Proof of a Conjecture About the Exponent of Primitive Matrices

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ABSTRACT

An \( n \times n \) nonnegative matrix \( A \) is called primitive if for some positive integer \( k \), every entry in the matrix \( A^k \) is positive (\( A^k > 0 \)). The exponent of primitivity of \( A \) is defined to be \( \gamma(A) = \min\{k \in \mathbb{Z}_+ : A^k > 0\} \), where \( \mathbb{Z}_+ \) denotes the set of positive integers. The upper bound on \( \gamma(A) \) due to Wielandt is \( \gamma(A) \leq (n - 1)^2 + 1 \), and a better bound for \( \gamma(A) \) due to Hartwig and Neumann is \( \gamma(A) \leq m(m - 1) \), where \( m \) is the degree of the minimal polynomial of \( A \). Also, Hartwig and Neumann conjecture that \( \gamma(A) \leq (m - 1)^2 + 1 \), which had been suggested in 1984. In this paper, we prove this conjecture.

1. INTRODUCTION AND NOTATION

For all terminology and notation used here we follow [1].

If \( A \) and \( B \) are \( n \times n \) nonnegative matrices, we shall use the notation \( A > 0 \), \( A \leq B \), and \( A \geq B \) to denote, respectively, that all the entries of \( A \) are positive, that \( A \) and \( B \) have nonzero entries in the same location, and that all entries which are zero in \( A \) are also zero in \( B \).

Associated with an \( n \times n \) nonnegative matrix \( A = (a_{ij}) \) we shall consider its directed graph \( D(A) \) which consists of a set \( V \) of \( n \) vertices, labeled conveniently 1, 2, \ldots, \( n \), and a set of directed edges \( E \) with a direct edge from vertex \( i \) to vertex \( j \) if and only if \( a_{ij} \neq 0 \). We shall use the notation \( i \rightarrow j \) and \( i \Rightarrow j \) to denote, respectively, that there is a direct edge from vertex \( i \) to vertex \( j \) and that there is no directed edge linking vertex \( i \) to vertex \( j \). Similarly,

\[ i \xrightarrow{d_1, \ldots, d_s} j \quad \text{and} \quad i \xrightarrow{d_1, \ldots, d_s} j \]

denote, respectively, that there are paths of length \( d_1, \ldots, d_s \) connecting vertex \( i \)
to vertex $j$ and that there are no paths of length $d_1, \ldots, d_s$ connecting vertex $i$ to vertex $j$. The distance $d(i, j)$ from vertex $i$ to vertex $j$ is the minimal length of a path linking vertex $i$ to vertex $j$. The symbols $D$ and $D_{Ak}$ denote, respectively, the diameter of $D(A)$ and the diameter of $D(A^k)$. We also designate the letters $m$ and $s$ to denote, respectively, the degree of the minimal polynomial and the length of the shortest circuit in $D(A)$.

The index of primitivity of $A$ is defined to be $\gamma(A) = \min\{k \in \mathbb{Z}^+: A^k \gg 0\}$, where $\mathbb{Z}^+$ denotes the set of positive integers.

Some well-known facts concerning nonnegative matrices which we shall use are the following:

\[
I + A + \cdots + A^D \gg 0, \\
A + A^2 + \cdots + A^{D+1} \gg 0, \\
(A^k)_{ij} > 0 \iff i \to j \text{ in } D(A), \\
A^k \gg 0 \iff i \to j \text{ for any } i, j \in V(D(A)).
\] (1.1)

Thus, in particular, if $(A^k)_{ij} > 0$, then the vertex $i$ lies on a closed path of length $k$ in $D(A)$. Also it is known that $D \leq m - 1$.

Suppose $a_1, \ldots, a_t$ is a set of distinct positive integers with $\gcd(a_1, \ldots, a_t) = 1$. Then we define $\Phi(a_1, \ldots, a_t)$ to be the least integer $m$ such that every integer $k \geq m$ can be expressed in the form $k = c_1a_1 + \cdots + c_ta_t$, where $c_1, \ldots, c_t$ are some nonnegative integers. A well-known result due to Schur shows that $\Phi(a_1, \ldots, a_t)$ is well defined when $\gcd(a_1, \ldots, a_t) = 1$.

2. SOME RESULTS FROM NUMBER THEORY

**Lemma 2.1** [12]. If $\gcd(a, b) = 1$, where $a, b$ are positive integers, then

\[\Phi(a, b) = (a - 1)(b - 1).\]

**Lemma 2.2** [7]. Let $0 < a_1 < a_2 < \cdots < a_k, k \geq 3$ and $\gcd(a_1, a_2, \ldots, a_k) = 1$. If it is impossible to choose $a_l$ and $a_m$ from the set $\{a_1, a_2, \ldots, a_k\}$ such that for any $1 \leq i \leq k, a_i$ can be expressed in the form $a_i = c_{i1}a_1 + c_{i2}a_m$, where $c_{i1}, c_{i2}$ are some nonnegative integers, then

\[\Phi(a_1, a_2, \ldots, a_k) \leq \left\lfloor \frac{1}{2}a_1 \right\rfloor (a_k - 2).\]

**Lemma 2.3** [6]. If $\gcd(p_1, \ldots, p_k) = 1$, let $r_1 = p_1, r_i = \gcd(p_1, \ldots,$
$p_{i-1}, p_i) = \gcd(r_{i-1}, r_i)$ for $i = 2, 3, \ldots, k$. Then

\[ \Phi(p_1, \ldots, p_k) \leq \sum_{i=1}^{k-1} \frac{r_i}{r_{i+1}} - p_{i+1} + 1 - \sum_{i=1}^{k} p_i. \]

**Lemma 2.4** [10]. If $a_0, a_1, \ldots, a_s$ is an arithmetic progression such that $a_r = a_0 + rd$ and $\gcd(a_0, d) = 1$, then

\[ \Phi(a_0, a_1, \ldots, a_s) = \left\lfloor \frac{a_0 - 2}{s} + 1 \right\rfloor a_0 + (a_0 - 1)(d - 1). \]

**Theorem 2.1.** Let $D \geq 4$ and $D + 3 \leq l \leq 2D + 1$. Then

\[ \Phi(D + 1, D + 2, l) \leq D^2 - D - 2 \quad \text{except for} \quad \Phi(5, 6, 9) = 14. \]

**Proof.** By Lemma 2.1, it is sufficient to show that, for any $1 \leq k \leq 2D + 2$, $D(D + 1) - k$ can be expressed in the form $D(D + 1) - k = c_1(D + 1) + c_2(D + 2) + c_3 l$, where $c_1, c_2, c_3$ are nonnegative integers except for $D = 4, l = 9$. We have

\[ D(D + 1) - 1 = (l - D - 3)(D + 1) + (2D + 1 - l)(D + 2) + l; \]
\[ D(D + 1) - k = (k - 2)(D + 1) + (D + 1 - k)(D + 2), \]

where $2 \leq k \leq D + 1$;
\[ D(D + 1) - k = (l + k - 2D - 6)(D + 1) + (3D + 3 - l - k)(D + 2) + l, \]

where $k - D + 2, D + 4 \leq l \leq 2D + 1$ or
\[ k = D + 3, D + 3 \leq l \leq 2D; \]
\[ D(D + 1) - k = (2l + k - 3D - 8)(D + 1) + (4D + 4 - k - 2l)(D + 2) + 2l, \]

where $k = D + 2, l = D + 3, D \geq 4$;
\[ D(D + 1) - k = (2l + k - 4D - 10)(D + 1) + (5D + 5 - 2l - k)(D + 2) + 2l, \]

where $k = D + 3, l = 2D + 1, D \geq 5$;
\[ D(D + 1) - k = (k - D - 4)(D + 1) + (2D + 2 - k)(D + 2), \]

where $D + 4 \leq k \leq 2D + 2$. Combining these, we have $\Phi(D + 1, D + 2, l) \leq D^2 - D - 2$ except for $\Phi(5, 6, 9) = 14$. \[ \blacksquare \]

**Theorem 2.2.** Suppose $\gcd(D + 1, l) = 1$, $D/2 + 1 \leq l \leq D$, and there exists some integer $j \geq 0$ such that $3 < (j + 1)l - j(D + 1) \leq D - l + 4$. Then:
When \( j \geq 1 \),

\[
\Phi(D + 1, 2(D + 1) - l, (j + 3)(D + 1) - (j + 2)l) \leq D^2 - D,
\]

\[
\Phi(D + 1, 2(D + 1) - l, (j + 4)(D + 1) - (j + 3)l) \leq D^2 - D.
\]

When \( j = 0 \),

\[
\Phi(D + 1, 2(D + 1) - l, 3(D + 1) - 2l, 4(D + 1) - 3l) \leq D^2 - D + 1.
\]

**Proof.** (1): We only prove \( \Phi(D + 1, 2(D + 1) - l, (j + 3)(D + 1) - (j + 2)l) \leq D^2 - D \); the other inequality can be proved similarly.

**Case 1.** \( l = D \). Since \( 3 < (j + 1)l - j(D + 1) \leq D - l + 4 \), then \( (j + 1)D - j(D + 1) = 4 \), i.e., \( j = D - 4 \). By Theorem 2.4, \( \Phi(D + 1, 2(D + 1) - l, (j + 3)(D + 1) - (j + 2)l) = \Phi(D + 1, D + 2, 2D - 1) \leq D^2 - D - 2 \).

**Case 2.** \( l \leq D - 1 \), i.e., \( D + 1 - l \geq 2 \). From \( 3 < (j + 1)l - j(D + 1) \), we have \( D > (j + l)(D + 1) - 2 \).

Since \( \text{gcd}(D + 1, l) = 1 \), for any integer \( n \) the equation \((D + 1)r - ls = n\) has a pair of solutions \( r, s \) such that \( 0 \leq s \leq D \). Let \( z = \lfloor s/(j + 2) \rfloor \), \( y = s - (j + 2)z \), and \( x = r - 2y - z(i + 3) \); it is sufficient for us to show \( x \geq 0 \) for any \( n \geq D^2 - D \).

We have

\[
(D + 1)x = n - y\{(2(D + 1) - l) - z\{(j + 3)(D + 1) - (j + 2)l\}
\]

\[
\geq D^2 - D - (D + 1 - l)\{y + (j + 2)z\} - (D + 1)(y + z)
\]

\[
\geq D^2 - D - D(D + 1 - l) - (D + 1)\left( j + 1 + \frac{D - j - 1}{j + 2} \right)
\]

\[
\geq \frac{jD^2}{j + 1} + \frac{D}{j + 1}\{(j + 1)(D + 1 - l) + 3\}
\]

\[
-D - D(D + 1 - l) - D + 1 \left( j + \frac{D + 1}{j + 2} \right)
\]

\[
= \frac{j^2 + j - 1}{(j + 1)(j + 2)}D^2 - \left( j + \frac{2}{j + 2} - \frac{3}{j + 1} \right)D - j + \frac{j + 1}{j + 2}
\]

\[
-D + 1 \{j + 1)(D + 1 - l) + 3\}
\]
So $x \geq 0$, from which (1) follows.

(2): Since $j = 0$, we have $3 < l \leq D$. If $D = 4$, then $l = 4$. If $D = 5$, by $\gcd(D + 1, l) = 1$, then $l = 5$. By Lemma 2.4,

$$\Phi (D + 1, 2(D + 1) - l, 3(D + 1) - 2l, 4(D + 1) - 3l)$$

$$= \left[ \frac{D + 2}{3} \right] (D + 1) + D^2 - D l$$

$$\leq \frac{D + 2}{3} (D + 1) + D^2 - D \left( \frac{D}{2} + 1 \right) \leq D^2 - D + 1, \quad D \geq 6,$$

$$= 10, \quad D = 4, \quad l = 4,$$

$$= 12, \quad D = 5, \quad l = 5,$$

So Theorem 2.2 follows.

3. SOME RESULTS ABOUT THE CONJECTURE $\gamma(A) \leq D^2 + 1$

Prior to [1], R. E. Hartwig [9] conjectured that $\gamma(A) \leq D^2 + 1$, which is stronger than $\gamma(A) \leq (m - 1)^2 + 1$, because $D \leq m - 1$. In this section we consider the conjecture $\gamma(A) \leq D^2 + 1$.

**Theorem 3.1.** Suppose $A$ is primitive, $s$ is an integer, and $D$ and $D_A$ are, respectively, the diameters of $D(A)$ and $D(A^s)$. Then

$$D_A \leq D.$$

**Proof.** For any $A_0, A_s \in V(D(A))$, $A_0 \neq A_s$, by the primitivity of $A$, we have $A_0 \xrightarrow{k_0} A_s$, where $k$ is large enough. Let $k_0 = \min\{k : A_0 \xrightarrow{k_0} A_s\}$. We will prove that $k_0 \leq D$. 

Otherwise, we suppose \( k_0 \geq D + 1 \), so there are \( s - 1 \) vertices \( A_1, A_2, \ldots, A_{s-1} \) in \( D(A) \) such that

\[
A_0 \xrightarrow{k_0} A_1 \xrightarrow{k_0} A_2 \xrightarrow{k_0} \cdots \xrightarrow{k_0} A_{s-1} \xrightarrow{k_0} A_s.
\]

For any \( 1 \leq i \leq s \), let \( x_i = d(A_{i-1}, A_i) \) if \( A_{i-1} \neq A_i \), and let \( x_i = 0 \) if \( A_{i-1} = A_i \), so \( 0 \leq x_i \leq D < k_0 \). Let \( y_i \equiv k_0 - x_i \pmod{s} \), where \( 0 \leq y_i \leq s - 1 \). Consider the set \( \{ \sum_{i=1}^{T} y_i : l = 1, 2, \ldots, s \} \), it is easy to see that at least one of the following two cases is true:

\begin{itemize}
\item **Case 1.** There exists some \( l_0 \) such that \( \sum_{i=1}^{l_0} y_i \equiv 0 \pmod{s} \)
\item **Case 2.** There exist \( l_1 < l_2 \) such that \( \sum_{i=1}^{l_1} y_i \equiv \sum_{i=1}^{l_2} y_i \pmod{s} \), i.e., \( \sum_{i=l_1+1}^{l_2} y_i \equiv 0 \pmod{s} \).
\end{itemize}

So without loss of generality, we assume that \( \sum_{i=1}^{m} y_i \equiv 0 \pmod{s} \) for two integers \( l \leq m \). Consider the following path

\[
A_0 \xrightarrow{k_0} A_1 \xrightarrow{k_0} \cdots \xrightarrow{k_0} A_{l-1} \xrightarrow{x_{l+1}} A_l \xrightarrow{x_{l+2}} A_{l+1} \xrightarrow{x_{l+2}} \cdots
\]

\[
x_{l} A_{m} \xrightarrow{k_0} A_{m+1} \xrightarrow{k_0} \cdots \xrightarrow{k_0} A_s,
\]

the length of which, less than \( k_0s \), is \( k_0s - (m + l + 1)k_0 + \sum_{i=1}^{m} x_i = k_0s - \sum_{i=1}^{m} (k_0 - x_i) \equiv k_0s - \sum_{i=1}^{m} y_i \equiv 0 \pmod{s} \), contradicting the choice of \( k_0 \). Therefore \( A_1 \xrightarrow{k_0} A_s \) for some \( k_0 \leq D \), so in \( D(A^s) \) we have \( A_0 \xrightarrow{k_0} A_s \), i.e., \( D(A^s) \leq D \).

**Theorem 3.2.** Suppose \( A \) is primitive and \( D \) is the diameter of \( D(A) \). Then

\[
\gamma(A) \leq D(D + 1).
\]

**Proof.** For any \( i, j \in V(D(A)) \), by (1.1) we know that \( i \) lies on a circuit in \( D(A) \) whose length is \( s \leq D + 1 \); then \( D(A^s) \) has a loop at \( i \). By Theorem 3.1, it is easy to see that \( i \xrightarrow{D} j \) in \( D(A^s) \), i.e., \( i \xrightarrow{D} j \) in \( D(A) \). Then the arbitrariness of \( j \) leads to \( i \xrightarrow{D(D+1)} j \), from which Theorem 3.2 follows.

**Theorem 3.3.** Suppose \( A \) is primitive and \( D \) is the diameter of \( D(A) \). If the length of the shortest circuit in \( D(A) \) is \( s \leq D - 1 \), then

\[
\gamma(A) \leq D^2.
\]

**Proof.** Suppose \( k' \) is a vertex which lies on the shortest circuit, named \( C \), in \( D(A) \). For any \( i, j \in V(D(A)) \), there exists some integer \( m, 0 \leq m \leq D \), such
that \( i \xrightarrow{m} k' \). From \( k' \) we walk \( D - m \) steps along the shortest circuit \( C \) to \( k \), i.e., \( k' \xrightarrow{D-m} k \). Because \( k \) also lies on \( C \), \( D(A') \) has a loop at \( k \), which means \( k \xrightarrow{(D-1)D} j \) in \( D(A') \), i.e., \( k \xrightarrow{(D-1)D} j \) in \( D(A) \). By the arbitrariness of \( j \), \( i \xrightarrow{(D-1)D} j \), i.e., \( i \xrightarrow{D} j \), from which Theorem 3.3 follows.

**Theorem 3.4.** Suppose \( A \) is primitive and \( D \) is the diameter of \( D(A) \). If the length of the shortest circuit in \( D(A) \) is \( s = D \), then

\[
\gamma(A) \leq D^2 + 1.
\]

**Proof.** For any \( a, b \in V(D(A)) \), if \( a \) lies on a circuit whose length is \( D \), then in the same fashion as in the proof of Theorem 3.2 we have \( a \xrightarrow{D^2} b \), and by the arbitrariness of \( b \) we have \( a \xrightarrow{D^2+1} b \). Similarly, if \( b \) lies on a circuit whose length is \( D \), then we also have \( a \xrightarrow{D^2+1} b \).

Now we suppose that neither \( a \) nor \( b \) lies on a circuit whose length is \( D \). By (1.1), \( a \) and \( b \) respectively lie on a circuit whose length is \( D + 1 \). Suppose \( k_1 \rightarrow k_2 \rightarrow \cdots \rightarrow k_D \rightarrow k_1 \) is the shortest circuit in \( D(A) \). Since for any \( 1 \leq i \leq D \) we have \( k_i \neq a, b \), then there exist \( 1 \leq x_i, y_i \leq D \) such that \( a \xrightarrow{x_i} k_i \xrightarrow{y_i} b \), i.e., from \( a \) to \( b \) there is a path with length \( p = x_i + y_i \), which meets at least one circuit of each length \( D \) and \( D + 1 \).

**Case 1.** There exists some \( i \) and \( x_i, y_i \) such that \( p = x_i + y_i \leq D + 1 \). Since

\[
\Phi(D, D + 1) + p \leq D(D - 1) + D + 1 = D^2 + 1,
\]

then \( a \xrightarrow{D^2+1} b \).

**Case 2.** There exists some \( i \) and \( x_i, y_i \) such that \( D + 3 \leq p = x_i + y_i \leq 2D+1 \). Since \( (2D - p + 1)(D + 1) + (p - D - 3)D + p = D^2 + 1 \), there is \( a \xrightarrow{D^2+1} b \).

**Case 3.** For any \( i \) and any \( x_i, y_i \), \( p = x_i + y_i = D + 2 \) is always true. Since \( a \xrightarrow{x_i} k_1 \xrightarrow{y_2} b \), i.e., \( a \xrightarrow{x_i+1} k_2 \xrightarrow{y_2} b \) and \( x_1 + 1 + y_2 \leq 2D + 1 \), if \( x_1 + 1 + y_2 \neq D+2 \), then we have case 1 or case 2. So \( x_1 + 1 + y_2 = D + 2 = x_2 + y_2 \), i.e., \( x_2 = x_1 + 1 \). By using induction, we have \( x_{i+1} = x_i + 1 \) for any \( 1 \leq i \leq D - 1 \), so \( x_D = x_1 + D - 1 \). Note that \( 1 \leq x_i, y_i \leq D \) and we have \( x_1 = 1 \) and \( y_1 = D + 2 - x_1 = D + 1 \), which contradicts the fact that \( y_1 \leq D \). So case 3 belongs to case 1 or case 2.

Combining cases 1 to 3, we have \( a \xrightarrow{D^2+1} b \), from which Theorem 3.4 follows.

**Theorem 3.5.** Suppose \( A \) is primitive and \( D \) is the diameter of \( D(A) \). If the length of the shortest circuit in \( D(A) \) is \( s = D + 1 \) and the diameter of \( D(A^{D+1}) \) is...
\[ D_{A^{D+1}} \leq D - 1, \text{ then} \]
\[ \gamma(A) \leq D^2 - 1. \]

**Proof.** For any \( i, j \in V(D(A)) \), \( i \) lies on a circuit whose length is \( D + 1 \); then \( D(A^{D+1}) \) has a loop at \( i \). Since \( D_{A^{D+1}} \leq D - 1 \), we know \( i \xrightarrow{D-1} j \) in \( D(A^{D+1}) \), i.e., \( i \xrightarrow{D^2-1} j \) in \( D(A) \), from which Theorem 3.5 follows.

**Theorem 3.6.** Suppose \( A \) is primitive and \( D \) is the diameter of \( D(A) \). If the length of the shortest circuit in \( D(A) \) is \( s = D + 1 \) and the diameter of \( D(A^{D+1}) \) is \( D_{A^{D+1}} = D \), i.e., there exist \( r, t \in V(D(A)) \) such that the distance from \( r \) to \( t \) in \( D(A^{D+1}) \) is \( d_{A^{D+1}}(r, t) = D \), then \( r \xrightarrow{D^2-1} t \) and there exists some \( k, 1 \leq k \leq D \), \( \gcd(k, D + 1) = 1 \), such that

\[ r = A_0 \xrightarrow{D+1,k} A_1 \xrightarrow{D+1,k} A_2 \xrightarrow{D+1,k} \cdots \xrightarrow{D+1,k} A_D = t, \]

where \( A_i \in V(D(A)) \) for any \( 0 \leq i \leq D \). Furthermore for any \( k' \) such that \( k' \neq k \) and \( 1 \leq k' \leq D \), we have \( A_i \xrightarrow{k'} A_{i+1} \) for any \( 0 \leq i \leq D \).

**Proof.** Since \( d_{A^{D+1}}(r, t) = D \), then \( r \xrightarrow{D^2-1} t \) follows, and also we have

\[ r = A_0 \xrightarrow{D+1} A_1 \xrightarrow{D+1} A_2 \xrightarrow{D+1} \cdots \xrightarrow{D+1} A_{D-1} \xrightarrow{D+1} A_D = t, \]

where \( A_i \in V(D(A)) \) for any \( 0 \leq i \leq D \). If \( A_{i-1} = A_i \) for some \( i \) then \( r \xrightarrow{D^2-1} t \), which cannot hold. So for any \( 1 \leq i \leq D \), we have \( A_{i-1} \neq A_i \), and there exists \( k_i, 1 \leq k_i \leq D \), such that

\[ r = A_0 \xrightarrow{k_i} A_1 \xrightarrow{k_2} A_2 \xrightarrow{k_3} \cdots \xrightarrow{k_{D-1}} A_{D-1} \xrightarrow{k_D} A_D = t. \]

Now we define the set

\[ T(k_1, k_2, \ldots, k_D) = \left\{ \sum_{i=1}^{l} k_i : l = 1, 2, \ldots, D \right\}. \]

Suppose either of the following two cases is true:

**Case 1.** \( \sum_{i=1}^{l_1} k_i \equiv 0 \pmod{D + 1} \) for some \( l_1 \).

**Case 2.** \( \sum_{i=1}^{l_1} k_i \equiv \sum_{j=1}^{l_2} k_j \pmod{D + 1} \) for some \( l_1 \neq l_2 \).

Then there exists some \( l_1 \leq l_2 \) such that \( \sum_{i=l_1}^{l_2} k_i \equiv 0 \pmod{D + 1} \). We consider the path

\[ r = A_0 \xrightarrow{D+1} A_1 \xrightarrow{D+1} \cdots \xrightarrow{D+1} A_{l_1-1} \xrightarrow{k_i} A_{l_1} \xrightarrow{k_i+1} A_{l_1+1} \xrightarrow{k_i+2} \cdots \xrightarrow{k_2} A_{l_2} \xrightarrow{D+1} A_{l_2+1} \xrightarrow{D+1} \cdots \xrightarrow{D+1} A_D = t, \]
the length of which is less than \( D^2 + D \), but is \((D + 1)D - (D + 1)(l_2 - l_1 + 1) + \sum_{i=1}^{l_1} k_i \equiv 0 \pmod{D + 1} \). This is a contradiction to \( d_{A^{i+1}}(r, t) = D \).

So neither case 1 nor case 2 is true, i.e., \( T(k_1, k_2, \ldots, k_D) \equiv \{1, 2, \ldots, D - 1, D\} \pmod{D + 1} \). Similarly, we have \( T(k_1, k_2, \ldots, k_{i-1}, k_{i+1}, k_{i+2}, \ldots, k_D) \equiv \{1, 2, \ldots, D - 1, D\} \pmod{D + 1} \). Comparing elements of these two sets, we have \( k_i = k_{i+1} \), i.e., \( k_1 = k_2 = \cdots = k_D \equiv k \), i.e.,

\[
\begin{align*}
& r = A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_D = t.
\end{align*}
\]

Since \( T(k_1, k_2, \ldots, k_D) = \{ l k (\mod D + 1) : l = 1, 2, \ldots, D \} \equiv \{1, 2, \ldots, D\} \pmod{D + 1} \), from number theory, it is easy to prove that \( \gcd(D + 1, k) = 1 \). If there exists some \( k', k' \neq k, 1 \leq k \leq D \), such that \( A_{i-1} A_i \), we replace \( k_i \) by \( k' \), and from the above, \( T(k_1, k_2, \ldots, k', \ldots, k_D) \equiv \{1, 2, \ldots, D\} \pmod{D + 1} \) cannot hold. Then we know \( A_{i-1} \not\rightarrow A_i \) for any such \( k' \). So Theorem 3.6 follows.

Finally, we close this section with the statement that the only case for which we cannot prove the conjecture \( \gamma(A) \leq D^2 + 1 \) is when \( s = D + 1, D_{A^{i+1}} = D \), for any \( A_0, A_D \in V(D(A)) \) such that \( A_0 \rightarrow A_D \) there exists some \( k = k(A_0, A_D) \), \( 1 < k < D \), \( \gcd(k, D + 1) = 1 \), such that \( A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_D \), and for any \( k' \) such that \( k' \neq k, 1 \leq k \leq D \), we always have \( A_{i-1} \not\rightarrow A_i \) for any \( 1 \leq i \leq D \).

4. PROOF OF THE CONJECTURE \( \gamma(A) \leq (m - 1)^2 + 1 \)

If \( m > D + 1 \), then by Theorem 3.2, \( \gamma(A) \leq D(D + 1) \leq (m - 2)(m - 1) \leq (m - 1)^2 + 1 \).

If \( s = D + 1, D_{A^{i+1}} = D \), and \( A_0 \rightarrow A_D \), by Theorem 3.6 there exists some \( k(A_0, A_D) \) satisfying Theorem 3.6. Now we define \( K = \max\{k(A_0, A_D) : A_0 \rightarrow A_D \} \), which is a constant of such \( D(A) \). Note that if \( \gamma(A) \leq D^2 + 1 \) is true, then \( \gamma(A) \leq (m - 1)^2 + 1 \) is also true, so in this section it is reasonable for us to make the following hypothesis: (*) \( s = D + 1, D_{A^{i+1}} = D \), and there exist \( A_i, 0 \leq i \leq D \), such that \( A_0 \rightarrow A_D \) and \( A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_D \), where \( K \), defined as above, is a constant such that \( \gcd(D + 1, K) = 1 \); furthermore, \( A_{i-1} \not\rightarrow A_i \) for any \( 1 \leq i \leq D \) and any \( k \neq K \) such that \( 1 \leq k \leq D \); also, \( D \geq 4 \) and \( m = D + 1 \), where \( m \) is the degree of the minimal polynomial of \( A \).

A path \( i \rightarrow i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_i \rightarrow j \) is called closed if \( i = j \). By the hypothesis (*), including \( s = D + 1 \) and \( m = D + 1 \), we know that if \( D(A) \) has a closed path whose length is \( c \), i.e., \( A^c = a_0 A^c + a_{D-1} A^{D-1} + \cdots + a_0 l \), where \( a_0 > 0 \), then
for any vertex \( i \) in \( D(A) \), \( i \) lies on some closed path whose length is \( c \).

For any integer \( k \) and \( A^k = a_0 A^0 + a_{D-1} A^{D-1} + \cdots + a_1 A + a_0 I \), we delete the term \( a_i A^i \) if and only if \( a_i = 0 \). The uniform expression of \( A^k \) is defined as that \( A^k = a_i A^i + a_{i_2} A^{i_2} + \cdots + a_i A^i \) such that \( D \geq i_1 > i_2 > i_3 > \cdots > i_l > \cdots \geq 0 \) and \( a_i \neq 0 \) for any \( i \) which is possible. We denote that \( A^k \cong a_i A^i + a_{i_2} A^{i_2} + \cdots + a_i A^i \). By the definition of minimal polynomial, the uniform expression of \( A^k \) is unique.

**Lemma 4.1 [1].** Suppose \( m \) is the degree of the minimal polynomial of the primitive matrix \( A \). If \( m \leq 4 \), then \( \gamma(A) \leq (m - 1)^2 + 1 \).

**Lemma 4.2 [1].** Suppose \( m \) is the degree of the minimal polynomial of the primitive matrix \( A \). If there exists some \( p, 1 \leq p \leq m - 1 \), such that \( A^m \supseteq A^p \), then

\[
\gamma(A) \leq p + (m - 1)(m - p) \leq (m - 1)^2 + 1.
\]

**Theorem 4.1.** Suppose the hypothesis (\( \ast \)) holds and \( A^{D+1} = a_D A^D + a_{D-1} A^{D-1} + \cdots + a_1 A + a_0 I \). Then:

1. \( a_k > 0, a_0 > 0 \).
2. If there exists a closed path whose length is \( c \) in \( D(A) \), then every vertex \( i \) in \( D(A) \) lies on a closed path whose length is \( c \).
3. If \( a_i \) is the first nonzero number in the sequence \( a_D, a_{D-1}, \ldots, a_1, a_0 \), then \( a_i > 0 \) and there exists a closed path whose length is \( 2(D + 1) - 1 \).

**Proof.** (1): By (\( \ast \)), let \( i = A_0 \) and \( j = A_1 \); we have \( i \neq j \) and \( 0 < (A^{D+1})_{ij} = a_D(A^D)_{ij} + a_{D-1}(A^{D-1})_{ij} + \cdots + a_1 A_{ij} + a_0 I_{ij} = a_K(A^K)_{ij}, \) which leads to \( a_K > 0 \). Since the length of the shortest circuit in \( D(A) \) is \( D + 1 \), we have \( 0 < (A^{D+1})_{ii} = a_D(A^D)_{ii} + a_{D-1}(A^{D-1})_{ii} + \cdots + a_1 A_{ii} + a_0 I_{ii} = a_0 \).

2. If \( A^r = b_D A^D + b_{D-1} A^{D-1} + \cdots + b_1 A + b_0 I \), then \( b_0 > 0 \) as \( D(A) \) has a closed path with the length \( c \). So for any \( i \in V(D(A)) \), \( (A^r)_{ii} = b_D(A^D)_{ii} + b_{D-1}(A^{D-1})_{ii} + \cdots + b_0 I_{ii} = b_0 > 0 \), i.e., \( i \) lies on some closed path with the length \( c \).

3. Since \( A^{D+1} = a_0 A^0 + a_{D-1} A^{D-1} + \cdots + a_1 A + a_0 I \), then \( A^{2(D+1)-l} = a_0 A^{2D+1} + a_{D-1} A^{D-1} + \cdots + a_1 A + a_0 I \), where \( c_0 = a_0 a_l \). Note that \( (A^{2(D+1)-l})_{ii} = c_D(A^D)_{ii} + \cdots + c_0 I_{ii} = a_0 a_l \). Since \( a_0 > 0 \) and \( a_l \neq 0 \), we have \( a_i > 0 \) and \( (A^{2(D+1)-l})_{ii} > 0 \), i.e., \( D(A) \) has some closed path with length \( 2(D + 1) - l \).

**Theorem 4.2.** Suppose the hypothesis (\( \ast \)) holds. Then \( A^{D+1} \cong a_i A^i + a_{i_2} A^{i_2} + \cdots + a_i A^i \) and \( A^k \cong b_{k_1} A^{k_1} + b_{k_2} A^{k_2} + \cdots + b_{k_r} A^{k_r} \), where \( k \geq D+1 \) is some integer. If there exist some \( r \) and \( s \) such that \( l_s \geq D+1-k_1+k_r \).
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and $k_i \equiv k_r \pmod{d'}$ for any $1 \leq i < r$, where $d'$ is an integer such that $d' | d = \gcd(D + 1 - l_1, l_1 - l_2, l_2 - l_3, \ldots, l_{s-1} - l_s)$, then $b_{k_r} > 0$ and $D(A)$ has a closed path whose length is $k + D - k_r + 1$.

**Proof.** For any integer $t$ satisfying $t \geq k$, and the uniform expression $A^t \equiv c_{p_1}A^{p_1} + c_{p_2}A^{p_2} + \cdots + c_{p_u}A^{p_u} + \cdots$, we define the set

$$T_t = \{(c_{p_i}, p_i) : c_{p_i}A^{p_i} \text{ is a term in the uniform expression of } A^t \text{ and } p_i \not\equiv p_j \pmod{d'} \text{ for any } 1 \leq i < j \}.$$  

By the definition, we know $(b_{k_r}, k_r) \in T_k$. Now we will prove $(b_{k_r}, k_r + 1) \in T_{k+1}$.

**Case 1.** $k_1 < D$. Then $A^{k+1} \equiv b_{k_1}A^{k_1+1} + b_{k_2}A^{k_2+1} + \cdots + b_{k_r}A^{k_r+1} + \cdots = b_{k_1}(a_{l_1}A^{l_1} + a_{l_2}A^{l_2} + \cdots + a_{l_s}A^{l_s} + \cdots) + b_{k_2}A^{k_2+1} + \cdots + b_{k_r}A^{k_r+1} + \cdots$. Since $(b_{k_r}, k_r) \in T_k$, then for any $1 \leq i < j$ we have $k_i \equiv k_r \pmod{d'}$, i.e., $k_i + 1 \not\equiv k_r + 1 \pmod{d'}$, from which $(b_{k_r}, k_r + 1) \in T_{k+1}$ follows.

**Case 2.** $k_1 = D$. Then $A^{k+1} = b_{k_1}A^{D+1} + b_{k_2}A^{k_2+1} + \cdots + b_{k_r}A^{k_r+1} + \cdots = b_{k_1}(a_{l_1}A^{l_1} + a_{l_2}A^{l_2} + \cdots + a_{l_s}A^{l_s} + \cdots) + b_{k_2}A^{k_2+1} + \cdots + b_{k_r}A^{k_r+1} + \cdots$. Since $(b_{k_r}, k_r) \in T_k$, we have $k_1 \not\equiv k_r + 1 \pmod{d'}$ for any $2 \leq i < r$. If $a_{l_s}A^{l_s}$ is a term in the uniform expression of $A^{D+1}$ such that $l_s \geq k_r + 1$, then by $l_s \geq D + 1 - k_1 + k_r > l_{s+1}$, i.e., $l_s \geq l_r + 1$, we have $l_i \geq l_s$, i.e., $i \leq s$, from which we know $D + 1 = l_i \pmod{d'}$ and $k_r + 1 \not\equiv k_r + 1 \pmod{d'}$. So $(b_{k_r}, k_r + 1) \in T_{k+1}$.

Combining case 1 and case 2, we have $(b_{k_r}, k_r + 1) \in T_{k+1}$. Now we suppose

$$A^{k+1} = c_{p_1}A^{p_1} + c_{p_2}A^{p_2} + \cdots + c_{p_u}A^{p_u} + \cdots,$$

where $c_{p_u} = b_{k_r}$ and $p_u = k_r + 1$, then $p_i \not\equiv p_u \pmod{d'}$ for any $1 \leq i < u$.

If there exists an integer $s'$ such that $l_{s'} \geq D + 1 - p_1 + p_u > l_{s'+1}$, since $p_1 \leq k_1 + 1$ and $l_s \geq D + 1 - k_1 + k_r > l_{s+1}$, then $l_{s'} \geq l_s$, i.e., $s' \leq s$, from which it follows that $d' | \gcd(D + 1 - l_1, l_1 - l_2, \ldots, l_{s-1} - l_s)$. So by using induction, we can easily prove that $(b_{k_r}, k_r + 2) \in T_{k+2}$, $(b_{k_r}, D) \in T_{k+D-k_r}$, i.e., $A^{k+D-k_r} \equiv b_{k_r}A^D + \cdots$. By Theorem 4.1(3), we have $b_{k_r} > 0$, and $D(A)$ has a closed path with length $k + D - k_r + 1$.

**Remark.** In Theorem 4.2, if there is no $s$ satisfying $l_s \geq D + 1 - k_1 + k_r > l_{s+1}$, then we define $\gcd(D + 1 - l_1, l_1 - l_2, \ldots, l_{s-1} - l_s) = 0$, and so $d' | d$ is always true.

**Corollary 4.2.1.** Suppose the hypothesis (*) holds and $A^{D+1} \equiv a_{l_1}A^{l_1} + a_{l_2}A^{l_2} + \cdots + a_{l_r}A^{l_r} + \cdots$. If there exists an integer $r$ such that $d' = \gcd(D + 1, l_1, l_2, \ldots, l_{r-1}, l_r) > \gcd(D + 1, l_1, \ldots, l_{r-1}, l_r) = \gcd(d', l_r)$, then $a_{l_r} > 0$ and $D(A)$ has a closed path whose length is $c = 2(D + 1) - l_r$.
Proof. In Theorem 4.2, let \( k = D + 1 \); then \( k_i = l_i \) and \( b_{ki} = a_{li} \) for any \( i \). Since \( d' > \gcd(d', l_r) \), we have \( l_i \neq l_r \pmod{d'} \) for any \( 1 \leq i < r \). If \( l_s \geq D + 1 - l_1 + l_r > l_{s+1}, \) i.e., \( l_s > l_r \), then \( s \leq r - 1 \), so \( d' = \gcd(D + 1, l_1, l_2, \ldots, l_{r-1})|\gcd(D + 1 - l_1, l_1 - l_2, \ldots, l_{r-2} - l_{r-1})\gcd(D + 1 - l_1, l_1 - l_2, \ldots, l_{s-1} - l_s) \). By Theorem 4.2, \( b_{lr} > 0 \) and \( D(A) \) has a closed path whose length is \( 2(D + 1) - l_r \).

**Corollary 4.2.2.** Suppose the hypothesis (*) holds and \( A^{D+1} \cong a_{l_1}A^{l_1} + a_{l_2}A^{l_2} + \cdots + a_{l_K}A^{l_K} + \cdots \). If there exists an integer \( t \) such that \( (D + 1 - l_1) \notin (D + 1 - l_i) \), then \( D(A) \) has a closed path whose length is \( c \leq 2(D + 1) - l_i \) and \( c \neq D + 1, 2(D + 1) - l_1 \).

Proof. In Theorem 4.2, let \( k = D + 1 \); then \( k_i = l_i \) and \( b_{ki} = a_{li} \) for any \( i \). Also we let \( d' = D + 1 - l_i \) and \( r = \min\{i : (D + 1 - l_i) \notin (D + 1 - l_i)\} \); then \( 1 < r \leq t \), i.e., \( l_1 > l_r \geq l_i \). If \( l_1 \geq D + 1 - l_1 + l_r \), then \( l_1 > l_r \), i.e., \( s \leq r - 1 \). By the choice of \( r \), we have \( \gcd(D + 1 - l_1, l_1 - l_2, \ldots, l_{r-1} - l_r) = \gcd(D + 1 - l_1, D + 1 - l_2, \ldots, D + 1 - l_s) = D + 1 - l_1 = d' \). If there exist some \( 1 \leq i < r \) such that \( li \equiv l_r \pmod{d'} \), then \( D + 1 - l_r - (D + 1 - l_i + l_r - l_i) \equiv 0 \pmod{d'} \), which contradicts the choice of \( r \). So \( li \neq l_r \pmod{d'} \) for any \( 1 \leq i < r \). By Theorem 4.2, \( D(A) \) has a closed path whose length is \( c = 2(D + 1) - l_r \leq 2(D + 1) - l_1 \), and \( c \neq D + 1, 2(D + 1) - l_1 \).

**Theorem 4.3.** Suppose the hypothesis (*) holds and \( A^{D+1} \cong a_{l_1}A^{l_1} + a_{l_2}A^{l_2} + \cdots + a_{l_K}A^{l_K} + \cdots \). If \( \gcd(D + 1, l_i) \neq 1 \), then \( D(A) \) has a sequence of closed paths with the length \( c_1, c_2, \ldots, c_t \), where \( \gcd(D + 1, c_1, c_2, \ldots, c_t) = 1 \), \( t \geq 2 \), \( c_1 = 2(D + 1) - l_1 \), and \( c_1 < c_2 < \cdots < c_t \leq 2(D + 1) - K \). Furthermore, if we let \( r_0 = D + 1, r_1 = \gcd(D + 1, c_1), \) and \( r_{i+1} = \gcd(D + 1, c_i, c_{i+1}) = \gcd(r_i, c_{i+1}) \), then \( 1 = r_t < r_{t-1} < \cdots < r_1 < r_0 = D + 1 \).

Proof. Let \( r'_0 = D + 1, r'_1 = \gcd(D + 1, l_1), r'_{i+1} = \gcd(D + 1, l_1, l_2, \ldots, l_{i+1}) = \gcd(r'_i, l_{i+1}) \), so \( D + 1 = r'_0 \geq r'_1 \geq r'_2 \geq \cdots \). Now we select \( l'_1, l'_2, \ldots, l'_t \) from \( l_1, l_2, \ldots, K, \ldots \) such that

\[
l'_j = l_j \quad \Leftrightarrow \quad j \text{ is the } i \text{th } j \text{ satisfying } r'_{j-1} > r'_j.
\]

By \( \gcd(D + 1, K) = 1 \) in (*), this selection is practicable; also we have \( l_1 = l'_1 > l'_2 > \cdots > l'_t \geq K \) and \( D + 1 = r_0 > r_1 > r_2 > \cdots > r_t = 1 \), where \( t \geq 2 \), \( r_1 = \gcd(D + 1, l'_1) \), and \( r_{i+1} = \gcd(D + 1, l'_1, l'_2, \ldots, l'_{i+1}) = \gcd(r_i, l'_{i+1}) \) for any \( 1 \leq i \leq t - 1 \). So by Corollary 4.2.1 and the choice of \( l'_j \), \( D(A) \) has a closed path whose length is \( c_1 = 2(D + 1) - l'_1 \) for any \( 1 \leq i \leq t \), and also \( 2(D + 1) - l_1 = c_1 < c_2 < \cdots < c_t \leq 2(D + 1) - K \), \( \gcd(D + 1, c_1, c_2, \ldots, c_t) = r_t = 1 \).
**Theorem 4.4.** Suppose the hypothesis (*) holds and $K = 1$. Then

$$\gamma(A) \leq D^2 + 1.$$ 

**Proof.** Case 1: $D(A)$ has a closed path with length $D + 1 + p$, where $p$ is an integer such that $1 \leq p \leq D - 2$. By Theorem 4.1(2), we know the vertex $A_D$ lies on a closed path with the length $D + 1 + p$. Consider the path

$$A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_{D+1-p} \rightarrow A_{D+2-p} \rightarrow \cdots \rightarrow A_{D} \rightarrow A_{D-1} \rightarrow \cdots$$

the length of which is $D + 1 - p + (p - 1)(D + 1) + D + 1 + p + (D + 1)(D - p - 2) = D^2 - 1$, so $A_0 \circ A_D$, contradicting (*).

Case 2: $D(A)$ has no closed path with length $D + 1 + p$, where $1 \leq p \leq D - 2$. By Theorem 4.1, we have $A^{D+1} = a_2A^2 + a_1A + a_0I$, where $a_2 \geq 0$, $a_0 > 0$, and $a_1 = a_K > 0$, so $A^{D+1} \geq A$. By Lemma 4.2 $\gamma(A) \leq D^2 + 1$ follows. \(\blacksquare\)

**Theorem 4.5.** Suppose the hypothesis (*) holds and $A^{D+1} \geq a_iA^{l_i} + a_iA^{l_2} + \cdots + a_0I$. If $2l_1 \leq D + 1$, then

$$\gamma(A) \leq (D + 1 - l_1)D + l_1 \leq D^2 + 1.$$ 

**Proof.** Similarly to Theorem 4.2, for any integer $t \geq D + 1$ and the uniform expression $A^t \geq c_{p_1}A^{p_1} + c_{p_2}A^{p_2} + \cdots + c_{p_r}A^{p_r} + \cdots$, we define the set

$$T_t = \{(c_{p_j}, p_j) : c_{p_j}A^{p_j} \text{ is a term in the uniform expression of } A^t\}.$$ 

We will prove $a_i > 0$ for any $a_i$. If this is not true, then let $a_i = \{a_i : a_i < 0$ and $a_i > 0$ for any $1 \leq i \leq r\}$. Note that

$$A^{2(D+1)-l_i} = a_iA^{D+1} + a_iA^{D+1-l_i+l_2} + \cdots$$

$$+ a_iA^{D+1-l_i+l_2 - l_i} + \cdots + a_0A^{D+1-l_i}$$

$$= a_i(a_lA^{l_i} + a_2A^{l_2} + \cdots + a_0I) + a_iA^{D+1-l_i+l_2 - l_i}$$

$$+ a_iA^{D+1-l_i + l_2} + \cdots + a_0A^{D+1 - l_i}. \quad (4.1)$$
Since $2l_i \leq D + 1$, we have $D + 1 - l_i + l_r > l_i$, i.e., the degree of any term of the uniform expression of $A^{D+1}$ is less than $D + 1 - l_i + l_r$, which is to say that, after amalgamating the same type of terms in (4.1), $a_{i_r}A^{D+1-l_i+l_r}$ is a term in the uniform expression of $A^{2(D+1)-l_i}$, i.e., $(a_{i_r}, D + 1 - l_i + l_r) \in T_{2(D+1)-l_i}$. Similarly we also have $(a_{i_r}, D + 1 - l_i + l_r) \in T_{2(D+1)-l_i}$ for any $2 \leq i \leq r$. Suppose $A^{2(D+1)-l_i} \cong a_{i_1}A^{D+1-l_1+l_2} + a_{i_2}A^{D+1-l_2+l_3} + \ldots + a_{i_l}A^{D+1-l_l+l_l} + \ldots$; then

$$A^{2(D+1)-l_2} = a_{i_1}A^{D+1} + a_{i_2}A^{D+1-l_2+l_3} + \ldots + a_{i_l}A^{D+1-l_l+l_l} + \ldots = a_{i_1}(a_{i_1}A^{D+1} + a_{i_2}A^{D+1-l_2+l_3} + \ldots + a_{i_l}A^{D+1-l_l+l_l} + \ldots + a_{i_r}A^{D+1-l_r+l_r} + \ldots) + a_{i_2}A^{D+1-l_2+l_3} + \ldots + a_{i_l}A^{D+1-l_l+l_l} + \ldots.$$

As above, we have $(a_{i_1}, D + 1 - l_2 + l_3) \in T_{2(D+1)-l_2}$ for any $3 \leq i \leq r$. By using induction, we have $(a_{i_1}, D + 1 - l_r-1 + l_r) \in T_{2(D+1)-l_r-1}$ and also $A^{2(D+1)-l_r-1} \cong a_{i_1}A^{D+1-l_r-1+l_r} + \ldots$. By Theorem 4.2(3), we have $a_{i_r} > 0$, contradicting the choice of $a_{i_r}$. So $a_{i_r} > 0$ for any $a_{i_r}$; then $A^{D+1} \geq A^{D+1} + \ldots$, and from Lemma 4.2 the result follows.

**Theorem 4.6.** Suppose the hypothesis (*) holds and $A^{D+1} \cong a_{i_1}A^{l_1} + a_{i_2}A^{l_2} + \ldots + a_{i_l}A^{l_l}$. If $K \neq 1$, $2l_i > D + 1$, and $\gcd(D + 1, l_i) \neq 1$, then

$$\gamma(A) \leq D^2 + 1.$$

**Proof.** By Theorem 4.3, $D(A)$ has a sequence of closed paths with lengths $c_1, c_2, \ldots, c_t$ such that $\gcd(D + 1, c_1, c_2, \ldots, c_t) = 1$, where $t \geq 2$ and $D + 1 < 2(D + 1) - c_i = c_1 < c_2 < \ldots < c_t \leq 2(D + 1) - K$, and also we have $1 = r_t < r_{t-1} < \cdots < r_1 < r_0$, where $r_0 = D + 1$, $r_1 = \gcd(D + 1, r_1)$, $r_{i+1} = \gcd(D + 1, c_i, \ldots, c_{i+1}) = \gcd(r_i, c_{i+1})$ for any $0 \leq i \leq t - 1$. Now we will prove $\Phi(D + 1, c_1, c_2, \ldots, c_t) \leq D^2 - D + 1$.

**Case 1.** $K = 2$. Since $\gcd(D + 1, K) = 1$, $D$ is even. By Lemma 2.2,

$$\Phi(D + 1, c_1, c_2, \ldots, c_t) \leq [(D + 1)/2](c_1 - 2) \leq (D/2)(2(D + 1) - 4) = D^2 - D.$$

**Case 2.** $K \geq 4$. By Lemma 2.2,

$$\Phi(D + 1, c_1, c_2, \ldots, c_t) \leq \frac{D + 1}{2} \{2(D + 1) - 6\} = D^2 - D - 2.$$

**Case 3.** $K = 3$. Since $1 = r_t | r_{t-1} | \cdots | r_2 | r_1 | D + 1$ and $1 = r_t < r_{t-1} < \cdots < r_2 < r_1$, by using induction on $t (t \geq 2)$, it is easy to prove that
\[ \sum_{i=1}^{t-1} \frac{r_i}{r_{i+1}} \leq \frac{r_1}{2^{t-2} + 2(t - 2)}. \] By Lemma 2.3,

\[
\Phi(D + 1, c_1, c_2, \ldots, c_t) 
\leq \frac{D + 1}{r_1} c_1 + \frac{r_1}{r_2} c_2 + \cdots + \frac{r_{t-1}}{r_t} c_t + 1 - \left( D + 1 + \sum_{i=1}^{t} c_i \right)
\]

\[
= \frac{D + 1}{r_1} \left\{ 2(D + 1) - l_1 \right\} + \left( \frac{r_1}{r_2} - 1 \right) c_2 + \cdots 
\]

\[
+ \left( \frac{r_{t-1}}{r_t} - 1 \right) c_t - D - \{ 2(D + 1) - l_1 \}
\]

\[
\leq \frac{D + 1}{r_1} \left\{ 2(D + 1) - l_1 \right\} + \left( \sum_{i=1}^{t-1} \frac{r_i}{r_{i+1}} - t + 1 \right) \{ 2(D + 1) - K \}
\]

\[
- D - \{ 2(D + 1) - l_1 \}
\]

\[
\leq \frac{D + 1}{r_1} \left\{ 2(D + 1) - l_1 \right\} + \left( \frac{r_1}{2^{t-2} + t - 3} \right) \{ 2(D + 1) - 3 \}
\]

\[
- D - \{ 2(D + 1) - l_1 \}
\]

\[
\leq \frac{D + 1}{r_1} \left\{ 2(D + 1) - l_1 \right\} + (r_1 - 1)(2D - 1)
\]

\[
- D - (2D + 2 - l_1) \quad \text{(as } 2 \leq t, \ 2^{t-1} \leq r_1) \]

\[
= \frac{D + 1}{r_1} \left\{ 2(D + 1) - l_1 \right\} + r_1(2D - 1) - 5D + l_1 - 1
\]

\[
\leq 2\{ 2(D + 1) - l_1 \} + \frac{D + 1}{2} \{ 2D - 1 \} - 5D + l_1 - 1
\]

\[
\left( \text{as } 2 \leq r_1 \leq \frac{D + 1}{2}, \ l_1 > K = 3 \right)
\]

\[
= D^2 - D + 2 + \frac{D + 1 - 2l_1}{2}
\]

\[
< D^2 - D + 2 \quad \text{(as } 2l_1 > D + 1). \]

So \( \Phi(D + 1, c_1, c_2, \ldots, c_t) \leq D^2 - D + 1. \) Combining cases 1 to 3, we have \( \Phi(D + 1, c_1, c_2, \ldots, c_t) \leq D^2 - D + 1 \) for any \( K \neq 1. \) By Theorem 4.1(2), every vertex in \( D(A) \) lies on closed paths whose length is \( D + 1, c_1, c_2, \ldots, c_t, \) from which we find \( \gamma(A) \leq \Phi(D + 1, c_1, c_2, \ldots, c_t) + D \leq D^2 + 1. \)
Theorem 4.7. Suppose the hypothesis (*) holds and $A^{D+1} \cong a_1 A^{l_1} + a_2 A^{l_2} + \cdots + a_l I$. If $2l_1 > D + 1$ and $\gcd(D + 1, l_1) = 1$, then

$$\gamma(A) \leq D^2 + 1.$$ 

Proof. By the hypothesis, $D \geq 4$; then $l_1 \geq D/2 + 1 = 3$.

Case 1. $l_1 = 3$. Then $D = 4$, and we suppose that $A^2 = A^3 + aA^2 + bA + cI$, where $c > 0$. By Theorem 4.1, $D(A)$ has closed paths with lengths 5 and 7.

Subcase 1.1: $a \neq 0$. Then $A^7 = aA^4 + (1 + b)A^3 + (a + c)A^2 + bA + cI$; by Theorem 4.1(3), $a > 0$ and $D(A)$ has a closed path with the length 8, so every vertex in $D(A)$ lies on closed paths with the lengths 5, 7, and 8, from which we know $\gamma(A) \leq \Phi(5, 7, 8) + D = 12 + 4 = 16 \leq 4^2 + 1$.

Subcase 1.2: $a = 0$. If $b \geq 0$, then $A^5 \geq A^3$, from which the result follows by Lemma 4.2. So we suppose $b < 0$; then

$$A^7 = (1 + b)A^3 + ca^2 + ba + cI,$$  \hspace{1cm} (4.2)

$$A^9 = cA^4 + (1 + 2b)A^3 + cA^2 + (b + b^2)A + (bc + c)I,$$  \hspace{1cm} (4.3)

$$A^{10} = (1 + 2b)A^4 + 2cA^3 + (b + b^2)A^2 + (2bc + c)A + c^2 I,$$  \hspace{1cm} (4.4)

$$A^{11} = 2cA^4 + (b^2 + 3b + 1)A^3 + (2bc + c)A^2 + (c^2 + 2b^2 + b)A + (1 + 2b)cI,$$  \hspace{1cm} (4.5)

By Theorem 4.1(3) and (4.2) to (4.5), we know that $1 + b \geq 0$ and $1 + 2b \geq 0$, so $1 + b > 0$; then $bc + c > 0$. By (4.3) $D(A)$ has a closed path with length 9. By the definition of minimal polynomial, $b$ is a rational number, from which $b^2 + 3b + 1 \neq 0$; by (4.5) we know $D(A)$ has a closed path with length 13. So in all, $D(A)$ has closed paths with lengths 5, 7, 9, and 13, from which $\gamma(A) < \Phi(5, 7, 9, 13) + 4 = 13 + 4 = 4^2 + 1$.

Combining subcase 1.1 and subcase 1.2, we know $\gamma(A) \leq D^2 + 1$ when $l_1 = 3$.

Case 2. $l_1 \geq 4$. If $D(A)$ has a closed path whose length is $c$, where $c \leq 2D - 2$ and $c \neq D + 1, 2(D + 1) - l_1$, then by Lemma 2.2, $\gamma(A) \leq \Phi(D + 1, 2(D + 1) - l_1, c) + D \leq \frac{1}{2}(D + 1)(c - 2) + D \leq D^2 - 2$. So we suppose that for any $c$ satisfying $c \leq 2D - 2$ and $c \neq D + 1, 2(D + 1) - l_1, D(A)$ has
no closed path whose length is \( c \). Since \( A^{D+1} \cong a_0 A + a_1 A^2 + \cdots + a_0 I \), by Corollary 4.2.2 we have \((D + 1 - l_i)(D + 1 - l_i)\) for any \( l_i \geq 4 \). So \( A^{D+1} = \sum_{i=0}^{j} a_{l_i+1} A^{l_i-(D+1-l_i)} + a_3 A^3 + a_2 A^2 + a_1 A + a_0 I \), where \( a_{l_i} > 0 \), other \( a_i \) may be 0, and \( 3 < (j + 1)l_i - j(D + 1) \leq D + 1 - l_i + 3 \). Then

\[
A^{2(D+1)-l_i} = \sum_{i=0}^{j-1} (a_{l_i} a_{l_i+1} + a_{l_i+2}) A^{l_i-(D+1-l_i)} + a_{l_i} a_{l_i+1} A^{l_i+1-j(D+1)} + a_3 A^{D+1-l_i+3} + \cdots + a_0 A^{D+1-l_i} + a_{l_i} a_{3} A^3 + \cdots + a_{l_i} a_{0} I.
\]

Since \( D(A) \) has no closed path whose length is \( c \), where \( c \leq 2D - 2 \) and \( c \neq D + 1, 2(D + 1) - l_i \), by Theorem 4.1(3) we have \( a_{l_i} a_{l_i+1} + a_{l_i+2} = 0 \) for any \( 0 \leq t < j - 1 \). By using induction on \( t \), we have \( a_{l_i+2} = (-1)^{j+1} a_{l_i+2} \) for any \( 0 \leq t \leq j - 1 \). So

\[
A^{2(D+1)-l_i} = (-1)^{j} a_{l_i+2} A^{(j+1)l_i-j(D+1)} + a_3 A^{D+1-l_i+3} + \cdots + a_0 A^{D+1-l_i} + a_{l_i} a_{3} A^3 + \cdots + a_{l_i} a_{0} I. \tag{4.6}
\]

Let \((j + 1)l_i - j(D + 1) = D + 1 - l_i + t, \) so \( 3 - (D + 1 - l_i) < t < D + 1 \). If \( t = 0 \), then \( l_i = (j + 1)(D + 1 - l_i), \) \( D + 1 = (D + 1 - l_i) + l_i = (j + 2)(D + 1 - l_i) \), and \( 1 = \gcd(D + 1, l_i) = D - l_i, \) i.e., \( l_i = D \). Then \((j + 1)l_i - j(D + 1) = D + 1 - l_i = 1, \) contradicting \( 3 < (j + 1)l_i - j(D + 1) \). So \( t \neq 0 \). If \( 1 \leq t \leq 3 \), then amalgamating the same type of terms, \( \{a_{l_i} + (-1)^{j} a_{l_i+2}\} A^{D+1-l_i+3} + \cdots + a_0 A^{D+1-l_i} + a_{l_i} a_{3} A^3 + \cdots + a_{l_i} a_{0} I \) are two terms in (4.6). Note meanwhile that \( a_{l_i} + (-1)^{j} a_{l_i+2} \) and \( a_{l_i} a_{3} \) cannot all be 0, because \( a_{l_i} > 0 \). In Theorem 4.2, let \( k = 2(D + 1) - l_i, \) \( d' = D + 1 - l_i, \) and \( b_k \), be the first nonzero number of \( a_{l_i} + (-1)^{j} a_{l_i+2} \) and \( a_{l_i} a_{3} \). By Theorem 4.2, we know that if \( a_{l_i} + (-1)^{j} a_{l_i+2} \neq 0 \), then \( a_{l_i} + (-1)^{j} a_{l_i+2} = 0 \) and \( D(A) \) has a closed path with length \((j + 3)(D + 1) - (j + 2)l_i, \) and that if \( a_{l_i} + (-1)^{j} a_{l_i+2} = 0 \), then \( a_{l_i} a_{3} > 0 \) and \( D(A) \) has a closed path with length \((j + 4)(D + 1) - (j + 3)l_i \). If \( t < 0 \), then after amalgamating the same type of terms in (4.6), \( (-1)^{j} a_{l_i+2} A^{D+1-l_i+t} \) is a term in (4.6). In Theorem 4.2, let \( k = 2(D + 1) - l_i, d' = D + 1 - l_i, \) and \( b_k = (-1)^{j} a_{l_i+2}, \) then \( (-1)^{j} a_{l_i+2} > 0, \) and \( D(A) \) has a closed path with length \((j + 3)(D + 1) - (j + 2)l_i \). All in all, we have proved \( D(A) \) always has a closed path whose length is \( c \), where \( c = (j + 3)(D + 1) - (j + 2)l_i \) or \( c = (j + 4)(D + 1) - (j + 3)l_i, 3 < (j + 1)l_i - j(D + 1) \leq D + 1 - l_i + 3 \).

**Subcase 2.1:** \( j \geq 1 \). By Theorem 2.2, \( \gamma(A) \leq (D + 1, 2(D + 1) l_i, c) + D \leq D^2 - D + D = D^2 \).

**Subcase 2.2:** \( j = 0 \). Then \( 1 \leq t = 2l_i - (D + 1) \leq 3 \). Note that \( D \geq l_i \geq 4 \). If \( D + 1 - l_i = 1, \) then \( D = l_i = 4, \) so \( \gamma(A) \leq 4 + \max(\Phi(5, 6, 7), \Phi(5, 6, 8)) = 4 + \max(9, 9) = 18 \).
14 < 4^2 + 1. If D + 1 - l_1 = 2, since gcd(D + 1, l_1) = 1, then D = 6 and l_1 = 5, so \( \gamma(A) \leq 6 + \max(\Phi(7, 9, 11), \Phi(7, 9, 13)) = 33 < 6^2 + 1 \). If D + 1 - l_1 \geq 3, let \( d' = D + 1 - l_1 \geq 3 \) and \( A^{d+1} = a_i A^{l_1} + a_3 A^3 + a_2 A^2 + a_1 A + a_0 I \), where \( a_0, a_i > 0 \). Note that \( d' \geq 3 \) and \( \gcd(D + 1, l_1) = 1 \). It is easy to see that \( i \not\equiv t \pmod{d'} \) for any \( i \) such that \( i \not= t \) and \( 0 \leq i \leq 3 \). So after amalgamating the same type of terms, we have

\[
A^{2(D+1)-h} = (a_i^2 + a_i) A^{l_1} + a_i a_1 A' + \cdots,
\]

\[
A^{3(D+1)-2h} = a_i (a_i^2 + 2a_i) A^{l_1} + a_t (a_t^2 + a_t) A' + \cdots.
\]

By Theorem 4.2, the first nonzero number among \( a_i^2 + a_i \) and \( a_i a_t \) must be positive, and also the first nonzero number among \( a_t (a_t^2 + 2a_t) \) and \( a_t (a_t^2 + a_t) \) must be positive. Note that \( a_i > 0 \). If \( a_i^2 + a_i = 0 \), then \( a_i (a_i^2 + 2a_i) < 0 \), which is a contradiction. If \( a_t (a_t^2 + 2a_t) = 0 \), then \( a_t (a_t^2 + a_t) < 0 \), which is also a contradiction. So \( a_i^2 + a_i \not= 0 \) and \( a_t (a_t^2 + 2a_t) \not= 0 \); by Theorem 4.2, \( a_i^2 + a_i > 0 \), \( a_t (a_t^2 + 2a_t) > 0 \), and \( D(A) \) has closed paths with length \( 3(D + 1) - 2l_1, 4(D + 1) - 3l_1 \). By Theorem 2.2(2), \( \gamma(A) \leq D + \gamma(D + 1, 2(D + 1) - l_1, 3(D + 1) - 2l_1, 4(D + 1) - 3l_1) \leq D^2 + 1 \). Combining case 1 and case 2, Theorem 4.7 follows. 

**Main Theorem.** Suppose \( A \) is an \( n \times n \) nonnegative and primitive matrix whose minimal polynomial is of degree \( m \). Then

\[
\gamma(A) \leq (m - 1)^2 + 1.
\]
**Proof.** In fact, we have proved the main theorem as can be seen clearly from Figure 1.

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**REFERENCES**


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