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Limits of Tame Automorphisms of $k[x_1,..., x_N]$

DAVID J. ANICK*

Department of Mathematics, University of California, Berkeley, California 94720

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Let J consist of all endomorphisms ψ of $S = k[x_1,..., x_N]$ whose Jacobian determinant, $\det(\partial\psi(x_i)/\partial x_j)$, is a non-zero constant. Let G be the group of tame automorphisms of S. In this paper we prove that G is dense in J in the "formal power series topology." In particular, every automorphism of S is a limit of tame automorphisms. We also demonstrate that certain subquotients of G have a natural Lie algebra structure.

INTRODUCTION

Let $N \geqslant 2$ be a fixed positive integer. Let k be any field of characteristic zero and let $S = k[x_1,..., x_N]$. Let $R = \mathrm{End}(S)$ be the set of all endomorphisms of S which leave k fixed. Any $f \in R$ is completely determined by $f(x_i)$, $1 \leqslant i \leqslant N$. By identifying f with $(f(x_1),..., f(x_N))$, we may view R as consisting of all N-vectors with components in S. R has a ring structure given by componentwise addition and multiplication.

Take each x_i to have degree one. Let P_n be the k-vector space of all N-vectors whose components are homogeneous nth degree polynomials in $\{x_1,..., x_N\}$. We may write any $f \in R$ as the sum of its homogeneous parts: $f = \sum_{n=0}^{M} f_{(n)}$, where $f_{(n)} \in P_n$. The *height* of f is defined by $\mathrm{ht}(f) = \inf\{n \mid f_{(n)} \neq 0\}$; $\mathrm{ht}(0) = \infty$. Let R_0 denote $\{f \in R \mid \mathrm{ht}(f) > 0\}$. R_0 is also the kernel of the augmentation $\varepsilon_0: R \to k^N$ given by $\varepsilon_0(f) = f(0)$. Note that $\mathrm{ht}(f) \geqslant n$ iff $f \equiv 0 \pmod{R_0^n}$.

Let G be the group of all tame automorphisms of S. G is generated by the invertible affine automorphisms, $L = \{f \in R \mid f_{(n)} = 0$ for $n > 1$ and f^{-1} exists$\}$ and by $\{f \in R \mid f(x_1) = x_1 + g(x_2,..., x_N),\ f(x_i) = x_i$ for $i > 1\}$. Although we view G as a subset of R, the group operation on G, which is function composition, bears little connection with the commutative multiplication in R. When $f, g \in R$, gf will denote the composition $g \circ f$ and $g \cdot f$ will denote the product in R. For scalar or matrix quantities, however, juxtaposition will denote multiplication.

* Current address: Department of Mathematics, MIT, Cambridge, Massachusetts 02139.

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R is given the R_0-adic topology. We could also define a metric on R by $d(f, g) = \exp(-\mathrm{ht}(f-g))$, which would induce the same topology. We call this the "formal power series topology." R has this topology when it is viewed as a subset of its R_0-adic completion, which is isomorphic to $(k[[x_1,..., x_N]])^N$. $f_n \to f$ in R iff $\mathrm{ht}(f_n - f) \to \infty$ iff for every m there is an n_0 such that $f_n \equiv f \pmod{R_0^m}$ for all $n \geqslant n_0$. G, L, and other subsets of R are given the subspace topology.

Henceforth, for any N-vector quantity, a subscript of i will designate the ith component. For $f \in R$, f_i denotes $f(x_i)$. In keeping with this, we let x be the vector $(x_1,..., x_N)$. $\{e_i\}_{1 \leqslant i \leqslant N}$ are the standard basis vectors of k^N. Let id be the identity of G, $\mathrm{id}(x) = x$.

Lastly, for $f \in R$, let f' denote the Jacobian of f, the N by N matrix $(\partial f_i/\partial x_j)$. (If k is not $\mathbb{R}$ or $\mathbb{C}$, $\partial/\partial x_j$ still denotes the unique derivation satisfying $\partial x_i/\partial x_j = \delta_{ij}$ and $\partial(\text{constant})/\partial x_j = 0$.) If $S = k[x_1,..., x_N]$ is also given the formal power series topology, the function $D: R \to S$ given by $D(\psi) = \det(\psi')$ is continuous.

Statement of Results

We are ready to state the main theorem of this paper.

THEOREM 1. *Let $J = \{\psi \in R \mid D(\psi)$ is a non-zero constant$\}$. Then J is closed and G is dense in J.*

In fact, we will prove a much stronger result. For $\psi \in R_0$ define the *rank* of ψ by $\mathrm{rk}(\psi) = \mathrm{ht}(\psi - \mathrm{id}) - 1$. An automorphism of rank n agrees with the identity in degrees $\leqslant n$. Let $R_n = \{\psi \in R_0 \mid \mathrm{rk}(\psi) \geqslant n\}$; this agrees with the old R_0 for $n = 0$. Each R_n is closed under composition. Let $G_n = G \cap R_n$ and $J_n = J \cap R_n$ for $n \geqslant 0$. Each G_n is a subgroup of G, and $D(\psi) = 1$ for $\psi \in J_n$ if $n \geqslant 1$. It is well known that $G \subseteq J$, hence $G_n \subseteq J_n$.

Let $B \subseteq G_0$ be the subgroup generated by $L_0 = L \cap R_0$ and w, where $w(x) = x + x_2^2 e_1$. Let A be the subgroup $B \cap G_1$ and let A_n be the nth term in the lower central series for A, starting with $A_1 = A$. Then we will prove

THEOREM 2. *$\bar{A}_n = J_n$ for $n \geqslant 1$, where $\bar{A}_n$ is the closure of A_n.*

Let us first see how our main theorem will be a corollary of this result.

Proof of Theorem 1 from Theorem 2. $k^* = k - \{0\}$ is a closed subset of $S = k[x_1,..., x_N]$ in the formal power series topology. Since D is continuous, $J = D^{-1}(k^*)$ is also closed.

Let $\psi \in J$ and let $n > 0$ be arbitrary. We wish to find $\theta \in G$ such that $\mathrm{ht}(\psi - \theta) \geqslant n$. ψ may be written uniquely as $\tau\rho$, where τ is a translation and

$\rho \in J_0$ with $D(\rho) = D(\psi) = c \in k^*$. $\rho_{(1)}$ is invertible because $D(\rho_{(1)}) = c$; let $\sigma = \rho\rho_{(1)}^{-1}$. Then $\mathrm{rk}(\sigma) \geqslant 1$, so by Theorem 2, we may choose $\phi \in A_1 \subseteq G$ with $\mathrm{ht}(\phi - \sigma) \geqslant n$. Let $\theta = \tau\phi\rho_{(1)}$. $\mathrm{Ht}(\theta - \psi) = \mathrm{ht}(\phi\rho_{(1)} - \rho) = \mathrm{ht}(\phi - \sigma) \geqslant n$, as desired.

The last equality is a special case of the following lemma, of which we will make frequent use.

LEMMA 1. *If* $\mathrm{ht}(\psi - \phi) \geqslant n$, then $\mathrm{ht}(\psi\theta - \phi\theta) \geqslant n$ *for any* $\theta \in R_0$.

Proof. This is clear since $(\psi\theta - \phi\theta)(x) = (\psi - \phi)(\theta(x))$ is a sum of terms of degree n and higher only.

We postpone the proof of Theorem 2 until after we have developed some more theory.

THE STRUCTURE OF G_n/G_{n+1}

We prove next a series of lemmas concerning the behavior of functions of rank n under composition. Recall that $\psi \in R_{n-1}$, $\mathrm{rk}(\psi) \geqslant n-1$, $\psi \equiv \mathrm{id} (\mathrm{mod}\, R_0^n)$, and $\psi \equiv \mathrm{id} + \psi_{(n)} (\mathrm{mod}\, R_0^{n+1})$ are all equivalent for $n > 1$.

LEMMA 2. *Let* $m \geqslant 2$, $n \geqslant 2$. *If* $\mathrm{rk}(f) \geqslant 1$, $\mathrm{rk}(g) \geqslant m-1$, *and* $\mathrm{ht}(h) \geqslant n$, *then*

$$(g)(f+h) \equiv gf + h + g'_{(m)} h_{(n)} \qquad (\mathrm{mod}\, R_0^{m+n}) \tag{1}$$

$$(g)(f+h) \equiv gf + h \qquad (\mathrm{mod}\, R_0^{m+n-1}). \tag{2}$$

Proof. The ith component of $(g)(f+h)$ may be written as the sum of a series (a finite series because g is a polynomial):

$$\begin{aligned} &((g)(f(x)+h(x)))_i \\ &\quad = g_i(f(x)+h(x)) \\ &\quad = g_i(f(x)) + \sum_{p_1=1}^{N} \frac{\partial g_i}{\partial x_{p_1}}(f(x))\, h_{p_1} \\ &\qquad + \sum_{j\geqslant 2} \left(\sum_{p_1,\ldots,p_j} \left(\frac{\partial^j g_i}{\partial x_{p_1} \cdots \partial x_{p_j}} \right) (f(x))\, h_{p_1} \cdots h_{p_j} \right). \end{aligned}$$

Since $\mathrm{ht}(h) \geqslant n$ and $f \in R_0$, the jth term of this series has height $\geqslant m - j + nj$ for $j \geqslant 2$. For $j \geqslant 2$ and $n \geqslant 2$, $m - j + nj = m + n + (n-1)(j-1) - 1 \geqslant m + n$. It follows that

$$(g)(f+h)(x) \equiv g(f(x)) + g'(f(x))\, h(x) \qquad (\mathrm{mod}\, R_0^{m+n}) \tag{3}$$

Because $\mathrm{rk}(g) \geqslant m-1$ and $\mathrm{ht}(h) \geqslant n$,

$$g'(f(x))\, h(x) = \sum_{s \geqslant n} \left(h_{(s)}(x) + \sum_{t \geqslant m} g'_{(t)}(f(x))\, h_{(s)}(x) \right).$$

The term $g'_{(t)}(f(x))\, h_{(s)}(x)$ has height $\geqslant t-1+s$, which is smaller than $m+n$ only when $t=m$ and $s=n$. Hence

$$\begin{aligned} g'(f(x))\, h(x) &\equiv \sum_{s \geqslant n} h_{(s)}(x) + g'_{(m)}(f(x))\, h_{(n)}(x) \\ &\equiv h(x) + g'_{(m)}(x)\, h_{(n)}(x) \qquad (\mathrm{mod}\, R_0^{m+n}). \end{aligned}$$

Plugging this into (3) gives the desired result (1). Equation (2) then follows from (1) by observing that $g'_{(m)} h_{(n)}$ has degree $m+n-1$.

LEMMA 3. (a) *If* $\mathrm{rk}(\psi) \geqslant n-1 \geqslant 1$ *and* $\mathrm{rk}(\phi) \geqslant m-1 \geqslant 1$, *then*

$$\phi\psi \equiv \psi\phi \equiv \psi + \phi - \mathrm{id} \qquad (\mathrm{mod}\, R_0^{n+m-1}).$$

(b) *If* $\phi \in G_1$ *and* $\psi \equiv \phi$ $(\mathrm{mod}\, R_0^n)$, *then*

$$\phi^{-1}\psi \equiv \psi\phi^{-1} \equiv \psi - \phi + \mathrm{id} \qquad (\mathrm{mod}\, R_0^{n+1}).$$

Proof. (a) Apply Lemma 2 with $g=\phi$, $f=\mathrm{id}$, and $h=\psi-\mathrm{id}$. Then $\phi\psi \equiv g+h = \phi+\psi-\mathrm{id}$ $(\mathrm{mod}\, R_0^{m+n-1})$. A symmetrical argument works for $\psi\phi$.

(b) By Lemma 1, $\psi \equiv \phi$ $(\mathrm{mod}\, R_0^n)$ yields $\psi\phi^{-1} \equiv \mathrm{id}$ $(\mathrm{mod}\, R_0^n)$. By part (a) we get $\psi = (\psi\phi^{-1})\,\phi \equiv \psi\phi^{-1} + \phi - \mathrm{id}$ $(\mathrm{mod}\, R_0^{n+1})$. Solve this for $\psi\phi^{-1}$. For $\phi^{-1}\psi$, let $g=\phi^{-1}$, $f=\phi$, and $h=\psi-\phi$ in Lemma 2. We get $\phi^{-1}\psi = \phi^{-1}(\phi + (\psi-\phi)) \equiv \phi^{-1}\phi + (\psi-\phi) = \mathrm{id} + \psi - \phi$ $(\mathrm{mod}\, R_0^{n+1})$.

LEMMA 4. *Suppose* $\psi \in G_m$, $\phi \in G_n$ *for* $m \geqslant 1$, $n \geqslant 1$, *and let* $\theta = \psi^{-1}\phi^{-1}\psi\phi$. *Then* $\theta \in G_{m+n}$ *and* $\theta_{(m+n+1)} = \psi'_{(m+1)}\phi_{(n+1)} - \phi'_{(n+1)}\psi_{(m+1)}$.

Proof. Let $\alpha = \psi_{(m+1)}$, $\beta = \phi_{(n+1)}$. Setting $g=\psi$, $f=\mathrm{id}$, and $h=\psi-\mathrm{id}$ in lemma 2 gives $\psi\phi \equiv \psi + \phi - \mathrm{id} + \alpha'\beta$ $(\mathrm{mod}\, R_0^{m+n+2})$. Switching ψ and ϕ gives $\phi\psi \equiv \psi + \phi - \mathrm{id} + \beta'\alpha$ $(\mathrm{mod}\, R_0^{m+n+2})$. By Lemma 3(b) we obtain $\theta = (\phi\psi)^{-1}(\psi\phi) \equiv \mathrm{id} + \alpha'\beta - \beta'\alpha$ $(\mathrm{mod}\, R_0^{m+n+2})$, which is equivalent to the desired result.

This lemma shows that there is a strong connection between the lower central series for G_1 and the groups $G_n = G_1 \cap R_n$. We are particularly interested in this connection for the subgroup $A \subseteq G_1$. Recall that A_n is defined to be the nth term in the lower central series for A, i.e., $A_1 = A$ and $A_n = [A, A_{n-1}]$.

LEMMA 5. $A_n \subseteq G_n$ for $n \geqslant 1$.

Proof. By induction on n. $f \in A = A_1$ implies $f \in G_1$. A_n is generated by $\{f^{-1}g^{-1}fg \mid f \in A, g \in A_{n-1}\} \subseteq \{f^{-1}g^{-1}fg \mid f \in G_1, g \in G_{n-1}\} \subseteq G_n$.

Since $G_n \subseteq J_n$, one corollary of Lemma 5 is that $A_n \subseteq J_n$. This is a step toward our goal (Theorem 2) of showing that $\bar{A}_n = J_n$.

LEMMA 6. *Let $f \in G_n$, $n \geqslant 1$, and let $c \in k$. There exists $g \in G_n$ such that $g_{(n+1)} = cf_{(n+1)}$.*

Proof. Let $K = \{c \in k \mid \forall f \in G_n, \exists g \in G_n$ such that $g_{(n+1)} = cf_{(n+1)}\}$. Taking $g = \mathrm{id}$ gives $0 \in K$. If g, $h \in G_n$ and $g_{(n+1)} = cf_{(n+1)}$ and $h_{(n+1)} = df_{(n+1)}$, then by Lemma 3(b), $(gh^{-1})_{(n+1)} = (c-d)f_{(n+1)}$, so K is an abelian subgroup of k. K is closed under multiplication and $1 \in K$, so K is a subring of k and $\mathbb{Z} \subseteq K$. If λ_c denotes multiplication by c (i.e., $\lambda_c(x) = cx$), then $\lambda_{c^{-1}} f \lambda_c \in G_n$ for $f \in G_n$. $(\lambda_{c^{-1}} f \lambda_c)_{(n+1)} = c^n f_{(n+1)}$, so K contains all nth powers of elements of k. Since $\mathrm{char}(k) = 0$, $\mathbb{Q} \subseteq k$, so for any rational r/s, $r/s = (rs^{n-1})(s^{-1})^n \in K$, i.e., $\mathbb{Q} \subseteq K$. Finally, note that any $c \in k$ may be written as a linear combination of c^n, $(c+1)^n, \ldots, (c+n)^n$ with rational coefficients, hence $K = k$.

PROPOSITION 1. *Let $H_n = \{f_{(n+1)} \mid f \in G_n\}$. Then H_n is a k-module and is isomorphic as an abelian group to G_n/G_{n+1}.*

Proof. By Lemma 4, G_{n+1} contains all commutators of elements of G_n, so G_n/G_{n+1} is abelian. By lemma 3(b) it is clear that the correspondence $\Phi: G_n/G_{n+1} \to H_n$, given by $\Phi(\bar{f}) = f_{(n+1)}$, is well-defined, one-to-one, and onto. Also by Lemma 3, Φ is an isomorphism of abelian groups. By Lemma 6, H_n is closed under scalar multiplication and it is trivial to check the axioms for a vector space. Indeed, H_n is a submodule of P_{n+1}.

Lemma 4 suggests that a much stronger structure exists on the $\{H_n\}$. A *connected graded Lie algebra over* k is a collection $F = \{F_n\}_{n \geqslant 1}$ of k-modules, together with a bilinear operation $[\,,\,]: F_m \otimes_k F_n \to F_{m+n}$ which satisfies the Jacobi identities:

$$[f, g] = -[g, f], \tag{4}$$

$$[[f, g], h] + [[g, h], f] + [[h, f], g] = 0. \tag{5}$$

PROPOSITION 2. *For $\alpha \in H_n$, $\beta \in H_m$, define $[\alpha, \beta] = \alpha'\beta - \beta'\alpha$. Then $[\,,\,]$ makes $H = \{H_n\}_{n \geqslant 1}$ into a connected graded Lie algebra over k.*

Proof. $[\,,\,]$ is bilinear, the dimensions work, and (4) holds. To check (5), let $\alpha \in H_n$, $\beta \in H_m$, $\gamma \in H_p$. Observe that $(\alpha'\beta)'\gamma = \alpha''(\beta, \gamma) + \alpha'\beta'\gamma$, where $\alpha''(\beta, \gamma)_i = \sum_{s=1}^{N} \sum_{t=1}^{N} (\partial^2 \alpha_i / \partial x_s \partial x_t) \beta_s \gamma_t = \alpha''(\gamma, \beta)_i$. Hence $\alpha''(\beta, \gamma) = \alpha''(\gamma, \beta)$. Likewise for $(\beta'\gamma)'\alpha$, etc. With this simplifying tool we obtain

$$\begin{aligned}
&[[\alpha,\beta],\gamma]+[[\beta,\gamma],\alpha]+[[\gamma,\alpha],\beta]\\
&\quad=\alpha''(\beta,\gamma)+\alpha'\beta'\gamma-\beta''(\alpha,\gamma)-\beta'\alpha'\gamma-\gamma'\alpha'\beta+\gamma'\beta'\alpha\\
&\qquad+\beta''(\gamma,\alpha)+\beta'\gamma'\alpha-\gamma''(\beta,\alpha)-\gamma'\beta'\alpha-\alpha'\beta'\gamma+\alpha'\gamma'\beta\\
&\qquad+\gamma''(\alpha,\beta)+\gamma'\alpha'\beta-\alpha''(\gamma,\beta)-\alpha'\gamma'\beta-\beta'\gamma'\alpha+\beta'\alpha'\gamma=0.
\end{aligned}$$

THEOREM 3. *Define a k-module structure on G_n/G_{n+1} by $c\cdot\bar{f}=\Phi^{-1}(c\cdot\Phi(\bar{f}))$, where $\Phi\colon G_n/G_{n+1}\to H_n$ is the isomorphism of Proposition* 1. *Define $[\,,\,]\colon(G_n/G_{n+1})\times(G_m/G_{m+1})\to(G_{m+n}/G_{m+n+1})$ by $[\bar{f},\bar{g}]=\overline{f^{-1}g^{-1}fg}$. Then $\{G_n/G_{n+1}\}_{n\geqslant 1}$ is a connected graded Lie algebra over k.*

Proof. This follows immediately from Proposition 1, Proposition 2, and Lemma 4.

A BASIS FOR H_n

We now return to the proof of Theorem 2. Briefly, our plan is as follows. Because each $H_n=\{f_{(n+1)}\mid f\in G_n\}$ is a finite-dimensional vector space over k, it is possible to write down an explicit basis. Using this basis we will show that we get the same vector space when G_n is replaced by J_n or by A_n. This will lead naturally to a proof of Theorem 2. The reader is warned that, because of the explicit calculational nature of these results, their demonstrations are rather technical and some straightforward but lengthy computations have been omitted.

For $n\geqslant 2$, let $V_n=\{\psi_{(n)}\mid\psi\in J_{n-1}\}$. Then $H_{n-1}\subseteq V_n$. Let $W_n=\{\alpha\in P_n\mid\nabla\alpha=0\}$, where $\nabla\alpha=\sum_{i=1}^N\partial\alpha_i/\partial x_i$.

PROPOSITION 3. *For $n\geqslant 2$, $V_n\subseteq W_n$.*

Proof. Of course $V_n\subseteq P_n$. For $\psi\in J_{n-1}$, $D(\psi)=1$, so in particular, the homogeneous degree $n-1$ part of the polynomial $\det(\psi')$ is zero. We see that this is just $\nabla\psi_{(n)}$. Every entry of $\psi'-I$ has height $\geqslant n-1$. The only term contributing to the determinant in degrees smaller than $2(n-1)$ is the product of the entries on the main diagonal. In degree $n-1$, this product contributes precisely $\nabla\psi_{(n)}$.

Let $T_n=\{a=(a_1,\ldots,a_N)\mid$each a_i is a non-negative integer and $\sum_{i=1}^N a_i=n\}$. For $a\in T_n$, let x^a denote the monomial $x_1^{a_1}\cdots x_N^{a_N}$.

LEMMA 7. *Let $n\geqslant 2$. A basis for W_n is $Y_n=\{x^ae_i\mid a\in T_n, a_i=0\}\cup\{b_i(a)\mid a\in T_{n-1}, 1<i\leqslant N\}$, where $b_i(a)=-(a_i+1)\,x^{a+e_1}e_1+(a_1+1)\,x^{a+e_i}e_i$.*

Proof. This set is linearly independent. To see this, note that the monomials $\{x^a e_i \mid a \in T_n,\ 1 \leqslant i \leqslant N\}$ are independent in P_n and that each of these monomials appears in exactly one element of Y_n. Thus no non-trivial combination of the Y_n may be zero without also contradicting the independence of $\{x^a e_i\}$.

It is also an easy check that $Y_n \subseteq W_n$, so it suffices to show that Y_n spans W_n.

Let $\alpha \in W_n$. For each i, write

$$\alpha_i = \sum_{\substack{a \in T_n \\ a_i = 0}} c_{a,i} x^a + \sum_{a \in T_{n-1}} d_{a,i} x^{a+e_i}.$$

$$\frac{\partial \alpha_i}{\partial x_i} = \sum_{a \in T_{n-1}} d_{a,i}(a_i + 1)\, x^a.$$

Since $\nabla \alpha = 0$, we have $\sum_{i=1}^N d_{a,i}(a_i + 1) = 0$ for each $a \in T_{n-1}$. Solving for $d_{a,1}$ gives

$$d_{a,1} = -\sum_{i=2}^{N} (a_1 + 1)^{-1} d_{a,i}(a_i + 1). \tag{6}$$

Let

$$\beta = \sum_{i=1}^{N} \sum_{\substack{a \in T_n \\ a_i = 0}} c_{a,i} x^a e_i + \sum_{i=2}^{N} \sum_{a \in T_{n-1}} (a_1 + 1)^{-1} d_{a,i} b_i(a) \in \operatorname{Span}(Y_n).$$

We assert that $\alpha = \beta$. First note that $\alpha_i = \beta_i$ for $i > 1$. As to the first components,

$$\begin{aligned}
\beta_1 &= \sum_{\substack{a \in T_n \\ a_1 = 0}} c_{a,1} x^a - \sum_{i=2}^{N} \sum_{a \in T_{n-1}} (a_1 + 1)^{-1} d_{a,i}(a_i + 1)\, x^{a+e_1} \\
&= \sum_{\substack{a \in T_n \\ a_1 = 0}} c_{a,1} x^a + \sum_{a \in T_{n-1}} \left(-\sum_{i=2}^{N} (a_1 + 1)^{-1} d_{a,i}(a_i + 1) \right) x^{a+e_1} \\
&= \sum_{\substack{a \in T_n \\ a_1 = 0}} c_{a,1} x^a + \sum_{a \in T_{n-1}} d_{a,1} x^{a+e_1} = \alpha_1 \quad \text{(by (6))}.
\end{aligned}$$

Thus $\alpha = \beta \in \operatorname{Span}(Y_n)$.

Recall that the subgroup A of G is defined by $A = B \cap G_1$, where B is generated by $w(x) = x + x_2^2 e_1$ and the linear automorphisms L_0. A is closed under conjugation by elements of L_0. To certain automorphisms we give specific names. For $i \neq j$, $v_{ij}(x) = x + x_j e_i$, i.e., v_{ij} adds the jth component to the ith. For $i \neq j$, τ_{ij} switches coordinates i and j. λ_c is scalar multiplication by c. Last, let $w_{ij}(x) = x + x_j^2 e_i$ for $i \neq j$. Since $w = w_{12} \in A$ and we may conjugate by any τ_{ij}, any $w_{ij} \in A$.

For $n \geqslant 2$, let $U_n = \{\psi_{(n)} | \psi \in A_{n-1}\}$. We have already observed that this is a subset of H_{n-1} and hence a subset of V_n.

PROPOSITION 4. $W_2 = V_2 = U_2$.

Proof. Since $U_2 \subseteq V_2 \subseteq W_2$, it is enough to show that $W_2 \subseteq U_2$. As in the proof of Lemma 6, $(\lambda_{c^{-1}} \psi \lambda_c \phi)_{(2)} = c\psi_{(2)} + \phi_{(2)}$ for $\psi, \phi \in A$ and $c \in k^*$, so U_2 is a vector space over k. It suffices to show that U_2 contains the basis Y_2. Y_2 may be divided into the five subsets $\{x_i^2 e_m | m \neq i\}$; $\{x_i x_j e_m | i \neq j \neq m \neq i\}$; $\{b_i(e_1) | i > 1\}$; $\{b_i(e_i) | i > 1\}$; and $\{b_i(e_j) | 1 < j \neq i > 1\}$. We do a case by case consideration. For $i \neq j$, let $\mu_{ij} = \lambda_2 w_{ij}^2 v_{ij} w_{ji}^{-1} v_{ij}^{-2} w_{ji} v_{ij} \lambda_{1/2}$. $\mu_{ij} \in A$ and $\mu_{ij}(x) \equiv x - x_i^2 e_i + 2x_i x_j e_j \pmod{R_0^3}$.

Case 1. We have already seen that $w_{mi} \in A$.

Case 2. $(\lambda_2 w_{mj}^{-1} w_{mi}^{-1} v_{ij}^{-1} w_{mi} v_{ij} \lambda_{1/2})(x) = x + x_i x_j e_m$.

Case 3. $\mu_{1i}(x) \equiv x + b_i(e_1) \pmod{R_0^3}$.

Case 4. $\mu_{i1}^{-1}(x) \equiv x + b_i(e_i) \pmod{R_0^3}$.

Case 5. $(\lambda_2 \mu_{j1}^{-1} \mu_{ji} \lambda_{1/2})(x) \equiv x + b_i(e_j) \pmod{R_0^3}$.

PROPOSITION 5. $W_n = V_n = U_n$ *for* $n \geqslant 2$.

Proof. Since $U_n \subseteq V_n \subseteq W_n$, it is enough to show that $W_n \subseteq U_n$. We proceed by induction on n. Assuming that $W_n = U_n$, we show that $W_{n+1} \subseteq U_{n+1} = \{\psi_{(n+1)} | \psi \in [A, A_{n-1}]\}$. By Lemma 4 and Proposition 4, it suffices to show that any $\theta \in W_{n+1}$ is a sum of the expressions $[\alpha, \beta] = \alpha'\beta - \beta'\alpha$ for $\alpha \in W_2$ and $\beta \in W_n$. The set of finite sums of $[\alpha, \beta]$ for $\alpha \in W_2, \beta \in W_n$, is a k-module because W_2 is. It is consequently enough to demonstrate that the basis elements Y_{n+1} may be written in this form. Again we do a case-by-case consideration.

The basis elements $x^a e_i$ for $a \in T_{n+1}$, $a_i = 0$, are covered by one of the following:

Case 1. $i = 1$. Pick j such that $a_j > 0$. Then $[b_j(e_j), \; x^{a - e_j} e_1] = (a_j + 1) x^a e_1$.

Case 2. $i > 1$ and $a_1 > 0$. Then $[b_i(e_1), x^{a - e_1} e_i] = (a_1 + 1) x^a e_i$.

Case 3. $i > 1$ and $a_1 = 0$. Pick j such that $a_j > 0$. Then $[b_i(e_j), x^{a-e_i}e_i] = x^a e_i$.

The basis elements $b_i(a)$, $a \in T_n$, $i > 1$, are covered by one or more of the following:

Case 4. $a_1 = 0$. Then $[b_i(e_1), x^a e_1] = 2b_i(a)$.

Case 5. $a_i = 0$. Then $[b_i(e_i), x^a e_i] = 2b_i(a)$.

Case 6. $a_1 > 0$ and $a_1 \neq 2a_i + 2$. Then $[b_i(e_1), b_i(a - e_1)] = (a_1 - 2a_i - 2)\, b_i(a)$.

Case 7. $a_i > 0$ and $a_i \neq 2a_1 + 2$. Then $[b_i(e_i), b_i(a - e_i)] = (2a_1 - a_i + 2)\, b_i(a)$.

We are at last able to complete the proof of the main theorem.

Proof of Theorem 2. $J_n = R_n \cap D^{-1}(1)$ is closed and $A_n \subseteq J_n$, so it is enough to show that $J_n \subseteq \bar{A}_n$.

Given $\psi \in J_m$ and $M > m$, we must find $\phi \in A_m$ with $\mathrm{ht}(\psi - \phi) > M$. Use induction on $M - m$. If $M - m = 1$, this is the claim that $\psi_{(m+1)}$ occurs among $\{\phi_{(m+1)} | \phi \in A_m\}$, which is just what Proposition 5 tells us. Suppose the result is true when $M - m = j$ and let $M = m + j + 1$. Use the inductive assumption to choose $\theta \in A_m$ with $\mathrm{ht}(\psi - \theta) \geqslant M$. By Lemma 1, $\mathrm{ht}(\psi\theta^{-1} - \mathrm{id}) \geqslant M$, i.e. $rk(\psi\theta^{-1}) \geqslant M - 1$. $\psi\theta^{-1} \in J_{M-1}$ so by Proposition 5 there is some $\sigma \in A_{M-1} \subseteq A_m$ with $\sigma_{(M)} = (\psi\theta^{-1})_{(M)}$. $\mathrm{Ht}(\psi\theta^{-1} - \sigma) > M$ and $\mathrm{ht}(\psi - \sigma\theta) > M$ with $\sigma\theta \in A_m$, as desired.

Conclusion

We have shown that any element of J is a limit of elements of G. In fact we have shown much more, namely, that J is the closure of a rather sparsely generated subgroup of G.

The main theorem enables us to recast two major outstanding problems in ring theory into topological terms. The tame automorphism problem, i.e., whether all automorphisms of S are tame, becomes the question of whether or not G is closed in $\mathrm{Aut}(S)$. The Jacobian determinant problem, i.e., whether $D(\psi) = c \in k^*$ implies $\psi \in \mathrm{Aut}(S)$, becomes the question of whether or not limits of invertible endomorphisms are invertible.

A few words are in order concerning the subgroup A of G_1. Is A a proper subgroup of G_1? If not, we can hardly justify using A_n in Theorem 2 instead of the conceptually simpler G_n. When $N = 2$ we have a canonical way of writing any element of G_1 as a product of "shear" automorphisms and τ_{12}'s [6], and this decomposition shows that A is indeed much smaller than G_1.

For $N = 2$, A consists of all automorphisms of rank one such that f_1 and f_2 have (highest) degrees which are powers of two. However, no such decomposition is known for $N > 2$ and, at this writing, it is unknown whether $A = G_1$ for $N > 2$. Determining which automorphisms belong to A seems to be as hard as the tame automorphism problem itself. The case $N = 2$ illustrates that it is possible for the closure of a group of automorphisms to be strictly larger than the group itself. It also shows that very little of the full "strength" of G is needed in order to approximate any element of J.

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