The ignition problem for a scalar nonconvex combustion model

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Abstract

The ignition problem for the scalar Chapman–Jouguet combustion model without convexity is considered. Under the pointwise and global entropy conditions, we constructively obtain the existence and uniqueness of the solution and show that the unburnt state is stable (unstable) when the binding energy is small (large), which is the desired property for a combustion model. The transitions between deflagration and detonation are shown, which do not appear in the convex case.

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1. Introduction

In Lagrangian coordinates, the simplest Chapman–Jouguet combustion model [2,5] is expressed as

\[
\begin{align*}
(u + q)_t + f(u)_x &= 0, \\
q(x, t) &= \begin{cases} 
q(x, 0), & \sup_{0 \leq \tau \leq t} u(x, \tau) \leq u_i, \\
0, & \text{otherwise},
\end{cases}
\end{align*}
\]

(1)
where \( u \) is a lumped quantity representing density, velocity, pressure or temperature and \( q \), which denotes the binding energy of the reactive gas, equals a constant \( q_0 > 0 \) for unburnt gas and zero for burnt gas. The combustible gas is inviscid and has an infinite rate of reaction or, equivalently, the infinitely thin reaction region, which implies that a gas particle releases all of its binding energy once its temperature exceeds \( u_i \) (ignition temperature). Though this model is physically simplified and does not incorporate all types of combustion waves, its theory is capable of describing a rich variety of wave phenomena including nonlinear stability and instability of reaction fronts.

The simplest combustion model (1) has been studied since 1979 [2,5]. In 1984, Ying and Teng [7] solved the Riemann problem for the Zeldovich–von Neumann–Döring (ZND) model

\[
\begin{align*}
(u + q)_t + f(u)_x &= 0, \\
q_t &= -k\varphi(u)q,
\end{align*}
\]

where \( k \) is the rate of reaction for combustible gas, and \( \varphi(u) \) is the Heaviside function: \( \varphi(u) = 0 \) as \( u \leq u_i \), \( \varphi(u) = 1 \) as \( u > u_i \). Furthermore, they obtained the limit of the Riemann solution as \( k \) tends to infinity and found that the limit function is a solution of the Riemann problem for (1). Based on the work of Ying and Teng [7], Liu and Zhang [4] summarized a set of entropy conditions, with which the existence and uniqueness of the Riemann solution for (1) can be obtained constructively. The results aforementioned were all obtained under the assumption that the flux function \( f(u) \) is strictly convex.

Since a genuine two-dimensional conservation law must be nonconvex in certain directions [1,10], it is interesting to investigate a scalar combustion model with a nonconvex flux \( f(u) \), which is the indispensable preparation for the study of multidimensional combustion problems. There is another motivation to study the nonconvex model (1). A well-known phenomenon in combustion theory is the transition from deflagration to detonation. However, this phenomenon cannot occur in the convex case because detonation and deflagration waves cannot propagate in the same direction (forward or backward). While in the nonconvex case, the phenomenon can be observed [6].


\[
\begin{align*}
(u + q)_t + f(u)_x &= 0, \\
q_t &= -\frac{k}{7}\varphi(u)q,
\end{align*}
\]

as the rate of reaction goes to infinity. However, through the study of the structure stability of combustion solutions, Sheng and Zhang [6] found their entropy conditions are incomplete since some cases were not included in their discussion. They made a modification to these entropy conditions and constructed the Riemann solutions for (1) uniquely.

For reactive gas flow, it is very interesting to study the nonlinear stability and instability of flows with combustion waves. In particular, by the study of ignition problem, our model (1) exhibits instability for unburnt states if the binding energy is sufficiently large. In the present
paper, our attention is focused on the nonconvex case. For simplicity, we assume \( f(u) \) is the simplest nonconvex function, i.e.,

\[
 f(u) \text{ has only one inflection point } \tilde{u}, \quad \text{and} \quad f'(\pm \infty) = +\infty. \tag{A}
\]

The case for \( f'(\pm \infty) = -\infty \) can be treated similarly without substantial difficulties. Under the entropy conditions in [6], solutions of the ignition problem for (1) can be constructed uniquely. We can observe the ignition and termination of combustion waves and the transitions between deflagration and detonation. For the CJ gas dynamic combustion, the transition from deflagration to detonation has been investigated in [9].

This paper is organized as follows. In Section 2, some preliminaries containing elementary waves and the pointwise and global entropy conditions are presented. Then in Section 3, the ignition problem for (1) are solved constructively, provided that \( f(u) \) satisfies the above assumption (A). The transitions of some different kinds of combustion waves are shown in the solutions.

2. Preliminaries

We begin by considering the ignition problem for (1) with initial data

\[
 (u, q)(x, 0) = \begin{cases}
 (u_-, q_-), & -\infty < x < -\varepsilon, \\
 (\tilde{u}, 0), & -\varepsilon < x < \varepsilon, \\
 (u_+, q_+), & \varepsilon < x < +\infty,
\end{cases}
\]

where \( u_- = u_+ := u_0 \leq u_i, \quad q_- = q_+ := q_0 > 0, \quad \tilde{u} > u_i \) and \( \varepsilon > 0 \) is small. The state \((\tilde{u}, 0)\) is viewed as an ignited state through small energy input, and the data (4) as a perturbation of the unburnt state \((u_0, q_0)\). Obviously, we should seek piecewise smooth solution \((u, q)(x, t)\).

It is easy to show that \( q(x, t) \) is piecewise constant, 0 or \( q_0 \). Smooth solutions \( u(x, t) \) are, besides the constant state \( u \equiv \) constant, rarefaction waves (abbr. R).

At jumps \( \frac{dx}{dt} = \sigma \), Rankine–Hugoniot condition

\[
 \sigma = \frac{[f]}{[u + q]} \tag{5}
\]

must be true, where \([f] = f(u_r) - f(u_l), \quad u_l = u(\sigma - 0), \quad u_r = u(\sigma + 0), \) etc.

The following three kinds of noncombustion discontinuities are admissible.

1. \([q] = 0, [u] \neq 0 \Rightarrow \sigma = \frac{[f]}{[q]}, \) it is a generalized shock (abbr. S);
2. \([q] \neq 0, [u] = 0 \Rightarrow \sigma = 0, \) it is a contact jump (abbr. J);
3. \([q] \neq 0, [u] \neq 0, \sigma = 0, \) it is a combination of S and J (abbr. SJ), in which S and SJ satisfy the generalized Lax entropy condition

\[
 \frac{f(u) - f(u_l)}{u - u_l} \geq \frac{f(u_r) - f(u_l)}{u_r - u_l} \quad \text{for } (u - u_l)(u - u_r) \leq 0. \tag{6}
\]

We next investigate the combustion wave, which has nonzero speed \( \sigma \neq 0 \), and across which \( q \) jumps from \( q_0 \) to zero. Let \( u_l \) and \( u_r \) be the limit values of \( u \) in the combustion wave front and
wave back respectively, i.e., \( q_l > 0 = q_r \) and \( u_l \leq u_i < u_r \), which implies \( \sigma = \frac{f(u_r) - f(u_l)}{u_r - (u_l + q_0)} < 0 \). Then the following six kinds of combustion waves satisfying the pointwise entropy conditions [6] are admissible.

2.1. Pointwise entropy conditions

a. If there exists a \( u_R \in [u_l, u_r) \), such for all \( u \in (u_l, u_r) \),

\[
\sigma = \frac{f(u_R) - f(u_l)}{u_R - (u_l + q_0)} = \frac{f(u_R) - f(u_l)}{u_R - u_l} \leq \frac{f(u) - f(u_l)}{u - u_l},
\]

the discontinuity \( \sigma \) is called deflagration. Furthermore, it can be divided into three subcases:

1. \( f'(u_l) = \sigma < f'(u_r) \): CJ deflagration (abbr. CJDF);
2. \( f'(u_l) > \sigma < f'(u_r) \): weak deflagration (abbr. WDF);
3. \( f'(u_l) = \sigma = f'(u_r) \): double contact combustion (abbr. DCC).

b. If there exists a \( u_R \in [u_r, +\infty) \) satisfying (7) for \( u \in (u_l, u_R) \), \( \sigma \) is called detonation. Also, it can be divided into three subcases:

4. \( f'(u_l) > \sigma = f'(u_r) \): CJ detonation (abbr. CJDT);
5. \( f'(u_l) > \sigma > f'(u_r) \): strong detonation (abbr. SDT);
6. \( f'(u_l) = \sigma > f'(u_r) \): contact detonation (abbr. CDT).

We call R, S, J, SJ, CJDF, WDF, DCC, CJDT, SDT and CDT elementary waves for (1) without convexity.

The aforementioned entropy conditions are not sufficient to guarantee the uniqueness and structure stability of the Riemann solutions for (1). Hence, global entropy conditions [6] are needed.

2.2. Global entropy conditions

If the Riemann problem for (1) has several solutions, we choose the one which satisfies the following two conditions:

a. The propagation speed of the interface between unburnt and burnt states is as low as possible.

b. If \( U = \{ u \mid \exists u < u_i, \text{s.t.} \ f(u) = f(u + q_0) \text{ and} \ f'(u + q_0) f'(u_i) > 0 \} \neq \emptyset \), take

\[
u_l = \max\{ u \mid u \in U \}.
\]

For the case that \( q_l = 0 < q_r \), \( u_l > u_i \geq u_r \), the pointwise entropy conditions for combustion waves can be easily defined by means of transformation \( \tilde{x} = -x \), \( \tilde{f} = -f \). Then in the next section, we show that the ignition problem subject to the above entropy conditions can be solved uniquely.
3. Solutions of the ignition problem

When discussing the ignition problem, we face the question of determining whether the burning process persist indefinitely or terminate in finite time or in other words, whether the unburnt state \((u_0, q_0)\) is stable or not. We will deal with this problem case by case along with constructing the solution.

By the assumption (A), there are two possibilities: \(f'(\tilde{u}) < 0\) or \(f'(\tilde{u}) \geq 0\). The latter can be treated as the special case of the former. Therefore, we suppose \(f'(\tilde{u}) < 0\) in the following without loss of generality. From \(f'(\tilde{u}) < 0\) and \(f'(\pm\infty) = +\infty\), we know that there exist \(u_1\) and \(u_2\) such that \(f'(u_1) = f'(u_2) = 0\) where \(u_1 > \tilde{u} > u_2\). Let \(u_3, u_4\) satisfy \(f(u_1) = f(u_3), f(u_2) = f(u_4)\), respectively (Fig. 1). Then in order to cover all the cases, our discussion should be divided into four parts: A. \(u_0 \in (-\infty, u_3]\); B. \(u_0 \in (u_3, u_2]\); C. \(u_0 \in (u_2, \tilde{u})\); D. \(u_0 \in [\tilde{u}, +\infty)\). For part D, \(f(u)\) is convex when \(u \in (\tilde{u}, +\infty)\). Hence the ignition problem becomes the convex case. In the following, we focus our attention mainly on part B and one subcase in part C for the reason that the others can be treated similarly to them.

3.1. Construction of solutions in part B: \(u_0 \in (u_3, u_2]\)

In general, the structure of the solution depends on the position of ignition point besides the values of \(u_0\) and \(q_0\). Therefore, with \(u_0 \in (u_3, u_2]\) in mind, we have three cases according to \(u_i\): 1. \(u_i \in (u_0, u_2]\); 2. \(u_i \in (u_2, u_1]\); 3. \(u_i \in [u_1, +\infty)\), which will be discussed with characteristic method in great detail. For convenience, in the following figures, we denote \((f(u_\pm), u_\pm)\) as \((\pm)\), \((f(u_\pm), u_\pm + q_0)\) as \((\pm^*)\), \((f(u_i), u_i)\) as \((i)\) in \((f, u)\) plane, and etc.

Case 1. \(u_i \in (u_0, u_2]\). For this case, there must be combustion solution of the Riemann problem at \((\epsilon, 0)\). Whether combustion waves emerge from \((-\epsilon, 0)\) or not depends on the value of \(\tilde{u}\). Through \((u_\pm, f(u_\pm))\), draw a vertical line that intersects \(f = f(u)\) at \(u = \tilde{u}_\pm (\tilde{u}_\pm < u_1)\). Then there are two subcases according to the value of \(\tilde{u}\).

Case 1.1. \(\tilde{u} \in (u_i, \tilde{u}_\pm]\). The Riemann solution corresponding to no reaction may exist at \((-\epsilon, 0)\). It consists of a contact discontinuity for \(q\) and a shock wave for \(u\) jumping from \(u_0\) to \(\tilde{u}\). The kind of combustion waves emanating from \((\epsilon, 0)\) is determined by the value of \(q_0\). Without loss of generality, we assume the existence of the common tangent point of \(f(u)\) and \(f(u - q)\) in this paper. Namely, there are such \(u_c \in (u_\pm, u_i)\) and \(q^* > 0\) that \(f'(u_c) = f'(u_c + q^*)\). Here we denote \(\hat{u}_c = u_c + q^*\). Then according to the assumption (A), \(u_m > \hat{u}_c\) and positive number \(q^{**}\) exist satisfying \(f'(u_m) = f'(u_\pm) = \frac{f(u_m) - f(u_\pm)}{u_m - (u_\pm + q^{**})}\) (Fig. 2).

Subcase 1.1.1. \(0 < q_0 < q^*\). In this case, the Riemann solution at \((\epsilon, 0)\) is

\[
(u, q) = \begin{cases} 
(U(\xi), 0), & -\infty < \xi \leq f'(u_i), \\
((f')^{-1}(\xi), q_0), & f'(u_i) < \xi \leq f'(u_+), \\
(u_+, 0), & f'(u_+) < \xi < +\infty,
\end{cases}
\]

where \(U(\xi)\) is a solution of

\[
\begin{align*}
-\xi u_\xi + f(u)u_\xi &= 0, \\
u(+\infty) &= u_+, \\
u(-\infty) &= u_{m_1},
\end{align*}
\]
in which $u_{m1}$ is defined by $f'(u_i) = f(u_{m1}) - f(u_i)$ and $\xi = \frac{x - \varepsilon}{t}$. Obviously the combustion wave here is CJDF: $x - \varepsilon = f'(u_i)t$. Drawing lines with slope $f'(u_i)$ through $(u_{\pm}, f(u_{\pm}))$, $(\tilde{u}_{\pm}, f(\tilde{u}_{\pm}))$, we have $q_1, q_2 > 0$ satisfying (Fig. 2)

$$f'(u_i) = \frac{f(u_{\pm}) - f(u_i)}{u_{\pm} - (u_i + q_1)} = \frac{f(\tilde{u}_{\pm}) - f(u_i)}{\tilde{u}_{\pm} - (u_i + q_2)}.$$ 

When $q_0$ is small ($0 < q_0 < q_1$), the shock wave from $(-\varepsilon, 0)$ penetrates the centered wave that lies behind CJDF in finite time. Then $S$ goes on to catch up with the deflagration wave and extinguishes it before canceling the rarefaction wave ahead of CJDF. The time-asymptotic state consists of two contact discontinuities $J$ separating the burnt and unburnt gases (Fig. 3). When $q_1 \leq q_0 \leq q_2$, the two shock waves intersect and then unify into a new shock propagating with speed $w(u_{m1}, u_-) = \frac{f(u_{m1}) - f(u_-)}{u_{m1} - u_-}$. Since $w(u_{m1}, u_-) < f'(u_i)$, the shock cannot overtake the deflagration wave, which means they both survive (Fig. 4).
For $q_2 < q_0 < q^*$, we have the following fact.

**Lemma 1.** When $q_0 > q_2$, the unburnt gas on the left-hand side will be ignited with a WDF by a shock $x = x(t)$ at finite time $t_0$ (see Figs. 5–7).

**Proof.** The interaction of the two shock waves from $(-\varepsilon, 0)$ and $(\varepsilon, 0)$ results a new shock wave $x = x(t)$, which can be determined by

\[
\begin{cases}
\frac{dx}{dt} = \frac{f(u) - f(u_-)}{u - u_-}, \\
\frac{x - \varepsilon}{t} = f'(u), \\
x(\hat{t}) = \bar{x},
\end{cases}
\]

(10)

where $\hat{t}$ is the time when the two shocks overtake each other, $u_{m2} \in (\bar{u}, u_{m1})$ satisfies

\[
f'(u_{m2}) = \frac{f(u_{m2}) - f(\hat{u})}{u_{m2} - \hat{u}}.
\]

Because $u_{m1} > \bar{u}_\pm$, it holds that $\frac{dx}{dt} < 0$ at some time $t > \hat{t}$. From

\[
\frac{d^2x}{dt^2} = \frac{f'(u) - (f(u) - f(u_-))/(u - u_-)}{u - u_-} < 0,
\]

we get that the shock wave must strike the unburnt gas ($x = -\varepsilon$) at some time $t_0 > \hat{t}$.

The temperature on the back of the shock is higher than the ignition temperature $u_\dagger$. For the instability of reactive gas flow and according to the entropy condition, this gives rise to a deflagration wave WDF: $x + \varepsilon = f'(u_5)(t - t_0)$ with the state $(u_{m3}, 0)$ behind. Here $u_5 \in (\bar{u}, u_1)$ and $u_{m3} > u_5$ satisfy

\[
f'(u_5) = \frac{f(u_5) - f(u_\pm)}{u_5 - u_\pm} = \frac{f(u_{m3}) - f(u_-)}{u_{m3} - (u_- + q_0)}.
\]

At this moment, the CJDF from $(\varepsilon, 0)$ persists since it is faster than the shock wave following the WDF. Thus the solution indefinitely contains two deflagration waves (Fig. 5).
Fig. 5. $q_2 < q_0 < q^*$. 

Fig. 6. $q^* \leq q_0 < q^{**}$. 

Fig. 7. $q^{**} \leq q_0$. 
Subcase 1.1.2. $q_0 \geq q^*$. Similarly to the case $q_2 < q_0 < q^*$, we have two combustion waves eventually in this case. However, the combustion wave emanating from $(\varepsilon, 0)$ is different from the previous one. When $q^* \leq q_0 < q^{**}$, a degenerate combustion wave DCC: $x - \varepsilon = f'(\tilde{u}_i)t$ occurs at $(\varepsilon, 0)$, in which $\tilde{u}_i \in (u_\pm, u_c]$ satisfying $f'(\tilde{u}_i) = f'(u_{m_2}) = f(u_{m_1} - u_{i} + q_0)$. (Fig. 6).

Instead, we have a detonation wave CJDT: $x - \varepsilon = f'(u_{m_3})t$, $f'(u_{m_3}) = f(u_{m_5} - u_{i} + q_0)$, for large $q_0$ ($q_0 \geq q^{**}$) (Fig. 7). They both survive since it is impossible for the shock $x = x(t)$ to cross the rarefaction wave behind the DCC and CJDT completely. It is easy to find that $\frac{dx}{dt} \rightarrow f'(u_{m_3})$ as $t \rightarrow \infty$.

Case 1.2. $\hat{u} \in (\tilde{u}_\pm, +\infty)$. This case is similar to Case 1.1 except that combustion wave forms at $(-\varepsilon, 0)$ in the beginning. We assume here $\hat{u} \in (\tilde{u}_\pm, u_5)$ without loss of generality. Let $q_2, q_3 > 0$ satisfy

$$f'(u_3) = \frac{f(\hat{u}) - f(u_\pm)}{\hat{u} - (u_\pm + q_3)} = \frac{f(u_{m_1} - u_{i} + q_0)}{u_{m_1} - (u_\pm + q_3)} = \frac{f(u_{m_1} - u_{i} + q_0)}{u_{m_3} - (u_{i} + q_0)},$$

where $\tilde{u}_m \in (u_2, \tilde{u}_\pm)$. The other symbols have the same meanings as above (Fig. 8).

The Riemann problem at $(-\varepsilon, 0)$ is resolved into a SDT when $q_0 \in (0, q_2]$, which is represented as $\frac{x+\varepsilon}{t} = \frac{f(\hat{u}) - f(u_\pm)}{\hat{u} - (u_\pm + q_0)}$. The SDT is terminated at some time and then a new forward shock wave forms, which goes on to penetrate the rarefaction (or shock) wave. Finally, the CJDF from $(\varepsilon, 0)$ is extinguished by the shock in the case $q_0 < q_1$, and it survives in the other case $q_1 \leq q_0 \leq q_2$ (Figs. 9–10).

When $q_0 > q_2$, the structure of the Riemann solution at $(-\varepsilon, 0)$ can be denoted by $(u_0, q_0) + WDF + (u_{m_3}, 0) + S + (\hat{u}, 0)$, where “+” means “following” and WDF: $x + \varepsilon = f'(u_3)t$. At last the solution is $(u_0, q_0) + J + (u_0, 0) + S + (u_{m_1}, 0) + CJDF + R + (u_0, q_0)$ in the case $q_2 < q_0 < q_3$ (Fig. 11), while the WDF and the combustion wave from $(\varepsilon, 0)$ will go to infinity in the case $q_0 \geq q_3$.

In a word, the combustion wave emanating from $(-\varepsilon, 0)$ is exterminated for $q_0 < q_3$, while persists for $q_0 \geq q_3$. Here we just give the case $q_3 \leq q_0 < q^*$, in which we have two deflagration waves eventually (Fig. 12). For strong binding energy, persistence of WDF and DCC ($q^* \leq q_0 < q^{**}$) or CJDT ($q^{**} \leq q_0$) are omitted here, since they can be depicted similarly to Figs. 6 and 7.
Fig. 9. $0 < q_0 < q_1$.

Fig. 10. $q_1 \leq q_0 \leq q_2$.

Fig. 11. $q_2 < q_0 < q_3$. 
So far we have finished the discussion of Case 1. It turns out that the unburnt state is stable if and only if the binding energy is small, namely, a perturbation would yield the time-asymptotic state close to \((u_0, q_0)\) with no combustion wave for \(0 < q_0 < q_1\). Note that as \(u_i\) approaches \(u_2\), \(q_1\) tends to infinity and so \((u_0, q_0)\) becomes stable since \(q_0 < q_1\) is satisfied.

**Case 2.** \(u_i \in (u_2, u_1)\). There are three subcases in this part: 1. \(\hat{u} \in (u_i, \tilde{u}_\pm]\); 2. \(\hat{u} \in (\tilde{u}_\pm, u_4]\); 3. \(\hat{u} \in (u_4, +\infty)\).

**Case 2.1.** \(\hat{u} \in (u_i, \tilde{u}_\pm]\). It can be treated as the convex case. Obviously there is no combustion wave and the solution tends to two contact discontinuities \(Js\) issuing from \((-\epsilon, 0)\) and \((\epsilon, 0)\). Details are omitted (see [4]).

In the following two subcases, we take into account \(u_i \in (u_2, u_5]\).
Case 2.2. \( \hat{u} \in (\tilde{u}_{\pm}, u_4) \). Clearly, there exists \( u^* \in (u_2, \tilde{u}) \) such that \( f'(u^*) = f'(u_5) \) from the assumption (A). Then through \((u^*, f(u^*))\) and \((u_4, f(u_4))\), draw lines with slope \( f'(u_5) \) that intersect the vertical line from \((u_\pm, f(u_\pm))\) at \( u = u_\pm + q_1 \) and \( u = u_\pm + q_2 \), respectively (Fig. 13).

We only discuss the case \( \hat{u} \in [u_5, u_4] \). For \( \hat{u} \in (\tilde{u}_{\pm}, u_5) \), the solution on the left is similar to Case 1.2 and the right is the same as we discuss here.

The Riemann problem at \((-\epsilon, 0)\) is resolved into a deflagration wave WDF and no reaction waves (R or S). At \((\epsilon, 0)\), the Riemann solution without reaction occurs, which contains a shock for \( u \) jumping from \( \hat{u} \) to \( u_{m_2} \). It is easy to see that for small binding energy \((q_0 < q_1)\), WDF is extinguished indefinitely due to its interaction with the shock wave having \( x - \epsilon = f'(u_{m_3}) t \) as its asymptote when \( t \to \infty \). Here \( u_{m_3} \in (u_2, \tilde{u}) \) satisfies \( f'(u_{m_3}) = f(u_{m_1}) - f(u_{m_3}) \), where \( u_{m_1} > u_5 \) is defined by \( f'(u_5) = \frac{f(u_{m_1}) - f(u_-)}{u_{m_1} - (u_- + q_0)} \) (Fig. 14). When \( q_0 \geq q_1 \), we have the persistence of the deflagration wave (Figs. 15–16).
Especially when the binding energy is strong enough ($q_0 > q_2$), the unburnt state on the right-hand side will be ignited with a SDT, which has the following properties.

**Lemma 2.** The detonation wave SDT penetrates the centered waves on two sides and has an asymptotic line $\frac{x - \varepsilon}{t - t_1} = f'(u_{m_4})$. Here $t_1$ is the time when the shock intersect the contact discontinuity from $(\varepsilon, 0)$ and $u_{m_4}$ satisfies $f'(u_{m_4}) = \frac{f(u_{\pm}) - f(u_{m_4})}{u_{m_4} - (u_2 + q_0)}$ (see Fig. 16).

**Proof.** In fact, the SDT: $x = x(t)$ is determined by

$$
\begin{align*}
\frac{dx}{dt} &= \left( f(u_r) - f(u_r) \right) \frac{u_l - (u_r + q_0)}{u_l - (u_r + q_0)}, \\
\frac{x - \varepsilon}{t} &= f'(u_r), \quad u_r \in [u_+, u_2], \\
\frac{x - \varepsilon}{t - t_1} &= f'(u_l), \quad u_l \in [u_{m_1}, u_{m_5}], \\
x(t_1) &= \varepsilon,
\end{align*}
$$

(11)

where $u_{m_5}$ satisfies $f'(u_{m_5}) = \frac{f(u_{m_5}) - f(u_2)}{u_{m_5} - (u_2 + q_0)}$.

Differentiating (11) with respect to $t$ along $x = x(t)$, one obtains

$$
(u_r + q_0 - u_l) \frac{d^2x}{dt^2} = \left( f'(u_r) \right) \frac{dx}{dt} \frac{du_r}{dt} - \left( f'(u_l) \right) \frac{dx}{dt} \frac{du_l}{dt},
$$

$$
t f''(u_r) \frac{du_r}{dt} = \frac{dx}{dt} - f'(u_r),
$$

$$
(t - t_1) f''(u_l) \frac{du_l}{dt} = \frac{dx}{dt} - f'(u_l).
$$
Substituting \( \frac{dx}{dt} = f'(u_l) \) into the above expressions since we know \( f'(u_r) < \frac{dx}{dt} = f'(u_l) \) at \((\varepsilon, t_1)\), it follows that
\[
\frac{d^2x}{dt^2} = \frac{[f'(u_r) - f'(u_l)]^2}{tf''(u_r)[u_l - (u_r + q_0)]} < 0, \quad t = t_1.
\]

Thus we have \( f'(u_r) < \frac{dx}{dt} < f'(u_l) \) for \( t = t_1 + 0 \). Further, we can prove that it always satisfies \( \frac{d^2x}{dt^2} < 0 \). Assuming to the contrary that \( \frac{d^2x}{dt^2} = 0 \) at some time, we have
\[
\left( f'(u_r) - \frac{dx}{dt} \right)^2 \frac{1}{tf''(u_r)} - \left( f'(u_l) - \frac{dx}{dt} \right)^2 \frac{1}{(t - t_1)f''(u_l)} = 0.
\]

It is impossible due to \( f''(u_r) < 0, \ f''(u_l) > 0 \).

By a similar argument, it can be verified that \( f'(u_r) < \frac{dx}{dt} < f'(u_l) \) always holds and the SDT penetrates the rarefaction waves on two sides.

Moreover, it can be shown that \( x = x(t) \) does not intersect with the characteristic \( \frac{x - \varepsilon}{t - t_1} = f'(u_{m4}) \). Suppose it does. Then \( \frac{dx}{dt} \leq f'(u_{m4}) \) must hold at the first intersection point, which implies
\[
\frac{f(u_{m4}) - f(u_r)}{u_{m4} - (u_r + q_0)} \leq f'(u_{m4}), \quad u_+ \leq u_r \leq u_2.
\]

This is obviously impossible. Therefore \( x = x(t) \) will intersect with the characteristic \( \frac{x - \varepsilon}{t - t_1} = f'(u_+) \), and it has \( \frac{x - \varepsilon}{t - t_1} = f'(u_{m4}) \) as asymptote when \( t \to \infty \), which can be proved easily. \( \square \)

**Case 2.3.** \( \hat{u} \in (u_4, +\infty) \). Firstly, we take \( q_1 > 0, u_{i_1} \in (u_\pm, u_2) \), satisfying \( f'(u_{i_1}) = \frac{f(u_\pm) - f(u_{i_1})}{u_\pm - (u_{i_1} + q_1)} \), and \( q_2 > 0, \ u_{i_2} \in (u_\pm, u_2), \hat{u}_m \in (u_2, u_{\hat{u}}) \), satisfying \( f'(u_5) = \frac{f(\hat{u}_m) - f(\hat{u})}{\hat{u}_m - (u_\pm + q_2)} \), \( f'(u_{i_2}) = \frac{f(\hat{u}_m) - f(u_{i_2})}{\hat{u}_m - (u_{i_2} + q_2)} \). For simplicity and without loss of generality, we take \( \hat{u} \) satisfying \( f'(u_\pm) = \frac{f(\hat{u}) - f(u_{\hat{u}})}{\hat{u} - (u_\pm + q_1)} \) (Fig. 17).

Similarly to Case 2.2, the Riemann solution at \((-\varepsilon, 0)\) is made up of \((u_0, q_0) + WDF + (u_{m4}, 0) + R + (\hat{u}, 0)\), in which the deflagration wave is extinguished when \( q_0 < q_2 \) (Figs. 18–19) and survives otherwise (Figs. 20–23). The structure of the solution at \((\varepsilon, 0)\) depends on \( q_0 \). When the binding energy is small \((0 < q_0 < q_1)\), detonation wave CDT appears in the beginning, which propagates with the speed \( f'(\bar{u}_i) = \frac{f(\hat{u}) - f(\bar{u}_i)}{\hat{u} - (\bar{u}_i + q_0)} \), \( \bar{u}_i \in (u_\pm, u_2) \) until the R from \((-\varepsilon, 0)\) catches up with it. Then the CDT slows down and finally dies out when its speed goes to zero (Fig. 18).

For large binding energy \( q_0 \geq q_1 \), \((\hat{u}, 0)\) and \((u_+, q_0)\) are connected by a strong detonation: \( \frac{x - \varepsilon}{t} = \frac{f(\hat{u}) - f(u_\pm)}{\hat{u} - (u_\pm + q_0)} \), which decelerates during the penetration of R. Especially when \( q_1 \leq q_0 < q^* \), the transition from SDT to CDT which finally transfers to be a CJDF for \( q_1 \leq q_0 < q^* \), can be observed. In all, there is a surviving deflagration \((q_1 \leq q_0 < q^*)\) or detonation \((q^* \leq q_0)\) wave on the right-hand side (Figs. 19–23).
Here $u_{m1}, u_{m3}$ satisfy the same representations as in Case 2.2. The intersection point of $f(u)$ and $f(u - q_0)$ ($q_1 \leq q_0 < q^*$, $u_\pm < u < u_2$) is denoted by $(u_{m4}, f(u_{m4}))$. And $u_{m5}, u_{m6}, u_{m7}$ are so defined that

$$f'(u_{m4} - q_0) = \frac{f(u_{m4}) - f(u_{m5})}{u_{m4} - u_{m5}}, \quad f'(u_{m6}) = f'(\tilde{u}_i) = \frac{f(u_{m6}) - f(\tilde{u}_i)}{u_{m6} - (\tilde{u}_i + q_0)},$$

$$f'(u_{m7}) = \frac{f(u_{m7}) - f(u_\pm)}{u_{m7} - (u_\pm + q_0)},$$

where $u_{m5} \in (u_2, \tilde{u}_c)$, $\tilde{u}_i \in (u_\pm, u_c)$.

Considering the case $u_i \in (u_5, u_1)$, we have the appearance of CJDF: $x + \varepsilon = f'(u_i)t$ from $(-\varepsilon, 0)$ instead of the WDF if $\hat{u}$ is greater than $u_5$. It can be discussed similarly to Cases 2.2 and 2.3. Thus in Case 2, the unburnt state $(u_0, q_0)$ is stable if the perturbation $\hat{u} \in (u_i, \tilde{u}_\pm)$ or the binding energy is small. We note that as $u_0$ is close to $u_2$, the unburnt state becomes unstable.
Fig. 19. $q_1 \leq q_0 < q_2$.

Fig. 20. $q_2 \leq q_0 < q^*$.  

Fig. 21. $q^* \leq q_0 < q^{**}$.  

Fig. 22. $q^{**} \leq q_0 < q_3$.

Fig. 23. $q_3 \leq q_0$: $f'(\hat{u}) = (f(\hat{u}) - f(u_\pm))/(\hat{u} - (u_\pm + q_3))$.

**Case 3.** $u_i \in [u_1, +\infty)$. It is easy to check that the ignition problem has the unique noncombustion solution when $\hat{u} \in (u_i, u_4)$. For $\hat{u} \in (u_4, +\infty)$, it is similar to Case 2.3. The difference is that there is no combustion wave issuing from $(-\varepsilon, 0)$ and the unburnt gas on the left-hand side will never be ignited. Details are omitted.

**3.2. Construction of solutions in part C: $u_0 \in (u_2, \hat{u})$**

In this section, we only deal with one subcase in part C, i.e.,

$u_0 \in (u_2, \hat{u}), \quad u_i \in (u_0, u^*) \quad \text{and} \quad \hat{u} \in [u_5, u_6],$

where $u^*, u_5$ have the same representations as before and $(u_6, f(u_6))$ is the intersection point of $f(u)$ ($u > u_0$) with the vertical line through $(u_\pm, f(u_\pm))$. Let $q_1, q_2, q_3 > 0$ be so defined that $f'(u_i) = f(u_i)/u_i - (u_\pm + q_1) > 0$, $f'(u^*) = f(u^*/u_i - (u_\pm + q_3))$ and $u_6 = u_\pm + q_3$. It is evident that $q_3 > q_2 > q_1$ (Fig. 24). When $q_1 \leq q_0 < q_2$, this case demonstrates the persistence of SDT transformed from WDF on the left-hand side, which does not occur in part B. So we should take into account this case as a supplement to part B.

Without loss of generality, we take $\hat{u}$ satisfying $f'(u_i) = f(\hat{u}) - f(u_i)/u_i - u_i$. Similarly to Case 2.2, it is easy to find the existence of the Riemann solution without combustion wave at $(\varepsilon, 0)$, in which the temperature jumps from $\hat{u}$ to $u_i$ in the burnt gas across $S$. At $(-\varepsilon, 0)$, deflagration wave WDF must happen, which connects two states $(u_0, q_0)$ and $(u_{m1}, 0)$ followed by a $R$ or $S$.

When $0 < q_0 < q_2$, the WDF transfers to be a SDT at the time the shock wave catches up with it and then the SDT slows down due to the interaction with $R$. For small $q_0$ ($0 < q_0 < q_1$), the SDT finally dies out when its speed arrives at $f(u_i)/u_i - (u_\pm + q_0)$, at the same time, turns to be a $J$. 

and a shock which will penetrate the rest part of $R$ at infinity (Fig. 25). While for $q_1 \leq q_0 < q_2$, we have a persisting SDT with the speed $\frac{dx}{dt} \to f'(\bar{u}_m)$ as $t \to \infty$, which is a new phenomenon (Fig. 26). Here $u_{m1}$, $u_{m3}$ have the same meanings as in Case 2.2 and $\bar{u}_m \in (u_i, u^*)$ satisfies $f'(\bar{u}_m) = f(\bar{u}_m) - f(u_{m1})$.

If $q_0$ is large ($q_0 \geq q_2$), the WDF will go to infinity (Figs. 27–28). Particularly, when $q_0$ is sufficiently large ($q_0 > q_3$), the temperature on the back bank of WDF will be raised so high that the unburnt gas on the right-hand side will be ignited to be a CJDT propagating with the speed $f'(u_{m4})$, where $u_{m4} > u_6$ satisfies $f'(u_{m4}) = \frac{f(u_{m4}) - f(u_{m1})}{u_{m4} - (u_{m1} + q_0)}$ (Fig. 28).

As we can see, the solutions display the transitions from deflagration to detonation and detonation to deflagration, which do not occur in convex cases.
Fig. 26. $q_1 \leq q_0 < q_2$.

Fig. 27. $q_2 \leq q_0 \leq q_3$.

Fig. 28. $q_0 > q_3$. 
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