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Orthogonal Polynomials with a Constant Recursion Formula and an Application to Harmonic Analysis

JOEL M. COHEN*

Department of Mathematics, University of Maryland, College Park, Maryland 20742

AND

ALICE R. TRENHOLME

Department of Mathematics, George Mason University, Fairfax, Virginia 22030

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The purpose of this paper is to calculate the measure for which a sequence of polynomials with a constant recursion formula is orthogonal. The answer is well known in many special cases, since these sequences include the Chebychev polynomials of the first and second kinds. Many other special cases are discussed in [2]. Our method uses nothing more than the classical case of Chebychev polynomials together with the converse to the classical translation theorem of Christoffel. The continuous part of the measure (there may be one or two mass points) is given as a quotient—the numerator from the Chebychev case and the denominator given by the Christoffel theorem.

Our interest stems from the fact that a special case of these polynomials, introduced in [3] for the study of harmonic analysis on free groups, has turned out to be very useful and interesting. The works [4, 6, 8, 11] used [3] to further develop the study of radial functions on discrete groups. (The polynomials had appeared in a different but related context earlier [1, 9].) The measure corresponding to the polynomials is the Plancherel measure of the radial functions on the group. In the study of other interesting groups (e.g., $SL(2, Z)$), other special cases likewise yield interesting information [6, 11].

All of the results proved previously about such Plancherel measures now follow as special cases of the general theorem proved here. In addition, we shall prove a generalization of the results of [7] in Section 4. Section 4,

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which is the only section to discuss groups, depends only on Theorem 3, and is otherwise independent of the work in the rest of the paper.

In its most general form we study the following sequence of polynomials:

$$p_0(x) = c, \qquad p_1(x) = x - \alpha, \tag{$*$}$$

and

$$p_{n+1}(x) = (x-a)\,p_n(x) - bp_{n-1}(x) \qquad \text{for} \quad n > 1,$$

where a and α are arbitrary real numbers and b and $c > 0$.

Given $(*)$ the Favard theorem [5] says that there exists a compact positive measure μ on $\mathbb{R}$, unique up to constant multiple, such that $\int_{\mathbb{R}} p_n(x)\,p_m(x)\,d\mu(x) > 0$ if $n = m$ and is 0 otherwise. It is this measure we seek. We shall refer to μ as the measure induced by the p_n.

In discussing the polynomials, we find it convenient to assume that $p_n(x)$ has 1 or 2^n as leading coefficient for $n > 0$. Hence they are not necessarily ortho*normal*. Thus we do not have to distinguish between $\{p_n\}$ and $\{c_n\,p_n\}$ or between μ and $c\mu$, where $c > 0$ and the c_n are constants.

Notation. (1) $(a)_+ = (a + |a|)/2$.

(2) Since we never manipulate nonreal numbers we take $\sqrt{a}$ to mean $\sqrt{(a)_+}$ throughout.

(3) $\delta(p)$ is the unit measure concentrated at the point p.

We shall prove

THEOREM 3. *Assume $\{p_n\}$ satisfies $(*)$. Let $f(x) = (1-c)(x-a)^2 + (c-2)(\alpha-a)(x-a) + (\alpha-a)^2 + bc^2$. Then the continuous part of the measure $d\mu$ induced by the p_n is given by*

$$d\mu_c = \frac{\sqrt{4b-(x-a)^2}}{\pi f(x)}\,dx$$

and the discrete part $d\mu_d$ is 0 except possibly in the following two cases:

Case 1. f has two real roots $y_1 \neq y_2$. Then

$$d\mu_d = \lambda_1\,\delta(y_1) + \lambda_2\,\delta(y_2),$$

where

$$\lambda_i = \frac{2}{\sqrt{(\alpha-a)^2 + 4b(c-1)}}\left(\frac{bc}{|y_i-a|} - \frac{|y_i-a|}{c}\right)_+.$$

Case 2. $c = 1$ and $\alpha \neq a$ so that f has one root $y = \alpha + bc^2/(\alpha - a)$. Then

$$d\mu_d = \left(2 - \frac{2b}{(\alpha-a)^2}\right)_+ \delta(y).$$

1. Classical Results

We begin by recalling the Chebychev polynomials of the second kind, $U_n(x)$, given by $U_n(\cos\theta) = \sin(n+1)\,\theta/\sin\theta$. Trigonometric sum formulas yield $U_0(x) = 1$, $U_1(x) = 2x$, and $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$. The substitution $x = \cos\theta$, $\theta \in [0, \pi]$ yields easily that the $U_n(x)$ are actually ortho*normal* with respect to $(\sqrt{1-x^2}/\pi)\,dx$.

Note. [10] is a reference for everything in this section.

To describe the Christoffel theorem, we first introduce some notation. Assume we are given a sequence of polynomials $\{p_n(x)\}$ with $\deg p_n = n$. Let $f(x)$ be a polynomial of degree t with distinct roots $x_1,..., x_t$. Notice that for any n, $k(x_0) = \det((p_{n+i}(x_j)))_{i,j=0,\ldots,t}$ is a polynomial in x_0 of degree $\leqslant(n+t)$, and $k(x_i) = 0$ for $i = 1,..., t$. Thus $k(t)/f(t)$ is a polynomial of degree $\leqslant n$.

If $f(x)$ has multiple roots we change the construction of $k(x)$ as follows: assume that $x_s = \cdots = x_{s+m}$ is a root of multiplicity $(m+1)$. Then for $n \leqslant r \leqslant n+t$ we replace the redundant $p_r(x_s)$, $p_r(x_{s+1})$, $p_r(x_{s+m})$ with $p_r(x_s)$, $p_r'(x_s),..., p_r^{(m)}(x_s)$. Thus $k(x)$ has the property that $k(x_s) = k'(x_s) = \cdots = k^{(m)}(x_s) = 0$, whence $f(x)$ divides $k(x)$. We still have $k(x)/f(x)$ a polynomial of degree $\leqslant n$. We set $q_n(x) = c_n(k(x)/f(x))$, where $c_n \neq 0$ is a constant chosen to give $q_n(x)$ a convenient leading coefficient (1 or 2^n as discussed earlier).

With the construction as above we write $T_f\{p_n\} = \{q_n\}$ and say that the q_n are the Christoffel translates of the $p_n(x)$ under $f(x)$. If f is linear with the root r we write $T_{(r)}$ for T_f. The following is very well-known [10, p. 30]:

Theorem A (Christoffel). *If $\{p_n(x)\}$ induces the measure $d\mu(x)$ and $f(x)$ is a polynomial which is $\geqslant 0$ on the support of $d\mu$, then $T_f\{p_n\}$ induces the measure $f(x)\,d\mu(x)$.*

Notice that the support of $d\mu(x)$ is the support of $f(x)\,d\mu(x)$ together with a subset of the zeroes of $f(x)$. Thus we get the following corollary-converse which is the direction in which we use the Christoffel theorem.

Theorem B. *Assume $\{q_n\}$ is a set of orthogonal polynomials inducing the measure $dv(x)$ and $f(x)$ is a polynomial $\geqslant 0$ on the support of dv. If $\{q_n\} = T_f\{p_n\}$ then the $\{p_n\}$ induce the measure $d\mu(x) = (1/f(x))\,dv(x) + \sum_{f(r)=0, r \text{ real}} \lambda_r \delta(r)$, where the λ_r are nonnegative constants which can be calculated by using the fact that $\int p_n(x)\,d\mu(x) = 0$ for $n \geqslant 1$.*

Remarks. (1) If f is a constant, T_f will be assumed to be the identity.

(2) The construction f can be defined even if the $\{p_n\}$ are not orthogonal polynomials or if f is an arbitrary complex polynomial. Of course it may then make no sense to talk about an induced measure.

2. Composition of Christoffel Translates

THEOREM 1. *If f and g are polynomials, then $T_f \circ T_g = T_{fg}$.*

Proof. Let $\{p_n\}$ be a sequence of polynomials orthogonal with respect to the compact measure $\mu(x)$. Let $x_1,\dots,x_t$, $y_1,\dots,y_s$ be real numbers which are all less than any element in the support of μ. Let $f(x)=\prod_{i=1}^{t}(x-x_i)$, $g(x)=\prod_{j=1}^{s}(x-y_j)$. Then $T_g\{p_n\}$ induces $g(x)\,d\mu(x)$ and $T_f\{T_g\{p_n\}\}$ induces $f(x)\,g(x)\,d\mu(x)$, but so does $T_{fg}\{p_n\}$. Hence $(T_f\circ T_g)\{p_n\}=T_{fg}\{p_n\}$. Now the mth polynomial of $(T_f\circ T_g)\{p_n\}$ and $T_{fg}\{p_n\}$ is a polynomial in x whose coefficients are polynomials in $x_1,\dots,x_t$, $y_1,\dots,y_s$. Since they are equal for all $x_1,\dots,x_t$, $y_1,\dots,y_s$ real and small, they are equal as polynomials in $x_1,\dots,x_t$, $y_1,\dots,y_s$ and hence are equal for all values of $x_1,\dots,x_t$, $y_1,\dots,y_s$, hence for all polynomials f and g.

This proves that $T_f\circ T_g=T_{fg}$ when applied to sequences which are orthogonal on some interval, but since $T_f\circ T_g$ and T_{fg} are both purely finite algebraic constuctions, the fact that they are equal on all sets of orthogonal polynomials means that they are equal as constructions, hence on all sets of polynomials.

Remark. In fact the construction T_f can be applied to *any* sequence of C^∞ functions $\{p_n\}$ and then $T_f\circ T_g=T_{fg}$ (up to constant multiples, of course).

3. Proof of the Main Theorem

Let us first consider the following sequence: $S(c,\beta)$: $P_0(x)=c>0$, $P_1(x)=2x-2\beta$, and $P_{n+1}(x)=2xP_n(x)-P_{n-1}(x)$.

If $p_n(x)=2^{-n}P_n(x)$ then $\{p_n(x)\}$ is the sequence $(*)$ with $a=0$, $\alpha=\beta$, and $b=\frac{1}{4}$. Notice that $S(1,0)=\{U_n(x)\}$, the Chebychev polynomials of the second kind. Let $f(x)=(1-c)\,x^2+(c-2)\,\beta x+(\beta^2+(c^2/4))$.

PROPOSITION 1. *$T_{f(x)}S(c,\beta)=\{U_n(x)\}$ and so the measure induced by $S(c,\beta)$ is given by*

$$d\mu=\frac{\sqrt{1-x^2}\,dx}{\pi f(x)}+\sum_{y\text{ a real root}}\lambda_y\delta(y).$$

Proof. Let y be a (not necessarily real) root of $f(x)$ and $r=(2y-2\beta)/c$. Then r is a root of $g(x)=(1-c)\,x^2-2\beta x+1$. By induction we see that

$P_n(y) = r^n c$. A straightforward calculation then shows that $T_{(y)} S(c, \beta) = S(1, r(1-c)/2)$. (Remember that $T_{(y)}$ is defined as T_{x-y}.) In particular, if $\hat{\beta}$ is some number and $\hat{y} = \hat{\beta} + (1/4\hat{\beta})$, then $T_{(\hat{y})} S(1, \hat{\beta}) = S(1, 0) = \{U_n(x)\}$ which induces the measures $\sqrt{1-x^2}/\pi\, dx$. Thus applying Theorem B, we see that $S(1, \hat{\beta})$ induces the measure

$$\frac{\sqrt{1-x^2}}{\pi|x-y|}\, dx + \hat{\lambda}\delta(\hat{y}).$$

Since this completes the case $c = 1$, we now take $c \neq 1$. Let $\hat{\beta} = r(1-c)/2$, $\hat{y} = \hat{\beta} + (1/4\hat{\beta}) = r(1-c)/2 + 1/2r(1-c)$, and $\hat{r} = (2\hat{y} - 2\hat{\beta})/c = 1/r(1-c)$. Now $T_{(y)} S(c, \beta) = S(1, \hat{\beta})$ and $T_{(\hat{y})} S(1, \hat{\beta}) = S(1, 0)$. It is easy to show that $\hat{y}$ is the second root of $f(x)$ (and $\hat{r}$ is the second root of $g(x)$). Thus by Theorem 1, $T_{f(x)} = T_{(\hat{y})} T_{(y)}$ giving us $T_{f(x)} S(c, \beta) = S(1, 0)$, proving the proposition.

We are left now with the computation of the discrete part of the measure. (In particular, we are done in the case that there are no real roots, i.e., if $\beta^2 + c - 1 < 0$.)

As in the proof of Proposition 1, let us first do the case $S(1, \hat{\beta})$. Thus the measure is

$$d\hat{\mu} = \frac{\sqrt{1-x^2}}{\pi\,|x-\hat{y}|}\, dx + \hat{\lambda}\delta(\hat{y}), \qquad \hat{y} = \hat{\beta} + \frac{1}{4\hat{\beta}}.$$

We calculate $\hat{\lambda}$ by using the fact that $\int p_1(x)\, d\hat{\mu} = 0$. Then using the formulae

$$\frac{1}{\pi}\int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{1}{2} \quad \text{and} \quad \frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-x^2}}{a-x}\, dx = a - \varepsilon\sqrt{a^2-1}$$

for $|a| > 1$, where $\varepsilon = \operatorname{sign}(a)$, we find that

$$\hat{\lambda} = \left(2\,|\hat{\beta}| - \frac{1}{2\,|\hat{\beta}|}\right)_+ = \left(\frac{1}{|\hat{r}|} - |\hat{r}|\right)_+,$$

$\hat{r} = 2\hat{y} - 2\hat{\beta}$. Thus

PROPOSITION 2. *$S(1, \hat{\beta})$ induces the measure*

$$d\mu = \frac{\sqrt{1-x^2}}{\pi\,|x-\hat{y}|}\, dx + \left(\frac{1}{|\hat{r}|} - |\hat{r}|\right)_+ \delta(\hat{y}), \quad \textit{where } \hat{y} = \hat{\beta} + \frac{1}{4\hat{\beta}}$$

and $\hat{r} = 2\hat{y} - 2\hat{\beta}$.

Note. Dividing by $|\beta|$ yields the measure

$$\frac{\sqrt{1-x^2}}{\pi f(x)}\,dx+\left(2-\frac{1}{2\hat{\beta}^2}\right)_+\delta(\hat{y}).$$

For $|\hat{\beta}|\leqslant\frac{1}{2}$ the discrete part vanishes and as $\hat{\beta}\to 0, f(x)\to$ constant. Hence even at $\hat{\beta}=0$ we get the correct expression. We continue, as before, to study the case of $S(c,\beta)$, $c\neq 1$, letting r, y, $\hat{\beta}$, $\hat{r}$, and $\hat{y}$ be as in the proof of Proposition 1. Since $T_{(y)}S(c,\beta)=S(1,\hat{\beta})$ we get, again by Theorem B, that $S(c,\beta)$ induces

$$\frac{\sqrt{1-x^2}}{\pi\,|x-y||x-\hat{y}|}\,dx+\frac{((1/|\hat{r}|)-|\hat{r}|)_+\,\delta(\hat{y})}{|y-\hat{y}|}+\lambda\delta(y).$$

(Of course we have to assume that $y\neq\hat{y}$, i.e., $\beta^2\neq 1-c$.) But y was originally chosen simply as a root of $f(x)$, hence y and $\hat{y}$ play completely symmetric roles. Thus

$$\lambda=\frac{((1/|r|)-|r|)_+}{|y-\hat{y}|}.$$

Note that if $|r|\geqslant 1$ then $\lambda=0$. Dividing the measure by $(1-c)$ we get

$$\begin{aligned} d\mu={}&\frac{\sqrt{1-x^2}}{\pi f(x)}\,dx+\frac{1}{\sqrt{\beta^2+c-1}}\\ &+\sum_{\text{rel roots } y}\left(\frac{1}{|(2y-2\beta|}-\frac{|2y-2\beta|}{c}\right)_+\delta(y). \end{aligned}$$

Let us observe finally that the discrete part is 0 in the case of y a root of multiplicity 2. This is the case that $0<\beta^2=1-c<1$. So let us fix c, $0<c<1$, and observe that for β near $\pm\sqrt{1-c}$, r is approximately

$$\frac{\beta}{1-c}\sim\frac{\pm\sqrt{1-c}}{1-c}=\frac{\pm 1}{\sqrt{1-c}},$$

so $|r|>1$ and the discrete part is always 0 for β in a neghborhood of $\pm\sqrt{1-c}$. Thus by continuity of all the data and the process, $d\mu_d=0$ at $\beta^2=1-c$. Thus we get

THEOREM 2. *$S(c,\beta)$ induces the measure whose continuous part is given by*

$$d\mu_c=\frac{\sqrt{1-x^2}}{\pi f(x)}\,dx$$

for $f(x) = (1-c)x^2 + (c-2)\beta x + (\beta^2 + c^2/4)$ *and whose discrete part vanishes unless*

(a) *f has two real distinct roots, where*

$$d\mu_d = \frac{1}{\sqrt{\beta^2 + c - 1}} \sum_{\text{roots } y} \left(\frac{1}{|2y - 2\beta|} - \frac{|2y - 2\beta|}{c} \right)_+ \delta(y), \quad or$$

(b) $c = 1$ *whence* $d\mu_d = (2 - (1/2\beta^2))_+ \delta(y)$, *where* $f(y) = 0$.

Now finally we make a simple change of variables to get the general case. Assume $\{p_n(x)\}$ satisfies $(*)$. Then let

$$P_n(x) = \left(\frac{1}{\sqrt{b}} \right)^n p_n(2\sqrt{b}\, x + a).$$

Then it is easy to show that $\{P_n(x)\} = S(c, \beta)$, where $\beta = (\alpha - a)/2\sqrt{b}$. If $\{P_n(x)\}$ induces some measure $d\mu(x)$, then $S(c, \beta)$ induces $d\mu(2\sqrt{b}\, z + a)$ which we can calculate by Theorem 2. It has continuous part

$$\frac{\sqrt{1 - z^2}\, dz}{\pi\{(1-c) z^2 + (c - z)\beta z + \beta^2 + (c^2/4)\}}.$$

We make the substitution $z = (x - a)/2\sqrt{b}$, i.e., $x = 2\sqrt{b}\, z + a$. This gives us

$$d\mu_c = \frac{\sqrt{4b - (x-a)^2}\, dx}{\pi f(x)},$$

where $f(x) = (1-c)(x-a)^2 + (c-2)(\alpha - a)(x - a) + (\alpha - a)^2 + bc^2$. Similarly, to calculate the discrete part $d\mu_d$ we make the appropriate substitutions to get

$$\lambda = \frac{1}{\sqrt{(\alpha - a)^2 + 4b(c-1)}} \left(\frac{bc}{|y - \alpha|} - \frac{|y - \alpha|}{c} \right)_+$$

and $d\mu_d = \sum_{\text{real roots } \lambda} \lambda \delta(y)$ in the case of 2 different roots and

$$d\mu_d = \left(2 - \frac{2b}{(\alpha - a)^2} \right)_+ \delta(y)$$

in the case of $c = 1$. This completes the proof of our main theorem, Theorem 3, as stated in the introduction.

4. Applications to Harmonic Analysis on Discrete Groups

In this section we will show how to compute the Plancherel measure of radial elements in the reduced C^*-algebra, $C^*(G)$, of certain discrete groups G.

We consider a discrete group G on which some length function has been defined. The identity element is assumed to be the only element of length 0 and the length is assumed invariant under inversion. (This is equivalent to choosing a finite set of generators.) We consider the C^*-algebra $C^*(G)$ as operators on $L^2(G)$ under the action of convolution. Let X_n be the characteristic function of the set of elements of length n. X_n is self-adjoint. Let $f\colon C^*(G) \to C$ assign to any function its value at the identity and consider the functional F which assigns to any polynomial $p(x)$ the value $f(p(X_1))$. This functional yields a positive measure $d\mu$ which we call the Plancherel measure of the group with respect to the given set of generators.

Notice that $X_n * X_m = 0$ at the identity if $n \neq m$ and equals the number of elements of length n if $n = m$. Assume that we can find a sequence of polynomials such that $X_n = p_n(X_1)$. Then $F(p_n(x)\, p_m(x)) = 0$ for $m \neq n$ and is positive for $m = n$. So the $p_n(x)$ are exactly the orthogonal polynomials of the Plancherel measure $d\mu$. In the case of discrete groups, this measure plays a very special role, and its evaluation has proved to be a very important tool in the study of the harmonic analysis of G. [6], for example, contains a very thorough account of the uses of radial functions.

A well-studied example is that of the free group of t generators (cf. [1, 3, 6, 8, 9]). It is easily shown that the sequence of polynomials described above is given by $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2 - 2t$, and $p_{n+1}(x) = xp_n(x) - (2t-1)\, p_{n-1}(x)$. Without affecting the measure, we can change the first polynomial to $p_0(x) = 2t/(2t-1)$. Then $(*)$ is satisfied for $\alpha = a = 0$, $b = 2t - 1$, and $c = 2t/(2t-1)$. Applying Theorem 3 gives us the measure

$$\frac{\sqrt{4(2t-1) - x^2}}{\pi(2t - x^2)}\, dx.$$

This result appears in several earlier articles, calculated by various means.

Now let G be the free product $G_1 * G_2 * \cdots * G_k$ of k groups, each of the same finite order n. No assumption of commutativity is made nor are the groups assumed to be isomorphic to one another. Each element other than the identity can be written uniquely as $g = g_1 \cdots g_s$, where each g_i is in one of the G_j with no two consecutive g_i in the same G_j. We define the length of g, $|g|$, to be s. Then $X_0 = 1$ and $X_1 = \sum_{g \neq 1, g \in G_i} g$. We wish to study the X_m from the above point of view. Let $G_i' = G_i - \{1\}$. Let $Y_i = \sum_{g \in G_i'} g$. Thus $X_1 = \sum_i Y_i$, and $X_m = \sum_{i_j \neq i_{j+1}} Y_{i_1} Y_{i_1} \cdots Y_{i_m}$.

Notice that for $g \in G_i$, $g(Y_i+1) = Y_i + 1$. Thus $\sum_{g \in G_i'} gY_i = \sum_{G_i'} Y_i + (n-1) - g = (n-2)Y_i + (n-1)$. That is, $T_i Y_i = (n-2)\,Y_i + (n-1)$. This leads to the following formulas: $X_1^2 = \sum_{i \neq j} Y_i Y_j + \sum_i Y_i^2 = X_2 + \sum_i (n-2)\,Y_i + k(n-1) + X_2 = X_2 + (n-1)\,X_1 + k(n-1)$.

Let $X_{m,i} = \sum_{i_1 \neq i} Y_{i_1} \cdots Y_{i_m}$, $X_m = X_{m,i} + Y_i X_{m-1,i}$, $X_m = \sum_i Y_i X_{m-1,i}$, and $\sum_i X_{m,i} = (k-1)\,X_m$. So $Y_i X_m = Y_i X_{m,i} + Y_i^2 X_{m-1,i} = Y_i X_{m,i} + (n-2)\, Y_i X_{m-1,i} + (n-1)\,X_{m-1,i}$. Thus $X_1 X_m = \sum_i Y_i X_m = \sum Y_i X_m = \sum Y_i X_{m,i} + (n-2)\; Y_i X_{m-1,i} + (n-1)\,X_{m-1,i} = X_{m+1} + (n-2)\,X_m + (n-1)(k-1)\,X_{m-1}$. This is for $m \geqslant 2$.

We then have the equations $X_0 = 1$, $X_1 = X_1, X_2 = X_1^2 - (n-2)\,X_1 - k(n-1)$, and for $m \geqslant 2$, $X_{m+1} = X_1 X_m - (n-2)\,X_m - (k-1)(n-1)\,X_{m-1}$. Let $p_0 = k/(k-1)$, $p_1(x) = x$, and for $m \geqslant 1$ let $p_{m+1}(x) = (x-(n-2))\,p_m(x) - (k-1)(n-1)\,p_{m-1}(x)$. Then for $i > 1$, $p_i(X_1) = X_i$. These are the same polynomials that appear in [7], which is a study of the case $G_1 = G_2 = \cdots = G_k$, a cyclic group of order n.

So we can apply Theorem 3, with $\alpha = 0$, $a = (n-2)$, $b = (k-1)(n-1)$, and $c = k/(k-1)$. We get

$$f(x) = \frac{1}{(k-1)}\,(k+x)(k(n-1)-x)$$

which has the real roots $y_1 = -k$ and $y_2 = k(n-1)$. This puts us in Case 1 of Theorem 3 and we see that λ_2 always $=0$ and $\lambda_1 = 0$ if $k \geqslant n$. Precisely, we have the following:

THEOREM 4. *The Plancherel measure of X_1 is*

$$\frac{(k-1)\sqrt{(4k-1)(n-1)-(x-n+2)^2}}{\pi(k+x)(k(n-1)-x)}\,dx + \frac{2(n-k)_+}{n}\,\delta(-k).$$

The continuous spectrum is $[n-2-r,\; n-2+r]$, *where* $r = 2\sqrt{(k-1)(n-1)}$.

This study can be extended to one other case by a trick shown to us by Massimo Picardello: Let N be a finite group of order n and K a finite group of order k. The $H = N * K$ is an infinite group. Let $p\colon H \to K$ be the projection and let $G = \ker p$. Then G is the free product of the k groups, hNh^{-1}, hK, each group of order n. So the previous analysis may be applied to G. Since G is normal in H of index k, the harmonic analysis on G is more-or-less that on H.

As in [6] this then provides the means for getting for G a principal series and a complementary series of representations, and leads to a great deal of information about the harmonic analysis of such groups.

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