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Two relations for median graphs

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Abstract

We generalize the well-known relation for trees $n-m=1$ to the class of median graphs in the following way. Denote by q_i the number of subgraphs isomorphic to the hypercube Q_i in a median graph. Then, $\sum_{i\geqslant 0}(-1)^i q_i = 1$. We also give an explicit formula for the number of Θ-classes in a median graph as $k = -\sum_{i\geqslant 0}(-1)^i\, i q_i$. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

A graph G (with distance function d) is called a *median graph*, if for any three vertices u, v, w of G there exists a unique vertex x such that $d(u,v) = d(u,x) + d(x,v)$, $d(u,w) = d(u,x) + d(x,w)$, and $d(v,w) = d(v,x) + d(x,w)$. Median graphs are beautiful generalization of trees and hypercubes. In the survey of Klavžar and Mulder [2] one can find many different characterizations of this class of graphs.

One of the most important results in the theory of median graphs is Mulder's convex expansion theorem [4] (see also [5]). Roughly speaking, this theorem says that if G is a median graph different from K_1, then there exist median graphs G', G'_0, G'_1, and G'_2 such that $G' = G'_1 \cup G'_2$ and $G'_0 = G'_1 \cap G'_2$ is not empty. Moreover, if we take disjoint copies of G'_1 and G'_2 and for every vertex from G'_0 connect by an edge the appropriate vertices from these two copies, then we obtain G.

Define a relation Θ on the edges of a connected graph G as follows. We say, that edges $e_1 = x_1y_1$ and $e_2 = x_2y_2$ are in relation Θ (and write $e_1\Theta e_2$) if and only if $d(x_1,x_2)+d(y_1,y_2) \neq d(x_1,y_2)+d(x_2,y_1)$. The relation Θ was introduced by Djoković [1] (in different notation). It is well known that Θ is an equivalence relation providing G is a median graph. Even more, the number of different Θ-classes (i.e. equivalence

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classes of Θ) is the smallest number k for which G has an isometric embedding in the hypercube Q_k.

Denote by n, m, and k the number of vertices, the number of edges, and the number Θ-classes of a median graph, respectively. In [3] authors prove an Euler-type relation $2n-m-k=2$ for median Q_3-free graphs. This result is a consequence of the following theorem. With this theorem, we generalize the well known relation for trees $n-m=1$ to the class of median graphs and also give an explicit formula for k.

Theorem. *Let G be a median graph and let q_i $(i \geqslant 0)$ be the number of subgraphs of G isomorphic to the hypercube Q_i. Denote by k the number of Θ-classes of G. Then the following holds*:

$$\sum_{i\geqslant 0}(-1)^i q_i = 1 \quad \textit{and} \quad k = -\sum_{i\geqslant 0}(-1)^i i\, q_i.$$

Proof. The proof is by induction on the number of vertices. The claim is obviously true for $G \cong K_1$. So, we may assume that G is the convex expansion of the median graph G' with respect to the subgraphs G_1' and G_2' with $G_0' = G_1' \cap G_2'$. By definition, G_0', G_1', and G_2' are median graphs. Denote by q_i^j the number of distinct subgraphs of G_j' isomorphic to the hypercube Q_i and denote by k^j the number of Θ-classes of G_j'. Since each G_j' $(j=0,1,2)$ has less vertices than G, we have

$$\sum_{i\geqslant 0}(-1)^i q_i^j = 1 \quad \text{and} \quad k^j = -\sum_{i\geqslant 0}(-1)^i i\, q_i^j.$$

It is easy to observe that $q_0 = q_0^1 + q_0^2$ and $q_i = q_i^1 + q_i^2 + q_{i-1}^0$ for every $i \geqslant 1$. Observe also that $k = k^1 + k^2 - k^0 + 1$. Thus,

$$\begin{aligned}\sum_{i\geqslant 0}(-1)^i q_i &= \sum_{i\geqslant 0}(-1)^i q_i^1 + \sum_{i\geqslant 0}(-1)^i q_i^2 + \sum_{i\geqslant 0}(-1)^{i+1} q_i^0\\ &= 1+1-1\\ &= 1.\end{aligned}$$

And, for the second equation,

$$\begin{aligned}k &= k^1 + k^2 - k^0 + 1\\ &= -\sum_{i\geqslant 0}(-1)^i i(q_i^1+q_i^2) + \sum_{i\geqslant 0}(-1)^i i\, q_i^0 + \sum_{i\geqslant 0}(-1)^i q_i^0\\ &= -\sum_{i\geqslant 0}(-1)^i i(q_i^1+q_i^2) - \sum_{i\geqslant 0}(-1)^{i+1}(i+1) q_i^0\\ &= -\sum_{i\geqslant 1}(-1)^i i(q_i^1+q_i^2+q_{i-1}^0)\\ &= -\sum_{i\geqslant 0}(-1)^i i\, q_i. \qquad \square\end{aligned}$$

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