Algebraic Multiplicity of the Eigenvalues of a Tournament Matrix

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ABSTRACT

Let $\mathscr{T}_n$ denote the set of irreducible $n \times n$ tournament matrices. Here are our main results: (1) For all $n \geqslant 3$, every matrix in $\mathscr{T}_n$ has at least three distinct eigenvalues; such a matrix has exactly three distinct eigenvalues if and only if it is a Hadamard tournament matrix. (2) For all $n \geqslant 3$ there is a matrix in $\mathscr{T}_n$ having n distinct eigenvalues. (3) If α_n denotes the maximum algebraic multiplicity of 0 as an eigenvalue of the matrices in $\mathscr{T}_n$, then $\lfloor n/2 \rfloor - 2 \leqslant \alpha_n \leqslant n - 6$ for all $n \geqslant 8$. Each algebraic multiplicity m with $1 \leqslant m \leqslant \lfloor n/2 \rfloor - 2$ is achieved for the eigenvalue 0 by some matrix in $\mathscr{T}_n$ for every $n \geqslant 6$. (4) If π_n is the minimum Perron value (i.e. spectral radius) of all matrices in $\mathscr{T}_n$, then $2 < \pi_n < 2.5$ for all $n \geqslant 8$.

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1. INTRODUCTION

A *tournament matrix of order* n is an $n \times n$ $(0,1)$ matrix M satisfying $M + M^t = J - I$, where J is the $n \times n$ matrix of all ones. It has been observed recently that the rank of a tournament matrix of order n is at least

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$n - 1$ (see de Caen and Hoffman [9]). Subsequently, Maybee and Pullman [15] showed that if λ is an eigenvalue of an $n \times n$ tournament matrix M, then rank$(M - \lambda I) = n - 1$ or Re $\lambda = -\frac{1}{2}$. Thus the geometric multiplicity of an eigenvalue is one except when its real part is $-\frac{1}{2}$; in this case, we show in Section 2 that the algebraic and geometric multiplicities coincide. What can be said about the algebraic multiplicities of the eigenvalues of a tournament matrix? That question is the subject of our paper.

Let T_n be the $n \times n$ $(0,1)$ matrix $[t_{ij}]$ having $t_{ij} = 1$ if and only if $i < j$; this is a *transitive* $n \times n$ tournament matrix. It is reducible when $n > 1$ and clearly has $\lambda = 0$ as an eigenvalue of algebraic multiplicity n and geometric multiplicity 1. Other reducible tournament matrices can easily be constructed having eigenvalues of various algebraic multiplicities. Therefore we confine ourselves to the irreducible case.

In Section 2, we note that irreducible tournament matrices of order $n \geqslant 3$ must have at least three distinct eigenvalues. In Section 3 we characterize those having exactly three; specifically, in Theorem 3.2 we show that for $n \geqslant 3$, such matrices are precisely the $n \times n$ *Hadamard* tournament matrices—namely, the tournament matrices H of order $n \equiv 3 \pmod 4$ satisfying $HH^t = [(n + 1)/4]I + [(n - 3)/4]J$. Hadamard tournament matrices of order n are known to be coexistent with the skew Hadamard matrices of order $n + 1$. Such matrices exist for infinitely many n.

In contrast to Section 3, we study in Section 4 a sequence of $n \times n$ irreducible tournament matrices U_n having n distinct eigenvalues. The matrix U_n is obtained from the transitive tournament T_n by exchanging the entries in the $(1, n)$ and $(n, 1)$ positions. In Theorem 4.1 we compute the characteristic polynomial of U_n and deduce in Theorem 4.2 that U_n has n distinct eigenvalues. The U_n have the interesting property that they have only one real eigenvalue, the Perron value (also called the Perron root) when n is odd, and only two real eigenvalues (the Perron value and a negative one) when n is even. We calculate the eigenvectors of U_n as functions of its eigenvalues, and show that no eigenvector has a zero entry.

The result of Section 3 shows that for infinitely many n, the algebraic multiplicity of a nonreal eigenvalue of an irreducible tournament matrix of order n can be $(n - 1)/2$, the maximum possible for nonreal eigenvalues. How large can the algebraic multiplicity of a *real* eigenvalue be? In Section 5, we study this question by presenting, for each $n \geqslant 6$ and each $1 \leqslant m \leqslant \lfloor n/2 \rfloor - 2$, irreducible tournament matrices of order n having 0 as an eigenvalue of multiplicity m (Theorem 5.3). Letting α_n denote the maximum algebraic multiplicity of 0 as an eigenvalue of an $n \times n$ irreducible tournament matrix, the previous result shows that $\lfloor n/2 \rfloor - 2 \leqslant \alpha_n$ for $n \geqslant 6$. Further, we show in Corollary 5.1.1 that $\alpha_n \leqslant n - 6$ for all $n \geqslant 8$, by investigating the minimum Perron value, π_n, of the irreducible tournament matrices of

order n. Brauer and Gentry [3] showed that $\sqrt[3]{2}$ is a lower bound on π_n for all $n \geqslant 4$. We show that, in fact, $2 < \pi_n < 2.5$ for all $n \geqslant 8$ (see Theorem 5.1).

Throughout this paper, we will employ some of the terminology and results of both nonnegative matrix theory and graph theory. The relevant background information on these topics can be found in Horn and Johnson [11] and in Bondy and Murty [1], respectively.

2. GENERAL REMARKS

In what follows, we will use some basic facts concerning the eigenvalues of tournament matrices. These results can be found in [2] and [17], but for the sake of completeness, we will sketch the proofs here. Given an $n \times n$ tournament matrix M, let x be an eigenvector of M corresponding to the eigenvalue λ. Pre- and postmultiplying the equation $M + M^t = J - I$ by x^* and x respectively yields $(2 \operatorname{Re} \lambda + 1)x^*x = x^*jj^tx$, where j is the all ones vector. Thus we have $\operatorname{Re} \lambda \geqslant -\frac{1}{2}$ (see [2]), with equality holding if and only if $x^*j = 0$ (see [17]). Applying the Cauchy-Schwartz inequality to x^*jj^tx gives $(2 \operatorname{Re} \lambda + 1)x^*x = x^*jj^tx \leqslant j^tjx^*x = nx^*x$, with equality holding if and only if x is a scalar multiple of j. It follows that the Perron value of a tournament matrix is at most $(n - 1)/2$ (see [2]), with equality holding if and only if the tournament matrix is *regular*, that is, each of its row sums is $(n - 1)/2$ (see [17]).

It has been shown by Maybee and Pullman [15] that if λ is an eigenvalue of a tournament matrix, then its geometric multiplicity is one whenever $\operatorname{Re} \lambda \neq -\frac{1}{2}$. On the other hand, given a tournament matrix M with eigenvalue λ and $\operatorname{Re} \lambda = -\frac{1}{2}$, the following argument shows that the geometric and algebraic multiplicities of λ are the same.

If the two multiplicities are different, then there is an eigenvector x corresponding to λ and a generalized eigenvector y satisfying $My = \lambda y + x$. Then $x^*My + x^*M^ty = x^*Jy - x^*y$. Since $x^*J = 0$, we have $x^*(\lambda y + x) + \bar{\lambda}x^*y = -x^*y$, which yields $x^*x = 0$, a contradiction. Thus no such y can exist, so that the algebraic and geometric multiplicities of λ must coincide.

Next we note that if M is an irreducible tournament matrix of order $n \geqslant 3$, then M has at least three distinct eigenvalues. This follows from the fact that the trace of M^2 is zero, so not all of the eigenvalues of M can be real. Thus M has at least one conjugate pair of nonreal eigenvalues, as well as a Perron value, for a total of at least three distinct eigenvalues. Evidently the algebraic multiplicity of a nonreal eigenvalue of an $n \times n$ tournament

matrix is at most $(n-1)/2$. In the next section, we characterize the class of such matrices that achieve equality.

3. TOURNAMENT MATRICES WITH THREE DISTINCT EIGENVALUES

A tournament matrix H of order n is a *Hadamard tournament matrix* if it satisfies the equation $HH^t = [(n+1)/4]I + [(n-3)/4]J$. Note that necessarily $n \equiv 3 \pmod 4$. We remark that the question of the existence of such matrices is a difficult one, since Hadamard tournament matrices of order n are known to be coexistent with skew Hadamard matrices of order $n+1$. See Geramita and Seberry [10] for a discussion.

PROPOSITION 3.1. *Let H be a Hadamard tournament matrix of order n. Then H has eigenvalues $(n-1)/2$ (with algebraic multiplicity one) and $-\frac{1}{2} \pm i\sqrt{n}/2$ (each with algebraic multiplicity $(n-1)/2$).*

Proof. Let $p(\lambda) = \det(\lambda I - H)$ and $c = (n-3)/4$. Then $p^2(\lambda) = \det(\lambda I - H)\det(\lambda I - H^t) = \det[\lambda^2 I - \lambda(H + H^t) + HH^t] = \det[(c-\lambda)J + (\lambda^2 + \lambda + c + 1)I] = [\lambda^2 + \lambda + c + 1 + n(c-\lambda)](\lambda^2 + \lambda + c + 1)^{n-1}$, the last equality following from the general fact that $\det(aI + bJ) = (a + nb)a^{n-1}$. Thus we have

$$p^2(\lambda) = \left(\lambda - \frac{n-1}{2}\right)^2 \left(\lambda^2 + \lambda + \frac{n+1}{4}\right)^{n-1},$$

so that

$$p(\lambda) = \left(\lambda - \frac{n-1}{2}\right)\left(\lambda^2 + \lambda + \frac{n+1}{4}\right)^{(n-1)/2}. \qquad \blacksquare$$

Having seen that Hadamard tournament matrices have exactly three distinct eigenvalues (and hence their nonreal eigenvalues have maximum algebraic multiplicity), the question arises whether these are the only such irreducible tournament matrices. Our next result will help us to answer this question in the affirmative.

THEOREM 3.1. *Suppose that M is an irreducible nonnegative matrix of order $n \geqslant 4$, with integer entries and Perron value ρ. Further suppose that the trace of M^2 is zero and that M has exactly three distinct eigenvalues. Then n is odd, ρ is an integer, and $(n-1)/2$ divides ρ.*

Proof. Let the eigenvalues of M be ρ, λ_1, and λ_2. Since the trace of M^2 is zero, λ_1 and λ_2 must be a complex conjugate pair, say $\alpha \pm i\beta$, with $\beta \neq 0$. Further, ρ is an algebraically simple eigenvalue, so λ_1 and λ_2 must both have algebraic multiplicity $(n-1)/2$; in particular, n is odd. The trace of M^2 is zero. Since M is nonnegative, its diagonal entries are zero, so the trace of M is also zero. It follows from these two facts that $\alpha = -\rho/(n-1)$ and $\beta = \rho\sqrt{n}/(n-1)$, so that the eigenvalues of M are ρ and $\rho(-1 \pm i\sqrt{n})/(n-1)$.

Note that the traces of M^3 and M^4 are both integers. Since $[\rho/(n-1)]^3(-1+i\sqrt{n})^3 = \rho^3[-1+3n+i\sqrt{n}(3-n)]/(n-1)^3$, we see that the trace of M^3 is $\rho^3 + (n-1)\rho^3(-1+3n)/(n-1)^3 = \rho^3 n(n+1)/(n-1)^2 = i_1 \in \mathbb{N}$. Also $[\rho/(n-1)]^4(-1+i\sqrt{n})^4 = \rho^4[1-6n+n^2+i\sqrt{n}(4n-4)]/(n-1)^4$, and hence the trace of M^4 is $\rho^4 + (n-1)\rho^4(1-6n+n^2)/(n-1)^4 = \rho^4 n(n-3)(n+1)/(n-1)^3 = i_2 \in \mathbb{N}$. Taking quotients, we find that $i_2/i_1 = \rho(n-3)/(n-1)$, so that ρ is a rational number. Since ρ is a rational root of $\det(\lambda I - M)$, which is a monic polynomial with integer coefficients, ρ must be an integer. Since λ_1 and λ_2 are algebraic integers, so is $\lambda_1 + \lambda_2 = -2\rho/(n-1)$. It follows that $-2\rho/(n-1)$ must be an integer, so that $(n-1)/2$ divides ρ.   ■

THEOREM 3.2. *Suppose that M is an irreducible tournament matrix of order $n \geqslant 3$. Then M has exactly three eigenvalues if and only if M is a Hadamard tournament matrix.*

Proof. The sufficiency is just Proposition 3.1. To see the other implication, we may assume that $n > 3$, since

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and its transpose are the only irreducible 3×3 tournament matrices, and both are Hadamard tournament matrices. Consider an irreducible tournament matrix M of order $n \geqslant 4$ with three distinct eigenvalues. Since Theorem 3.1 applies to M, its Perron value is at least $(n-1)/2$. But the Perron value of any tournament matrix of order n is at most $(n-1)/2$, so the

maximum eigenvalue of M is $(n-1)/2$, and hence M is a regular tourna-
ment matrix; i.e., $MJ = JM = [(n-1)/2]J$. Note that M is a normal matrix,
since $MM^t = MJ - M - M^2 = JM - M - M^2 = M^tM$. From the proof of
Theorem 3.1, we see that the eigenvalues of M are $(n-1)/2$ and $-\frac{1}{2} \pm i\sqrt{n}/2$.

Since M is normal, there is an orthonormal basis of $\mathbb{C}^n$ consisting of
eigenvectors of M, say j (which is a Perron vector) and w_k, $k = 1, 2, \ldots, n-1$.
As noted above, $MM^t = MJ - M - M^2 = [(n-1)/2]J - M - M^2$. Thus
$MM^tj = [(n-1)/2]Jj - Mj - M^2j = [(n-1)^2/4]j = \{[(n-3)/4]J + [(n+1)/4]I\}j$. Since $j^tw_k = 0$ for $1 \leqslant k \leqslant n-1$, we have $MM^tw_k = MJw_k - Mw_k - M^2w_k = [(n+1)/4]w_k = \{[(n-3)/4]J + [(n+1)/4]I\}w_k$. Thus MM^t and
$[(n-3)/4]J + [(n+1)/4]I$ agree on the orthonormal basis, so M is a
Hadamard tournament matrix.   ∎

REMARK 3.1. The last part of Theorem 3.2 can be deduced at least two
other ways: by noting that $\{M - [(n-1)/2]I\}\{M^2 + M + [(n+1)/4]I\} = O$
and using the fact that the null space of $M - [(n-1)/2]I$ is spanned by j; or
by applying a theorem of Ryser (see [18]) characterizing $(0,1)$ matrices with
maximum determinant as design matrices.

4. A FAMILY OF TOURNAMENT MATRICES WITH DISTINCT EIGENVALUES

Recall that U_n is obtained from the transitive tournament matrix T_n by
exchanging its $(1,n)$ and $(n,1)$ entries. Each U_n is irreducible, since its
directed graph has the Hamilton cycle $(1, 2, \ldots, n-1, n, 1)$. Let $p_n(\lambda) = \det(\lambda I - U_n)$.

We will use the following technique for calculating $p_n(\lambda)$ and other
characteristic polynomials (see [8, p. 32]; [14]). Given an $n \times n$ $(0,1)$ matrix
with 0 diagonal, we denote its associated directed graph by D. Let S be a
union of vertex disjoint directed cycles in D; such an S is called a *disjoint
cycle union*. Let $\nu(S)$ be the number of vertices in S, and let $c(S)$ be the
number of cycles in S. Then

$$\det(\lambda I - M) = \sum_S (-1)^{c(S)} \lambda^{n - \nu(S)}, \qquad (4.1)$$

where the sum is taken over all disjoint cycle unions in D (including the
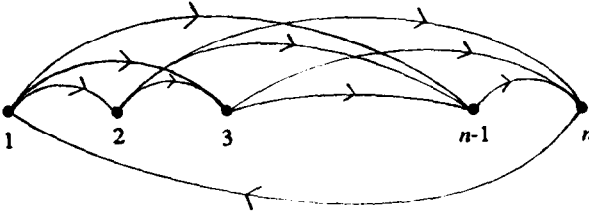empty one).

FIG. 1.

THEOREM 4.1. *For each $n \geqslant 3$, $p_n(\lambda) = \lambda^{n-2}(\lambda^2 + 1) - (\lambda + 1)^{n-2}$.*

Proof. The directed graph associated with U_n is shown in Figure 1. Each nonempty disjoint cycle union in the graph is a single cycle of the form $(n, 1, i_1, i_2, \ldots, i_k, n)$, where $2 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n - 1$. Thus the nonempty disjoint cycle unions are in one-to-one correspondence with the subsets of $\{2, 3, \ldots, n - 1\}$. It follows that

$$p_n(\lambda) = \lambda^n - \sum_{k=1}^{n-2} \binom{n-2}{k} \lambda^{n-k-2} = \lambda^n + \lambda^{n-2} - (\lambda + 1)^{n-2}. \qquad \blacksquare$$

THEOREM 4.2. *The eigenvalues of U_n all have algebraic multiplicity one. Further, when n is odd, the only real eigenvalue of U_n is the Perron value, while if n is even, there is only one more real eigenvalue, a negative one which is strictly greater than $-\frac{1}{2}$.*

Proof. Since $p_n(0) \neq 0$, we have $p_n(\lambda) = 0$ if and only if $\lambda^2 + 1 = (1 + \lambda^{-1})^{n-2}$. Set $g(\lambda) = \lambda^2 + 1$ and $h_n(\lambda) = (1 + \lambda^{-1})^{n-2}$. Since g is increasing (without bound) for positive λ while h_n decreases from ∞ to 1 for positive λ, we see that their graphs have a unique intersection for $\lambda > 0$, which corresponds to the Perron value of U_n. When $\lambda < 0$, note that when n is odd, $h_n(\lambda) < 1$ while $g(\lambda) > 1$, so that p_n has no negative roots for odd n. If n is even, then h_n increases without bound from 1, and g decreases from $\frac{5}{4}$ to 1 as λ runs between $-\frac{1}{2}$ and 0; note that if $\lambda < -\frac{1}{2}$ then $p_n(\lambda)$ cannot be 0. Thus the graphs of h_n and g have a unique intersection for negative λ, and it occurs for some λ between $-\frac{1}{2}$ and 0. Hence p_n has a simple negative root [in the interval $(-\frac{1}{2}, 0)$] when n is even.

To show that all of the complex roots of p_n are also simple, consider what happens if p_n has a complex root, say $\lambda = \mu + i\nu$ ($\nu \neq 0$) of multiplicity at

least two. Certainly $n \geq 5$ and $p_n(\lambda) = p'_n(\lambda) = 0$. This yields both $(1 + \lambda^{-1})^{n-2} = \lambda^2 + 1$ and $(1 + \lambda^{-1})^{n-3} = -2\lambda^3/(n-2)$. Thus $\lambda^2 + 1 = (1 + \lambda^{-1})[-2\lambda^3/(n-2)]$, which can be rearranged as $\lambda^3 + (n/2)\lambda^2 + (n-2)/2 = 0$. Substituting $\lambda = \mu + i\nu$ and taking real and imaginary parts (respectively) of the last equation, we have $\mu^3 - 3\mu\nu^2 + (n/2)(\mu^2 - \nu^2) + (n-2)/2 = 0$ and $\nu(3\mu^2 + n\mu - \nu^2) = 0$. Thus $\nu^2 = 3\mu^2 + n\mu$, which upon substitution into the latter yields $8\mu^3 + 4n\mu^2 + (n^2/2)\mu = (n-2)/2$, or $\mu(\mu + n/4)^2 = (n-2)/16$. If $\mu < 0$, then $\mu < -n/3$, because $\nu^2 = 3\mu^2 + n\mu > 0$, contradicting the necessity of $\mu \geq -\frac{1}{2}$ if λ is an eigenvalue of a tournament matrix (see [2]). Hence, writing $4\mu + n = \delta$, μ must be the unique positive solution to $\mu\delta^2 = n-2$, from which it follows that $\mu < 1/n$.

Now $|\lambda^2 + 1|^2 = |1 - 2\mu^2 - n\mu + i2\nu\mu|^2 = 16\mu^4 + 8n\mu^3 + (n^2 - 4)\mu^2 - 2n\mu + 1$. We have $|\lambda^2 + 1|^2 < 2 + 16/n^4 + 4/n^2 \leq 2.1856$ because $\mu < 1/n$ and $n \geq 5$. On the other hand, $|1 + \lambda^{-1}|^2 = |1 + \bar{\lambda}/(\mu\delta)|^2 = 1 + 2/\delta + 4/\delta^2 + n/(\mu\delta^2) > 1 + (n/\delta)^2$, the last because $\mu\delta^2 = n-2$ and $\delta > n$. Thus $|1 + \lambda^{-1}|^2 > 1 + (1 + 4\mu/n)^{-2} > 1 + (1 + \frac{4}{25})^{-2} = 1.743\ldots$, since $\mu^{-1} \geq n \geq 5$. As a result, $|(1 + \lambda^{-1})^{n-2}|^2 > (1.74)^2 > 2.1856 > |\lambda^2 + 1|^2$, and hence λ cannot be a solution to $(1 + \lambda^{-1})^{n-2} = \lambda^2 + 1$. Consequently p_n has no nonreal roots of multiplicity two or more. ∎

THEOREM 4.3.   *The right eigenvectors of U_n corresponding to the eigenvalue λ are scalar multiples of $y = (\lambda^2, \alpha^{n-3}, \alpha^{n-4}, \ldots, \alpha, 1, \lambda)^t$, where $\alpha = 1 + \lambda^{-1}$.*

Proof.   Let $z = U_n y$; we will show that $z = \lambda y$. Let the ith components of z and y be z_i and y_i respectively, $1 \leq i \leq n$. Multiplying y by U_n, we have $z_1 = \sum_{j=0}^{n-3}\alpha^j, z_{n-1} = \lambda, z_n = \lambda^2$, and $z_r = \lambda + \sum_{j=0}^{n-2-r}\alpha^j$ for $2 \leq r \leq n-2$. Evidently $z_{n-1} = \lambda y_{n-1}$ and $z_n = \lambda y_n$. For $2 \leq r \leq n-2$, we have $z_r = \lambda + (1 - \alpha^{n-1-r})/(1 - \alpha) = \lambda\alpha^{n-1-r} = \lambda y_r$. Similarly, $z_1 = (1 - \alpha^{n-2})/(1 - \alpha) = \lambda(1 + \lambda^{-1})^{n-2} - \lambda = \lambda y_1$, since λ is a root of the characteristic polynomial of U_n. ∎

A similar argument shows that a left eigenvector of U_n corresponding to λ is a scalar multiple of $w = (\lambda, 1, \alpha, \alpha^2, \ldots, \alpha^{n-3}, \lambda^2)$. In particular when n is even, the right and left eigenvectors, y and w respectively, which are associated with the negative root of $p_n(\lambda)$ have the sign pattern $\text{sgn}(y^t) = (+, -, +, -, \ldots, +, -) = -\text{sgn}(w)$. Moreover, Theorem 4.3 also implies that no entry in any eigenvector of U_n is 0 (since 0 and -1 are never eigenvalues of U_n). See [13] for a discussion of zero entries in eigenvectors.

Tournaments and directed graphs whose adjacency matrices have distinct eigenvalues have been discussed in the graph theory literature. Specifically, Cameron [7] points out that the automorphism groups of such directed

graphs must be abelian. As the referee for this work has noted, the fact that the automorphism group of the tournament corresponding to U_n is abelian also follows from its degree sequence.

5. MINIMUM PERRON VALUE AND THE MULTIPLICITY OF ZERO

Let π_n denote the minimum possible Perron value over the class of $n \times n$ irreducible tournament matrices. Let α_n be the maximum possible algebraic multiplicity of zero over the same class (we remark that $\alpha_n = 0$ if $3 \leqslant n \leqslant 5$). Our first result relates these two quantities.

LEMMA 5.1. *Let M be an irreducible tournament matrix of order $n \geqslant 3$ with Perron value ρ. Let m_0 be the algebraic multiplicity of 0 as an eigenvalue of M. Then $m_0 \leqslant n - 1 - 2\rho$, and in particular $\alpha_n \leqslant n - 1 - 2\pi_n$.*

Proof. Let the spectrum of M be $\{\rho, \lambda_2, \lambda_3, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_n\}$, where $\mathrm{Re}\,\lambda_i \neq 0$ if and only if $i \leqslant r$. Since the trace of M is zero, we have $\rho = -\sum_{i=2}^r \mathrm{Re}\,\lambda_i$. But $\mathrm{Re}\,\lambda_i \geqslant -\frac{1}{2}$ for $2 \leqslant i \leqslant n$, so $r \geqslant 2\rho + 1$. Thus we have $m_0 \leqslant n - r \leqslant n - 1 - 2\rho$, from which it follows that $\alpha_n \leqslant n - 1 - 2\pi_n$. ∎

The lemma above suggests that in order to find an upper bound on α_n, we should try to estimate π_n. In [3], Brauer and Gentry showed that $\pi_n > \sqrt[3]{2}$ for $n \geqslant 3$. We will improve that bound and show that the sequence of π_n's is bounded above in our next theorem.

THEOREM 5.1. *For all $n \geqslant 8$, $2 < \pi_n < 2.5$.*

Proof. We consider the family of tournament matrices defined by $S_n = [s_{ij}]$, $i, j = 1, 2, \ldots, n$, with $s_{ij} = 1$ if and only if $j > i + 1$ or $j = i - 1$. Note that S_n is obtained from the transitive tournament T_n by exchanging the entries in the first sub- and superdiagonals. Thus

$$S_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_4 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$
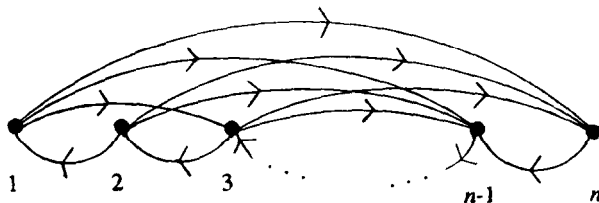
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and

$$S_n = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The transposes of these matrices were studied by Katzenberger and Shader [12]. Note that each S_n of order $n \geqslant 3$ is irreducible, since its digraph has the Hamilton cycle $(n, n-1, \ldots, 2, 1, n)$. The directed graph D_n associated with S_n is pictured in Figure 2.

Let $l_1(\lambda) = \lambda$ and $l_n(\lambda) = \det(\lambda I - S_n)$ for $n \geqslant 2$; we claim that for $n \geqslant 4$

$$l_n(\lambda) = (\lambda + 1)l_{n-1}(\lambda) - \lambda l_{n-2}(\lambda) - l_{n-3}(\lambda). \tag{5.1}$$

To show the claim, first partition the disjoint cycle unions in D_n into sets T_1, T_2, and T_3, where a disjoint cycle union S in D_n is in T_1 if the arc $(n, n-1)$ is not in S, in T_2 if the path $(n-2, n, n-1)$ is in S, and in T_3 if the arc $(n, n-1)$ is in S but the arc $(n-2, n)$ is not. We have

$$l_n(\lambda) = \sum_{S \in T_1} (-1)^{c(S)} \lambda^{n-\nu(S)} + \sum_{S \in T_2} (-1)^{c(S)} \lambda^{n-\nu(S)} + \sum_{S \in T_3} (-1)^{c(S)} \lambda^{n-\nu(S)}.$$

$$\tag{5.2}$$

Since each S in T_1 is in D_{n-1}, the first sum in (5.2) is $\lambda l_{n-1}(\lambda)$. If S is in T_2, then it consists of the 3-cycle $(n, n-1, n-2, n)$ along with a disjoint cycle union in D_{n-3}, so that the second sum in (5.2) is $-l_{n-3}(\lambda)$. Finally,

the disjoint cycle unions S in T_3 are in one-to-one correspondence with the disjoint cycle unions in D_{n-1} which involve the arc $(n-1, n-2)$ (i.e. those not in T_1 in D_{n-1}). It follows that the third sum in (5.2) is $l_{n-1}(\lambda) - \lambda l_{n-2}(\lambda)$, which yields (5.1). Note that setting $\lambda = 0$ in Eq. (5.1) yields the recurrence relation for the determinants of the S_n's that was found by Katzenberger and Shader in [12].

We will now show by induction that if $\lambda \geqslant 2.5$ and $n \geqslant 1$, then $l_n(\lambda) \geqslant 2l_{n-1}(\lambda) > 0$; this can be verified directly if $n \leqslant 3$. If $\lambda \geqslant 2.5$ and $n \geqslant 4$, then by the induction hypothesis we have

$$l_n(\lambda) = (\lambda + 1)l_{n-1}(\lambda) - \lambda l_{n-2}(\lambda) - l_{n-3}(\lambda)$$

$$\geqslant \left(\frac{\lambda}{2} + \frac{3}{4}\right)l_{n-1}(\lambda) \geqslant 2l_{n-1}(\lambda) > 0.$$

It follows that the Perron value of S_n, γ_n say, is less than 2.5 for any $n \geqslant 3$, so that $\pi_n < 2.5$ for $n \geqslant 3$. A result of Moser and Harary (see [16, Theorem 3, p. 6]) implies that when $n > 3$, every irreducible tournament matrix of order n contains an irreducible principal submatrix of order $n-1$ (also a tournament matrix). It follows from a theorem of Wielandt (see [11, Theorem 8.4.5, p. 509]) that irreducible principal submatrices of an irreducible matrix A have Perron values at most that of A. Thus we see that $\pi_n \leqslant \pi_{n+1}$ for $n \geqslant 3$ (indeed, a similar argument shows that $\gamma_n \leqslant \gamma_{n+1}$ for $n \geqslant 3$). A computer search of all irreducible tournament matrices of order 8 (using a disk listing all tournament matrices of orders $n \leqslant 8$ kindly supplied by Professor R. Read) showed that $\gamma_8 = \pi_8 = 2.0606\ldots$ and hence $2 < \pi_n < 2.5$ for all $n \geqslant 8$. ∎

Both the π_n's and the γ_n's are nondecreasing and bounded above, so it is natural to wonder about their respective limits. Led by an observation of the referee on the characteristic equation of the recurrence for l_n, we have been able to show that

$$\gamma_n \to \frac{-1 + \sqrt{13 + 16\sqrt{2}}}{2} = 2.4844353\ldots \qquad \text{as} \quad n \to \infty.$$

Brualdi and Li [6] conjecture that $\gamma_n = \pi_n$ for any $n \geqslant 3$; numerical results using Professor Read's disk show that the latter statement holds when $3 \leqslant n \leqslant 8$.

Applying the lower bound on π_n and Lemma 5.1, we have the following result.

COROLLARY 5.1.1.   *For* $n \geqslant 8$, $\alpha_n \leqslant n - 6$.

We now examine a class of irreducible tournament matrices that yield a lower bound on α_n.

For $p, q \geqslant 1$, denote the $p \times q$ matrix of ones by $J_{p,q}$ and of zeros by $O_{p,q}$. We define

$$
M_{k,l} = \left[\begin{array}{ccccc|ccccc}
 & & U_k & & & \multicolumn{5}{c}{J_{k-1,l}} \\
\hline
 & & & & & 0 & 1 & 1 & \cdots & 1 \\
\hline
0 & 0 & \cdots & 0 & 1 & \multicolumn{5}{c}{} \\
\hline
\multicolumn{5}{c|}{O_{l-1,k}} & \multicolumn{5}{c}{U_l}
\end{array} \right] \qquad \text{for all} \quad k, l \geqslant 3.
$$

Evidently, each $M_{k,l}$ is a tournament matrix. In the terminology of Brualdi and Li [5], $M_{k,l}$ is the *join* of U_k, U_2, and U_l. A result of Katzenberger and Shader [12] ensures that $M_{k,l}$ is singular for $l, k \geqslant 3$; the theorem below will find the algebraic multiplicity of 0 as an eigenvalue of $M_{k,l}$.

THEOREM 5.2.   *For each* $l, k \geqslant 3$, $M_{k,l}$ *is an irreducible tournament matrix and*

$$
\det(\lambda I - M_{k,l}) = \lambda^{k+l-4}(1 + \lambda^2)^2 - (1 + \lambda + \lambda^2)
$$

$$
\times \left[\lambda^{k-2}(1 + \lambda)^{l-2} + \lambda^{l-2}(1 + \lambda)^{k-2} \right].
$$

Proof. $M_{k,l}$ is irreducible, since its associated directed graph D has the Hamilton cycle $(k + l, k + 1, k, 1, 2, \ldots, k - 1, k + 2, k + 3, \ldots, k + l - 1, k + l)$. The subgraph of D induced by the vertices 1, k, $k + 1$, and $k + l$ is shown in Figure 3.

We partition the disjoint cycle unions in D into sets T_1, T_2, T_3, and T_4, where a disjoint cycle union S is in T_1 if the arc $(k + 1, k)$ is not in S, in T_2 if
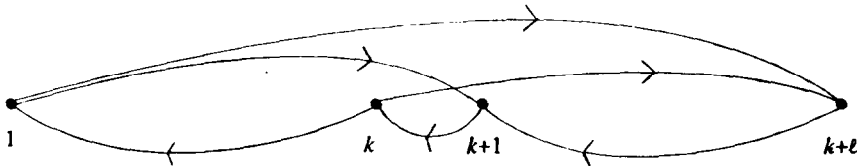

FIG. 3.

the arcs $(k + 1, k)$ and $(k, 1)$ are in S but the arc $(k + l, k + 1)$ is not, in T_3 if the arcs $(k + 1, k)$ and $(k + l, k + 1)$ are in S but the arc $(k, 1)$ is not, and in T_4 if the arcs $(k + 1, k)$, $(k + l, k + 1)$ and $(k, 1)$ are in S. Thus we have

$$\det(\lambda I - M_{k,l}) = \sum_{S \in T_1} (-1)^{c(S)} \lambda^{n - \nu(S)} + \sum_{S \in T_2} (-1)^{c(S)} \lambda^{n - \nu(S)}$$

$$+ \sum_{S \in T_3} (-1)^{c(S)} \lambda^{n - \nu(S)} + \sum_{S \in T_4} (-1)^{c(S)} \lambda^{n - \nu(S)}. \quad (5.3)$$

The disjoint cycle unions in T_1 are unions of a disjoint cycle union in the graph of U_k and disjoint cycle union in the graph of U_l; thus the first sum in (5.3) is $p_k(\lambda)p_l(\lambda)$. If S is in T_2, then it is $(k + 1, k, 1, k + 1)$ or a single cycle of the form $(k + 1, k, 1, i_1, \ldots, i_q, k + 1)$, where $2 \leqslant i_1 < \cdots < i_q \leqslant k - 1$. It follows that the second sum in (5.3) is

$$-\sum_{q=0}^{k-2} \binom{k-2}{q} \lambda^{k+l-q-3} = -\lambda^{l-1}(1 + \lambda)^{k-2}.$$

A similar argument shows that the third sum in (5.3) is $-\lambda^{k-1}(1 + \lambda)^{l-2}$. Finally, if S is in T_4, then it is $(k + l, k + 1, k, 1, k + l)$, or a single cycle of the form

$$(k + l, k + 1, k, 1, i_1, \ldots, i_q, j_1, \ldots, j_p, k + l),$$

where $2 \leqslant i_1 < \cdots < i_q \leqslant k - 1$ and $k + 2 \leqslant j_1 < \cdots < j_p \leqslant k + l - 1$.

It follows that the last sum in (5.3) is $-(1 + \lambda)^{k+l-4}$. Hence we find that $\det(\lambda I - M_{k,l}) = p_k(\lambda)p_l(\lambda) - \lambda^{l-1}(1 + \lambda)^{k-2} - \lambda^{k-1}(1 + \lambda)^{l-2} - (1 + \lambda)^{k+l-4}$, and applying the formula for $p_n(\lambda)$ in Theorem 4.1 yields the result. ∎

REMARK 5.1. It is clear from its definition that $M_{k,l}$ is obtained from the reducible tournament matrix

$$\tilde{M} = \left[\begin{array}{c|c} U_k & J_{k,l} \\ \hline O_{l,k} & U_l \end{array} \right]$$

by exchanging the $(k, k + 1)$ and $(k + 1, k)$ entries. Note that $\det(\lambda I - \tilde{M}) = \det(\lambda I - U_k)\det(\lambda I - U_l)$. Thus the effect of reversing the $(k, k + 1)$ and

$(k + 1, k)$ entries in $\lambda I - \tilde{M}$ (to obtain $\lambda I - M_{k,l}$) is to introduce the term $-[\lambda^{k-1}(1 + \lambda)^{l-2} + \lambda^{l-1}(1 + \lambda)^{k-2} + (1 + \lambda)^{k+l-4}]$ into the determinant.

THEOREM 5.3.   *For every $n \geqslant 6$ and every $1 \leqslant m \leqslant \lfloor n/2 \rfloor - 2$ there is an irreducible tournament matrix of order n having 0 as an eigenvalue with algebraic multiplicity m.*

Proof.   From our formula in Theorem 5.2, we see that $M_{k,l}$ has 0 as an eigenvalue of algebraic multiplicity $\min(k, l) - 2$. Thus, for each $n \geqslant 6$ and each $1 \leqslant m \leqslant \lfloor n/2 \rfloor - 2$, $M_{m+2, n-m-2}$ has 0 as an eigenvalue of algebraic multiplicity m.   ∎

COROLLARY 5.3.1.   *For $n \geqslant 8$, $\lfloor n/2 \rfloor - 2 \leqslant \alpha_n \leqslant n - 6$.*

Using Professor Read's disk, we found that $\alpha_6 = \alpha_7 = 1$ and $\alpha_8 = 2$. Further, the 9×9 tournament matrix below in Example 5.1 is irreducible and has 0 as an eigenvalue of algebraic multiplicity 3. Thus, $\alpha_9 = 3$ from Corollary 5.3.1.

EXAMPLE 5.1.   Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Its directed graph has the Hamilton cycle $(9, 7, 6, 4, 1, 2, 3, 5, 8, 9)$, so A is irreducible. Its characteristic polynomial is $\lambda^9 - 7\lambda^6 - 8\lambda^5 - 9\lambda^4 - 3\lambda^3$. Thus we see that when $n = 9$, the lower bound on α_n (given in Corollary 5.3.1) is too small, while the upper bound is sharp.

REMARK 5.2.   Each of the matrices in Theorems 4.1, 5.1, and 5.2 and in Example 5.1 has the row sum vector $(n - 2, n - 2, n - 3, n - 4 \ldots 3, 2, 1, 1)$. Because of a characterization of tournament matrices with this row sum vector given by Brualdi and Li (see [5]), the disjoint cycle union technique for finding characteristic polynomials works well for such matrices.

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