \) is the axis of \( \Pi \) and (2) follows. \( \square \)

As a consequence in the proof of Lemma 4.4, \( F \) forms a cone with the axis \( L \) as the vertex and a \((0, 3, 9)\)-plane or a \((0, 9, 9)\)-plane as a base. If one wishes to prove this directly as for \( t = 3 \), the characterizations of \((\theta_{t-2}, \phi_1(t), 3 \times 4^t-1)_{t}\) flats are needed, which will require more spaces than the above proof.

Table 3.3 and Lemma 4.4(1) yield Theorem 1.3(2). Note that \( \theta_{t-3} + 2 \times 4^t-2 + 3 \times 4^t-2 = \theta_{t-1} \). Since \( c_{\theta_{t-2},2\times4^t-2,3\times4^t-2} \geq 0 \) holds for \( t \geq 2 \) when \( \phi_0(t) = \theta_{t-3} \) by Table 3.3 and Lemma 4.4, Theorem 1.3(3) follows from Lemma 2.3.

The following lemma can be proved using Tables 3.2–3.4 similarly to the proof of Lemma 4.4.

**Lemma 4.5.** Let \( \Pi \) be a \( t \)-flat with diversity \((\theta_{t-2}, \phi_1(t), \phi_2(t))\).

1. When \((\phi_1(t), \phi_2(t)) = (4^t-1, 4^t-1)\), the spectrum of \( \Pi \) is
\[
\left( c_{\theta_{t-2},0,0}^{(t-1)}, c_{\theta_{t-3},3,4^t-2}^{(t-1)}, c_{\theta_{t-2},4^t-1,0}^{(t-1)}, c_{\theta_{t-2},0,4^t-1}^{(t-1)} \right) = (3, \theta_t - \theta_1, 1, 1)
\]
and \( \Pi \) has a \((\theta_{t-2}, 0, 0)_{t-2}\) flat which is the axis of \( \Pi \) of type \((\theta_{t-3}, 4^t-2, 4^t-2)\).

2. When \((\phi_1(t), \phi_2(t)) = (0, 3 \times 4^t-1)\), the spectrum of \( \Pi \) is
\[
\left( c_{\theta_{t-2},0,0}^{(t-1)}, c_{\theta_{t-3},0,3\times4^t-2}^{(t-1)}, c_{\theta_{t-2},0,4^t-1}^{(t-1)} \right) = (2, \theta_t - \theta_1, 3)
\]
and \( \Pi \) has a \((\theta_{t-2}, 0, 0)_{t-2}\) flat which is the axis of \( \Pi \) of type \((\theta_{t-3}, 0, 3 \times 4^t-2)\).

3. When \((\phi_1(t), \phi_2(t)) = (2 \times 4^t-1, 3 \times 4^t-1)\), the spectrum of \( \Pi \) is
\[
\left( c_{\theta_{t-2},4^t-1,0}^{(t-1)}, c_{\theta_{t-2},2\times4^t-2,3\times4^t-2}^{(t-1)}, c_{\theta_{t-2},0,4^t-1}^{(t-1)} \right) = (2, \theta_t - \theta_1, 3)
\]
and \( \Pi \) has a \((\theta_{t-2}, 0, 0)_{t-2}\) flat which is the axis of \( \Pi \) of type \((\theta_{t-3}, 2 \times 4^t-2, 3 \times 4^t-2)\).
(4) When \((\varphi_1^{(t)}, \varphi_2^{(t)}) = (3 \times 4^{t-1}, 4^{t-1})\), the spectrum of \(\Pi\) is
\[
\left( e_{\theta_{t-2},0,0}^{(t-1)}, e_{\theta_{t-3},3 \times 4^{t-2},4^{t-2}}, e_{\theta_{t-2},0,4^{t-1}}, e_{\theta_{t-3},4^{t-1},0}^{(t-1)} \right) = (1, \theta_t - \theta_1, 1, 3)
\]
and \(\Pi\) has a \((\theta_{t-2}, 0, 0)_{t-2}\) flat which is the axis of \(\Pi\) of type \((\theta_{t-3}, 3 \times 4^{t-2}, 4^{t-2})\).

Lemma 4.6.

(1) A \((\theta_{t-2}, 2 \times 4^{t-1}, 3 \times 4^{t-1})_{t}\) flat contains a \((0, 2, 3)\)-line.
(2) None of \((\theta_{t-2}, 4^{t-1}, 4^{t-1})_{t}\) flats, \((\theta_{t-2}, 0, 3 \times 4^{t-1})\), flats and \((\theta_{t-2}, 3 \times 4^{t-1}, 4^{t-1})_{t}\) flats contain a \((0, 2, 3)\)-line.

Proof. When \(t = 2\), the lemma follows from Table 3.2. When \(t \geq 3\), it can be proved using Lemma 4.5 by induction on \(t\).

Proof of Theorem 1.5. (i) Assume that \(C\) satisfies \(\Phi_0 = \theta_{k-4}\). With \(t = k - 1\), \(\Sigma^*\) is a \((\theta_{t-3}, \Phi_1, \Phi_2)_{t}\) flat. By Lemma 4.4, \(\Sigma^*\) contains a \((\theta_{t-3}, 0, 0)_{t-3}\) flat which is the axis of \(\Pi\), say \(L\). From the spectrum of \(\Sigma^*\), one can take a \((0, 2, 3)\)-line \(l\) by Lemma 4.6(1). Note that \(L \cap l = \emptyset\). From the spectrum of \(\Sigma^*\) and Lemma 4.6, \(\langle L, l \rangle\) is a \((\theta_{t-3}, 2 \times 4^{t-2}, 3 \times 4^{t-2})_{t-1}\) flat. Let \(P_1, \ldots, P_{t-2}\) be the points of \(F_0\) corresponding to \(a_1, \ldots, a_{t-2}\) and let \(Q_1, Q_2\) be the points of \(F_1 \cup F_2\) corresponding to \(a_{t-1}, a_{t-2}\). Then, it follows from the conditions that \(L = \langle P_1, \ldots, P_{t-2} \rangle\) and that \(\langle Q_1, Q_2 \rangle\) is a \((0, 2, 3)\)-line. Hence, the vector \(h\) corresponds to a \((\theta_{t-3}, 2 \times 4^{t-2}, 3 \times 4^{t-2})_{t-1}\) flat, and the extended matrix \([G, h]\) generates an extension of \(C\) by Lemma 2.2.

(ii) Assume that \(C\) satisfies \(\Phi_0 + \Phi_2 = \theta_{k-2} + 2 \times 4^{k-2}\). With \(t = k - 1\), \(\Sigma^*\) forms a \((\Phi_0, \Phi_1, \Phi_2)_{t}\) flat with \(\Phi_0 + \Phi_2 = \theta_{t-1} + 2 \times 4^{t-1}\). Let \(P_1, \ldots, P_t\) be the points of \(F_0 \cup F_2\) corresponding to \(a_1, \ldots, a_t\). Then \(\langle P_1, \ldots, P_t \rangle\) is an \((h, i, j)_{t-1}\) flat with \(h + j = \theta_{t-1}\). The existence of such a \((t - 1)\)-flat is guaranteed by Lemma 4.1. Hence the vector \(h\) corresponds to \(\langle P_1, \ldots, P_t \rangle\), and \([G, h]\) generates an extension of \(C\).

References

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