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Refined Kato Inequalities and Conformal Weights in Riemannian Geometry

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We establish refinements of the classical Kato inequality for sections of a vector bundle which lie in the kernel of a natural injectively elliptic first-order linear differential operator. Our main result is a general expression which gives the value of the constants appearing in the refined inequalities. These constants are shown to be optimal and are computed explicitly in most practical cases. © 2000 Academic Press

Key Words: Kato inequality; conformal weight; elliptic operator.

1. INTRODUCTION

The Kato inequality is an elementary and well-known estimate in Riemannian geometry, which has proved to be a powerful technique for linking vector-valued and scalar-valued problems in analysis on manifolds [3, 5, 6, 16, 20, 28]. Its content may be stated as follows: for any section ξ of any Riemannian

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(or Hermitian) vector bundle E endowed with a metric connection ∇ over a Riemannian manifold (M, g), and at any point where ξ does not vanish,

$$|d\,|\xi|| \leqslant |\nabla\xi|. \tag{0.1}$$

This estimate is easily obtained by applying the Schwarz inequality to the right-hand side of the trivial identity: $d(|\xi|^2) = 2\,\langle \nabla\xi, \xi\rangle$. Hence equality is achieved at a given point x if and only if $\nabla\xi$ is a multiple of ξ at x, i.e., if and only if there is a 1-form α such that

$$\nabla\xi = \alpha \otimes \xi. \tag{0.2}$$

The present work is motivated by circumstances in which more subtle versions of the Kato inequality appear. Examples include the treatment of the Bernstein problem for minimal hypersurfaces in $\mathbb{R}^n$ by R. Schoen, *et al.* [26], where it is shown that the second fundamental form h of any minimal immersion satisfies

$$|d\,|h|| \leqslant \sqrt{\frac{n}{n+2}}\,|\nabla h|, \tag{0.3}$$

(see also [4]); the study by S. Bando *et al.* of Ricci flat and asymptotically flat manifolds [1], where a key role is played by the inequality

$$|d\,|W|| \leqslant \sqrt{\frac{n-1}{n+1}}\,|\nabla W| \tag{0.4}$$

for the Weyl curvature W of any Einstein metric; and the proof given by J. Rade of the classical decay at infinity of any Yang–Mills field F on $\mathbb{R}^4$ [23], which relies on the estimate

$$|d\,|F|| \leqslant \sqrt{\frac{2}{3}}\,|\nabla F|. \tag{0.5}$$

Other examples may be found in the work of S. T. Yau on the Calabi conjecture [29], or more recently in the work of P. Feehan [11] and M. Gursky and C. LeBrun [15]. For a survey of these techniques see also [21].

In all of these examples, the classical Kato inequality (0.1) is insufficient to obtain the desired results. Moreover, the knowledge of the best constant involved between the two terms of the inequality seems to be a key element of all the proofs. For instance, in the case of Yang–Mills fields on $\mathbb{R}^4$, the classical Kato inequality (0.1) gives only the decay estimate $|F| = O(r^{-2})$ at infinity, whereas the (optimal) refined inequality (0.5) yields the expected $|F| = O(r^{-4})$ and thus paves the way for proving that any finite energy Yang–Mills field on flat space is induced from one on the sphere.

These examples suggest that it is an interesting question to determine when such a refined Kato inequality may occur and to compute its optimal constant. A convincing explanation of the principle underlying this phenomenon was first provided by J. P. Bourguignon in [7]. He remarked that in all the cases quoted above the sections under consideration are solutions of a natural linear first-order injectively elliptic system, and that in such a situation equality cannot occur in (0.1) except at points where $\nabla\xi=0$. To see this, suppose that equality is achieved (at a point) by a solution ξ of such an elliptic system. At that point, $\nabla\xi=\alpha\otimes\xi$ for some 1-form α. Now a natural first-order linear differential operator may be written as $\Pi\circ\nabla$, where Π is a projection onto a (natural) subbundle of $T^*M\otimes E$. Hence $\Pi(\alpha\otimes\xi)$ vanishes and so, by ellipticity, $\alpha\otimes\xi$ vanishes.

Hence it is reasonable to expect that a refined Kato constant might appear in this situation, i.e., that there should exist a constant $k_P<1$, depending only on the choice of elliptic operator P, such that

$$|d\,|\xi||\leqslant k_P\,|\nabla\xi| \tag{0.6}$$

if ξ lies in the kernel of P.

In this paper we attack the task of establishing explicitly the existence of refined Kato constants for the injectively elliptic linear first-order operators naturally defined on bundles associated to a Riemannian (spin) manifold by an irreducible representation of the special orthogonal group $\mathrm{SO}(n)$ or its nontrivial double-cover $\mathrm{Spin}(n)$. We devise a systematic method to obtain the values of the refined constants k_P and we compute the constants explicitly in a large number of cases. We express the constants in terms of the *conformal weights* of *generalized gradients* (those operators given by projection on an *irreducible* component of the tensor product above) which are numbers canonically attached to any such operator and which can be easily computed from representation–theoretic data (see Section 2 for details). As a byproduct of our approach, we obtain a number of representation-theoretic formulae, relating conformal weights to higher Casimirs of $\mathfrak{so}(n)$, some of which appear to be new.

The structure of the paper is as follows. In the first section, we present the basic definitions and strategy that will be followed to obtain the Kato constants. Then, in Section 2, we review the representation-theoretic background that will be needed for our study. We do this in part for the benefit of the reader with a limited knowledge of representation theory, but also to set up some notation and to demonstrate that the conformal weights used in the following are easy to compute. Most importantly, we discuss the question of which first order natural operators are injectively elliptic. This question has been settled by Branson [8], whose result we restate in the notation of this paper.

Before developing the main machinery, we use some elementary computations to give the Kato constants when the number N of irreducible components of $T^*M \otimes E$ is 2. Although this is entirely straightforward, the results are sufficient to obtain a new proof of the Hijazi inequality in spin geometry, which we sketch. For more complicated representations we need more tools, which we develop in Section 4. Building on the work of Perelomov and Popov [22], and also on more recent ideas of Diemer and Weingart [30] we study higher Casimir elements in the universal enveloping algebra of $\mathfrak{so}(n)$ and obtain formulae relating them to conformal weights. The main result in this direction is Theorem 4.8. We use this in Section 5 to prove our main theorem, which reduces the search for Kato constants to linear programming. Section 6 gives some explicit constants for N odd, whereas Section 7 deals with the case that N is even. In each we give the Kato constants for a large number of operators and we detail the precise values for $N=3$ and $N=4$. We also deal with the sharpness of our inequalities by giving the (algebraic) equality case. Finally, as an appendix, we present tables listing all of the Kato constants in dimensions 3 and 4.

1. STRATEGY

We consider an irreducible natural vector bundle E over a Riemannian (spin) manifold (M, g) of dimension n with scalar product $\langle \cdot, \cdot \rangle$ and a metric connection ∇. By assumption, E is attached to an irreducible representation λ of $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$ on a vector space V. If τ is the standard representation on $\mathbb{R}^n$, then the (real) tensor product $\tau \otimes \lambda$ splits in N irreducible components as

$$\tau \otimes \lambda = \bigoplus_{j=1}^{N} \mu^{(j)}$$

$$\mathbb{R}^n \otimes V = \bigoplus_{j=1}^{N} W_j.$$

This induces a decomposition of $T^*M \otimes E$ into irreducible subbundles F_j associated to the representations $\mu^{(j)}$. Projection on the jth summand (of $\mathbb{R}^n \otimes V$ or $T^*M \otimes E$) will be denoted Π_j.

Following [12, 14, 18], we can describe this decomposition in terms of the equivariant endomorphism B: $\mathbb{R}^n \otimes V \to \mathbb{R}^n \otimes V$ defined by

$$B(\alpha \otimes v) = \sum_{i=1}^{n} e_i \otimes d\lambda(e_i \wedge \alpha)\, v, \tag{1.1}$$

where $e_1, ..., e_n$ is an orthonormal basis of $\mathbb{R}^n$ and $d\lambda$ is the representation of $\mathfrak{so}(n)$ induced by λ.

1.1. *Notation.* For a linear map $T\colon \mathbb{R}^n \otimes V \to \mathbb{R}^n \otimes V$ we write $\alpha \otimes \beta \mapsto T_{\alpha \otimes \beta}$ for the unique linear map $\mathbb{R}^n \otimes \mathbb{R}^n \to \mathrm{End}(V)$ satisfying

$$T(\alpha \otimes v) = \sum_{i=1}^{n} e_i \otimes T_{e_i \otimes \alpha}(v). \tag{1.2}$$

Note that $(S \circ T)_{\alpha \otimes \beta} = \sum_{i=1}^{n} S_{\alpha \otimes e_i} \circ T_{e_i \otimes \beta}$.

Observe that $B_{\alpha \otimes \beta} = d\lambda(\alpha \wedge \beta)$ is a skew endomorphism of V which is skew in $\alpha \otimes \beta$, and that B itself is symmetric. Therefore, the eigenvalues of B are real and so, by Schur's lemma, on the irreducible summands W_j it acts by scalar multiples w_j of the identity, called *conformal weights*. The conformal weights are all distinct, except in the case that V is an representation of $\mathrm{SO}(n)$ such that $\mathbb{R}^n \otimes V$ contains two irreducible components whose sum is an irreducible representation of $O(n)$. Therefore, apart from this exceptional situation, the decomposition of $\mathbb{R}^n \otimes V$ into irreducibles corresponds precisely to its eigenspace decomposition under B. We shall adopt the convention that irreducible representations of $O(n)$ in $\mathbb{R}^n \otimes V$ will *not be split* under $\mathrm{SO}(n)$, so that the conformal weights w_j of W_j are *always distinct*. Henceforth, therefore, W_j will denote the eigenspaces of B arranged so that the conformal weights w_j are (strictly) decreasing, and N will denote the number of eigenspaces, i.e., the number of (distinct) conformal weights.

The origin of this terminology is the following fact [12, 14]: when the connection ∇ on E is induced by the Levi–Civita connection of (M, g), the natural first order operators $P_j = \Pi_j \circ \nabla$, sometimes called *generalized gradients*, are conformally invariant with conformal weight w_j.

The operators of interest in this paper are the first order linear differential operators $P_I := \sum_{i \in I} \Pi_i \circ \nabla$ acting on sections of E, where I is a subset of $\{1, ..., N\}$. Such operators are called *Stein–Weiss operators* [27]. The operator P_I is said to be (injectively, i.e., possibly overdetermined) *elliptic* iff its symbol $\Pi_I := \sum_{i \in I} \Pi_i$ does not vanish on any nonzero decomposable elements $\alpha \otimes v$ of the tensor product $\mathbb{R}^n \otimes V$. Note that P_I is (injectively) elliptic if and only if $P_I^* \circ P_I$ is elliptic in the usual sense.

We could consider, more generally, the operators $\sum_{i \in I} a_i P_i$ for any nonzero coefficients a_i: such an operator will be elliptic iff P_I is, and the methods of this paper can be adapted to apply to this situation. Also note that throughout the paper ∇ can be an *arbitrary* metric connection on E, i.e., it need not be induced by the Levi–Civita connection of M.

We shall obtain refined Kato inequalities from refined Schwarz inequalities of the form

$$\frac{|\langle \Phi, v\rangle|}{|v|} \leqslant k\,|\Phi|, \tag{1.3}$$

where $\Phi \in \mathbb{R}^n \otimes V$ and $v \in V$. For $k=1$, this holds for any Φ and nonzero v, with equality if $\Phi = \alpha \otimes v$ for some $\alpha \in \mathbb{R}^n$. Recall that the classical Kato inequality (0.1) is obtained from this by lifting it to the associated bundles and putting $v = \xi$, $\Phi = \nabla\xi$ for a section ξ of E. If ξ lies in the kernel of the operator P_I then $\nabla\xi$ is a section of $\ker \Pi_I = W_{\hat{I}}$, where $\hat{I}$ is the complement of I in $\{1, ..., N\}$ and $W_{\hat{I}}$ denotes the image of $\Pi_{\hat{I}}$. Hence to obtain a Kato inequality for the operator P_I, we only need an estimate of the form (1.3) for $\Phi \in W_{\hat{I}}$ and $v \in V$. The supremum, over all nonzero v, of the left-hand side of (1.3) is the *operator norm* $|\Phi|_{\mathrm{op}}$ of $\langle \Phi, \cdot\rangle$, viewed as a linear map from V to $\mathbb{R}^n$. Now observe that for any $\Phi \in W_{\hat{I}}$, we have:

$$|\Phi|_{\mathrm{op}} = \sup_{|v|=1} |\langle \Phi, v\rangle| = \sup_{|\alpha|=|v|=1} |\langle \Phi, \alpha\otimes v\rangle| = \sup_{|\alpha|=|v|=1} |\langle \Phi, \Pi_{\hat{I}}(\alpha\otimes v)\rangle$$
$$\leqslant \left(\sup_{|\alpha|=|v|=1} |\Pi_{\hat{I}}(\alpha\otimes v)| \right) |\Phi|.$$

This gives a refined Schwarz inequality with $k = \sup_{|\alpha|=|v|=1} |\Pi_{\hat{I}}(\alpha\otimes v)|$,

$$\frac{|\langle \Phi, v\rangle|}{|v|} = \frac{|\langle \Phi, \alpha_0 \otimes v\rangle|}{|v|} = \frac{|\langle \Phi, \Pi_{\hat{I}}(\alpha_0\otimes v)\rangle|}{|v|}$$
$$\leqslant \frac{|\Pi_{\hat{I}}(\alpha_0\otimes v)|}{|v|}\,|\Phi| \leqslant \left(\sup_{|\alpha|=|v|=1} |\Pi_{\hat{I}}(\alpha\otimes v)| \right) |\Phi|,$$

where α_0 is any unit 1-form such that $\langle \Phi, v\rangle = c\alpha_0$ for some $c \in \mathbb{R}$.

We therefore have the following Ansatz which reduces the search for refined Kato inequalities to a purely algebraic problem.

1.2. Ansatz. *Consider the operator P_I on the natural vector bundle E over (M, g). Then, for any section ξ in the kernel of P_I, and at any point where ξ does not vanish, we have*

$$|d\,|\xi|| \leqslant k_I\,|\nabla\xi|,$$

where the constant k_I is defined by

$$k_I = \sup_{|\alpha|=|v|=1} |\Pi_{\hat{I}}(\alpha\otimes v)|.$$

Furthermore equality holds at a point if and only if $\nabla\xi = \Pi_{\hat{I}}(\alpha\otimes\xi)$ *for a* 1-*form* α *at that point such that* $|\Pi_{\hat{I}}(\alpha\otimes\xi)| = k_I\,|\alpha\otimes\xi|$.

1.3. *Remark.* Equality holds in this Kato inequality if and only if it holds in the refined Schwarz inequality with $v=\xi$, $\Phi=\nabla\xi$. Hence the above Ansatz is algebraically sharp: the supremum $\sup_{|\alpha|=|v|=1}|\Pi_{\hat{I}}(\alpha\otimes v)|$ is attained by compactness. We also deduce that the Kato inequality is sharp in the flat case: equality is attained by a suitable chosen affine solution of $P_I\xi=0$.

In order to turn this Ansatz into a useful result, we must:

(i) Find when P_I is elliptic.

(ii) Show that when P_I is elliptic, k_I is less than one.

(iii) Give a formula for k_I in terms of easily computable data.

(iv) Obtain a more explicit description of the equality case.

The first question has been answered by T. Branson [8]. We shall discuss his result at the end of the next section. Also in that section we shall give a more explicit description of the operators and representations involved, together with the associated conformal weights. The conformal weights are easy to compute and so our guiding philosophy will be: *find* k_I *in terms of the conformal weights.*

Since $k_I = \sup_{|\alpha|=|v|=1}|\Pi_{\hat{I}}(\alpha\otimes v)|$ and

$$|\Pi_{\hat{I}}(\alpha\otimes v)|^2 = \sum_{j\in\hat{I}} |\Pi_j(\alpha\otimes v)|^2,$$

a key step in our task is to find a convenient formula for $|\Pi_j(\alpha\otimes v)|$ for each $j=1, ..., N$.

To do this, note that Π_j is the projection onto an eigenspace of B, and so Lagrange interpolation gives the standard formulae,

$$\Pi_j = \prod_{k\neq j}\frac{B-w_k\,\mathrm{id}}{w_j-w_k} = \frac{\sum_{k=0}^{N-1} w_j^{N-1-k}\sum_{\ell=0}^{k}(-1)^\ell\,\sigma_\ell(w)\,B^{k-\ell}}{\prod_{k\neq j}(w_j-w_k)}, \tag{1.4}$$

where $\sigma_i(w)$ denotes the ith elementary symmetric function in the eigenvalues w_j. We define A_k to be the operators

$$A_k = \sum_{\ell=0}^{k}(-1)^\ell\,\sigma_\ell(w)\,B^{k-\ell} \tag{1.5}$$

appearing in this formula, which are manifestly symmetric in the conformal weights. Using these operators, we have:

$$|\Pi_j(\alpha\otimes v)|^2=\langle \Pi_j(\alpha\otimes v), \alpha\otimes v\rangle=\frac{\sum_{k=0}^{N-1} w_j^{N-1-k}\,\langle A_k(\alpha\otimes v), \alpha\otimes v\rangle}{\prod_{k\neq j}(w_j-w_k)}. \tag{1.6}$$

This formula for the N quantities $|\Pi_j(\alpha\otimes v)|$ in terms of the N quantities $q_k=\langle A_k(\alpha\otimes v), \alpha\otimes v\rangle$ lies at the heart of our method. Note first that $A_0=1$, and so $q_0=|\alpha\otimes v|^2$, which we set equal to 1. Second, the formula (1.1) for B implies that

$$\langle B(\alpha\otimes v), \alpha\otimes v\rangle=0, \qquad \forall\alpha\in\mathbb{R}^n, \quad v\in V. \tag{1.7}$$

Hence q_1 is also computable. These two observations alone will allow us to find the Kato constants for $N\leqslant 4$. For larger N we shall need to obtain more information about the operators A_k.

We shall find that approximately half of the q_k's can be eliminated. The remainder can then be estimated from above and below using the non-negativity of $|\Pi_j(\alpha\otimes v)|$. These bounds can in turn be used to estimate $|\Pi_{\hat{\jmath}}(\alpha\otimes v)|$.

2. REPRESENTATION THEORETIC BACKGROUND

The description of representations of the special orthogonal group $\mathrm{SO}(n)$, or its Lie algebra $\mathfrak{so}(n)$, differs slightly according to the parity of n. We write $n=2m$ if n is even and $n=2m+1$ if n is odd; m is then the *rank* of $\mathfrak{so}(n)$.

We fix an oriented orthonormal basis $(e_1, ..., e_n)$ of $\mathbb{R}^n$, so that $e_i\wedge e_j$ (for $i<j$) is a basis of the Lie algebra $\mathfrak{so}(n)$, identified with $\Lambda^2\mathbb{R}^n$. We also fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}(n)$ by the basis $E_1=e_1\wedge e_2, ..., E_m=e_{2m-1}\wedge e_{2m}$, and denote the dual basis of $\mathfrak{h}^*$ by $(\varepsilon_1, ..., \varepsilon_m)$. We normalize the Killing form so that this basis is orthonormal. For further information on this, and the following, see [13, 24, 25].

An irreducible representation of $\mathfrak{so}(n)$ will be identified with its dominant weight $\lambda\in\mathfrak{h}^*$. Roots and weights can be given by their coordinates with respect to the orthonormal basis ε_i. Then the weight $\lambda=(\lambda_1, \lambda_2, ..., \lambda_m)$, whose coordinates are all integers or all half-integers, is dominant iff

$$\lambda_1\geqslant\cdots\geqslant\lambda_{m-1}\geqslant|\lambda_m|, \qquad n=2m,$$
$$\lambda_1\geqslant\cdots\geqslant\lambda_{m-1}\geqslant\lambda_m\geqslant 0, \qquad n=2m+1.$$

In this notation, the standard representation τ is given by the weight $(1, 0, ..., 0)$, the weight $\lambda = (1, 1, ..., 1, 0, ..., 0)$ (with k ones) corresponds to the k-form representation $\Lambda^k \mathbb{R}^n$, the weights $\lambda = (1, 1, ...1, \pm 1)$ (for $n = 2m$) correspond to the selfdual and antiselfdual m-forms, and the weights $\lambda = (\frac{1}{2}, ..., \frac{1}{2}, (\pm)\frac{1}{2})$ correspond to the spin or half-spin representations $\Delta_{(\pm)}$. The *Cartan product* of two representations is the subrepresentation $\lambda \odot \mu$ of highest weight $\lambda + \mu$ in $\lambda \otimes \mu$. If λ and μ are integral then $\lambda \odot \mu$ is the subrepresentation of "alternating-free, trace-free" tensors in $\lambda \otimes \mu$; for instance, the k-fold Cartan product $\bigodot^k \mathbb{R}^n$ is the representation $S^k_0 \mathbb{R}^n$ of totally symmetric traceless tensors, with weight $(k, 0, ..., 0)$.

Note that we take the real form of the representations wherever possible: in particular, when discussing elements of the tensor product $\tau \otimes \lambda$ only real elements of the standard representation will be used, even if λ is complex.

The decomposition of the tensor product $\tau \otimes \lambda$ into irreducibles is described by the following rule: an irreducible representation of weight μ appears in the decomposition if and only if

(i) $\mu = \lambda \pm \varepsilon_j$ for some j, or $n = 2m + 1$, $\lambda_m > 0$, and $\mu = \lambda$

(ii) μ is a dominant weight.

Weights μ satisfying (i) will be called *virtual* weights associated to λ. We shall say that μ is *effective* if it also satisfies (ii). It will be convenient to have a notation for the virtual weights which is compatible with the outer automorphism equivalence of representations of $\mathfrak{so}(2m)$. We define $\mu^0 = \lambda$ and $\mu^{i, \pm} = \lambda \pm \varepsilon_i$, unless $n = 2m$, $j = m$, and $\lambda_m \neq 0$, in which case we define $\mu^{m, \pm}$ to be the virtual weights such that $|\mu^{m,+}_m| = |\lambda_m| + 1$ and $|\mu^{m,-}_m| = |\lambda_m| - 1$. This notation allows us to assume, without loss of generality, that $\lambda_m = |\lambda_m|$, and we shall omit the modulus signs in the following.

The *Casimir number* of a representation λ is given by

$$c(\lambda) = \langle \lambda + \delta, \lambda + \delta \rangle - \langle \delta, \delta \rangle = \langle \lambda, \lambda \rangle + 2 \langle \lambda, \delta \rangle, \tag{2.1}$$

where δ is the half-sum of positive roots, i.e., $\delta_i = (n - 2i)/2$.

The *conformal weight* associated to a component μ of $\tau \otimes \lambda$ may be computed explicitly by the formula

$$w(\mu, \lambda) = \tfrac{1}{2}\,(c(\mu) - c(\lambda) - c(\tau)), \tag{2.2}$$

which continues to make sense for virtual weights. We let w^0 and $w^{i, \pm}$ denote the (virtual) conformal weights of μ^0 and $\mu^{i, \pm}$. Expanding the definition of the Casimir, and applying some Euclidean geometry in $\mathfrak{h}^*$, we obtain the explicit formulae (assuming $\lambda_m = |\lambda_m|$):

$$w^0 = (1-n)/2 \tag{2.3}$$

$$w^{i,+} = 1 + \lambda_i - i \tag{2.4}$$

$$w^{i,-} = 1 - n - (\lambda_i - i). \tag{2.5}$$

These formulae show that conformal weights are simple to compute in practice, which is one of our motivations for using them. We note that the virtual conformal weights $w^{i,\pm}$ satisfy

$$w^{1,+} > w^{2,+} > \cdots > w^{m,+} \geqslant w^{m,-} > \cdots > w^{2,-} > w^{1,-} \tag{2.6}$$

with equality in the middle if and only if $n = 2m$ and $\lambda_m = 0$. If $n = 2m+1$ and $\lambda_m > 0$, then the conformal weight w^0 lies strictly between $w^{m,+}$ and $w^{m,-}$. This verifies our earlier claim that the conformal weights are almost always distinct.

For effective weights, we remind the reader of our convention of not splitting subrepresentations with the same conformal weight. This means that we write $\tau \otimes \lambda = \bigoplus_{j=1}^{N} \mu^{(j)}$, where the representations $\mu^{(j)}$ are all irreducible, unless $n = 2m$ and $\lambda_m = 0$, in which case one of the components is taken to be $\mu^{m,+} \oplus \mu^{m,-}$.

In order to say which of the weights are effective (and hence, which representations occur in $\tau \otimes \lambda$), it is useful to make explicit any repetitions among the coordinates λ_j by writing λ in the form

$$\lambda = (k_1, \ldots, k_1, k_2, \ldots, k_2, \ldots, k_\nu, \ldots, k_\nu),$$

with $k_1 > k_2 > \cdots > k_{\nu-1} > k_\nu \geqslant 0$. If $k_\nu \neq 0$ and $n = 2m$, we write

$$\lambda = (k_1, \ldots, k_1, k_2, \ldots, k_2, \ldots, k_\nu, \ldots, \pm k_\nu)$$

for the two possible signs of the last entry. Here ν is the number of groups of equal entries and we let p_1 denote the number of k_1's, $p_2 - p_1$ the number of k_2's, etc., so that p_j is the number of entries greater than or equal to k_j.

We first note that the following $2\nu - 1$ weights, at least, are effective for any representation λ, and are associated with the conformal weights listed.

$$\begin{array}{ll}
\mu^{1,+} & w^{1,+} = k_1 \\
\mu^{p_1+1,+} & w^{p_1+1,+} = k_2 - p_1 \\
\vdots & \vdots \\
\mu^{p_{\nu-1}+1,+} & w^{p_{\nu-1}+1,+} = k_\nu - p_{\nu-1} \\
\mu^{p_{\nu-1},-} & w^{p_{\nu-1},-} = p_{\nu-1} - k_{\nu-1} + 1 - n \\
\vdots & \vdots \\
\mu^{p_1,-} & w^{p_1,-} = p_1 - k_1 + 1 - n.
\end{array}$$

If $k_\nu = 0$ there are no further effective weights unless $n = 2m$ and $p_{\nu-1} = m-1$, in which case $\mu^{m,\pm}$ are both effective with the same conformal weight. Hence, by convention, if $k_\nu = 0$ then $N = 2\nu - 1$.

If $k_\nu > 0$ and $n = 2m$ then $\mu^{m,-}$ is effective and $N = 2\nu$. If $k_\nu > 0$ and $n = 2m+1$ then μ^0 is a possible target; furthermore $\mu^{m,-}$ is effective for $k_\nu > 1/2$.

We therefore see that the number of components N in the decomposition $\mathbb{R}^n \otimes V = \bigoplus_{j=1}^N W_j$ is either $2\nu - 1$, 2ν, or $2\nu + 1$.

The case $N = 2\nu - 1$ arises when $\lambda_m = 0$. The representations occurring, in order of decreasing conformal weight, are as follows:

$$\mu^{(1)} = \mu^{1,+}, \qquad \mu^{(2)} = \mu^{p_1+1,+}, \ldots, \mu^{(\nu-1)} = \mu^{p_{\nu-2}+1,+},$$
$$\mu^{(\nu)} = \mu^{p_{\nu-1}+1,+} \quad \text{or} \quad \mu^{m,+} \oplus \mu^{m,-},$$
$$\mu^{(\nu+1)} = \mu^{p_{\nu-1},-}, \qquad \mu^{(\nu+2)} = \mu^{p_{\nu-2},-}, \ldots, \mu^{(2\nu-1)} = \mu^{p_1,-}.$$

The case $N = 2\nu$ arises when $n = 2m+1$ and $\lambda_m = 1/2$ or when $n = 2m$ and $\lambda_m \neq 0$. The representations occurring, in order of decreasing conformal weight, are as follows:

$$\mu^{(1)} = \mu^{1,+}, \qquad \mu^{(2)} = \mu^{p_1+1,+}, \ldots, \mu^{(\nu)} = \mu^{p_{\nu-1}+1,+},$$
$$\mu^{(\nu+1)} = \mu^{m,-} \quad \text{or} \quad \mu^0, \qquad \mu^{(\nu+2)} = \mu^{p_{\nu-1},-}, \ldots, \mu^{(2\nu)} = \mu^{p_1,-}.$$

The case $N = 2\nu + 1$ arises when $n = 2m+1$ and $\lambda_m > 1/2$. The representations occurring, in order of decreasing conformal weight, are as follows:

$$\mu^{(1)} = \mu^{1,+}, \qquad \mu^{(2)} = \mu^{p_1+1,+}, \ldots, \mu^{(\nu)} = \mu^{p_{\nu-1}+1,+},$$
$$\mu^{(\nu+1)} = \mu^0,$$
$$\mu^{(\nu+2)} = \mu^{m,-}, \qquad \mu^{(\nu+3)} = \mu^{p_{\nu-1},-}, \ldots, \mu^{(2\nu+1)} = \mu^{p_1,-}.$$

Note that for "most" representations (e.g., if $\lambda_m \neq 0$) N and n have the same parity. Indeed, if $\lambda_1 > \lambda_2 > \cdots > |\lambda_m| > 0$ we see that $N = n$. However, the representations arising in practice are not at all generic: N is usually very small.

We are now in a position to describe T. Branson's classification of the elliptic operators [8]. First, note that if J is a subset of I such that P_J is elliptic, then P_I is elliptic. Hence it suffices to find the *minimal elliptic operators* P_I, i.e., the elliptic P_I such that P_J is not elliptic for any proper subset J of I.

2.1. THEOREM (Branson [8]). *Let λ be an irreducible representation of* SO(n) *or* Spin(n). *Then the minimal elliptic operators associated to λ are* *either elementary or the sum of two elementary operators. The elementary elliptic operators are*:

(i) P_1 *with target* $\mu^{1,+}$.

(ii) *For* $N=2\nu$, $P_{\nu+1}$ *with target* $\mu^{m,-}$ *or* μ^0.

(iii) *For* $N=2\nu+1$, $P_{\nu+1}$ *with target* μ^0, *provided* λ *is properly half-integral.*

The other minimal elliptic operators are:

(iv) $P_{\{j, N+2-j\}}$ *with target* $\mu^{p_{j-1}+1,+} \oplus \mu^{p_{j-1},-}$ *or* $\mu^{m,+} \oplus \mu^{m,-} \oplus \mu^{m-1,-}$ *for all* $j \in \{2, ..., \nu\}$. (*For* $N=2\nu-1$, $j=\nu$, *and* $p_{\nu-1}=m-1$, $P_{\{\nu,\nu+1\}}$ *is obtained by combining the operators with targets* $\mu^{m,\pm} \oplus \mu^{m-1,-}$, *which are both elliptic.*)

(v) *For* $N=2\nu+1$: $P_{\{\nu+1,\nu+2\}}$ *with target* $\mu^0 \oplus \mu^{m,-}$, *provided* λ *is integral.*

Notice that the subsets of N corresponding to the minimal elliptic operators partition N (where we combine the operators with targets $\mu^{m,\pm} \oplus \mu^{m-1,-}$), unless $N=2\nu+1$ and λ is properly half-integral, in which case there is one "useless" operator $P_{\nu+2}$. This means that there are non-elliptic operators with relatively large targets. Indeed, the above theorem may equivalently be viewed as a description of the maximal non-elliptic operators. These play an important role in our later work, so we shall describe them explicitly here.

2.2. Definition. Let $\mathcal{NE}$ denote the set of subsets of $\{1, ..., N\}$ whose elements are obtained by choosing exactly one index in each of the sets $\{j, N+2-j\}$ for each j with $2 \leqslant j \leqslant \nu$ if $N=2\nu-1, 2\nu$ (giving $2^{\nu-1}$ elements in $\mathcal{NE}$) and for each j with $2 \leqslant j \leqslant \nu+1$ if $N=2\nu+1$ (giving 2^ν elements in $\mathcal{NE}$).

Branson's theorem implies that the set $\mathcal{NE}$ is precisely the set of subsets of $\{1, ..., N\}$ corresponding to the maximal non-elliptic operators, unless $N=2\nu+1$ and λ is properly half-integral, in which case the maximal non-elliptic operators correspond to the elements of $\mathcal{NE}$ which do not contain $\nu+1$. This last case will cause us problems because there are not enough non-elliptic subsets.

Branson proves Theorem 2.1 by reducing the problem to the study of the spectrum of the operator on the sphere $M=S^n$, which he computes by applying powerful techniques from harmonic analysis. For the benefit of the reader not familiar with these global techniques, we remark that there are some cases in which ellipticity or non-ellipticity can be established by elementary local arguments.

Since ellipticity depends only on the symbol Π_I on $\mathbb{R}^n \otimes V$ and since $\mathrm{SO}(n)$ is transitive on the unit sphere in $\mathbb{R}^n$, it follows that P_I is elliptic if and only if the linear map $v \to \Pi_I(e_n \otimes v)$ is injective (for a fixed unit vector e_n).

First note that this map is $\mathrm{SO}(n-1)$-equivariant and so we have the following necessary (but not sufficient) condition for ellipticity.

2.3. Lemma. *P_I cannot be elliptic unless every subrepresentation of V under the group $\mathrm{SO}(n-1)$ occurs as a subrepresentation of W_j for some $j \in I$.*

To use this lemma, one must apply the standard branching rule for restricting a representation of $\mathrm{SO}(n)$ to $\mathrm{SO}(n-1)$—see, for example [13, p. 426]. For $N = 2\nu - 1$ and $N = 2\nu$ it is straightforward to verify the non-ellipticity of the maximal non-elliptic operators and hence obtain most of the non-ellipticity results in Branson's theorem. For $N = 2\nu + 1$ this naive method does not cover all the cases: $P_{\nu+1}$ is *not* elliptic if λ is an integral weight, even though λ itself is the target representation.

Second, note the following sufficient (but not necessary) condition for ellipticity.

2.4. Lemma. *If the space of local solutions of P_I on $\mathbb{R}^n$ is finite-dimensional, then P_I is elliptic.*

Proof. If P_I is not elliptic then for some $v \in V$, $e_n \otimes v$ belongs to $\ker \Pi_I \leqslant \mathbb{R}^n \otimes V$. Now consider the operator P_I on $\mathbb{R}^n$ (with respect to the trivial connection on $E \cong \mathbb{R}^n \times V$). If L_v denotes the line subbundle of E corresponding to the span of $v \in V$ then any section of L_v which is independent of $x_1, ..., x_{n-1}$ belongs to the kernel of P_I. Hence the kernel of P_I is infinite dimensional on $\mathbb{R}^n$.

As observed (for instance) in [19], this second lemma shows that the *highest gradient* is always elliptic. This is the operator P_1 with the highest conformal weight w_1 whose target $\mu^{(1)}$ is the highest weight subrepresentation of $\tau \otimes \lambda$. We shall also refer to P_1 as the *Penrose* or *twistor operator*, since it reduces to the usual Penrose twistor operator if one views the representation λ as a subrepresentation of a tensor product of spinor representations. The kernel of a twistor operator on S^n (or any simply connected open subset) is well-known to be a finite-dimensional representation space for $\mathrm{SO}(n+1, 1)$: the twistor operator is the first operator in the Bernstein–Gelfand–Gelfand resolution of this representation (see for instance [2]).

Finally in this section, we recall the following ellipticity result:

2.5. PROPOSITION [14]. *P_I is elliptic in either of the following cases:*

(i) *I contains all j with $w_j \geqslant 0$*

(ii) *I contains all j with $w_j \leqslant 0$.*

These operators are of special interest because of a simple Weitzenböck formula relating them [14].

4. REFINED KATO INEQUALITIES WITH $N=2$

The case $N=2$ often arises in spin geometry and in two- and four-dimensional differential geometry. It occurs in the following two cases:

(i) When the dimension n is even, $\lambda=(k, ..., k, \pm k)$ with k an arbitrary integer or half-integer, i.e., $V=\bigodot^{2k} \Delta_+$ or $V=\bigodot^{2k} \Delta_-$. Therefore the bundle E is either $\bigodot^k \Lambda^m_\pm M$ or, if M is spin, $\bigodot^{k-1/2} \Lambda^m_\pm M \odot \Sigma^\pm$ ($\Sigma^\pm$ denotes positive and negative spinor bundles of M); one thus gets $w_1=k>w_2=1-\frac{n}{2}-k$.

(ii) When the dimension n is odd, $\lambda=(\frac{1}{2}, ..., \frac{1}{2})$, i.e., $V=\Delta$, E is the spinor bundle Σ and $w_1=\frac{1}{2}>w_2=\frac{1-n}{2}$.

Note that the operators P_1 and P_2 are both elliptic.

3.1. THEOREM. *Let E be associated to a representation λ with $N=2$.*

(i) *For any nonvanishing section ξ of E in the kernel of the twistor operator P_1,*

$$|d\,|\xi|| \leqslant \sqrt{\frac{w_1}{w_1-w_2}}\,|\nabla\xi| = \sqrt{\frac{k}{2k+(n/2)-1}}\,|\nabla\xi| \tag{3.1}$$

with equality if and only if, for some 1-form α,

$$\nabla\xi = \Pi_2(\alpha\otimes\xi).$$

(ii) *For any section ξ of E in the kernel of P_2,*

$$|d\,|\xi|| \leqslant \sqrt{\frac{-w_2}{w_1-w_2}}\,|\nabla\xi| = \sqrt{\frac{k+(n/2)-1}{2k+(n/2)-1}}\,|\nabla\xi| \tag{3.2}$$

with equality if and only if, for some 1-form α,

$$\nabla\xi = \Pi_1(\alpha\otimes\xi).$$

Proof. From the Ansatz 1.2, we have to estimate the norms of $\Pi_j(\alpha\otimes v)$ for $j=1, 2$. The crucial ingredient here is Eq. (1.7), which gives the following system of equations for the components of a unit length vector $\alpha\otimes v$ in $\mathbb{R}^n\otimes V$:

$$\begin{aligned} |\Pi_1(\alpha\otimes v)|^2+|\Pi_2(\alpha\otimes v)|^2&=1,\\ w_1\,|\Pi_1(\alpha\otimes v)|^2+w_2\,|\Pi_2(\alpha\otimes v)|^2&=0. \end{aligned} \tag{3.3}$$

The solution is a special case of Eq. (1.6),

$$|\Pi_1(\alpha\otimes v)|^2=\frac{w_2}{w_2-w_1}, \qquad |\Pi_2(\alpha\otimes v)|^2=\frac{w_1}{w_1-w_2}, \tag{3.4}$$

and moreover this is valid for *any* choice of unit α and v. These formulae easily yield the refined Kato inequalities and their equality cases.

3.2. *Remark.* The calculations above also yield some (possibly not optimal) refined Kato inequalities for any N and the operators

$$P_+=\sum_{w_j>0} P_j \qquad \text{or} \qquad P_-=\sum_{w_j<0} P_j$$

(for simplicity, we consider here only the case when conformal weights do not vanish). The reasoning for P_+ relies on the system of equations

$$\begin{aligned} |\Pi_+(\alpha\otimes v)|^2+|\Pi_-(\alpha\otimes v)|^2&=1\\ w_1\,|\Pi_+(\alpha\otimes v)|^2+w^{\max}_{<0}\,|\Pi_-(\alpha\otimes v)|^2&\geqslant 0, \end{aligned} \tag{3.5}$$

where $w^{\max}_{<0}=\max_{w_j<0} w_j$ and $\Pi_\pm$ are the projections associated to both operators. One easily gets the refined Kato inequality

$$|d\,|\xi||\leqslant\sqrt{\frac{w_1}{w_1-w^{\max}_{<0}}}\,|\nabla\xi| \tag{3.6}$$

for any section ξ in the kernel of P_+ and similarly

$$|d\,|\xi||\leqslant\sqrt{\frac{w_N}{w_N-w^{\min}_{>0}}}\,|\nabla\xi|, \qquad \text{with} \quad w^{\min}_{>0}=\min_{w_j>0} w_j \tag{3.7}$$

for any section ξ in the kernel of P_-.

3.3. *Remark.* As an application of these results, we give a new proof of the *Hijazi inequality* in spin geometry relating the first eigenvalue of the Dirac operator on a Riemannian spin manifold to the first eigenvalue of its conformal Laplacian. This application is due to Christian Bär and Andrei Moroianu [31] and we thank them for their permission to reproduce it in this work.

3.4. Proposition (Hijazi [17]). *Let M be a compact Riemannian spin manifold of dimension $n \geqslant 3$. Then the first eigenvalue λ_1 of the Dirac operator and the first eigenvalue μ_1 of the conformal Laplacian $4\,\frac{n-1}{n-2}\,\Delta + scal$ satisfy*:

$$\lambda_1^2 \geqslant \frac{n}{4(n-1)}\,\mu_1. \tag{3.8}$$

Proof. If ψ is an eigenspinor with eigenvalue λ, then ψ lies in the kernel of the Dirac operator given by the Friedrich connection $\tilde{\nabla}_X \psi = \nabla_X \psi + (\lambda/n)\, X \cdot \psi$, which is a metric connection on spinors. Hence we have the following refined Kato inequality for ψ, wherever it is nonzero:

$$|d\,|\psi||^2 \leqslant \frac{n-1}{n}\,|\tilde{\nabla}\psi|^2. \tag{3.9}$$

We next consider the conformal Laplacian of $|\psi|^{2\alpha}$ where $\alpha = \frac{n-2}{2(n-1)}$: the conformal Laplacian is invariant on scalars of weight $\frac{2-n}{2}$ and so this power is natural in view of the conformal weight $\frac{1-n}{2}$ for the Dirac operator. Using the Lichnerowicz formula and the elementary identity $d^*d(f^\alpha) = \alpha f^{\alpha-1} d^*df - \alpha(\alpha-1)\, f^{\alpha-2}\,|df|^2$ with $f = |\psi|^2$, we obtain the equalities on the open set where ψ is nonzero,

$$\begin{aligned}
&\frac{1}{2\alpha}\, d^*d(|\psi|^{2\alpha}) + \frac{1}{4}\, scal\, |\psi|^{2\alpha} - \frac{n-1}{n}\,\lambda^2\,|\psi|^{2\alpha}\\
&\quad= \frac{1}{2}\,\frac{1-\alpha}{2}\,|\psi|^{2\alpha-4}\,|d(|\psi|^2)|^2 + \frac{1}{2}\,|\psi|^{2\alpha-2}\, d^*d(|\psi|^2)\\
&\qquad+ \left(\frac{1}{4}\, scal - \frac{n-1}{n}\,\lambda^2\right) |\psi|^{2\alpha}\\
&\quad= |\psi|^{2\alpha-2} \Bigg(2(1-\alpha)\,|d\,|\psi||^2 + \langle \nabla^*\nabla\psi, \psi\rangle\\
&\qquad- |\nabla\psi|^2 + \left(\frac{1}{4}\, scal - \frac{n-1}{n}\,\lambda^2\right) |\psi|^2 \Bigg)\\
&\quad= |\psi|^{2\alpha-2} \left(2(1-\alpha)\,|d\,|\psi||^2 + \left(1 - \frac{n-1}{n}\right) \lambda^2\,|\psi|^2 - |\nabla\psi|^2 \right)\\
&\quad= |\psi|^{2\alpha-2} \left(\frac{n}{n-1}\,|d\,|\psi||^2 - |\tilde{\nabla}\psi|^2 \right),
\end{aligned}$$

since $|\tilde{\nabla}\psi|^2 = |\nabla\psi|^2 + \frac{1}{n}\,\lambda^2\,|\psi|^2$. This is nonpositive by (3.9). Note that this gives a local version of the Hijazi inequality, with equality iff $\tilde{\nabla}\psi$ is the projection of $\alpha \otimes \psi$ onto the kernel of Clifford multiplication, for some

1-form α. If the eigenvalue λ is nonzero, then differentiating and commuting derivatives shows in fact that $\tilde{\nabla}\psi = 0$. The case $\lambda = 0$ is distinguished by conformal invariance and the fundamental solutions $\psi(x) = c(x)\,\phi/|x|^n$ give examples with $\nabla\psi \neq 0$.

In order to globalize, we consider the Rayleigh quotient for the first eigenvalue μ_1 of the conformal Laplacian:

$$\mu_1 \leqslant \frac{\int_M 4\,\frac{n-1}{n-2}\,|d\varphi|^2 + scal\,\varphi^2}{\int_M \varphi^2}.$$

We can estimate the integral in the numerator by setting $\varphi = |\psi|^{2\alpha}$ on the open set where ψ is nonzero and writing

$$\begin{aligned}
&4\,\frac{n-1}{n-2}\,|(d\,|\psi|^{2\alpha})|^2 + scal\,|\psi|^{4\alpha}\\
&\quad = 4\,|\psi|^{2\alpha}\left(\frac{n-1}{n-2}\,d^*d(|\psi|^{2\alpha}) + \frac{1}{4}\,scal\,|\psi|^{2\alpha}\right) - \frac{2(n-1)}{n-2}\,d^*d(|\psi|^{4\alpha})\\
&\quad \leqslant \frac{4(n-1)}{n}\,\lambda^2\,|\psi|^{4\alpha} - \frac{2(n-1)}{n-2}\,d^*d(|\psi|^{4\alpha}).
\end{aligned}$$

Taking $\lambda = \lambda_1$, integrating over $\{x \in M : |\psi|(x) \geqslant \varepsilon\}$, and letting $\varepsilon \to 0$ gives (3.8). The equality case is also easy to establish.

A similar argument can be used to provide an alternative proof of the $N = 2$ vanishing theorems of Branson–Hijazi [10].

4. CASIMIR NUMBERS AND CONFORMAL WEIGHTS

One way to understand the powers $B^\ell\colon \mathbb{R}^n \otimes V \to \mathbb{R}^n \otimes V$ of the operator B is to relate them to invariants of V. Let $\operatorname{ptr} B^\ell = \sum_i (B^\ell)_{e_i \otimes e_i}\colon V \to V$ be the *partial trace* of B obtained by contracting over $\mathbb{R}^n$. Since V is irreducible and B is symmetric and equivariant, this partial trace must be a scalar multiple of the identity. The explicit expression (1.1) for B yields the following formula:

$$\begin{aligned}
\operatorname{ptr} B^\ell = \sum_{i_1, \ldots, i_\ell} & d\lambda(e_{i_1} \wedge e_{i_2}) \circ d\lambda(e_{i_2} \wedge e_{i_3})\\
&\circ \cdots \circ d\lambda(e_{i_{\ell-1}} \wedge e_{i_\ell}) \circ d\lambda(e_{i_\ell} \wedge e_{i_1}).
\end{aligned} \tag{4.1}$$

This is the action on V of an element of the centre of the universal enveloping algebra $\mathcal{U}(\mathfrak{so}(n))$ called a *higher Casimir*, since it reduces to the Casimir element when $\ell = 2$ (and vanishes when $\ell = 1$). The (scalar) action of the Casimir element on V is the Casimir number $c(\lambda)$ of V, and it is of some interest to compute the *higher Casimir numbers*. This computation was carried out by A. Perelomov and V. Popov in [22], where a generating series for the higher Casimir numbers in terms of polynomials in λ is given.

Our aim in this section is to obtain instead relations between higher Casimirs and conformal weights. These relations will enable us to find a more convenient basis for the higher Casimirs in terms of certain linear combinations of the B^{ℓ}.

In fact, it is more natural to work with the translated operator $\widetilde{B} = B + \frac{1}{2}(n-1)\,\mathrm{id}$ and its eigenvalues, the *translated conformal weights* $\tilde{w}_j = w_j + \frac{1}{2}(n-1) = \frac{1}{2}(c(\mu^{(j)}) - c(\lambda))$. The translated virtual conformal weights are then $\tilde{w}^{i,\pm} = \frac{1}{2} \pm (\lambda_i + \frac{n}{2} - i) = \frac{1}{2} \pm x_i$ where $x = \lambda + \delta$. These translated conformal weights are more convenient because if $\lambda_i = \lambda_{i+1}$ then

$$\tilde{w}^{i+1,+} + \tilde{w}^{i,-} = 0, \tag{4.2}$$

which is a useful cancellation property for non-effective weights. In particular, there is the following immediate consequence, which already suggests that (translated) conformal weights are a convenient tool for handling Casimir numbers.

4.1. Proposition. *Let P_ℓ be the polynomial on (the dual of) the Cartan subalgebra defined by*

$$P_\ell(\lambda) = \sum_{i=1}^{m} \left(\frac{1}{2} + x_i\right)^{\ell} + \sum_{i=1}^{m} \left(\frac{1}{2} - x_i\right)^{\ell} \qquad \textit{for} \quad \ell \in \mathbb{N},$$

where $x = \lambda + \delta$. Then:

(i) *if N is odd,*

$$\sum_{j=1}^{N} \tilde{w}_j^{2k+1} - \left(\frac{n-1}{2}\right)^{2k+1} = P_{2k+1}(x) - P_{2k+1}(\delta) \qquad \forall k \in \mathbb{N}; \tag{4.3}$$

(ii) *if N is even*

$$\sum_{j=1}^{N} \tilde{w}_j^{2k+1} - \left(\frac{n-1}{2}\right)^{2k+1} - \left(\frac{1}{2}\right)^{2k+1} = P_{2k+1}(x) - P_{2k+1}(\delta) \qquad \forall k \in \mathbb{N}. \tag{4.4}$$

Proof. The starting point is the trivial formula

$$P_{2k+1}(x) = \sum (\tilde{w}^{i,\pm})^{2k+1} \qquad \forall k \in \mathbb{N},$$

where the summation is over all virtual weights (it does not matter whether we include μ^0 as $\tilde{w}^0 = 0$). However, by the cancellation formula (4.2), almost all of the non-effective weights cancel. Examining the cases, we find that

$$P_{2k+1}(x) = \sum_j \tilde{w}_j^{2k+1} \qquad N \equiv n \mod 2$$

$$P_{2k+1}(x) = \sum_j \tilde{w}_j^{2k+1} + (-1)^n \left(\tfrac{1}{2}\right)^{2k+1} \qquad N \not\equiv n \mod 2.$$

If we now apply this formula to the trivial representation, where $N = 1$ and $x = \delta$, we readily obtain the statement of the proposition.

4.2. Corollary. *For N odd,*

$$\sum_{j=1}^{N} \tilde{w}_j - \frac{n-1}{2} = 0, \qquad \sum_{j=1}^{N} (\tilde{w}_j)^3 - \left(\frac{n-1}{2}\right)^3 = 3c(\lambda), \tag{4.5}$$

and for N even,

$$\sum_{j=1}^{N} \tilde{w}_j - \frac{1}{2} - \frac{n-1}{2} = 0, \qquad \sum_{j=1}^{N} (\tilde{w}_j)^3 - \left(\frac{1}{2}\right)^3 - \left(\frac{n-1}{2}\right)^3 = 3c(\lambda). \tag{4.6}$$

4.3. *Remark.* The distinction based on the parity of N (which coincides, for generic representations, with the parity of the dimension n) can be removed by adding a "dummy" conformal weight to the sum: one can either add a translated conformal weight $\tilde{w} = -1/2$ when N is even or, following Branson [8], a translated conformal weight $\tilde{w} = 1/2$ when N is odd. This remark remains true for all the results proved in this section, provided care is taken in exceptional cases where the dummy conformal weight already occurs as an effective conformal weight.

We now obtain a generating series for the higher Casimirs. These are similar to the expressions of Perelomov and Popov [22], but differ in two significant ways: first, we compute ptr $\widetilde{B}^{\ell}$, rather than ptr B^{ℓ}; and second, we give the generating series in terms of translated conformal weights, rather than in terms of coordinates of λ.

4.4. Proposition. *The partial traces of* $\widetilde{B}^{\ell}$ *are given by the following generating series*:

$$1+\sum_{\ell \geqslant 0} \operatorname{ptr} \widetilde{B}^{\ell} t^{\ell+1}=\frac{t}{2}+\left(1-(-1)^{N} \frac{t}{2}\right) \prod_{j=1}^{N} \frac{1+\tilde{w}_{j} t}{1-\tilde{w}_{j} t}.$$

Proof. For each ℓ,

$$\operatorname{ptr} \widetilde{B}^{\ell}=\frac{\operatorname{tr} \widetilde{B}^{\ell}}{\operatorname{dim} V}=\sum (\tilde{w}_{j})^{\ell} \frac{\operatorname{dim} W_{j}}{\operatorname{dim} V},$$

since the partial traces act by scalars on V. The relative dimensions $\dim W_j/\dim V$ may be computed as follows.

4.5. Lemma. *Let* $\operatorname{Res}_{z=\tilde{w}_j}(\cdot)$ *denote the residue at* $\tilde{w}_j$ *of the rational function within parentheses. Then*:

(i) *if* N *is odd*,

$$\frac{\operatorname{dim} W_{j}}{\operatorname{dim} V}=(2 \tilde{w}_{j}+1) \prod_{k \neq j} \frac{\tilde{w}_{j}+\tilde{w}_{k}}{\tilde{w}_{j}-\tilde{w}_{k}}=\operatorname{Res}_{z=\tilde{w}_{j}}\left(\frac{z+\frac{1}{2}}{z} \prod_{k=1}^{N} \frac{z+\tilde{w}_{k}}{z-\tilde{w}_{k}}\right),$$

(ii) *if* N *is even*,

$$\frac{\operatorname{dim} W_{j}}{\operatorname{dim} V}=(2 \tilde{w}_{j}-1) \prod_{k \neq j} \frac{\tilde{w}_{j}+\tilde{w}_{k}}{\tilde{w}_{j}-\tilde{w}_{k}}=\operatorname{Res}_{z=\tilde{w}_{j}}\left(\frac{z-\frac{1}{2}}{z} \prod_{k=1}^{N} \frac{z+\tilde{w}_{k}}{z-\tilde{w}_{k}}\right).$$

Proof of the Lemma. Weyl's dimension formula (see for instance [13, 25]) gives

$$\operatorname{dim} W_{j}=\prod_{\alpha \in \mathscr{R}^{+}} \frac{\langle\mu^{(j)}+\delta, \alpha\rangle}{\langle\delta, \alpha\rangle}, \qquad \operatorname{dim} V=\prod_{\alpha \in \mathscr{R}^{+}} \frac{\langle\lambda+\delta, \alpha\rangle}{\langle\delta, \alpha\rangle}, \tag{4.7}$$

where $\mathscr{R}^+$ is the set of positive roots of $\mathfrak{so}(n)$, hence

$$\frac{\operatorname{dim} W_{j}}{\operatorname{dim} V}=\prod_{\alpha \in \mathscr{R}^{+}} \frac{\langle\mu^{(j)}+\delta, \alpha\rangle}{\langle\lambda+\delta, \alpha\rangle}. \tag{4.8}$$

Unless the dominant weight $\mu^{(j)}$ of W_j is equal to λ, $\mu^{(j)}$ is one of the $2m$ virtual weights $\mu^{i,\pm}=\lambda\pm\varepsilon_i$. Hence

$$\frac{\operatorname{dim} W_{j}}{\operatorname{dim} V}=\prod_{\alpha \in \mathscr{R}^{+}}\left(1 \pm \frac{\alpha_{i}}{\langle\lambda+\delta, \alpha\rangle}\right) \tag{4.9}$$

so that

$$\frac{\dim W_j}{\dim V} = \prod_{\{\tilde{w}^{k,\pm}: k \neq i(j)\}} \frac{\tilde{w}^{i,\pm} + \tilde{w}^{k,\pm}}{\tilde{w}^{i,\pm} - \tilde{w}^{k,\pm}}, \tag{4.10}$$

if $n = 2m$ is even, and

$$\frac{\dim W_j}{\dim V} = \frac{\tilde{w}^{i,\pm} + \frac{1}{2}}{\tilde{w}^{i,\pm} - \frac{1}{2}} \prod_{\{\tilde{w}^{k,\pm}: k \neq i(j)\}} \frac{\tilde{w}^{i,\pm} + \tilde{w}^{k,\pm}}{\tilde{w}^{i,\pm} - \tilde{w}^{k,\pm}}, \tag{4.11}$$

if $n = 2m+1$ is odd. Applying the cancellation rule (4.2) and analyzing each case in turn completes the proof.

Proof of Proposition 4.4 (Continued). It follows from the lemma that

$$\begin{aligned}\frac{\operatorname{tr} \tilde{B}^{\ell}}{\dim V} &= \sum_{j=1}^{N} \operatorname*{Res}_{z = \tilde{w}_j} \left(z^{\ell-1} \left(z - \frac{(-1)^N}{2} \right) \prod_{k=1}^{N} \frac{z + \tilde{w}_k}{z - \tilde{w}_k} \right) \\ &= \operatorname*{Res}_{t=0} \left(t^{-2} t^{1-\ell} \left(\frac{1}{t} - \frac{(-1)^N}{2} \right) \prod_{k=1}^{N} \frac{1/t + \tilde{w}_k}{1/t - \tilde{w}_k} \right)\end{aligned}$$

by the residue theorem. It is straightforward to check that this residue is the coefficient of $t^{\ell+1}$ in the desired rational expression of Proposition 4.4.

4.6. Corollary. *The partial traces of* $\tilde{B}^{\ell}$ *are given by the generating series*

$$1 + \sum_{\ell \geqslant 0} \operatorname{ptr} \tilde{B}^{\ell} t^{\ell+1} = \frac{t}{2} + \left(1 - (-1)^N \frac{t}{2} \right) S(t)$$

where $S'(t)/S(t) = 2 \sum s_{2k+1}(\tilde{w})\, t^{2k+1}$ *and* $s_{2k+1}(\tilde{w})$ *are the power sum symmetric functions in the translated conformal weights. In particular, by Proposition* 4.1, *the partial traces can be computed from the polynomials* $P_{2k+1}(x)$.

We recover from these generating functions the results of Perelomov and Popov for the orthogonal Lie algebras [22]. The generating functions are not too complicated, but they suggest that the operators $\tilde{A}_k$ defined by

$$\tilde{A}_k = \sum_{\ell=0}^{k} (-1)^{\ell} \sigma_{\ell}(\tilde{w})\, \tilde{B}^{k-\ell}, \tag{4.12}$$

where $\sigma_\ell(\tilde{w})$ denotes the ℓ th elementary symmetric function in the translated conformal weights, will have much simpler traces. This is indeed the case.

4.7. PROPOSITION 5.7. *The partial trace of $\tilde{A}_j$ is*

$$\operatorname{ptr} \tilde{A}_j = (1+(-1)^j)\, \sigma_{j+1}(\tilde{w}) + \tfrac{1}{2}\,((-1)^j - (-1)^N)\, \sigma_j(\tilde{w}). \tag{4.13}$$

Proof. We compute the generating function

$$\begin{aligned}
\sum_{j\geqslant 0} \operatorname{ptr} \tilde{A}_j t^{j+1} &= \sum_{j\geqslant 0} \sum_{k=0}^{j} (-1)^k\, \sigma_k(\tilde{w}) \operatorname{ptr} \tilde{B}^{j-k} t^{j+1} \\
&= \sum_{k\geqslant 0} \sum_{j\geqslant k} (-1)^k\, \sigma_k(\tilde{w}) \operatorname{ptr} \tilde{B}^{j-k} t^k t^{j-k+1} \\
&= \sum_{k\geqslant 0} (-1)^k\, \sigma_k(\tilde{w})\, t^k \sum_{\ell\geqslant 0} \operatorname{ptr} \tilde{B}^\ell t^{\ell+1} \\
&= \left(1 - \frac{(-1)^N t}{2}\right) \prod_{j=1}^{N} (1+\tilde{w}_j t) - \left(1 - \frac{t}{2}\right) \prod_{j=1}^{N} (1-\tilde{w}_j t) \\
&= \sum_{j\geqslant 0} \left((1-(-1)^j) + \frac{t}{2}((-1)^j - (-1)^N)\right) \sigma_j(\tilde{w})\, t^j.
\end{aligned}$$

This yields the stated formula.

We are now ready for the main result of this section.

4.8. THEOREM. *Define $\tilde{C}_j = \tilde{A}_j + \frac{1}{4}((-1)^N - (-1)^j)\, \tilde{A}_{j-1}$, where $\tilde{A}_{-1} = 0$ by convention. Then $(\tilde{C}_j)_{\alpha\otimes\beta} = (-1)^j\, (\tilde{C}_j)_{\beta\otimes\alpha}$.*

4.9. COROLLARY. *If N is odd then*

$$\langle \tilde{A}_{2j+1}(\alpha\otimes v), \alpha\otimes v\rangle = 0, \tag{4.14}$$

while if N is even

$$\langle \tilde{A}_{2j+1}(\alpha\otimes v), \alpha\otimes v\rangle + \tfrac{1}{2}\,\langle \tilde{A}_{2j}(\alpha\otimes v), \alpha\otimes v\rangle = 0. \tag{4.15}$$

The idea of looking for polynomials in B with symmetry properties was first suggested to the authors by T. Diemer and G. Weingart [30]. One of their key results is the following:

4.10. THEOREM (Diemer–Weingart). *Let $q_j(B)$ be a sequence of polynomials in B with $q_j(B)=0$ for $j<0$, $q_0(B)=1$, and for $j\geqslant 0$,*

$$q_{j+1}(B)_{\alpha\otimes\beta}=\left(\left(B+\frac{n-1+(-1)^j}{2}\,\mathrm{id}\right)\circ q_j(B)\right)_{\alpha\otimes\beta}$$

$$-\frac{1}{2}\langle\alpha,\beta\rangle\operatorname{ptr} q_j(B)+\sum_{k\geqslant 1} a_{jk}q_{j+1-2k}(B)_{\alpha\otimes\beta} \tag{4.16}$$

for some $a_{jk}\in\mathbb{R}$. Then

$$q_j(B)_{\alpha\otimes\beta}=(-1)^j\,q_j(B)_{\beta\otimes\alpha}. \tag{4.17}$$

Proof. We give the proof of Diemer and Weingart, which is by complete induction on j: clearly (4.17) holds for $j\leqslant 0$ and we have an inductive formula for q_{j+1}. Introducing the temporary notation $(c_j)_{\alpha\otimes\beta}=\frac{1}{2}\langle\alpha,\beta\rangle(\operatorname{ptr} q_j(B))$ we have

$$\begin{aligned}
&2(q_{j+1}(B)_{\alpha\otimes\beta}-(-1)^{j+1}\,q_{j+1}(B)_{\beta\otimes\alpha})\\
&\quad=((2B+(n-1+(-1)^j)\,\mathrm{id})\circ q_j(B)-c_j)_{\alpha\otimes\beta}\\
&\qquad+(-1)^j\,((2B+(n-1+(-1)^j)\,\mathrm{id})\circ q_j(B)-c_j)_{\beta\otimes\alpha}\\
&\quad=(B\circ q_j(B))_{\alpha\otimes\beta}+(-1)^j\,(q_j(B)\circ B)_{\beta\otimes\alpha}\\
&\qquad+(-1)^j\,((B\circ q_j(B))_{\beta\otimes\alpha}+(-1)^j\,(q_j(B)\circ B)_{\alpha\otimes\beta})\\
&\qquad+((n-1+(-1)^j)\,q_j(B)-c_j)_{\alpha\otimes\beta}\\
&\qquad+(-1)^j\,((n-1+(-1)^j)\,q_j(B)-c_j)_{\beta\otimes\alpha}
\end{aligned}$$

since $q_j(B)$ commutes with B. The result follows by observing that

$$\begin{aligned}
&(B\circ q_j(B))_{\alpha\otimes\beta}+(-1)^j\,(q_j(B)\circ B)_{\beta\otimes\alpha}\\
&\quad=\sum_i\,(B_{\alpha\otimes e_i}\circ q_j(B)_{e_i\otimes\beta}+(-1)^j\,q_j(B)_{\beta\otimes e_i}\circ B_{e_i\otimes\alpha})\\
&\quad=\sum_i\,(B_{\alpha\otimes e_i}\circ q_j(B)_{e_i\otimes\beta}-q_j(B)_{e_i\otimes\beta}\circ B_{\alpha\otimes e_i})\\
&\quad=\sum_i\,[d\lambda(\alpha\wedge e_i),q_j(B)_{e_i\otimes\beta}]=\sum_i q_j(B)_{\alpha\wedge e_i\cdot(e_i\otimes\beta)}
\end{aligned}$$

by equivariance of $q_j(B)$, where $\alpha \wedge e_i \cdot (e_i \otimes \beta)$ is defined using the action of $\mathfrak{so}(n)$ on $\mathbb{R}^n \otimes \mathbb{R}^n$. This gives, finally,

$$\begin{aligned}(B \circ q_j(B))_{\alpha \otimes \beta} &+ (-1)^j (q_j(B) \circ B)_{\beta \otimes \alpha} \\ &= q_j(B)_{\alpha \otimes \beta} - n q_j(B)_{\alpha \otimes \beta} + \langle \alpha, \beta \rangle \operatorname{ptr} q_j(B) - q_j(B)_{\beta \otimes \alpha} \\ &= ((1 - n - (-1)^j)\, q_j(B) + c_j)_{\alpha \otimes \beta},\end{aligned}$$

which completes the proof.

By taking $a_{jk} = 0$ (for all j, k), Diemer and Weingart obtain an inductive definition of a sequence of polynomials with the desired symmetry properties. Unfortunately, the task of computing these polynomials explicitly is formidable because of the complexity of the traces of the powers of B.

The polynomials $\tilde{C}_j$ defined here are completely explicit and because they have simple traces we are able to prove that they satisfy the inductive conditions of Theorem 4.10. More precisely, we have:

4.11. Lemma. *For $j \geqslant 0$,*

$$\begin{aligned}\tilde{C}_{j+1} = \left(\tilde{B} + \frac{(-1)^j}{2} \operatorname{id} \right) \circ \tilde{C}_j - \frac{1}{2} \operatorname{ptr} \tilde{C}_j + \frac{1}{8} (1 - (-1)^{N+j})\, \tilde{C}_{j-1} \\ + \frac{1}{2} (1 - (-1)^j) \left(\sigma_{j+1}(\tilde{w}) - \frac{1}{2} (1 - (-1)^N)\, \sigma_j(\tilde{w}) \right) \operatorname{id}.\end{aligned}$$

Proof. Note that $\tilde{C}_j = \tilde{A}_j + \frac{1}{4}((-1)^N - (-1)^j)\, \tilde{C}_{j-1}$ and so

$$\begin{aligned}\tilde{C}_{j+1} &- \tilde{B}\tilde{C}_j - \tfrac{1}{2}(-1)^j\, \tilde{C}_j \\ &= \tilde{A}_{j+1} - \tilde{B}\tilde{A}_j - \tfrac{1}{2}(-1)^j\, \tilde{C}_j \\ &\quad + \tfrac{1}{4}((-1)^N + (-1)^j)\, \tilde{C}_j - \tfrac{1}{4}((-1)^N - (-1)^j)\, \tilde{B}\tilde{C}_{j-1} \\ &= \tilde{A}_{j+1} - \tilde{B}\tilde{A}_j + \tfrac{1}{4}((-1)^N - (-1)^j)(\tilde{C}_j - \tilde{B}\tilde{C}_{j-1}) \\ &= \tilde{A}_{j+1} - \tilde{B}\tilde{A}_j + \tfrac{1}{4}((-1)^N - (-1)^j)(\tilde{C}_j - \tilde{B}\tilde{C}_{j-1} - \tfrac{1}{2}(-1)^j\, \tilde{C}_{j-1}) \\ &\quad + \tfrac{1}{8}(1 - (-1)^{N+j})\, \tilde{C}_{j-1} \\ &= \tilde{A}_{j+1} - \tilde{B}\tilde{A}_j + \tfrac{1}{4}((-1)^N - (-1)^j)(\tilde{A}_j - \tilde{B}\tilde{A}_{j-1}) \\ &\quad + \tfrac{1}{8}(1 - (-1)^{N+j})\, \tilde{C}_{j-1}.\end{aligned}$$

Now, by definition, we have $\widetilde{A}_{j+1}-\widetilde{B}\widetilde{A}_j=(-1)^{j+1}\,\sigma_{j+1}(\tilde{w})$ id and so

$$\begin{aligned}\widetilde{C}_{j+1}-\widetilde{B}\widetilde{C}_j-\tfrac{1}{2}(-1)^j\,\widetilde{C}_j\\ &=\tfrac{1}{8}(1-(-1)^{N+j})\,\widetilde{C}_{j-1}-(-1)^j\,\sigma_{j+1}(\tilde{w})\,\mathrm{id}\\ &\quad-\tfrac{1}{4}(1-(-1)^{N+j})\,\sigma_j(\tilde{w})\,\mathrm{id}.\end{aligned} \tag{4.18}$$

Finally, observe that

$$\operatorname{ptr}\widetilde{C}_j=(1+(-1)^j)(\sigma_{j+1}(\tilde{w})+\tfrac{1}{4}(1-(-1)^{N+j})\,\sigma_j(\tilde{w}))\,\mathrm{id}.$$

Adding one half of this onto (4.18) completes the proof.

Theorem 4.8 follows immediately from this lemma and Theorem 4.10.

5. REFINED KATO INEQUALITIES

In the last section we learnt that by working with $\widetilde{B}$ and $\widetilde{A}_j$ instead of B and A_j, we could obtain some explicit formulae. Of course $\widetilde{B}=B+\frac{1}{2}(n-1)$ id has the same eigenspaces as B and so we can rewrite (1.6) as:

$$|\Pi_j(\alpha\otimes v)|^2=\frac{\sum_{k=0}^{N-1}\tilde{w}_j^{N-1-k}\,\langle\widetilde{A}_k(\alpha\otimes v),\alpha\otimes v\rangle}{\prod_{k\neq j}(\tilde{w}_j-\tilde{w}_k)}. \tag{5.1}$$

If N is odd, Corollary 4.9 implies that the terms with k odd vanish, while for N even, we have

$$\langle\widetilde{A}_{2j+1}(\alpha\otimes v),\alpha\otimes v\rangle+\tfrac{1}{2}\langle\widetilde{A}_{2j}(\alpha\otimes v),\alpha\otimes v\rangle=0.$$

Our main result will readily follow from this.

5.1. Main Theorem. *Let I a subset of $\{1,...,N\}$ corresponding to an operator P_I acting on E. Then a Kato constant k_I for sections in the kernel of P_I is given by the following expressions.*

If N is odd, then

$$\begin{aligned}k_I^2&=\max_{J\in\mathcal{N}\mathcal{E}}\left(\sum_{i\in\hat{I}\cap\hat{J}}\frac{\prod_{j\in J}(\tilde{w}_i+\tilde{w}_j)}{\prod_{j\in\hat{J}\setminus\{i\}}(\tilde{w}_i-\tilde{w}_j)}\right)\\ &=1-\min_{J\in\mathcal{N}\mathcal{E}}\left(\sum_{i\in I\cap\hat{J}}\frac{\prod_{j\in J}(\tilde{w}_i+\tilde{w}_j)}{\prod_{j\in\hat{J}\setminus\{i\}}(\tilde{w}_i-\tilde{w}_j)}\right).\end{aligned} \tag{5.2}$$

If N is even, then

$$k_I^2 = \max_{J \in \mathcal{NE}} \left(\sum_{i \in \hat{I} \cap \hat{J}} \frac{(\tilde{w}_i - 1/2) \prod_{j \in J} (\tilde{w}_i + \tilde{w}_j)}{\prod_{j \in \hat{J} \setminus \{i\}} (\tilde{w}_i - \tilde{w}_j)} \right)$$

$$= 1 - \min_{J \in \mathcal{NE}} \left(\sum_{i \in I \cap \hat{J}} \frac{(\tilde{w}_i - 1/2) \prod_{j \in J} (\tilde{w}_i + \tilde{w}_j)}{\prod_{j \in \hat{J} \setminus \{i\}} (\tilde{w}_i - \tilde{w}_j)} \right). \tag{5.3}$$

These constants are sharp, unless $N = 2v + 1$, λ is properly half-integral, and the set J achieving the extremum contains $v + 1$.

Recall that $\mathcal{NE}$ denotes the set of subsets of $\{1, ..., N\}$ whose elements are obtained by choosing exactly one index in each of the sets $\{j, N+2-j\}$ for each j with $2 \leqslant j \leqslant v$ if $N = 2v - 1$, $2v$ and for each j with $2 \leqslant j \leqslant v + 1$ if $N = 2v + 1$. These correspond to the maximal non-elliptic operators unless $N = 2v + 1$ and λ is properly half-integral, when there are also some elliptic subsets in $\mathcal{NE}$.

Explicit values of the constants for a number of cases, including all minimal elliptic operators, will be given in Sections 6 and 7, and in the Appendix. Note that $k_I = 1$ for non-elliptic operators, as one would expect.

Proof of the Main Theorem. We let first $N = 2v - 1$ and denote $Q_k = (-1)^{k-1} \langle \widetilde{A}_{2k-2}(\alpha \otimes v), \alpha \otimes v \rangle$. We have

$$|\Pi_j(\alpha \otimes v)|^2 = \frac{\sum_{k=1}^{v} \tilde{w}_j^{2(v-k)} (-1)^{k-1} Q_k}{\prod_{k \neq j} (\tilde{w}_j - \tilde{w}_k)} = \frac{\tilde{w}_j^{2(v-1)} - \sum_{k=2}^{v} \tilde{w}_j^{2(v-k)} (-1)^k Q_k}{\prod_{k \neq j} (\tilde{w}_j - \tilde{w}_k)} \tag{5.4}$$

since $Q_1 = 1$. We can now obtain bounds on $Q_2, ..., Q_v$ using the non-negativity of the norms. Since the denominator in (5.4) has sign $(-1)^{j-1}$ these inequalities are

$$\sum_{k=2}^{v} (-1)^{j+k} \tilde{w}_j^{2(v-k)} Q_k \geqslant (-1)^j \tilde{w}_j^{2(v-1)} \tag{5.5}$$

with equality iff $|\Pi_j(\alpha \otimes v)|^2 = 0$.

This system of linear inequalities confines the values of the Q_k's to a convex region in $\mathbb{R}^{v-1}$. Our first goal is to show that this region is compact, hence polyhedral, and to identify its vertices. For this we let π_j denote the affine functions of $Q = (Q_2, ..., Q_v)$ given by $|\Pi_j(\alpha \otimes v)|^2$ and note the following.

5.2. LEMMA. *Let J be a subset of $\{1, ..., N\}$ with $v - 1$ elements. Then the intersection of the $v - 1$ affine hyperplanes $\pi_j = 0$ for all $j \in J$ consists of the* *single point* $Q_J = (Q_2, ..., Q_v)$ *with* $Q_k = \sigma_{k-1}((\tilde{w}_j^2)_{j\in J})$. *At this point the affine functions* π_j *take the values*

$$\pi_j(Q_J) = \frac{\prod_{k\in J}(\tilde{w}_j^2 - \tilde{w}_k^2)}{\prod_{k\neq j}(\tilde{w}_j - \tilde{w}_k)} = \frac{\prod_{k\in J, k\neq j}(\tilde{w}_j + \tilde{w}_k)}{\prod_{k\in \hat{J}, k\neq j}(\tilde{w}_j - \tilde{w}_k)}\, \varepsilon_j(J) \tag{5.6}$$

where $\varepsilon_j(J) = 0$ *if* $j \in J$ *and* $=1$ *otherwise.*

This lemma follows by simply observing that the affine function π_j is obtained by evaluating a polynomial independent of j on $\tilde{w}_j^2$ and then using the fact that the coefficients of a polynomial are the elementary symmetric functions of the roots.

Compactness of the convex region is obtained by taking $J = \{2, ..., v\}$ and $J = \{v+1, ..., 2v-1\}$. The inverse of the Vandermonde system of inequalities for $J = \{2, ..., v\}$ has non-negative entries, while for $J = \{v+1, ..., 2v-1\}$ it has non-positive entries.

5.3. Proposition. *Let* $N = 2v-1$. *Then for* $k = 2, ..., v$,

$$\sigma_{k-1}(\tilde{w}_2^2, ..., \tilde{w}_v^2) \leqslant Q_k \leqslant \sigma_{k-1}(\tilde{w}_{v+1}^2, ..., \tilde{w}_{2v-1}^2). \tag{5.7}$$

The lower bounds are all attained if and only if $\Pi_{\{2,...,v\}}(\alpha\otimes v) = 0$, *while the upper bounds are all attained if and only if* $\Pi_{\{v+1,...,2v-1\}}(\alpha\otimes v) = 0$. *These bounds are sharp by non-ellipticity of* $P_{\{2,...,v\}}$ *and* $P_{\{v+1,...,2v-1\}}$.

When $N = 2v-1 = 3$, the case most commonly occurring in practice, it is now straightforward to obtain sharp Kato constants. However, for $N \geqslant 5$, the upper bound for some Q_k and the lower bound for another (as given in this proposition) will not be simultaneously attained: the convex region is smaller. We illustrate this in the case $N = 5$ $(v = 3)$.

In Fig. 1, the numbered lines represent the conditions on Q_2 and Q_3 for the norms of $\Pi_1, ..., \Pi_5$ to vanish. The shaded region represents the range of possible values for (Q_2, Q_3), while the dotted rectangle represents the bounds on (Q_2, Q_3) we have found. We have circled the points corresponding to the non-elementary minimal elliptic operators.

According to Ansatz 1.2, in order to find a sharp Kato constant for P_I we must maximize (for $|\alpha\otimes v| = 1$) the projection $|\Pi_{\hat{I}}(\alpha\otimes v)|^2 = 1 - |\Pi_I(\alpha\otimes v)|^2$, which is equivalent to minimizing $|\Pi_I(\alpha\otimes v)|^2 = \sum_{i\in I}\pi_i$.

Since these norms are affine in the Q_k's, it follows that to minimize or maximize them on the polyhedral region of admissible values of the Q_k's we must find the supporting hyperplanes associated to the linear part of the function. Such a supporting hyperplane certainly contains a vertex of the polyhedron, and so it suffices to minimize or maximize over the set of vertices.

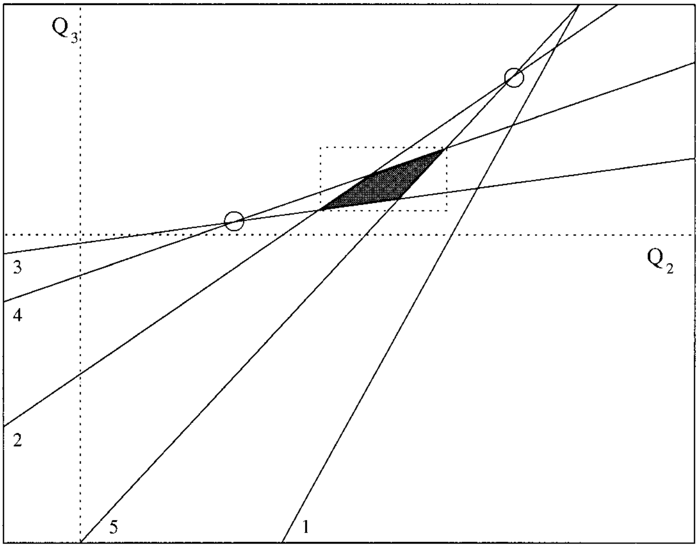

FIGURE 1

We claim that these vertices are the points Q_J with $J \in \mathscr{N}\mathscr{E}$. Certainly these points are vertices, since if $J \in \mathscr{N}\mathscr{E}$ then P_J is non-elliptic (this part of the argument will fail when $N = 2\nu + 1$) and so there is some $\alpha \otimes v$ of norm one with $\Pi_j(\alpha \otimes v) = 0$ for each j in J. Therefore it remains to eliminate the points Q_J with $J \notin \mathscr{N}\mathscr{E}$ as possible vertices, which we do by showing that a point Q_J with P_J elliptic does not lie in the polyhedral region. This is done by proving that there is, for every such J, an index i such that the affine function π_i assumes a (strictly) negative value at Q_J. Equation (5.6) tells us that for $i \notin J$, $\pi_i(Q_J)$ is nonzero and has the sign $(-1)^{i-1}\rho_i$ where ρ_i is the sign of $\prod_{j \in J}(\tilde{w}_i^2 - \tilde{w}_j^2)$. If P_J is elliptic, J contains a minimal elliptic set, and hence either the index 1 or a couple of indices of the form $(j, N+2-j)$. In any case, since J has length $\nu - 1$ and there are exactly $\nu - 1$ couples of type $(\ell, N+2-\ell)$, there is at least one such couple *outside* J. One readily checks that $\tilde{w}_\ell^2$ and $\tilde{w}_{N+2-\ell}^2$ are adjacent in the ordering of the squares of the conformal weights, and so $\rho_\ell = \rho_{N+2-\ell}$. Since N is odd, ℓ and $N+2-\ell$ have the opposite parity, and so one of $i = \ell$ or $i = N+2-\ell$ yields a negative sign for π_i. This proves the claim, and now maximizing or minimizing over the vertices using (5.6) proves the main theorem for $N = 2\nu - 1$.

The argument for the case $N = 2\nu + 1$ is completely analogous, by replacing ν with $\nu - 1$. When λ is properly half-integral, the lower bounds in the analogue of (5.7) will not be sharp since $P_{\{2, \ldots, \nu+1\}}$ is elliptic. However, we only used these bounds to establish compactness of the convex region defined by the non-negativity of the norms, so this does not matter. The ellipticity of $P_{\nu+1}$ means that some of the vertices of this polyhedral region are not possible values for the Q_k's. More precisely, the index sets corresponding to the vertices are still contained in the set $\mathcal{NE}$, and so we can maximize or minimize over $\mathcal{NE}$, but we will not obtain sharp results if the extremum is obtained at a vertex corresponding to an index set containing $\nu+1$.

Now suppose $N=2\nu$ and let $Q_k=(-1)^{k-1}\langle\tilde{A}_{2k-2}(\alpha\otimes v), \alpha\otimes v\rangle$. We have

$$
\begin{aligned}
|\Pi_j(\alpha\otimes v)|^2 &= \frac{\tilde{w}_j-\frac{1}{2}}{\prod_{k\neq j}(\tilde{w}_j-\tilde{w}_k)}\sum_{k=1}^{\nu}\tilde{w}_j^{2(\nu-k)}(-1)^{k-1}Q_k \\
&= \frac{\tilde{w}_j-\frac{1}{2}}{\prod_{k\neq j}(\tilde{w}_j-\tilde{w}_k)}\left(\tilde{w}_j^{2(\nu-1)}-\sum_{k=2}^{\nu}\tilde{w}_j^{2(\nu-k)}(-1)^k Q_k\right)
\end{aligned}
\tag{5.8}
$$

since $Q_1=1$. Our strategy is now the same as before: we obtain the polyhedron using the non-negativity of the norms and its vertices by looking at maximal length non-elliptic operators. Since the denominator in (5.8) has sign $(-1)^{j-1}$ these inequalities are:

$$
\begin{aligned}
&\sum_{k=2}^{\nu}(-1)^{j+k}\tilde{w}_j^{2(\nu-k)}Q_k\geqslant(-1)^j\tilde{w}_j^{2(\nu-1)} \qquad \text{for} \quad j\leqslant\nu \\
&\sum_{k=2}^{\nu}(-1)^{j+k}\tilde{w}_j^{2(\nu-k)}Q_k\leqslant(-1)^j\tilde{w}_j^{2(\nu-1)} \qquad \text{for} \quad j\geqslant\nu+1.
\end{aligned}
\tag{5.9}
$$

Lemma 5.2 is unchanged except that the formula for $\pi_j(Q_J)$ has an additional $\tilde{w}_j-\frac{1}{2}$. To obtain compactness, we consider $J=\{2, ..., \nu\}$ and $J=\{\nu+2, ..., 2\nu\}$ and again observe that the inverses of these Vandermonde systems have entries all of one sign.

5.4. Proposition. *Let $N=2\nu$. Then for $k=2, ..., \nu$,*

$$\sigma_{k-1}(\tilde{w}_2^2, ..., \tilde{w}_\nu{}^2)\leqslant Q_k\leqslant\sigma_{k-1}(\tilde{w}_{\nu+2}^2, ..., \tilde{w}_{2\nu}^2). \tag{5.10}$$

The lower bounds are all attained if and only if $\Pi_{\{2, ..., \nu\}}(\alpha\otimes v)=0$, while the upper bounds are all attained if and only if $\Pi_{\{\nu+2, ..., 2\nu\}}(\alpha\otimes v)=0$. These bounds are sharp by the non-ellipticity of $P_{\{2, ..., \nu\}}$ and $P_{\{\nu+2, ..., 2\nu\}}$.

The vertices are identified with $\mathcal{N}\mathcal{E}$ in a way similar to the case $N=2\nu-1$. The only difference comes from the way the sign changes when passing from $i=\ell$ to $i=N+2-\ell$: the parity of i does not change but the sign of the factor $\tilde{w}_i-1/2$ does. This proves the main theorem for $N=2\nu$.

In the next two sections we shall calculate some of the constants more explicitly by finding the vertex at which the maximum or minimum is achieved. This is only feasible when the number of terms in the sum is small and, in general, the vertex depends on the coordinates of λ. Nevertheless, this is a worthwhile task, as explicit constants are of more practical use than extrema over exponentially large sets.

Our main tool is the order of the conformal weights, together with the fact that, for $j\in\{2, ..., \nu\}$, we have $\tilde{w}_j+\tilde{w}_{N+2-j}=k_j-k_{j-1}<0$. Similarly, for $N=2\nu+1$, $\tilde{w}_{\nu+1}+\tilde{w}_{\nu+2}=\tilde{w}_{\nu+2}=-\lambda_m<0$. Hence for any $i\in\{1, ..., N\}$ and $j\in\{2, ..., \nu\}$,

$$(\tilde{w}_i+\tilde{w}_j)(\tilde{w}_i-\tilde{w}_j)-(\tilde{w}_i+\tilde{w}_{N+2-j})(\tilde{w}_i-\tilde{w}_{N+2-j})$$
$$=-(\tilde{w}_j+\tilde{w}_{N+2-j})(\tilde{w}_j-\tilde{w}_{N+2-j})>0$$

and this also holds for $N=2\nu+1$ and $j=\nu+1$.

By considering the possible signs of the terms, we obtain:

5.5. Proposition. *For any $i\in\{1, ..., N\}$ and $j\in\{2, ..., \nu\}$ (or $j=\nu+1$ when $N=2\nu+1$) with $i\neq j$ and $i\neq N+2-j$, we have*:

$$\frac{\tilde{w}_i+\tilde{w}_j}{\tilde{w}_i-\tilde{w}_{N+2-j}}>\frac{\tilde{w}_i+\tilde{w}_{N+2-j}}{\tilde{w}_i-\tilde{w}_j}>0 \qquad \textit{iff} \quad i<j \quad \textit{or} \quad N+2-j<i$$

$$\frac{\tilde{w}_i+\tilde{w}_{N+2-j}}{\tilde{w}_i-\tilde{w}_j}>\frac{\tilde{w}_i+\tilde{w}_j}{\tilde{w}_i-\tilde{w}_{N+2-j}}>0 \qquad \textit{iff} \quad j<i<N+2-j.$$

6. REFINED KATO INEQUALITIES WITH N ODD

When N is odd, we have to minimize or maximize over $J\in\mathcal{N}\mathcal{E}$, a sum of a subset of the following terms:

$$\frac{\prod_{j\in J}(\tilde{w}_i+\tilde{w}_j)}{\prod_{j\in \hat{J}\setminus\{i\}}(\tilde{w}_i-\tilde{w}_j)}=\frac{\tilde{w}_i+\tilde{w}_{N+2-i}}{\tilde{w}_i-\tilde{w}_1}\prod_{\substack{j\in J\\ j\neq N+2-i}}\frac{\tilde{w}_i+\tilde{w}_j}{\tilde{w}_i-\tilde{w}_{N+2-j}} \qquad \text{for} \quad i\in\hat{J}\setminus\{1\}$$

$$\frac{\prod_{j\in J}(\tilde{w}_1+\tilde{w}_j)}{\prod_{j\in \hat{J}\setminus\{i\}}(\tilde{w}_1-\tilde{w}_j)}=\prod_{j\in J}\frac{\tilde{w}_1+\tilde{w}_j}{\tilde{w}_1-\tilde{w}_{N+2-j}}.$$

Using Proposition 5.5, the first expression is minimized (subject to $J \not\ni i$) by $J_i^{\min} = \{2, ..., i-1, N+2-v, ..., N+2-i\}$ (together with $v+2$ if $N=2v+1$) and is maximized by $J_i^{\max} = \{i+1, ..., v, N+2-i, ..., N\}$ (together with $v+1$ if $N=2v+1$). The second expression is minimized by $J_1^{\min} = \{N+2-v, ..., N\}$ (together with $v+2$ if $N=2v+1$) and maximized by $J_1^{\max} = \{2, ..., v\}$ (together with $v+1$ if $N=2v+1$).

This information suffices to find Kato constants for the elementary elliptic operators and the complements of generalized gradients. Note that $J_i^{\min} = J_{N-1-i}^{\min}$ and $J_i^{\max} = J_{N-1-i}^{\max}$, which will give a few more explicit results.

We shall now show how the values of the constants can be computed for the non-elementary (i.e., length 2) minimal elliptic operators.

Let $I = \{i, N+2-i\}$ for $i \in \{2, ..., v\}$ (or $i = v+1$ when $N = 2v+1$). Then for any $J \in \mathcal{NE}$, $J \cap I$ has precisely one element, and hence so does $J \cap \hat{I}$. Therefore, for each J, the sum has only one term, indexed by either i or $N-2-i$, and so the minimum, over all J, is given by the minimum over $J_i^{\min}$ and $J_{N+2-i}^{\min}$. Unfortunately, each of these two quantities may be the smallest, depending on the precise values of the conformal weights, so that we are forced to keep a minimum in our formulae. However, if $N = 2v-1$ and $i = v$, then the following argument, together with the fact that $\tilde{w}_1 - \tilde{w}_{v+1} > \tilde{w}_1 - \tilde{w}_v$, shows that the minimum is obtained by using $J_{v+1}^{\min}$.

6.1. LEMMA. *For each $k = 1, ..., v-2$*

$$(\tilde{w}_v + \tilde{w}_{k+1})(\tilde{w}_{v+1} - \tilde{w}_{2v-k}) > (\tilde{w}_{v+1} + \tilde{w}_{k+1})(\tilde{w}_v - \tilde{w}_{2v-k}) > 0.$$

Proof. Positivity holds because $2v - k > v+1$, while the inequality follows from the identity

$$\begin{aligned}(\tilde{w}_v + \tilde{w}_{k+1})(\tilde{w}_{v+1} - \tilde{w}_{2v-k}) &- (\tilde{w}_{v+1} + \tilde{w}_{k+1})(\tilde{w}_v - \tilde{w}_{2v-k}) \\ &= -(\tilde{w}_{k+1} + \tilde{w}_{2v-k})(\tilde{w}_v - \tilde{w}_{v+1})\end{aligned}$$

and the fact that $\tilde{w}_{k+1} + \tilde{w}_{2v-k} < 0$.

A similar argument works when $N = 2v+1$ and $i = v+1$.

We summarize these observations in the following results.

6.2. THEOREM. *Let E be associated to a representation λ with $N = 2v-1$ and let P_I be an elliptic operator on sections of E associated to a subset I of $\{1, ..., N\}$. Then in the following cases, a refined Kato inequality of the type $|d\,|\xi|| \leqslant k_I\, |\nabla \xi|$ holds outside the zero set of ξ for ξ in the kernel of P_I.*

(i) *For* $\{1\} \subseteq I \subseteq \{1, v+1, ..., 2v-1\}$, *we have*

$$k_I^2 = 1 - \frac{\prod_{k=v+1}^{2v-1}(\tilde{w}_1 + \tilde{w}_k)}{\prod_{k=2}^{v}(\tilde{w}_1 - \tilde{w}_k)}.$$

Equality case: $\nabla\xi = \Pi_{\{2, ..., v\}}(\alpha \otimes \xi)$ *for* α *with* $\Pi_{\{v+1, ..., 2v-1\}}(\alpha \otimes \xi) = 0$.

(ii) *For* $\{i, 2v+1-i\} \subseteq I \subseteq \{i, 2v+1-i\} \cup J_0$, *with* $i \in \{2, ..., v\}$ *and* $J_0 = \{j : 2 \leqslant j < i\} \cup \{2v+1-j : i < j \leqslant v\}$, *we have*

$$k_I^2 = 1 - \min(C_1, C_2),$$

where

$$C_1 = \frac{\tilde{w}_i + \tilde{w}_{2v+1-i}}{\tilde{w}_i - \tilde{w}_1} \prod_{k \in J_0} \frac{\tilde{w}_i + \tilde{w}_k}{\tilde{w}_i - \tilde{w}_{2v+1-k}},$$

$$C_2 = \frac{\tilde{w}_i + \tilde{w}_{2v+1-i}}{\tilde{w}_{2v+1-i} - \tilde{w}_1} \prod_{k \in J_0} \frac{\tilde{w}_{2v+1-i} + \tilde{w}_k}{\tilde{w}_{2v+1-i} - \tilde{w}_{2v+1-k}}.$$

Equality case: $\nabla\xi = \Pi_{J^0 \setminus \{i, 2v+1-i\}}(\alpha \otimes \xi)$ *for* α *with*

$$\Pi_{\{i\} \cup J_0}(\alpha \otimes \xi) = 0 \qquad \text{if} \quad C_2 < C_1 \quad \text{or}$$

$$\Pi_{\{2v+1-i\} \cup J_0}(\alpha \otimes \xi) = 0 \qquad \text{if} \quad C_1 < C_2.$$

Furthermore, $C_2 < C_1$ *if* $i = v$ (*and so* $\Pi_{\{2, ..., v\}}(\alpha \otimes \xi) = 0$).

(iii) *For* $I = \{2, ..., 2v-1\}$, *we have*

$$k_I^2 = \frac{\prod_{k=2}^{v}(\tilde{w}_1 + \tilde{w}_k)}{\prod_{k=v+1}^{2v-1}(\tilde{w}_1 - \tilde{w}_k)}.$$

Equality case: $\nabla\xi = \Pi_1(\alpha \otimes \xi)$ *for* α *with* $\Pi_{\{2, ..., v\}}(\alpha \otimes \xi) = 0$.

(iv) *For* $\hat{I} = \{i\}$ *with* $i \in \{2, ..., 2v-1\}$, *we have*

$$k_I^2 = \frac{\tilde{w}_i + \tilde{w}_{2v+1-i}}{\tilde{w}_i - \tilde{w}_1} \prod_{\substack{j \in J_i^{\max} \\ j \neq 2v+1-i}} \frac{\tilde{w}_i + \tilde{w}_j}{\tilde{w}_i - \tilde{w}_{2v+1-j}}.$$

Equality case: $\nabla\xi = \Pi_i(\alpha \otimes \xi)$ *for* α *with* $\Pi_{J_i^{\max}}(\alpha \otimes \xi) = 0$. *Here* $J_i^{\max} = \{i+1, ..., v, 2v+1-i, ..., 2v-1\}$.

(v) *For* $I = \{2, ..., 2v-2\}$, *we have*

$$k_I^2 = \frac{\prod_{k=2}^{v}(\tilde{w}_{2v-1} + \tilde{w}_k)}{(\tilde{w}_{2v-1} - \tilde{w}_1)\prod_{k=v+1}^{2v-2}(\tilde{w}_{2v-1} - \tilde{w}_k)} + \frac{\prod_{k=2}^{v}(\tilde{w}_1 + \tilde{w}_k)}{\prod_{k=v+1}^{2v-1}(\tilde{w}_1 - \tilde{w}_k)}.$$

Equality case: $\nabla\xi = \Pi_{\{1,2\nu-1\}}(\alpha\otimes\xi)$ *for* α *with* $\Pi_{\{2,\dots,\nu\}}(\alpha\otimes\xi)=0$. (*This is not a refined inequality when* $N=3$.)

(vi) *For* $\hat{I}=\{i, 2\nu-i\}$ *with* $i\in\{2,\dots,\nu-1\}$, *we have*

$$k_I^2 = \frac{\tilde{w}_i+\tilde{w}_{2\nu+1-i}}{\tilde{w}_i-\tilde{w}_1}\prod_{\substack{j\in J_i^{\max}\\ j\neq 2\nu+1-i}}\frac{\tilde{w}_i+\tilde{w}_j}{\tilde{w}_i-\tilde{w}_{2\nu+1-j}} + \frac{\tilde{w}_{i+1}+\tilde{w}_{2\nu-i}}{\tilde{w}_{2\nu-i}-\tilde{w}_1}\prod_{\substack{j\in J_i^{\max}\\ j\neq i+1}}\frac{\tilde{w}_{2\nu-i}+\tilde{w}_j}{\tilde{w}_{2\nu-i}-\tilde{w}_{2\nu+1-j}}.$$

Equality case: $\nabla\xi=\Pi_{\{i,2\nu-i\}}(\alpha\otimes\xi)$ *for* α *with* $\Pi_{J_i^{\max}}(\alpha\otimes\xi)=0$. *Here* $J_i^{\max}=\{i+1,\dots,\nu,2\nu+1-i,\dots,2\nu-1\}$.

Replacing ν *by* $\nu+1$ *gives analogous results for* $N=2\nu+1$, *but note that equality cases with* $\Pi_{\nu+1}(\alpha\otimes v)=0$ *will not be attained if* λ *is properly half-integral.*

We now give more detailed formulas when $N=3$, which is the most common case arising in practice: the representation $\tau\otimes\lambda$ splits into $N=3$ components when:

(i) $V=\bigodot^k \Lambda^p$ (k a positive integer) and $0<p\leqslant m-1$ ($p=m-1$ in even dimension belongs to this case only by virtue of our convention on distinctness of conformal weights). Then $\lambda=(k,\dots,k,0,\dots,0)$ where k is repeated p times and $w_1=k>w_2=-p>w_3=p-k+1-n$.

(ii) in odd dimensions, either $V=\bigodot^k\Lambda^m$ (k a positive integer) or $V=\bigodot^{k-1/2}\Lambda^m\odot\Delta$ ($k>1/2$ and half-integral), where Δ is the spin representation. This corresponds in both cases to $\lambda=(k,\dots,k)$ and $w_1=k>w_2=-\frac{n-1}{2}>w_3=-k-\frac{n-1}{2}$.

Note that P_1 and P_2+P_3 are elliptic, whereas P_2 and P_3 are non-elliptic, unless $\nu=1$ and λ is properly half-integral, when P_2 is elliptic, but the results above do not cover this case.

6.3. Theorem. *If* ξ *is a nonvanishing section in the kernel of one of the elliptic operators* P_1, P_2+P_3, P_1+P_3, *or* P_1+P_2, *we have a refined Kato inequality* $|d|\xi||\leqslant k_I|\nabla\xi|$ *with* k_I *given as follows.*

(i) *For* P_1 *or* P_1+P_3,

$$k_{\{1\}}^2=k_{\{1,3\}}^2=\frac{w_1}{w_1-w_2}=\frac{k}{k+p}.$$

Equality holds iff $\nabla\xi=\Pi_2(\alpha\otimes\xi)$ *for a* 1-*form* α *such that* $\Pi_3(\alpha\otimes\xi)=0$.

(ii) *For* P_2+P_3,

$$k_{\{2,3\}}^2=\frac{-w_3}{w_1-w_3}=\frac{k+n-p-1}{2k+n-p-1}.$$

Equality holds iff $\nabla\xi=\Pi_1(\alpha\otimes\xi)$ *for a* 1-*form* α *such that* $\Pi_2(\alpha\otimes\xi)=0$.

(iii) *For* P_1+P_2,

$$k_{\{1,2\}}^2=\frac{w_1}{w_1-w_3}=\frac{k}{2k+n-p-1}.$$

Equality holds iff $\nabla\xi=\Pi_3(\alpha\otimes\xi)$ *for a* 1-*form* α *such that* $\Pi_2(\alpha\otimes\xi)=0$.

When λ is properly half-integral, only the first constant is sharp and we do not get a nontrivial constant for P_2. Since this case sometimes arises in practice (e.g., the Rarita–Schwinger operator), we note briefly how the Kato constant can be found. Since $\tilde{w}_{\nu+1}=0$, the projection $\Pi_{\nu+1}=\Pi_2$ is equal to $\tilde{A}_{N-1}=\tilde{A}_2$ divided by $\tilde{w}_1\tilde{w}_3<0$. Hence we need to obtain a better upper bound on $\langle\tilde{A}_2(\alpha\otimes v),\alpha\otimes v\rangle=\langle(B^2-\tilde{w}_1\tilde{w}_3)(\alpha\otimes v),\alpha\otimes v\rangle$. Now for fixed $\alpha\neq 0$, say $\alpha=e_n$, we can break this up under $\mathfrak{so}(n-1)$ and use the fact, easily verified, that B^2 is the difference between the Casimir number of λ and the Casimir operator of $\mathfrak{so}(n-1)$. Applying the branching rule, we see that the eigenvalues of $(\tilde{A}_2)_{e_n\otimes e_n}$ are $-(k-\ell)^2$ for $\ell\in\mathbb{N}$ with $k-\ell\geqslant 0$. Hence if k is half-integral, $\langle\tilde{A}_2(\alpha\otimes v),\alpha\otimes v\rangle\leqslant -1/4$. This gives

$$k_{\{2\}}^2=1-\frac{1}{2k(2k+n-1)}$$

$$k_{\{2,3\}}^2=\frac{(2k+n-1)^2-1}{(2k+n-1)(4k+n-1)}\qquad k_{\{1,2\}}^2=\frac{k^2-1}{k(4k+n-1)}.$$

The analogues of these sharper results for larger $N=2\nu+1$ can be derived from Branson's minimization formula [9]. In particular, he gives the formula for $k_{\{\nu+1\}}$ explicitly there.

Most "uncomplicated" tensor bundles, such as vectors, forms, symmetric traceless tensors, and algebraic Weyl tensors, have $N=3$ (except in low dimensions, where N might be 2).

(i) For Λ^1, the constants are $\frac{1}{2}$ (conformal or Killing vector fields), $\frac{n-1}{n}$ (harmonic 1-forms), and $\frac{1}{n}$ (closed 1-forms dual to a conformal vector field). The last of these is trivial, since the only non-vanishing component of $\nabla\xi$ in this case is $\frac{1}{n}\operatorname{div}\xi\ \mathrm{id}$.

(ii) For Λ^2, the constants are $\frac{1}{3}$, $\frac{n-2}{n-1}$, and $\frac{1}{n-1}$. The second of these is the constant for harmonic 2-forms.

(iii) For S^2_0, the constants are $\frac{2}{3}$, $\frac{n}{n+2}$, and $\frac{2}{n+2}$. The second of these is the constant appearing in the work of Schoen *et al.* [26].

(iv) For $\Lambda^2 \odot \Lambda^2$, the constants are $\frac{1}{2}$, $\frac{n-1}{n+1}$, and $\frac{2}{n+1}$. The second of these is the constant for the second Bianchi identity appearing in the work of Bando *et al.* [1].

7. REFINED KATO INEQUALITIES WITH N EVEN

When $N=2v$ is even, we have to minimize or maximize over $J \in \mathcal{NE}$, a sum of a subset of the following terms:

$$\frac{(\tilde{w}_i - 1/2)(\tilde{w}_i + \tilde{w}_{N+2-i})}{(\tilde{w}_i - \tilde{w}_1)(\tilde{w}_i - \tilde{w}_{v+1})} \prod_{\substack{j \in J \\ j \neq N+2-i}} \frac{\tilde{w}_i + \tilde{w}_j}{\tilde{w}_i - \tilde{w}_{N+2-j}} \qquad \text{for} \quad i \in \hat{J} \setminus \{1, v+1\}$$

$$\frac{\tilde{w}_1 - 1/2}{\tilde{w}_1 - \tilde{w}_{v+1}} \prod_{j \in J} \frac{\tilde{w}_1 + \tilde{w}_j}{\tilde{w}_1 - \tilde{w}_{N+2-j}}$$

$$\frac{\tilde{w}_{v+1} - 1/2}{\tilde{w}_{v+1} - \tilde{w}_1} \prod_{j \in J} \frac{\tilde{w}_{v+1} + \tilde{w}_j}{\tilde{w}_{v+1} - \tilde{w}_{N+2-j}}.$$

Using Proposition 5.5, the first expression is minimized (subject to $J \not\ni i$) by $J_i^{\min} = \{2, ..., i-1, \ N+2-v, ..., N+2-i\}$ and is maximized by $J_i^{\max} = \{i+1, ..., v, N+2-i, ..., N\}$. The second expression is minimized by $J_1^{\min} = \{N+2-v, ..., N\}$ and maximized by $J_1^{\max} = \{2, ..., v)$, while the third expression is minimized by $J_{v+1}^{\min} = \{2, ..., v)$ and maximized by $J_{v+1}^{\max} = \{N+2-v, ..., N\}$.

We now proceed as in the odd-dimensional case, except that the analogue of Lemma 6.1 is no longer useful, due to the additional $\tilde{w} - \frac{1}{2}$ factors. The results are summarized below.

7.1. THEOREM. *Let E be associated to a representation λ with $N=2v$ and let P_I be an elliptic operator on sections of E associated to a subset I of $\{1, ..., N\}$. Then in the following cases, a refined Kato inequality of the type $|d\,|\xi|| \leqslant k_I\,|\nabla\xi|$ holds outside the zero set of ξ for ξ in the kernel of P_I.*

(i) *For $\{1\} \subseteq I \subseteq \{1, v+2, ..., 2v\}$, we have*

$$k_I^2 = 1 - \frac{(\tilde{w}_1 - 1/2) \prod_{k=v+2}^{2v} (\tilde{w}_1 + \tilde{w}_k)}{\prod_{k=2}^{v+1} (\tilde{w}_1 - \tilde{w}_k)}.$$

Equality case: $\nabla\xi = \Pi_{\{2, ..., v+1\}}(\alpha \otimes \xi)$ *for α with* $\Pi_{\{v+2, ..., 2v\}}(\alpha \otimes \xi) = 0$.

(ii) *For* $\{\nu+1\} \subseteq I \subseteq \{2, ..., \nu, \nu+1\}$, *we have*

$$k_I^2 = 1 - \frac{(\tilde{w}_{\nu+1} - 1/2) \prod_{k=2}^{\nu} (\tilde{w}_{\nu+1} + \tilde{w}_k)}{(\tilde{w}_{\nu+1} - \tilde{w}_1) \prod_{k=\nu+2}^{2\nu} (\tilde{w}_{\nu+1} - \tilde{w}_k)}.$$

Equality case: $\nabla\xi = \Pi_{\{1, \nu+2, ..., 2\nu\}}(\alpha \otimes \xi)$ *for* α *with* $\Pi_{\{2, ..., \nu\}}(\alpha \otimes \xi) = 0$.

(iii) *For* $\{i, 2\nu+2-i\} \subseteq I \subseteq \{i, 2\nu+2-i\} \cup J_0$, *with* $i \in \{2, ..., \nu\}$ *and* $J_0 = \{j : 2 \leqslant j < i\} \cup \{2\nu+2-j : i < j \leqslant \nu\}$, *we have*

$$k_I^2 = 1 - \min(C_1, C_2)$$

where

$$C_1 = \frac{(\tilde{w}_i + \tilde{w}_{2\nu+2-i})(\tilde{w}_i - 1/2)}{(\tilde{w}_i - \tilde{w}_{\nu+1})(\tilde{w}_i - \tilde{w}_1)} \prod_{k \in J_0} \frac{\tilde{w}_i + \tilde{w}_k}{\tilde{w}_i - \tilde{w}_{2\nu+2-k}},$$

$$C_2 = \frac{(\tilde{w}_i + \tilde{w}_{2\nu+2-i})(\tilde{w}_{2\nu+2-i} - 1/2)}{(\tilde{w}_{2\nu+2-i} - \tilde{w}_{\nu+1})(\tilde{w}_{2\nu+2-i} - \tilde{w}_1)} \prod_{k \in J_0} \frac{\tilde{w}_{2\nu+2-i} + \tilde{w}_k}{\tilde{w}_{2\nu+2-i} - \tilde{w}_{2\nu+2-k}}.$$

Equality case: $\nabla\xi = \Pi_{\hat{J}^0 \setminus \{i, 2\nu+2-i\}}(\alpha \otimes \xi)$ *for* α *with*

$$\Pi_{\{i\} \cup J_0}(\alpha \otimes \xi) = 0 \qquad \text{if} \quad C_2 < C_1$$

or

$$\Pi_{\{2\nu+2-i\} \cup J_0}(\alpha \otimes \xi) = 0 \qquad \text{if} \quad C_1 < C_2.$$

(iv) *For* $I = \{2, ..., 2\nu\}$, *we have*

$$k_I^2 = \frac{(\tilde{w}_1 - 1/2) \prod_{k=2}^{\nu} (\tilde{w}_1 + \tilde{w}_k)}{\prod_{k=\nu+1}^{2\nu} (\tilde{w}_1 - \tilde{w}_k)}.$$

Equality case: $\nabla\xi = \Pi_1(\alpha \otimes \xi)$ *for* α *with* $\Pi_{\{2, ..., \nu\}}(\alpha \otimes \xi) = 0$.

(v) *For* $I = \{1, ..., \nu, \nu+2, ..., 2\nu\}$, *we have*

$$k_I^2 = \frac{(\tilde{w}_{\nu+1} - 1/2) \prod_{k=\nu+2}^{2\nu} (\tilde{w}_{\nu+1} + \tilde{w}_k)}{\prod_{k=1}^{\nu} (\tilde{w}_{\nu+1} - \tilde{w}_k)}.$$

Equality case: $\nabla\xi = \Pi_{\nu+1}(\alpha \otimes \xi)$ *for* α *with* $\Pi_{\{\nu+2, ..., 2\nu\}}(\alpha \otimes \xi) = 0$.

(vi) *For* $\hat{I} = \{i\}$ *with* $i \in \{2, ..., \nu, \nu+2, ..., 2\nu\}$, *we have*

$$k_I^2 = \frac{(\tilde{w}_i - 1/2)(\tilde{w}_i + \tilde{w}_{2\nu+2-i})}{(\tilde{w}_i - \tilde{w}_1)(\tilde{w}_i - \tilde{w}_{\nu+1})} \prod_{\substack{j \in J_i^{\max} \\ j \neq 2\nu+2-i}} \frac{\tilde{w}_i + \tilde{w}_j}{\tilde{w}_i - \tilde{w}_{2\nu+2-j}}.$$

Equality case: $\nabla\xi = \Pi_i(\alpha\otimes\xi)$ *for* α *with* $\Pi_{J_i^{\max}}(\alpha\otimes\xi)=0$. *Here* $J_i^{\max}=\{i+1, ..., v, 2v+2-i, ..., 2v\}$.

(vii) *For* $I=\{2, ..., 2v-1\}$, *we have*

$$k_I^2=\frac{(\tilde{w}_1-1/2)\prod_{k=2}^{v}(\tilde{w}_1+\tilde{w}_k)}{\prod_{k=v+1}^{2v}(\tilde{w}_1-\tilde{w}_k)}+\frac{(\tilde{w}_{2v}-1/2)\prod_{k=2}^{v}(\tilde{w}_{2v}+\tilde{w}_k)}{(\tilde{w}_{2v}-\tilde{w}_1)\prod_{k=v+1}^{2v-1}(\tilde{w}_{2v}-\tilde{w}_k)}.$$

Equality case: $\nabla\xi=\Pi_{\{1, 2v\}}(\alpha\otimes\xi)$ *for* α *with* $\Pi_{\{2, ..., v\}}(\alpha\otimes\xi)=0$.

(viii) *For* $I=\{1, ..., v-1, v+2, ..., 2v\}$ *we have*

$$k_I^2=\frac{(\tilde{w}_v-1/2)\prod_{k=v+2}^{2v}(\tilde{w}_v+\tilde{w}_k)}{(\tilde{w}_v-\tilde{w}_{v+1})\prod_{k=1}^{v-1}(\tilde{w}_v-\tilde{w}_k)}+\frac{(\tilde{w}_{v+1}-1/2)\prod_{k=v+2}^{2v}(\tilde{w}_{v+1}+\tilde{w}_k)}{\prod_{k=1}^{v}(\tilde{w}_{v+1}-\tilde{w}_k)}.$$

Equality case: $\nabla\xi=\Pi_{\{v, v+1\}}(\alpha\otimes\xi)$ *for* α *with* $\Pi_{\{v+2, ..., 2v\}}(\alpha\otimes\xi)=0$.

(ix) *For* $\hat{I}=\{i, 2v+1-i\}$ *with* $i\in\{2, ..., v-1\}$, *we have*

$$\begin{aligned}k_I^2=&\frac{(\tilde{w}_i+\tilde{w}_{2v+2-i})(\tilde{w}_i-1/2)}{(\tilde{w}_i-\tilde{w}_1)(\tilde{w}_i-\tilde{w}_{v+1})}\prod_{\substack{j\in J_i^{\max}\\ j\neq 2v+2-i}}\frac{\tilde{w}_i+\tilde{w}_j}{\tilde{w}_i-\tilde{w}_{2v+2-j}}\\&+\frac{(\tilde{w}_{i+1}+\tilde{w}_{2v+1-i})(\tilde{w}_i-1/2)}{(\tilde{w}_{2v+1-i}-\tilde{w}_1)(\tilde{w}_{2v+1-i}-\tilde{w}_{v+1})}\\&\times\prod_{\substack{j\in J_i^{\max}\\ j\neq i+1}}\frac{\tilde{w}_{2v+1-i}+\tilde{w}_j}{\tilde{w}_{2v+1-i}-\tilde{w}_{2v+2-j}}.\end{aligned}$$

Equality case: $\nabla\xi=\Pi_{\{i, 2v+1-i\}}(\alpha\otimes\xi)$ *for* α *with* $\Pi_{J_i^{\max}}(\alpha\otimes\xi)=0$. *Here* $J_i^{\max}=\{i+1, ..., v, 2v+2-i, ..., 2v\}$.

We now give more detailed formulas when $N=4$, which is the generic case in four-dimensional differential geometry: the representation $\tau\otimes\lambda$ splits into $N=4$ components whenever

(i) if $n=2m$ is even and $V=\bigodot^{\ell}\Lambda_{\pm}\bigodot^{k-\ell}\Lambda^p$ or $V=\bigodot^{\ell-1/2}\Lambda_{\pm}\bigodot^{k-\ell}\Lambda^p\odot\Delta_{\pm}$ where $k>\ell>0$ are (simultaneously) integers or half-integers, $p<m$ are integers, $\Lambda^m_{\pm}$ stands for selfdual or anti-selfdual m-forms, and $\Delta_{\pm}$ for positive or negative spin representations. The associated weights are $\lambda=(k, ..., k, \ell, ..., \ell, \pm\ell)$, with k repeated p times. One gets $w_1=k>w_2=\ell-p>w_3=1-\frac{n}{2}-\ell>w_4=-k+p+1-n$.

(ii) $n=2m+1$ is odd, $V=\bigodot^{k-1/2}\Lambda^p\odot\Delta$ with $p<m$ integer and $k\geqslant\frac{1}{2}$ and half-integer, so that $\lambda=(k, ..., k, \frac{1}{2}, ..., \frac{1}{2})$. Conformal weights are a specialization of the previous formula with $\ell=\frac{1}{2}$: $w_1=k>w_2=\frac{1}{2}-p>w_3=\frac{(1-n)}{2}>w_4=-k+p+1-n$.

Note that that P_1, P_3, and P_2+P_4 are elliptic, whereas P_2 and P_4 are non-elliptic.

We give in the following theorem the Kato constants for the kernels of the minimal elliptic operators.

7.2. THEOREM. *If ξ is a non-vanishing section in the kernel of one of the elliptic operators P_1, P_3, or P_2+P_4, we have a refined Kato inequality $|d\,|\xi|| \leqslant k_I\,|\nabla\xi|$ with k_I given as follows.*

(i) *For P_1,*

$$k^2_{\{1\}} = 1 - \frac{\left(w_1 + \frac{n-2}{2}\right)(w_1 + w_4 + n - 1)}{(w_1 - w_2)(w_1 - w_3)}$$
$$= \frac{\left(k + \frac{n-2}{2}\right)(k-\ell) + \ell(k - \ell + p)}{(k - \ell + p)\left(k + \ell + \frac{n-2}{2}\right)}.$$

Equality holds iff $\nabla\xi = (\Pi_2 + \Pi_3)(\alpha \otimes \xi)$ for a 1-form α with $\Pi_4(\alpha\otimes\xi) = 0$.

(ii) *For P_3,*

$$k^2_{\{3\}} = 1 - \frac{\left(w_3 + \frac{n-2}{2}\right)(w_3 + w_2 + n - 1)}{(w_3 - w_4)(w_3 - w_1)}$$
$$= 1 - \frac{\ell\left(\frac{n}{2} - p\right)}{\left(k - \ell + \frac{n}{2} - p\right)\left(k + \ell + \frac{n-2}{2}\right)}.$$

Equality holds iff $\nabla\xi = (\Pi_1 + \Pi_4)(\alpha \otimes \xi)$ for a 1-form α with $\Pi_2(\alpha\otimes\xi) = 0$.

(iii) *For P_2+P_4, $k^2_{\{2,4\}}$ equals*

$$1 - \min\left\{\frac{\left(w_4 + \frac{n-2}{2}\right)(w_2 + w_4 + n - 1)}{(w_4 - w_1)(w_4 - w_3)}, \frac{\left(w_2 + \frac{n-2}{2}\right)(w_2 + w_4 + n - 1)}{(w_2 - w_1)(w_2 - w_3)}\right\}$$
$$= 1 - \min\left\{\frac{\left(k + \frac{n}{2} - p\right)(k - l)}{(2k + n - p - 1)\left(k - l + \frac{n}{2} - p\right)}, \frac{\left(l + \frac{n}{2} - p - 1\right)(k - l)}{(k - l + p)\left(2l + \frac{n}{2} - p - 1\right)}\right\}$$

Equality holds iff $\nabla\xi = (\Pi_1 + \Pi_3)(\alpha \otimes \xi)$ *for a* 1*-form* α *with* $\Pi_2(\alpha \otimes \xi) = 0$ *or* $\Pi_4(\alpha \otimes \xi) = 0$ *depending on which term is the minimum.*

APPENDIX: EXPLICIT CONSTANTS FOR DIMENSIONS 3 AND 4

Dimension 3. Irreducible representations of $\mathfrak{so}(3)$ are symmetric powers, denoted Δ^r, of the spin representation Δ (if r is even, Δ^r has a canonical real structure and from now on we denote by Δ^r its real part). The Clebsch–Gordan formulas show that we are in the case $N=2$ if $r=1$ and $N=3$ if $r \geqslant 2$. In the former case, the elliptic operators are the (Penrose) twistor operator P_1 and the Dirac operator P_2 corresponding to projections on the first and second part of

$$\mathbb{R}^3 \otimes \Delta = \Delta^2 \otimes \Delta = \Delta^3 \oplus \Delta.$$

In the latter case, the elliptic operators are the twistor operator P_1 and Dirac-type operator $P_2 + P_3$ corresponding to projections on the first or second-and-third part of

$$\mathbb{R}^3 \otimes \Delta^r = \Delta^2 \otimes \Delta^r = \Delta^{r+2} \oplus \Delta^r \oplus \Delta^{r-2}, \qquad r \geqslant 2. \tag{9.2}$$

If $r \geqslant 3$ and is odd, then P_2 is elliptic on its own: it is the Rarita–Schwinger operator when $r=3$ and so we denote it by R-S in general.

The following table sums up our formulae in three dimensions.

Operator	Conditions	Refined Constant
Twistor	all r	$\sqrt{\dfrac{r}{r+2}}$
Dirac	$r=1$	$\sqrt{\dfrac{2}{3}}$
Dirac-type	$r \geqslant 2$	$\sqrt{\dfrac{r+2}{2(r+1)}}$
R-S (r odd)	$r \geqslant 3$	$\sqrt{1 - \dfrac{1}{r(r+2)}}$

Minimal Elliptic Operators in Dimension 4. Irreducible representations of $\mathfrak{so}(4)$ are tensor products of symmetric powers, denoted $V^{r,s} = \Delta^r_+ \otimes \Delta^r_-$, of the positive and negative half-spin representation $\Delta_\pm$ (if $r+s$ is even, $V^{r,s}$

has a canonical real structure and, as above, $V^{r,s}$ will denote its real part). Assuming $r \geqslant s$, the Clebsch–Gordan formulas yield, for $r \geqslant s > 0$,

$$\mathbb{R}^4 \otimes V^{r,s} = V^{r+1,s+1} \oplus V^{r+1,s-1} \oplus V^{r-1,s+1} \oplus V^{r-1,s-1}$$

so that we are in the case $N = 4$ if $r > s > 0$ and the case $N = 3$ if $r = s > 0$ (the middle components have equal conformal weights here). If $r > s = 0$ then

$$\mathbb{R}^4 \otimes V^{r,0} = V^{r+1,1} \oplus V^{r-1,1},$$

and we are in the case $N = 2$.

Hence we have (at most) three minimal elliptic operators.

(i) The twistor operator, given by the projection on the first factor in every case.

(ii.a) The operator given by the projection onto $V^{r-1,s+1}$. It is the operator P_3 when $N = 4$ (i.e., if $r > s > 0$) or P_2 when $N = 2$ (i.e., if s vanishes). It defines the spin $r/2$ field equation in this last case and we shall call it a "spin $\frac{r+s}{2}$ field" in general.

(ii.b) The operator in (ii.a) is not elliptic if $N = 3$ (i.e., if $r = s > 0$). We shall replace it by the operator given by the projection onto $(V^{s+1,s-1} \oplus V^{s-1,s+1}) \oplus V^{s-1,s-1}$. The usual Hodge–de Rham belongs to this case, so that it seems reasonable to call it a Dirac-type operator.

(iii) The operator given by the projection onto $V^{r+1,s-1} \oplus V^{r-1,s-1}$ is the elliptic operator $P_2 + P_4$ if $N = 4$ (i.e., if $r > s > 0$). We shall again call it a Dirac-type operator.

The following table sums up our formulae in four dimensions.

Operator	Conditions	Refined Constant	$s = 0$	$r = s$
Twistor	$r \geqslant s \geqslant 0$	$\sqrt{\dfrac{2rs + r + s}{2(r+1)(s+1)}}$	$\sqrt{\dfrac{r}{2(r+1)}}$	$\sqrt{\dfrac{s}{s+1}}$
Spin $\dfrac{r+s}{2}$ field	$r > s \geqslant 0$	$\sqrt{\dfrac{2rs + r + 3s + 2}{2(r+1)(s+1)}}$	$\sqrt{\dfrac{r+2}{2(r+1)}}$	—
Dirac-type	$r \geqslant s > 0$	$\sqrt{\dfrac{s+2}{2(s+1)}}$	—	$\sqrt{\dfrac{s+2}{2(s+1)}}$

As an example, we can reobtain the value found by M. Gursky and C. LeBrun in [15] for a co-closed positive half Weyl tensor (outside its zero set),

$$|d\,|W^+||\leqslant\sqrt{\frac{3}{5}}\,|\nabla W^+|, \tag{7.1}$$

and note that equality occurs if and only if $\nabla W^+ = \Pi_2(\alpha\otimes W^+)$.

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During the course of this work it became clear that there is a close relationship between Kato constants and the spectral results of Branson [8]. Following the presentation of an early version of our results at a meeting in Luminy, Tom Branson has clarified this relationship very nicely [9] and independently obtained a general minimization formula for the Kato constants. We are very grateful to him for sharing his results with us. The formula that follows from our methods is slightly different from his and does not cover one special case. We present it in a similar way to permit easy comparison. We are also deeply indebted to Tammo Diemer and Gregor Weingart for informing us of their recent work, which plays a crucial role in our approach. Finally, we thank Christian Bär and Andrei Moroianu for the application of refined Kato inequalities to Hijazi's inequality.

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