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Mean-Bounded Operators and Mean Ergodic Theorems

R. EMILION

Université Pierre et Marie Curie, Laboratoire de Probabilités, Tour 56/66, 3ème étage, 4, Place Jussieu, 75230 Paris Cedex 05, France

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In any reflexive Banach (lattice), the resolvent (resp. the Césàro means) of a mean-bounded semi-group of operators (resp. of positive operators) is (resp. are) strongly convergent. The mean ergodic theorem fails for mean-bounded operators which are not necessarily positive. © 1985 Academic Press, Inc.

(I) RESOLVENT OF MEAN BOUNDED SEMI-GROUPS

Let B be a Banach space.

1.1. DEFINITION. A linear operator T (resp. a strongly continuous semi-group of linear operators $T=(T_t)_{t\geqslant 0}$) acting on B is said to be C-mean-bounded if

$$M=\sup_{\substack{n\geqslant 1\\ n\in\mathbb{N}}}\left\|\frac{S_n}{n}\right\|<+\infty \qquad \left(\text{resp. } M=\sup_{t>0}\left\|\frac{S_t}{t}\right\|<+\infty\right) \tag{1.2}$$

where $S_n=I+\cdots+T^{n-1}$ (resp. $S_tf=\int_0^t T_sf\,ds$ for any $f\in B$).

Remark. Since $T=(T_t)_{t>0}$ is strongly continuous there is no difficulty with the definition of $\int_0^t T_sf\,ds$ as the strong limit of Riemann sums.

1.3. DEFINITION. A resolvent on B is a family $V=(\lambda V_\lambda)_{\lambda>0}$ of linear operators acting on B such that $V_\lambda - V_\mu = -(\lambda-\mu)\,V_\lambda V_\mu$ for all $\lambda,\mu>0$.

V will be said bounded if $\sup_{\lambda>0}\|\lambda V_\lambda\|<+\infty$.

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1.4. DEFINITION. With the same notations as in (1.1), T (resp. $T=(T_t)_{t>0}$) is said to be A-mean-bounded if for each $\lambda > 0$ and $f \in B$

$$V_\lambda f = \sum_{n=0}^{\infty} \frac{T^n f}{(\lambda+1)^{n+1}} \qquad \left(\text{resp. } V_\lambda f = \int_0^\infty e^{-\lambda s} T_s f\, ds\right)$$

is defined and if $\sup_{\lambda>0} \|\lambda V_\lambda\| < +\infty$.

It is known that $(\lambda V_\lambda)_{\lambda>0}$ defined in (1.4) is necessarily a resolvent (in the sense of (1.3)) which is of course bounded.

Note that $\lambda \sum_{n=0}^{\infty} (T^n/(\lambda+1)^{n+1}) = (1-k)\sum_{n=0}^{\infty} k^n T^n$ when one of these terms is defined (put $k = 1/(\lambda+1)$, $0<k<1$).

1.5. Now, suppose that T (resp. T_t) is a positive operator on a Banach lattice B (i.e., $T(B^+) \subset B^+$, resp. $T_t(B^+) \subset B^+$), then "A-mean-bounded" implies "C-mean-bounded." Indeed $(1-k)\sum_{j=0}^{\infty} k^j T^j \geqslant (1-k)\, k^{n-1} S_n$ and thus $\|S_n\|/n \leqslant e \sup_{\lambda>0} \|\lambda V_\lambda\|$ (take $k = 1-1/n$). In the continuous case one has $\lambda \int_0^\infty e^{-\lambda s} T_s\, ds \geqslant \lambda e^{-\lambda t} S_t$ and thus $\|S_t\|/t \leqslant e \sup_{\lambda>0} \|\lambda V_\lambda\|$ (take $\lambda = 1/t$).

1.6. Our first result (1.7) shows that C-boundedness implies A-boundedness even if the operators are not necessarily positive.

Therefore in the next sections "mean-bounded" will mean A-mean-bounded. The two notions are equivalent if the operators are positive ((1.5) and (1.7)). The proof of (1.7) will show that any C-mean-bounded sequence (resp. continuous function) in B is A-mean-bounded, and the converse is true if the sequence (resp. the function) is positive (1.5).

1.7. THEOREM. *Let T (resp. $(T_t)_{t\geqslant 0}$) be a C-mean-bounded operator (resp. a C-mean-bounded semi-group) then for $\lambda > 0$ and $f \in B$*

$$V_\lambda f = \sum_{n\geqslant 0} \frac{T^n f}{(\lambda+1)^{n+1}} \qquad \left(\text{resp. } V_\lambda f = \int_0^\infty e^{-\lambda t} T_t f\, dt\right)$$

is well defined and defines a bounded resolvent which satisfies $\sup_{\lambda>0} \|\lambda V_\lambda\| \leqslant M$, where M is given by (1.2).

Proof. The continuous case. The semi-group is strongly continuous, thus, an integration by parts gives us

$$\int_0^A e^{-\lambda t} T_t f\, dt = e^{-\lambda A} S_A f + \int_0^A \lambda e^{-\lambda t} S_t f\, dt \qquad \text{for any } A > 0.$$

Since $\|S_t\| \leqslant Mt$, $\lim_{A\to+\infty} e^{-\lambda A} S_A f = 0$ and $\lim_{A\to+\infty} \int_0^A \lambda e^{-\lambda t} S_t f\, dt$ exists. Therefore $V_\lambda f = \int_0^\infty e^{-\lambda t} T_t f\, dt = \int_0^\infty \lambda e^{-\lambda t} S_t f\, dt$ exists and

$$\|\lambda V_\lambda f\| \leqslant \int_0^\infty \lambda^2 e^{-\lambda t} Mt\, dt = M \qquad \text{for any } \lambda > 0.$$

It is a classical result that $(\lambda V_\lambda)_{\lambda>0}$ is a resolvent.

The discrete case. Let $0 < k < 1$ and $S_n = I + \cdots + T^{n-1}$. One has

$$\sum_{n=p}^{p+q} k^n T^n = \sum_{n=p}^{p+q} k^n (S_{n+1} - S_n)$$
$$= -k^p S_p + \sum_{j=p}^{p+q-1} S_{j+1}(k^j - k^{j+1}) + k^{p+q} S_{p+q+1}.$$

By (1.2) one has

$$\left\| \sum_{n=p}^{p+q} k^n T^n \right\| \leqslant M \left(pk^p + \sum_{j=p}^{p+q-1} (j+1)(k^j - k^{j+1}) + (p+q+1)\, k^{p+q} \right)$$

and

$$\left\| \sum_{n=p}^{p+q} k^n T^n \right\| \leqslant M \left((2p+1)\, k^p + \sum_{j=p+1}^{p+q-1} k^j + k^{p+q} \right). \tag{1.8}$$

Therefore

$$\lim_{\substack{p\to\infty \\ q\to\infty}} \left\| \sum_{n=p}^{p+q} k^n T^n \right\| = 0 \qquad \text{and} \qquad \sum_{n=0}^{\infty} k^n T^n$$

is well defined. Moreover (1.8) shows that $\|\sum_{n=0}^\infty k^n T^n\| \leqslant M/(1-k)$. Now let $\lambda > 0$ and $k = 1/(\lambda + 1)$.

$$V_\lambda = \sum_{n=0}^\infty \frac{T^n}{(\lambda+1)^{n+1}} = k \sum_{n=0}^\infty k^n T^n = k(I - kT)^{-1}$$

is well defined and $\|\lambda V_\lambda\| \leqslant M$. It is a classical result that $(\lambda V_\lambda)_{\lambda>0}$ is a resolvent.

1.9. Note that if T is a mean-bounded positive operator on $L_1(\mu)$ (resp. on L_∞) such that $M \leqslant 1$ in (1.2) then T is a contraction. Indeed, $\int ((f + Tf)/2)\, d\mu \leqslant \int f\, d\mu$ for any $f \in L_1^+$ (resp. $(1 + T1)/2 \leqslant 1$ implies $T1 \leqslant 1$ and $\|Tf\|_\infty \leqslant \|T(\|f\|_\infty)\|_\infty \leqslant \|f\|_\infty$ for any $f \in L_\infty$).

This result is false in $L^2(X, \mathscr{F}, \mu)$. Indeed take $X = \{1, 2\}$ with $\mu(1) = \mu(2) = 1$. The positive operator T on $L^2(X, \mathscr{F}, \mu)$ defined by the matrix $\begin{pmatrix} 0 & 1+\varepsilon \\ 0 & 0 \end{pmatrix}$ is such that $T^2 = 0$. If $\varepsilon > 0$ is small enough we have $\|(I+T)/2\| \leqslant 1$ and

$$\sup_{\substack{n \geqslant 1 \\ n \in \mathbb{N}}} \frac{\|S_n\|}{n} \leqslant 1,$$

but T is not a contraction.

However, in the continuous case we obtain the

1.10. THEOREM. *Let $T = (T_t)_{t \geqslant 0}$ be a strongly continuous semi-group on a Banach space B, such that $M = \sup_{t>0}(1/t)\|S_t\| \leqslant 1$, then T_t is a contraction for any $t \geqslant 0$.*

Proof. If $V_\lambda = \int_0^\infty e^{-\lambda t} T_t \, dt$, Theorem (1.7) implies

$$\|\lambda V_\lambda\| \leqslant 1. \qquad \text{Note that } \|T_0\| = \left\| \text{strong} - \lim_{t \to 0^+} \frac{S_t}{t} \right\| \leqslant 1. \tag{1.11}$$

We have

$$\text{Strong} - \lim_{\lambda \to +\infty} \lambda V_\lambda = T_0. \tag{1.12}$$

Indeed, let $f \in B$, $\varepsilon > 0$ and $\delta > 0$ such that $\|(1/s)\, S_s f - T_0 f\| \leqslant \varepsilon$ for any $s: 0 < s \leqslant \delta$. An integration by parts shows that $\|\lambda V_\lambda f - T_0 f\| = \|\int_0^\delta \lambda^2 e^{-\lambda s}(S_s f - sT_0 f)\, ds + \int_\delta^{+\infty} \lambda^2 e^{-\lambda s}(S_s f - sT_0 f)\, ds\| \leqslant \varepsilon \int_0^\delta \lambda^2 e^{-\lambda s} s \, ds + (M\|f\| + \|T_0 f\|)\ e^{-\lambda(\delta/2)} \int_\delta^{+\infty} \lambda^2 e^{-\lambda(s/2)} s\, ds$. Therefore we have $\overline{\lim}_{\lambda \to +\infty} \|\lambda V_\lambda f - T_0 f\| \leqslant \varepsilon$ and $\lim_{\lambda \to +\infty} \lambda V_\lambda f = T_0 f$ for each $f \in B$.

Now, since $V_\lambda(T_0 f) = T_0(V_\lambda f) = V_\lambda f$, we see that $(\lambda V_\lambda)_{\lambda > 0}$ is a resolvent on the Banach space $H = T_0 B$ ($T_0^2 = T_0$ implies that $\overline{T_0 B} = T_0 B$). Moreover (1.11) and (1.12) imply $\|\lambda V_\lambda\|_H \leqslant 1$ and $\lim_{\lambda \to +\infty} \lambda V_\lambda h = h$ for any $h \in H$.

Hence, Hille–Yosida theorem ([2, p. 261]) shows that $(\lambda V_\lambda)_{\lambda > 0}$ is the resolvent of a uniquely determined semi-group of contractions on H, say U_t: $V_\lambda(T_0 f) = \int_0^\infty e^{-\lambda t} U_t(T_0 f)\, dt$ for any $f \in B$. But $V_\lambda(T_0 f) = \int_0^\infty e^{-\lambda t} T_t(T_0 f)\, dt$. Therefore $U_t = T_{t|H}$ and $\|T_t f\| = \|T_t T_0 f\| = \|U_t T_0 f\| \leqslant \|T_0 f\| \leqslant \|f\|$ implies that T_t is a contraction for any $t \geqslant 0$.

(II) STRONG CONVERGENCE OF ABELIAN MEANS

For early ergodic theorems related to the following, see Eberlein [2].

2.1. THEOREM. *Let T (resp. $(T_t)_{t \geqslant 0}$) be a mean-bounded operator (resp. a mean-bounded semi-group) on a reflexive Banach space B. Let*

$V_\lambda = \sum_{n\geq 0} (T^n/(\lambda+1)^{n+1})$ (resp. $V_\lambda = \int_0^\infty e^{-\lambda s} T_s\, ds$), then λV_λ is strongly convergent as $\lambda \to 0^+$ and also does as $\lambda \to +\infty$.

Remark. If we put $k = 1/(\lambda + 1)$ we see that

$$\lim_{\substack{k\to 1^- \\ (\text{resp. } k \to 0^+)}} (1-k) \sum_{n=0}^{\infty} k^n T^n f$$

exists for any $f \in B$.

Proof. We recall (1.6) that "mean-bounded" means A-mean-bounded and if T is C-mean bounded T is also A-mean bounded (1.7). In both cases $V = (\lambda V_\lambda)_{\lambda > 0}$ is a bounded resolvent (1.4 and 1.7). Therefore (2.1) holds as bounded resolvents on a reflexive Banach space are strongly convergent. Indeed, since V is bounded the set $X = \{f \in B \mid \lim \lambda V_\lambda f$ exists as $\lambda \to 0^+$ (resp. $\lambda \to +\infty$)$\}$ is closed and also weakly closed. Let $f \in B$ and $\lambda_n \to 0^+$ (resp. $\lambda_n \to +\infty$) such that $f^* = w - \lim \lambda_n V_{\lambda_n} f$ exists as $\lambda_n \to 0^+$ (resp. $\lambda_n \to +\infty$).

The case $\lambda \to 0^+$. Let $\lambda > 0$ and $\lambda_{n'}$ a subsequence of λ_n such that $h = w - \lim_{\lambda_{n'} \to 0^+} \lambda_{n'} V_{\lambda_{n'}} \lambda V_\lambda f$ exists. We have $\lambda V_\lambda f^* = \lambda V_\lambda (w - \lim_{\lambda_{n'} \to 0^+} \lambda_{n'} V_{\lambda_{n'}} f) = h$. But, since V is bounded $\lambda_{n'} V_{\lambda_{n'}} (I - \lambda V_\lambda) = \lambda_{n'} V_\lambda (I - \lambda_{n'} V_{\lambda_{n'}})$ converges strongly to 0 as $\lambda_{n'} \to 0^+$. Therefore $f^* - h = 0$ and for each $\lambda > 0$, $\lambda V_\lambda f^* = f^*$. Thus $f^* \in X$. Similarly $\lambda V_\lambda (I - \lambda_n V_{\lambda_n}) = \lambda V_{\lambda_n} (I - \lambda V_\lambda)$ converges strongly to 0 as $\lambda \to 0^+$. Hence, for each λ_n we have $f - \lambda_n V_{\lambda_n} f \in X$, and thus $f - f^* \in X$. Finally $f^* + f - f^* = f \in X$ and $X = B$.

The case $\lambda \to +\infty$. (See also (1.12) in the proof of (1.10)). Since V is bounded $\lambda V_\lambda \lambda_n V_{\lambda_n} = (\lambda \lambda_n / (\lambda - \lambda_n))(V_{\lambda_n} - V_\lambda)$ converges strongly to $\lambda_n V_{\lambda_n}$ as $\lambda \to +\infty$. Therefore $\lambda_n V_{\lambda_n} f \in X$ for each λ_n and $f^* \in X$. Now, again since V is bounded $\lambda V_\lambda (I - \lambda_n V_{\lambda_n}) = \lambda V_\lambda - (\lambda \lambda_n / (\lambda - \lambda_n))(V_{\lambda_n} - V_\lambda) = (1 + \lambda_n / (\lambda - \lambda_n))\, \lambda V_\lambda - (\lambda / (\lambda - \lambda_n))\, V_{\lambda_n}$ converges strongly to 0 as $\lambda_n \to +\infty$. Therefore $\lambda V_\lambda (f - f^*) = 0$ and $\lambda V_\lambda f = \lambda V_\lambda f^*$ for each $\lambda > 0$. Since $f^* \in X$, $f \in X$ and $X = B$.

(III) Tauberian Theorems

We can easily extend the classical real tauberian theorems [5] to B-valued functions where B is a Banach lattice.

The lattice property is needed only in the positive case and the fact that the space is complete is not used in the proofs.

Although the following proofs contain classical arguments due to

Karamata [5] we prefer to give the full details: the case of a positive sequence is a very slight modification of [5] but the case of a bounded sequence must be proved differently.

We state the discrete case. The continuous case is an immediate consequence of the discrete one. The limits are taken in the norm topology of B.

3.1. THEOREM. *Let (a_n) be a sequence of B such that $\sum_{i=0}^{\infty} e^{-\lambda i} a_i$ is defined for any $\lambda > 0$ and that $\lim_{\lambda \to 0^+} \lambda \sum_{i=0}^{\infty} e^{-\lambda i} a_i$ exists, then $\lim_{n \to +\infty} (1/n) \sum_{i=0}^{n} a_i$ exists if $a_n \geqslant 0$ (resp. a_n is bounded) and the two limits are equal.*

3.2. The hypotheses of (3.1) are equivalent to $\sum_{i=0}^{\infty} k^i a_i$ is defined for any k: $0 < k < 1$ and $\lim_{k \to 1^-} (1-k) \sum_{i=0}^{\infty} k^i a_i$ exists. (Put $k = e^{-\lambda}$.)

Note that (3.1) holds if $a_n \geqslant 0$ for any n large enough.

3.3. THEOREM. *Let $a: t \to a(t)$ be a B-valued function defined on $[0, +\infty[$, integrable on every finite interval and such that $\int_0^{\infty} e^{-\lambda t} a(t)\, dt$ exists for any $\lambda > 0$ and that $\lim_{\lambda \to 0^+} \lambda \int_0^{\infty} e^{-\lambda s} a(s)\, ds$ exists. Then $\lim_{t \to +\infty} (1/t) \int_0^t a(s)\, ds$ exists if a is positive (resp. is bounded) and the two limits are equal.*

(Apply (3.1) with $a_i = \int_i^{i+1} a(s)\, ds$).

Proof of 3.1. Put $\lambda = 1/n$; we have $(1/n) \sum_{i=0}^{n} a_i = \lambda \sum_{i=0}^{\infty} e^{-\lambda i} g(e^{-\lambda i})\, a_i$, where g is the real function defined on the interval $[0, 1]$ by $g(x) = 0$ if $0 \leqslant x < 1/e$ and $g(x) = 1/x$ if $1/e \leqslant x \leqslant 1$.

The theorem will be proved if $\lim_{\lambda \to 0^+} \lambda \sum_{i=0}^{\infty} e^{-\lambda i} g(e^{-\lambda i})\, a_i$ exists.

We are going to prove

$$\lim_{\lambda \to 0^+} \lambda \sum_{i=0}^{\infty} e^{-\lambda i} g(e^{-\lambda i})\, a_i = A \int_0^1 g(u)\, du \qquad (=A) \tag{3.4}$$

where $A = \lim_{\lambda \to 0^+} \lambda \sum_{i=0}^{\infty} e^{-\lambda i} a_i$ (which exists by hypothesis).

Let p be an integer.

If $\lim_{\lambda \to 0^+} \lambda \sum_{i=0}^{\infty} e^{-\lambda i} a_i = A$ and if we replace λ by $(p+1)\lambda$ we obtain

$$\lim_{\lambda \to 0^+} (p+1)\lambda \sum_{i=0}^{\infty} e^{-\lambda(p+1)i} a_i = A \qquad \text{or}$$

$$\lim_{\lambda \to 0^+} \lambda \sum_{i=0}^{\infty} e^{-\lambda i} (e^{-\lambda i})^p\, a_i = \frac{A}{p+1} = A \int_0^1 u^p\, du.$$

Therefore (3.4) holds if $g(x)$ is replaced by x^p and hence by any polynomial.

The case $a_i \geqslant 0$. Let $\varepsilon > 0$. Define a real continuous function h on $[0, 1]$ such that $g \leqslant h$ and $\int_0^1 (h-g)(u)\, du < \varepsilon/3$: take h affine on $[1/e - \delta, 1/e]$, with $\delta > 0$ small enough, and $h = g$ otherwise. Let Q be a polynomial such that $|h(x) - Q(x)| < \varepsilon/3$ for any $x: 0 \leqslant x \leqslant 1$. Let P be the polynomial $Q + \varepsilon/3$. Then $g \leqslant h \leqslant Q + \varepsilon/3 = P$ and $\int_0^1 (P-g)(u)\,du = \int_0^1 (P-Q)(u)\,du + \int_0^1 (Q-h)(u)\,du + \int_0^1 (h-g)(u)\,du \leqslant \varepsilon$. In the same way we can find a polynomial p such that $p \leqslant g$ and that $\int_0^1 (g-p)(u)\,du < \varepsilon$.

Then we have

$$\begin{aligned} X &= \lambda \sum_{i=0}^{\infty} e^{-\lambda i} p(e^{-\lambda i})\, a_i - A \int_0^1 g(u)\,du \\ \leqslant Y &= \lambda \sum_{i=0}^{\infty} e^{-\lambda i} g(e^{-\lambda i})\, a_i - A \int_0^1 g(u)\,du \\ \leqslant Z &= \lambda \sum_{i=0}^{\infty} e^{-\lambda i} P(e^{-\lambda i})\, a_i - A \int_0^1 g(u)\,du \end{aligned}$$

and thus $\|Y\| \leqslant \|X\| + \|Z\|$ by the lattice properties of B.

But

$$\|X\| \leqslant \left\| \lambda \sum_{i=0}^{\infty} e^{\lambda i} p(e^{-\lambda i})\, a_i - A \int_0^1 p(u)\,du \right\| + \|A\| \int_0^1 (g(u) - p(u))\,du$$

and since (3.4) holds for the polynomial p, we obtain $\overline{\lim}_{\lambda \to 0^+} \|X\| \leqslant \|A\|\,\varepsilon$.

In the same way $\overline{\lim}_{\lambda \to 0^+} \|Z\| \leqslant \|A\|\,\varepsilon$. ε being arbitrary we have $\lim_{\lambda \to 0^+} \|Y\| = 0$ or

$$\lim_{\lambda \to 0^+} \lambda \sum_{i=0}^{\infty} e^{-\lambda i} g(e^{-\lambda i})\, a_i = A \int_0^1 g(u)\,du = A.$$

This is (3.4).

The case a_i bounded. Let $K = \sup_{i \geqslant 0} \|a_i\|$.

Let $\varepsilon > 0$, let h and Q as above. Since

$$\begin{aligned} &\overline{\lim_{\lambda \to 0^+}} \left\| \lambda \sum_{i=0}^{\infty} e^{-\lambda i} h(e^{-\lambda i})\, a_i - \lambda \sum_{i=0}^{\infty} e^{-\lambda i} Q(e^{-\lambda i})\, a_i \right\| \\ &\leqslant \overline{\lim_{\lambda \to 0^+}} K \left(\lambda \sum_{i=0}^{\infty} e^{-\lambda i} \right) \frac{\varepsilon}{3} = K \frac{\varepsilon}{3} \end{aligned}$$

and $\left|\int_0^1 (h-Q)(u)\,du\right| \leqslant \int_0^1 |h-Q|\,(u)\,du \leqslant \varepsilon/3$ and since (3.4) holds for Q, (3.4) also holds for h.

To show that (3.4) holds for g it then suffices to consider the numbers

$\|\lambda \sum_{i=0}^{\infty} e^{-\lambda i}(h-g)(e^{-\lambda i})\, a_i\|$ and $\int_0^1 (h-g)(u)\, du$. The last one is dominated by $\varepsilon/3$. On the other hand, by the construction of h, we have

$$\begin{aligned}
\left\|\lambda \sum_{i=0}^{\infty} e^{-\lambda i}(h-g)(e^{-\lambda i})\, a_i\right\| &= \left\|\lambda \sum_{i:i\lambda \geqslant 1} e^{-\lambda i}h(e^{-\lambda i})\, a_i\right\| \\
&\leqslant K\lambda \sum_{i:i\lambda \geqslant 1} e^{-\lambda i}h(e^{-\lambda i}) \\
&\leqslant K\lambda(e^{-1}h(e^{-1}) + e^{-\lambda([1/\lambda]+1)}h(e^{-\lambda([1/\lambda]+1)})) \\
&\quad + K\lambda \sum_{i=[1/\lambda]+2}^{+\infty} e^{-\lambda i}h(e^{-\lambda i}) \\
&\leqslant K\lambda[1 + e^{-\lambda([1/\lambda]+1)}h(e^{-\lambda([1/\lambda]+1)})] \\
&\quad + K\lambda \int_{[1/\lambda]+1}^{+\infty} e^{-\lambda t}h(e^{-\lambda t})\, dt
\end{aligned}$$

(the construction of h shows that the function $t \to e^{-\lambda t}h(e^{-\lambda t})$ is decreasing on $[1/\lambda, +\infty[$)

$$\begin{aligned}
&\leqslant K\lambda[1 + e^{-\lambda([1/\lambda]+1)}h(e^{-\lambda([1/\lambda]+1)})] + K\lambda \int_{1/\lambda}^{+\infty} e^{-\lambda t}h(e^{-\lambda t})\, dt \\
&= K\lambda\, [(1 + e^{-\lambda([1/\lambda]+1)}h(e^{-\lambda([1/\lambda]+1)})] + K \int_0^{1/e} h(u)\, du \\
&\leqslant K\lambda[(1 + e^{-\lambda([1/\lambda]+1)}h(e^{-\lambda([1/\lambda]+1)})] + K\frac{\varepsilon}{3}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\overline{\lim_{\lambda \to 0^+}} \left\|\lambda \sum_{i=0}^{\infty} e^{-\lambda i}(h-g)(e^{-\lambda i})\, a_i\right\| \\
&\qquad \leqslant \overline{\lim_{\lambda \to 0^+}} K\lambda(1 + e^{-1}h(e^{-1})) + K\frac{\varepsilon}{3} = K\frac{\varepsilon}{3}.
\end{aligned}$$

Therefore, since (3.4) holds for h, (3.4) holds for g.

The theorem is completely proved.

3.5. In the non-positive case the conclusion of the tauberian theorem (3.1) fails if we replace the hypothesis a_n bounded by a_n C-mean-bounded (i.e., $\sup_{n \geqslant 1}(1/n)\,\|\sum_{i=1}^{n} a_i\| < \infty$) or a_n/n bounded.

Take a_n such that $S_n = a_1 + \cdots + a_n = n$ if n is even and $S_n = -n$ if n is odd. So $a_{2n} = 4n - 1$ and $a_{2n+1} = -4n - 1$ or $a_n = 4(-1)^n[n/2] - 1$. Clearly a_n/n (resp. a_n) is bounded (resp. C-mean-bounded) and S_n/n diverges, but $\lim_{k \to 1^-}(1-k) \sum_{n=0}^{\infty} k^n a_n$ exists.

(IV) Mean Ergodic Theorems

Let B be a reflexive Banach lattice (for example, $B = L^p(X, \mathscr{F}, \mu)$, $1 < p < \infty$).

Now, using Theorem (2.1) (see the remark) and Theorem (3.1) (or (3.2)) we immediately obtain the following mean ergodic theorems.

4.1. Theorem. *Let T be a mean-bounded operator on a reflexive Banach lattice B. Let $f \in B$ be such that $T^n f$ is positive for any n large enough (resp. $\sup_n \|T^n f\| < +\infty$), then $((I + \cdots + T^{n-1})/n)f$ converges in the norm topology as $n \to +\infty$ and $T^n f/n$ converges to 0.*

(Note that $T^n/n = ((n+1)/n)(S_{n+1}/(n+1)) - S_n/n$).

4.2. Theorem. *Let T be a positive operator on a reflexive Banach lattice, then $(I + \cdots + T^{n-1})/n$ is strongly convergent if (and only if) T is mean-bounded.*

4.3. Corollary. *Let T be a positive mean-bounded operator on a reflexive Banach lattice B then strong $- \lim_{n \to +\infty}(T^n/n) = 0$ and* $(+)\; B = \operatorname{Inv} T \oplus \overline{(I - T)(B)}$

($\operatorname{Inv} T = \{x \in B / Tx = x\}$. Write $f = f^* + f - f^*$ with $f^* = \lim_{n \to +\infty}((I + \cdots + T^{n-1})/n)f)$. By (2.1), (+) holds even if T is not positive. Indeed $\operatorname{Inv} T = \operatorname{Inv} \lambda V_\lambda$ and $(I - T)(B) = (I - \lambda V_\lambda)(B)$.

4.4. Corollary. *Let $T_1 \cdots T_k$ be commuting positive operators on a reflexive Banach lattice B then*

$$\lim_{n_i \to +\infty} \frac{1}{n_1 \cdots n_k} \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_k=0}^{n_k-1} T_1^{j_1} \cdots T_k^{j_k} f \qquad \textit{exists for any } f \in B$$

if and only if each T_i ($i = 1 \cdots k$) is mean-bounded.

Proof. The general case can be proved in the same way as the particular one $k = 2$.

Note that T_i ($i = 1, 2$) is mean-bounded if and only if

$$\sup_{n_i} \frac{1}{n_1 n_2} \left\| \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} T_1^{j_1} T_2^{j_2} \right\| < +\infty$$

$$(\text{take } n_j = 1 \text{ for } j \neq i). \tag{4.5}$$

So, the "only if" part is due to Banach–Steinhaus theorem.

Conversely suppose that each T_i is mean-bounded, then $B = \operatorname{Inv}(I - T_i) \oplus \overline{(I - T_i)(B)}$ (4.3). Hence the set $A = \{f + (g + h - T_1 h) -$

$T_2(g+h-T_1h)$ with $T_1f=f$ and $T_2g=g\}$ is dense in B. It is easy to check that the limit exists for f (if $T_1f=f$), for g (if $T_2g=g$), for $h-T_1h$ and thus for $T_2(g+h-T_1h)$. Therefore the set of convergence X contains A.

Since X is closed by (4.5), we have $X=B$.

4.6. If T is power-bounded (i.e., $\sup_n \|T^n\| < +\infty$) but not necessarily positive the conclusion of (4.1) holds for any $f\in B$.

4.7. The mean ergodic theorem (4.2) fails for mean-bounded operators which are not necessarily positive.

The following counterexample is due to I. Assani.

Take $B=\mathbb{R}^2$ and $T=\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$. Then, one has

$$T^n = \begin{pmatrix} \frac{(-1)^n}{0} & \frac{(-1)^{n+1}\,2n}{(-1)^n} \end{pmatrix}.$$

Clearly $T^n/n \not\to 0$ but

$$\sup_{\substack{n\leqslant 1\\ n\in\mathbb{N}}} \left\| \frac{I+\cdots+T^{n-1}}{n} \right\| < +\infty.$$

We can note that the abelian and the Cesaro means of T^nf/n converge in norm to 0 for any mean-bounded operator. Indeed, if T is C-mean-bounded, one has

$$\sup_{\substack{n\geqslant 1\\ n\in\mathbb{N}}} \left\| (I-T)\left(\frac{I+\cdots+T^{n-1}}{n}\right) \right\| < +\infty$$

and thus

$$\sup_{\substack{n\geqslant 1\\ n\in\mathbb{N}}} \left\| \frac{T^n}{n} \right\| < +\infty.$$

Now for any k: $0<k<1$, we see that

$$\sum_{n=1}^{\infty} k^n \frac{T^nf}{n} = \sum_{n=1}^{\infty} \frac{k^n}{n}(S_{n+1}-S_nf)$$
$$= -kS_1 + S_2\left(k-\frac{k^2}{2}\right) + S_3\left(\frac{k^2}{2}-\frac{k^3}{3}\right) + \cdots.$$

Therefore (1.2) yields

$$\left\| \sum_{n=1}^{\infty} k^n \frac{T^n f}{n} \right\| \leqslant M \left(k + 2\left(k - \frac{k^2}{2}\right) + 3\left(\frac{k^2}{2} - \frac{k^3}{3}\right) + \cdots \right)$$

$$= M\left(2k + k + \frac{k^2}{2} + \cdots + \frac{k^n}{n} \cdots\right)$$

and

$$\lim_{k \to 1^-} \left\| (1-k) \sum_{n=1}^{\infty} k^n \frac{T^n f}{n} \right\| = 0.$$

Thus (3.2) shows that the Cesaro means of $T^n f/n$ converge to 0.

Also note that if B is reflexive and T is C-mean-bounded then strong $-\lim_{n\to+\infty}(S_n/n)$ exists if (and only if) strong $-\lim_{n\to+\infty}(T^n/n)=0$ (same classical arguments as those used in the proof of: $\sup_{\lambda>0}\|\lambda V_\lambda\|<+\infty \Rightarrow$ strong $-\lim_{\lambda\to 0^+}\lambda V_\lambda$ exists (2.1)).

4.8. *The continuous case.* Let $T=(T_t)_{t\geqslant 0}$ be strongly continuous semi-group of B-positive operators. Assume that T is mean-bounded.

Although we have

$$\frac{n}{n+1}\frac{1}{n}\sum_{j=0}^{n-1} T_1^j S_1 f \leqslant \frac{S_t f}{t} \leqslant \frac{n+1}{n}\frac{1}{n+1}\sum_{j=0}^{n} T_1^j S_1 f$$

with $n=[t]$ and $f \in B^+$, we cannot apply the discrete result (4.1): It does not appear that T_1 is mean-bounded.

Therefore, we use (2.1) and (3.3) to obtain the

4.9. THEOREM. *Let $T=(T_t)_{t\geqslant 0}$ be a mean-bounded strongly continuous semi-group of linear operators on a reflexive Banach lattice B.*

Let $f \in B$ be such $T_t f \geqslant 0$ for any t large enough (resp. $\sup_{t>0}\|T_t f\|<+\infty$), then $\lim_{t\to+\infty}(S_t f/t)$ exists.

4.10. THEOREM. *Let $T=(T_t)_{t\geqslant 0}$ be a strongly continuous semi-group of positive operators on a reflexive Banach lattice B. $\lim_{t\to+\infty}(S_t f/t)$ exists for each $f\in B$ if (and only if) T is mean-bounded.*

4.11. *The local case.* If $T=(T_t)_{t\geqslant 0}$ is strongly continuous at $t=0$ we immediately see that strong $-\lim_{t\to 0^+}(S_t/t)=T_0$.

Note that if B is reflexive, then T is strongly continuous at $t=0$ (i.e., there exists T_0 such that $T_0=$ strong $-\lim_{t\to 0^+} T_t$) if and only if T is locally bounded (i.e., $\sup_{0<t<1}\|T_t\|<+\infty$). Indeed the set $X=\{f\in B/\lim_{t\to 0^+} T_t f$

exists} is closed and also weakly closed. Let $f \in B$ and $t_n \to 0^+$ such that $h = w - \lim_{t_n \to 0^+} T_{t_n} f$ exists. Since $T_{t_n} f \in X$ for each t_n, one has $h \in X$. But, since T is continuous at $t > 0$, we have $T_t h = T_t f$. Therefore $f \in X$ and $X = B$.

Now, suppose that T is not necessarily continuous at $t = 0$. We will say that T is locally mean-bounded if T is strongly integrable at 0 (i.e., $S_t f = \lim_{\varepsilon \to 0^+} \int_\varepsilon^t T_s f\, ds$ exists for each $f \in B$ and $t > 0$) and if $\sup_{0 < t < 1} \|S_t/t\| < +\infty$.

We obtain the following result which also holds for n-parameters semi-groups on a reflexive Banach space B:

4.12. THEOREM. *Strong* $- \lim_{t \to 0^+}(S_t/t)$ *exists if and only if* T *is locally mean-bounded.*

Proof. The "only if" part is clear.

Conversely, if T is locally mean-bounded the set $X = \{f \in B/ \lim_{t \to 0^+}(S_t f/t)$ exists} is closed and also weakly closed. For any $\alpha > 0$ and $f \in B$, we can show that $T_s S_\alpha f = S_{\alpha+s} f - S_s f$ and $\lim_{s \to 0^+} T_s(S_\alpha f) = S_\alpha f$. This implies that $\lim_{t \to 0^+}(1/t)\, S_t(S_\alpha f) = S_\alpha f$. Thus $(1/\alpha)\, S_\alpha f \in X$ for any $\alpha > 0$. Now, let $f \in B$ and $t_n \to 0^+$ such that $h = w - \lim_{t_n \to 0^+}(1/t_n)\, S_{t_n} f$ exists. Since X is weakly closed, $h \in X$. But, for any $s > 0$, we have

$$T_s h = w - \lim_{t_n \to 0^+} \frac{1}{t_n} T_s(S_{t_n} f)$$

$$= w - \lim_{t_n \to 0^+} \frac{1}{t_n} S_{t_n}(T_s f) = T_s f \qquad \text{and} \qquad S_s f = S_s h.$$

Therefore $f \in X$ and $X = B$.

(V) POINTWISE CONVERGENCE ON A DENSE SUBSET OF $L_p(1 < p < \infty)$

Now, suppose that $B = L_p(X, \mathscr{F}, \mu)$ $(1 < p < \infty)$, the usual reflexive Banach lattice. Suppose that T (resp. $(T_t)_{t \geqslant 0}$) is a mean-bounded positive operator (resp. semi-group of positive operators) on L_p. Then the abelian means $\lambda V_\lambda = \lambda \sum_{n=0}^\infty (T^n f/(\lambda+1)^{n+1})$ (resp. $\lambda V_\lambda = \lambda \int_0^\infty e^{-\lambda s} T_s f\, ds$) are pointwisely convergent as $\lambda \to 0^+$ (resp. $\lambda \to +\infty$) μ-a.e. on X for any f belonging to a dense subset of B.

Indeed $B = L_p = \text{Inv}\, V \oplus \overline{(I - \alpha V_\alpha)(B)}$, where $\text{Inv}\, V = \{f \in B/f - \alpha V_\alpha f = 0\}$. Note that $\text{Inv}\, V$ and $(I - \alpha V_\alpha)(B)$ do not depend of the particular α chosen:

$$\begin{aligned}(I-\beta V_\beta)-(I-\alpha V_\alpha)&=\alpha V_\alpha-\beta V_\beta=\alpha(V_\alpha-V_\beta)+(\alpha-\beta)V_\beta\\&=-\alpha(\alpha-\beta)V_\alpha V_\beta+(\alpha-\beta)V_\beta\\&=(\alpha-\beta)V_\beta(I-\alpha V_\alpha)\qquad\text{for any }\alpha,\beta>0.\end{aligned}$$

Also note that $(I-T)(B)=(I-\alpha V_\alpha)(B)$ and $\operatorname{Inv} T=\operatorname{Inv} V$. It is trivial that $\lim_{\lambda\to 0^+}\lambda V_\lambda f=f$ a.e. on X for any $f\in\operatorname{Inv} V$. On the other hand, if $f=(I-\alpha V_\alpha)(g)$ for some $g\in L_p$ and $\alpha>0$ then $\lambda V_\lambda f=\lambda V_\lambda(I-\alpha V_\alpha)g=\lambda V_\alpha(I-\lambda V_\lambda)(g)=\lambda V_\alpha g-\lambda^2 V_\lambda g'$ with $g'=V_\alpha g$.

We have $\lim_{\lambda\to 0^+}\lambda V_\alpha g=0$ a.e. on X (since $V_\alpha g\in L_p$, $V_\alpha g<+\infty$ a.e.). If $n=[1/\lambda]$, we also have

$$\left|\overline{\lim_{\lambda\to 0^+}}\lambda^2 V_\lambda g'\right|\leqslant\overline{\lim_{\lambda\to 0^+}}\lambda^2 V_\lambda|g'|\leqslant\overline{\lim_{n\to+\infty}}\frac{(n+1)^2}{n^2}\frac{1}{(n+1)^2}V_{1/(n+1)}|g'|.$$

Now, since $K=\sup_{\alpha>0}\|\alpha V_\alpha\|_p<+\infty$ (1.7), we obtain

$$\begin{aligned}\int\sum_{n\geqslant 0}\left(\left(\frac{1}{n+1}\right)^2 V_{1/(n+1)}|g'|\right)^p d\mu&=\sum_{n\geqslant 0}\int\left(\frac{1}{(n+1)^2}V_{1/(n+1)}|g'|\right)^p d\mu\\&\leqslant K^p\|g'\|_p^p\sum_{n\geqslant 0}\left(\frac{1}{n+1}\right)^p<+\infty.\end{aligned}$$

Therefore

$$\sum_{n\geqslant 0}\left(\frac{1}{(n+1)^2}V_{1/(n+1)}|g'|\right)^p<+\infty\qquad\text{a.e. on } X$$

and

$$\lim_{n\to+\infty}\frac{1}{(n+1)^2}V_{1/(n+1)}|g'|=0\qquad\text{a.e. on } X.$$

Finally

$$\lim_{\lambda\to 0^+}\lambda^2 V_\lambda g'=0\qquad\text{a.e. on } X$$

and

$$\lim_{\lambda\to 0^+}\lambda V_\lambda f=0\qquad\text{a.e. on } X \text{ if } f=(I-\alpha V_\alpha)(g).$$

Thus, $\lim_{\lambda\to 0^+}\lambda V_\lambda f$ exists a.e. on X for any f belonging to the dense subset $\operatorname{Inv} V\oplus(I-\alpha V_\alpha)(\beta)$.

Note that we have used above a well-known remark due to Akcoglu (see [3, p. 370]).

(VI) On an Example of Derrienic and Lin [1]

In [1], Derrienic and Lin construct an example of $L_1(\mathbb{N})$-positive operator induced by a transformation and which verifies

$$\sup_{\substack{n \geqslant 1 \\ n \in \mathbb{N}}} \left\| \frac{S_n}{n} \right\|_1 \leqslant 3$$

and $\sup_n \|T^n\|_1 = \infty$. Since $\|T\|_\infty \leqslant 1$, we also have

$$\sup_{\substack{n \geqslant 1 \\ n \in \mathbb{N}}} \left\| \frac{S_n}{n} \right\|_p \leqslant 3$$

(for any p: $1 \leqslant p \leqslant \infty$) and the construction of T shows that $\sup_n \|T^n\|_p \geqslant (2^k)^{1/p}$ for any $k \geqslant 1$ and any p: $1 \leqslant p < \infty$. Therefore we obtain $\sup_n \|T^n\|_p = \infty$ for any p: $1 \leqslant p < \infty$ with T positive and mean-bounded on $L_p(\mathbb{N})$.

Such an example cannot exist in a finite dimensional space because any mean-bounded positive matrix is necessarily power-bounded.

If θ is the transformation on $\mathbb{N}$ given in the previous example, define θ_t, $t \geqslant 0$ on $\mathbb{N} \times [0, 1[$ by $\theta_t(x, y) = (\theta^k x, y + t - k)$, where $k = [y + t]$; the semi-group $(T_t)_{t \geqslant 0}$ defined by $T_t f = f \circ \theta_t$ verifies $\sup_{t > 0} \|S_t/t\|_p \leqslant 3$ and $\sup_{t > 0} \|T_t\|_p = \infty$ $(1 \leqslant p < \infty)$.

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References

1. Y. Derrienic and M. Lin, On invariant measures and ergodic theorems for positive operators, *J. Funct. Anal.* **13** (1973), 252–267.
2. W. F. Eberlein, Abstract ergodic theorems and weakly almost periodic functions, *Trans. Amer. Math. Soc.* **67** (1949), 217–240.
3. A. Ionescu-Tulcea, Ergodic Properties of isometries in L^p spaces, *Bull. Amer. Soc.* **70** (1964), 366–371.
4. P. A. Meyer, "Probabilités et potentiels," Hermann, Paris.
5. D. Widder, "Introduction to Transform Theory," Academic Press, New York/London, 1971.