ELSEVIER

Discrete Applied Mathematics 54 (1994) 29–35

DISCRETE APPLIED MATHEMATICS

Graph homomorphisms with infinite targets

Gary MacGillivray

Department of Mathematics and Statistics, University of Victoria, Victoria, B.C., Canada V8W 3P4

Received 30 April 1991; revised 14 April 1992

Abstract

Let H be a fixed graph whose vertices are called colours. Informally, an H-colouring of a graph G is an assignment of these colours to the vertices of G such that adjacent vertices receive adjacent colours. We introduce a new tool for proving NP-completeness of H-colouring problems, which unifies all methods used previously. As an application we extend, to infinite graphs of bounded degree, the theorem of Hell and Nešetřil that classifies finite H-colouring problems by complexity.

1. Introduction

Let G and H be graphs. A *homomorphism* of G to H is a function $f: V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. If the vertices of H are regarded as colours, then f is an assignment of these colours to the vertices of G so that adjacent vertices receive adjacent colours. An n-colouring of a graph G is a homomorphism of G to K_n, thus the term *H-colouring* of G has been employed to describe a homomorphism of G to H. If there is an H-colouring of G, we say that G is *H-colourable.*

Let H be a fixed graph. The *H-colouring problem* is the decision problem defined below.

H-COL (H-colouring)
Instance: A finite graph G.
Question: Does there exist an *H-colouring* of G?

Hell and Nešetřil [11] have determined the complexity of the H-colouring problem for any finite undirected graph H. The H-colouring problem is NP-complete if H contains an odd cycle, and is polynomial otherwise. No such classification exists for finite digraphs, although many families have been classified [1–6, 10, 11, 13, 14]. A restricted version of H-COL is studied in [9].

SSDI 0166-218X(93)E0062-4

When H is a finite graph, the H-colouring problem is in NP. This is not always the case when H is infinite. Bauslaugh [7] has proved that for any recursive function f there is an infinite graph H such that H-COL has complexity at least f, that there exist graphs H with unsolvable H-colouring problems and, furthermore, that for any recursively enumberable degree of unsolvability A there is a graph H such that H-COL is exactly A-solvable. In [8] the same author proves that all recursive vertex-transitive graphs H have solvable H-colouring problems, but there are non-recursive vertex-transitive graphs H with unsolvable H-colouring problems.

Another contrast to the finite case (assuming $P \neq NP$) is that the presence of an odd cycle is not sufficient for NP-hardness of H-COL. For instance the graph

$$K = K_1 \cup K_2 \cup \cdots \cup K_n \cup \cdots$$

has an odd cycle, yet K-COL is polynomial because any finite graph is K-colourable. Similarly, the H-colouring problem is polynomial whenever H contains arbitrarily large cliques.

In the next section, we introduce a new generic polynomial-time transformation (the HSI construction, Lemma 2.1) which unifies and generalizes the three principal constructions previously used (see [1–6, 10, 11, 13, 14]). We hope this new tool will be helpful in simplifying some of the existing proofs (e.g. [11]). A slightly restricted variation of the HSI construction is used to extend the result of Hell and Nešetřil to infinite graphs of bounded degree. In Section 3 the aforementioned constructions are derived using the HSI construction.

2. The HSI construction and the HS construction

Let H be a subgraph of G. A *retraction* of G to H is a homomorphism r of G to H for which $r(h) = h$ for all vertices h of H. If there exists a retraction of G to H, we say H is a *retract* of G. There are then homomorphisms i of H to G (the inclusion) and r of G to H (a retraction), so a given graph is G-colourable if and only if it is H-colourable. Hence, for any retract H of G, the complexity of H-COL and G-COL is the same.

A graph is *retract-free* if it does not admit a retraction to a proper subgraph. Every finite graph G contains a unique (up to isomorphism) subgraph C, called the *core* of G, which is retract-free, and for which there is a retraction of G to C (see [11, 15]). It is an NP-complete problem to decide if a given finite graph is not retract-free [12].

We now describe the HSI construction. Let J be a fixed finite graph with specified vertices $u, v, j_1, j_2, \ldots, j_t$, such that some automorphism of J maps u to v and v to u. The *HSI construction* with respect to $(J, u, v, j_1, j_2, \ldots, j_t)$ transforms a given graph H into the graph $H^!$, defined as follows. Let $\mathcal{J}$ denoted the set of homomorphisms of J to H. Define an equivalence relation on $\mathcal{J}$ by $f \cong g$ just if $f(j_i) = g(j_i)$, $i = 1, 2, \ldots, t$. The vertex set $V(H^!)$ consists of a copy of $V(H)$ corresponding to each equivalence class of $\cong$. Let $f \in \mathcal{J}$, and W be the graph constructed from $H \cup J$ by identifying j_k and $f(j_k)$, $k = 1, 2, \ldots, t$. There is an edge in $V(H^!)$ joining vertices x and y in the copy of $V(H)$

corresponding to the equivalence class of f just if there is a retraction of W to H that maps u to x and v to y. (Our assumption about the symmetry of J guarantees that if such a mapping exists, then there is also a retraction of W to H that maps v to x and u to y; thus the edges of $H^{!}$ are undirected).

Lemma 2.1. *$H^{!}$-COL polynomially transforms to H-COL.*

Proof. Suppose an instance of $H^{!}$-COL, a finite graph D, is given. Without loss of generality D is connected, otherwise apply the construction below to each connected component. Construct a graph ${}^{!}D$ from $V(D)$ and $|E(D)|$ copies $J_1, J_2, \ldots, J_{|E(D)|}$ of J, as follows. For $i = 1, 2, \ldots, t$ identify all $|E(D)|$ copies of j_i. If xy is the kth edge of D, then identify the vertices u and v in the kth copy of J with the vertices x and y in the copy of $V(D)$ in ${}^{!}D$, respectively. All graphs involved in the construction are finite, and it may clearly be carried out in polynomial time.

We claim that ${}^{!}D \to H$ if and only if $D \to H^{!}$.

Suppose f is a homomorphism of ${}^{!}D$ to H. Let dd' be the nth edge of D. The restriction of f to the nth copy of J in ${}^{!}D$ can be extended to a retraction of W (as defined above, corresponding to the equivalence class of f) to H which maps u to $f(d)$ and v to $f(d')$ by mapping each vertex of H to itself. Therefore, for each edge dd' of D, $f(d)f(d')$ is an edge of $H^{!}$. Thus the restriction of f to $V(D)$ is a homomorphism of D to $H^{!}$.

Suppose f is a homomorphism of D to $H^{!}$. We must construct at homomorphism g of ${}^{!}D$ to H. Since D is connected, it maps to a connected component of $H^{!}$, and hence to the subgraph of $H^{!}$ induced by some copy F of $V(H)$. For each edge hh' of the subgraph of $H^{!}$ induced by F, there is a retraction of W (defined as above, corresponding to elements in the equivalence class corresponding to F) to H taking u to h and v to h'. For $j = 1, 2, \ldots, |E(D)|$, let xy be the jth edge of D, and r_j be a retraction of (the same) W to H that takes u to $f(x)$ and v to $f(y)$. The function g is defined as follows.

$$g(d) = f(d), \quad d \in V(D), \quad \text{and}$$

$$g(w) = r_j(w), \quad w \in V(J_j) - V(D), \quad j = 1, 2, \ldots, |E(D)|.$$

Then $g: V({}^{!}D) \to V(H)$. Since f and r_j ($j = 1, 2, \ldots, |E(D)|$) are homomorphisms which agree on the intersection of their domains. Thus g is a homomorphism. □

We now describe a slightly restricted variation of the HSI construction which we will use to prove the result regarding the complexity of infinite H-COL.

Let J be a fixed finite graph with specified vertices $x, j_1, j_2, \ldots, j_t$. The *HS-construction* with respect to $(J, x, j_1, j_2, \ldots, j_t)$ transforms a given graph H to the graph $H^{\#}$ defined as follows. Let $\mathscr{J}$ denote the set of homomorphisms of J to H. Define an equivalence relation on $\mathscr{J}$ by $f \cong g$ such that if $f(j_i) = g(j_i)$, $i = 1, 2, \ldots, t$. Let f be a representative of some equivalence class of $\cong$, and let W be the graph constructed from $H \cup J$ by identifying j_k and $f(j_k)$, $k = 1, 2, \ldots, t$. Let V_f be the set of vertices which are images of the vertex x under retractions of W to H, and H_f be the subgraph of H induced by V_f. The graph $H^{\#}$ is the disjoint union of all graphs H_f, over all equivalence classes of $\cong$.

Corollary 2.2. *$H^{\#}$-COL polynomially transforms to H-COL.*

Proof. Let J' be the graph constructed from two copies of $(J, x, j_1, j_2, \ldots, j_t)$ by identifying the corresponding vertices j_i, $i = 1, 2, \ldots, t$. Let u, v be the two copies of the vertex x, and add the edge uv, to J'. Let $H^{\#}$ and $H^{!}$, respectively, be the result of applying the HS construction with respect to $(J, x\, j_1, j_2, \ldots, j_t)$ and the HSI construction with respect to $(J', u, v, j_1, j_2, \ldots, j_t)$ to H. We claim that $E(H)^{\#} = E(H^{!})$. Let ab be an edge of $H^{!}$. Then there is a retraction of W' (constructed as above, using H and J') to H that maps u to a and v to b. Since J' was constructed using two copies of J, there are corresponding retractions of W to H that map x to a, and x to b. Thus ab is also an edge of $H^{\#}$. Conversely, if ab is an edge of $H^{\#}$, then there are retractions of W to H that map x to a, and x to y. These lead to a retraction of J' to H' that maps u to a and v to b. Hence ab is also an edge of $H^{\#}$. This proves the claim. Since, by definition, $V(H^{\#})$ contains $V(H^{!})$, it follows that $H^{\#}$ is a retract of $H^{!}$. We now have that $H^{\#}$-COL is equivalent to $H^{!}$-COL, which polynomially transforms to H-COL. □

We now use the HS construction to extend the result of Hell and Nešetřil to infinite graphs of bounded degree. We say that a graph H is of *bounded degree* if there is an integer B such that $d(v) \leqslant B$ for all $v \in V(H)$.

It is important to note why the HSI construction can be applied to infinite graphs. (Similar comments apply to the HS construction). Given an instance of $H^{!}$-COL, i.e., a finite graph D, the HSI construction produces a graph ${}^{!}D$ from $V(D)$, and $|E(D)|$ copies of $\mathcal{J}$. Since D and J are finite, so is ${}^{!}D$. The HSI construction therefore produces an instance of H-COL. By contrast, the subindicator construction and edge-subindicator construction [11] (also see Section 3) cannot be applied to infinite graphs since the transformed problem instance contains the graph H, and if H is infinite this is not an instance of H-COL.

Corollary 2.3. *Let H be an infinite graph of bounded degree. If H is bipartite, then H-COL is polynomial. Otherwise (H contains an odd cycle), H-COL is* NP-*hard.*

Proof. We prove only the second statement, the first statement being obvious. Let H be an infinite graph of bounded degree and having an odd cycle. We use the HS construction to find a finite non-bipartite graph G such that G-COL polynomially transforms to H-COL. The theorem of Hell and Nešetřil asserts that G-COL is NP-complete, hence H-COL is NP-hard.

Since H is fixed, it may be assumed that some odd cycle C, of length $2c + 1$, is known. We use P_n to denote the path of length n, with vertex set $V(P_n) = \{0, 1, \ldots, n\}$

and edge set $E(P_n) = \{i(i+1): i = 1, 2, \ldots, n-1\}$. Let $H^{\#}$ be the result of applying the HS construction with respect to $(P_{2c+1}, 0, 2c+1)$ to H. The graph $H^{\#}$ has infinitely many components, and each of them has maximum degree at most that of H, and diameter at most $4c+2$. Therefore each component of $H^{\#}$ is a finite graph. Furthermore, it is easy to see that each component of $H^{\#}$ that corresponds to an equivalence class in which $2c+1$ is mapped to a vertex of C contains an odd cycle isomorphic to C. Let G be the core of H. Since there are only finitely many graphs with diameter at most $4c+2$ and degrees bounded by the maximum degree in H, the graph G is finite. In addition, being a retract of a graph with an odd cycle, G also has an odd cycle. We now have that G-COL is equivalent to $H^{\#}$-COL, which polynomially transforms to H-COL. □

3. Derivation of the other principal constructions

We now illustrate how the HSI-construction unifies all of the principal constructions that have so far been employed, namely the indicator construction, the subindicator construction, and the edge-subindicator construction [1–6, 10, 11, 13, 14]. We describe each transformation, and then prove it correct using the HSI construction, or the HS construction.

Let I be a fixed graph, and let u and v be distinct vertices of I, such that some automorphism of I maps u to v and v to u. The *indicator construction with respect to* (I, u, v) transforms a given graph H into the graph H^*, defined to have the same vertex set as H, and to have as the edge set all pairs hh' for which there is a homomorphism of I to H taking u to h and v to h'.

Lemma 3.1 [11]. *H^*-COL polynomially transforms to H-COL.*

Proof. Let H^* be the result of applying the indicator construction with respect to (I, u, v) to H. Let $H^!$ be the result of applying the HSI construction with respect to $(I \cup \{y\}, u, v, y)$ to H, where y is an isolated vertex. Since u and v are not in the same component as y the graph $H^!$ consists of $|V(H)|$ disjoint copies of H^*, so H^* is a retract of $H^!$. □

Let J be a fixed graph with specified vertices x and $j_1, j_2, \ldots, j_t$. The *subindicator construction with respect to $(J, x, j_1, j_2, \ldots, j_t)$, and $h_1, h_2, \ldots, h_t$* transforms a given *retract-free* graph H with specified vertices $h_1, h_2, \ldots, h_t$, to its subgraph $H^{\sim}$ induced by the vertex set $V^{\sim}$ defined as follows. Let W be the graph obtained from the disjoint union of J and H by identifying j_i and h_i, $i = 1, 2, \ldots, t$. A vertex v of H belongs to $V^{\sim}$ such that if there is a retraction of W and H which maps x to v.

Lemma 3.2 [11]. *If H is finite and retract-free, then $H^{\sim}$-COL polynomially transforms to H-COL.*

Proof. Let H be a retract-free graph. Let $H^{\sim}$ be the result of applying the subindicator construction with respect to $(J, x, j_1, j_2, \ldots, j_t)$ and $h_1, h_2, \ldots, h_t$ to H, and let $H^{\#}$ be the result of applying the HS construction with respect to $(W, x, j_1, j_2, \ldots, j_t)$ to H, where W is defined as above. Since H is finite and retract-free, every homomorphism of H to itself is an automorphism. Thus the graph $H^{\#}$ consists of |Aut(H)| disjoint copies of $H^{\sim}$, so $H^{\sim}$ is a retract of $H^{\#}$. □

Similarly, let J be a fixed graph with a specified edge xy and specified vertices $j_1, j_2, \ldots, j_t$, such that some automorphism of J maps x to y and y to x. The *edge-subindicator construction with respect to* $(J, xy, j_1, j_2, \ldots, j_t)$, *and* $h_1, h_2, \ldots, h_t$ transforms a given *retract-free* graph H with specified vertices $h_1, h_2, \ldots, h_t$ into its subgraph $H^{\wedge}$ induced by the edges of H which are images of the edge xy under retractions of W (as defined above) to H.

Lemma 3.3 [11]. *If H is finite and retract-free, then $H^{\wedge}$-COL polynomially transforms to H-COL.*

Proof. Let H be a retract-free graph. Let $H^{\wedge}$ be the result of applying the edge-subindicator construction with respect to $(J, xy, j_1, j_2, \ldots, j_t)$ and $h_1, h_2, \ldots, h_t$ to H, and let $H^{!}$ be the result of applying the HSI construction with respect to $(W, x, y, j_1, j_2, \ldots, j_t)$, where W is defined as above, to H. As above, $H^{!}$ consists of |Aut(H)| disjoint copies of $H^{\wedge}$, so $H^{\wedge}$ is a retract of $H^{!}$. □

Finally, we mention the origin of the name HSI-construction. This construction can be viewed as having three phases: a Homomorphism phase in which the vertices $j_1, j_2, \ldots, j_t$ are "mapped" to some subset of $V(H)$, a Subindicator phase wherein the graph W is retracted to H, and an Indicator phase in which edges of $H^{!}$ are added between images of the specified vertices u and v. In the HS construction there is no "indicator" phase.

Acknowledgement

The author is grateful for the support of the Natural Sciences and Engineering Research Council of Canada.

References

[1] J. Bang-Jensen, On the complexity of generalized colourings of directed graphs, in: U. Faigle and C. Hoede, eds., Proc. Twente Workshop on Graphs and Combinatorial Optimization, Memorandum, No. 787, Universiteit Twente, Faculteit der Toegepaste Wiskunde (1989) 13–20.

[2] J. Bang-Jensen and P. Hell, The effect of two cycles on the complexity of colourings by directed graphs, Discrete Appl. Math. 26 (1990) 1–23.
[3] J. Bang-Jensen, P. Hell and G. MacGillivray, The complexity of colourings by semicomplete digraphs, SIAM J. Discrete Math. 1 (1988) 281–298.
[4] J. Bang-Jensen, P. Hell and G. MacGillivray, On the complexity of colourings by superdigraphs of bipartite graphs, Discrete Math. 109 (1992) 27–44.
[5] J. Bang-Jensen and G. MacGillivray, Further effects of two directed cycles on the complexity of H-colouring, J. Combin. Math. Combin. Comput. 10 (1991) 33–50.
[6] J. Bang-Jensen and G. MacGillivray, On the complexity of colouring by digraphs with at most one directed cycle, Ars Combin. 35-A (1993) 175–186.
[7] B. Bauslaugh, The complexity of infinite H-colouring, Simon Fraser University, Burnaby, Canada, preprint.
[8] B. Bauslaugh, The complexity of colouring by infinite vertex-transitive graphs, Discrete Math., to appear.
[9] R. Brewster and G. MacGillivray, A note on restricted H-colouring, Inform. Process. Lett. 40 (1991) 149–151.
[10] W. Gutjahr, E. Welzl and G. Woeginger, Polynomial graph colourings, Discrete Appl. Math. 35 (1992) 29–46.
[11] P. Hell and J. Nešetřil, On the complexity of H-colouring J. Combin. Theory Ser. B 48 (1990) 92–110.
[12] P. Hell and J. Nešetřil, The core of a graph, Discrete Math. 109 (1992) 117–126.
[13] G. MacGillivray, The complexity of colouring by vertex-transitive and arc-transitive digraphs, SIAM J. Discrete Math. 4 (1991) 397–408.
[14] H. A. Maurer, J. H. Sudborough and E. Welzl, On the complexity of the general colouring problem, Inform. and Control 51 (1981) 123–145.
[15] E. Welzl, Colour families are dense, Theoret. Comput. Sci. 17 (1982) 29–41.