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# Collapsing estimates and the rigorous derivation of the 2d cubic nonlinear Schrödinger equation with anisotropic switchable quadratic traps

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**Abstract**

We consider the 2d and 3d many body Schrödinger equations in the presence of anisotropic switchable quadratic traps. We extend and improve the collapsing estimates in Klainerman and Machedon (2008) [25] and Kirkpatrick, Schlein and Staffilani (2011) [23]. Together with an anisotropic version of the generalized lens transform in Carles (2011) [3], we derive rigorously the cubic NLS with anisotropic switchable quadratic traps in 2d through a modified Elgart–Erdös–Schlein–Yau procedure. For the 3d case, we establish the uniqueness of the corresponding Gross–Pitaevskii hierarchy without the assumption of factorized initial data.


**Résumé**

On considère les équations de Schrödinger à plusieurs corps en présence de pièges quadratiques anisotropes et commutables pour les dimensions 2 et 3. On étend et on améliore les estimations d'écroulement de Klainerman et Machedon (2008) [25] et de Kirkpatrick, Schlein et Staffilani (2011) [23]. En utilisant une version anisotrope de la transformation lenticulaire généralisée de Carles (2011) [3] on déduit rigoureusement, pour la dimension 2, la cubique NSL en présence de pièges quadratiques anisotropes et commutables par la méthode de Elgart–Erdös–Schlein–Yau modifiée. Pour la dimension 3 on établit l'unicité de la hiérarchie de Gross–Pitaevskii correspondante sans l'hypothèse d'une donnée initiale factorisée.


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*Keywords:* Gross–Pitaevskii hierarchy; Anisotropic switchable quadratic trap; Collapsing estimate; Metaplectic representation

## 1. Introduction

Bose–Einstein condensation (BEC) is the phenomenon that particles of integer spin ("Bosons") occupy a macroscopic quantum state. The first experimental observation of BEC in an interacting atomic gas occurred in 1995 [1,10]. Many similar experiments were performed later [9,22,29]. In these laboratory experiments, the particles are initially confined by traps, e.g., the magnetic fields in [1,10], then the traps are switched in order to enable observation. To be

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more precise about the word “switch”: in [1,10] the trap is removed, in [29] the initial magnetic trap is switched to an optical trap, in [9] the trap is turned off in 2 spatial directions to generate a 2d Bose gas. The dynamic during the period when the trap is shifting is sophisticated. To model the evolution in this process, we use a quadratic potential multiplied by a switch function in each spatial direction for analysis in this paper. This simplified yet reasonably general model is expected to capture the salient features of the actual traps: on the one hand the quadratic potential varies slowly and tends to $\infty$ as $|x| \to \infty$; on the other hand, the switch functions describe the space–time anisotropic properties of the confining potential. In the physics literature, Lieb, Seiringer and Yngvanson remarked in [26] that the confining potential is typically $\sim |x|^2$ in the available experiments. Mathematically speaking, the strongest trap we can deal with in the usual regularity setting of NLS is the quadratic trap since the work [30] by Yajima and Zhang points out that the ordinary Strichartz estimates start to fail as the trap exceeds quadratic.

Motivated by the above considerations, we aim to investigate the evolution of a many-body Boson system during the alteration of the trap. The $N$-body wave function $\psi_N(\tau, \overrightarrow{\mathbf{y}_N})$ solves the many body Schrödinger equation with anisotropic switchable quadratic traps:

$$i\partial_\tau \psi_N = \frac{1}{2} H_{\overrightarrow{\mathbf{y}_N}}(\tau)\psi_N + \frac{1}{N}\sum_{i<j} N^{n\beta} V\big(N^\beta(\mathbf{y}_i - \mathbf{y}_j)\big)\psi_N,$$

$$\psi_N(0, \overrightarrow{\mathbf{y}_N}) = \prod_{j=1}^{N} \phi_0(\mathbf{y}_j), \tag{1.1}$$

where $\tau \in \mathbb{R}$, $\overrightarrow{\mathbf{y}_N} = (\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_N) \in \mathbb{R}^{nN}$, $V$ is the interaction between particles, and

$$H_{\overrightarrow{\mathbf{y}_N}}(\tau) := \sum_{j=1}^{N} H_{\mathbf{y}_j}(\tau) := \sum_{j=1}^{N}\left(\sum_{l=1}^{n}\left(-\frac{\partial^2}{\partial y_{j,l}^2} + \eta_l(\tau) y_{j,l}^2\right)\right) \tag{1.2}$$

with the switch functions $\eta_l(\tau)$, $l = 1, \dots, n$. Throughout this paper, we only consider $n = 2$ or $3$ and we assume the switch functions $\eta_l \in C^1(\mathbb{R}_0^+ \to \mathbb{R}_0^+)$ satisfy the following conditions.

**Condition 1.** $\dot{\eta}_l(0) = 0$ *i.e. the trap is not at a switching stage initially.*

**Condition 2.** $\dot{\eta}_l$ *is supported in* $[0, T_0]$ *and* $T_0\sqrt{\sup_\tau |\eta_l(\tau)|} < \frac{\pi}{2}$.

When the trap is fully on, Lieb, Seiringer, Solovej and Yngvanson showed that the ground state of the Hamiltonian exhibits complete BEC in [27], provided that the trapping potential $V_{trap}(x)$ satisfies $\inf_{|x|>R} V_{trap}(x) \to \infty$ for $R \to \infty$ and the interaction potential is spherically symmetric. To be more precise, let $\psi_{N,0}$ be the ground state, then

$$\gamma_{N,0}^{(1)} \to |\phi_{GP}\rangle\langle\phi_{GP}| \quad \text{as } N \to \infty,$$

where $\gamma_{N,0}^{(1)}$ is the corresponding one particle marginal density defined via formula (1.3) and $\phi_{GP}$ minimizes the Gross–Pitaevskii energy functional

$$\int \big(|\nabla\phi|^2 + V_{trap}(x)|\phi|^2 + 4\pi a_0|\phi|^4\big)\, d\mathbf{x}.$$

Because we are now considering the evolution while the trap is changing, we start with a BEC state (factorized state) in Eq. (1.1).

However, $\psi_N$ does not remain a product of one-particle states i.e.

$$\psi_N(\tau, \overrightarrow{\mathbf{y}_N}) \neq \prod_{j=1}^{N} \phi(\tau, \mathbf{y}_j), \quad \tau > 0,$$

for some one particle state $\phi$. Moreover it is unrealistic to solve the $N$-body equation (1.1) for large $N$. Thence, to observe BEC, we have to show mathematically that $\psi_N$ is very close to $\prod_{j=1}^{N} \phi(\tau, \mathbf{y}_j)$, the mean field approximation, in an appropriate sense.

Notice that when $\phi \neq \phi'$

$$\left\| \prod_{j=1}^{N} \phi(\tau, \mathbf{y}_j) - \prod_{j=1}^{N} \phi'(\tau, \mathbf{y}_j) \right\|_2^2 \to 2 \quad \text{as } N \to \infty,$$

i.e. our desired limit (the BEC state) is not stable against small perturbations. One way to circumvent this difficulty is to use the concept of the $k$-particle marginal density $\gamma_N^{(k)}$ associated with $\psi_N$ defined as

$$\gamma_N^{(k)}\big(\tau, \overrightarrow{\mathbf{y}_k}; \overrightarrow{\mathbf{y}_k'}\big) = \int \psi_N(\tau, \overrightarrow{\mathbf{y}_k}, \overrightarrow{\mathbf{y}_{N-k}}) \overline{\psi_N\big(\tau, \overrightarrow{\mathbf{y}_k'}, \overrightarrow{\mathbf{y}_{N-k}}\big)}\, d\overrightarrow{\mathbf{y}_{N-k}}, \quad \overrightarrow{\mathbf{y}_k}, \overrightarrow{\mathbf{y}_k'} \in \mathbb{R}^{nk}. \tag{1.3}$$

Another way is to add a second order correction to the mean field approximation. See [8,20,21].

In this paper, we take the marginal density approach and establish the following theorem.

**Theorem 1.** *Consider the* 2*d case when* $\beta \in (0, \frac{3}{4})$. *Assume the interaction potential* $V$ *is nonnegative and belongs to* $L^1 \cap W^{2,\infty}$ *and the switch functions* $\eta_l$ *satisfy Conditions* 1 *and* 2. *Moreover, suppose the initial data has bounded energy per particle*

$$\sup_N \frac{1}{N} \langle \psi_N, H_N(\tau)\psi_N \rangle \big|_{\tau=0} < \infty,$$

*where the Hamiltonian* $H_N(\tau)$ *is*

$$H_N(\tau) = \frac{1}{2} \sum_{j=1}^{N} \left( \sum_{l=1}^{2} \left( -\frac{\partial^2}{\partial y_{j,l}^2} + \eta_l(\tau) y_{j,l}^2 \right) \right) + \frac{1}{N} \sum_{i<j} N^{2\beta} V\big(N^\beta(\mathbf{y}_i - \mathbf{y}_j)\big).$$

*If* $\{\gamma_N^{(k)}\}$ *are the marginal densities associated with* $\psi_N$, *the solution of the* $N$*-body Schrödinger equation* (1.1), *and* $\phi$ *solves the* 2*d Gross–Pitaevskii equation*:

$$i\partial_\tau \phi - \frac{1}{2} H_{\mathbf{y}}(\tau)\phi = b_0 |\phi|^2 \phi,$$
$$\phi(0, \mathbf{y}) = \phi_0(\mathbf{y}),$$

*where* $H_{\mathbf{y}}(\tau)$ *is the operator inside formula* (1.2) *and* $b_0 = \int V(x)\, dx$, *then* $\forall \tau \in [0, T_0]$ *and* $k \geqslant 1$, *we have the convergence*:

$$\left\| \gamma_N^{(k)}\big(\tau, \overrightarrow{\mathbf{y}_k}; \overrightarrow{\mathbf{y}_k'}\big) - \prod_{j=1}^{k} \phi(\tau, \mathbf{y}_j) \overline{\phi\big(\tau, \mathbf{y}_j'\big)} \right\|_{L^2(d\overrightarrow{\mathbf{y}_k} d\overrightarrow{\mathbf{y}_k'})} \to 0 \quad \text{as } N \to \infty.$$

**Example 1.** We give a simple example to explain the switching process we are considering here: say

$$\eta_1(\tau) = C_1 \quad \text{when } \tau \in \left(-\infty, \frac{1}{2}\right], \quad C_2 \quad \text{when } \tau \in [1, \infty),$$
$$\eta_2(\tau) = C_3 \quad \text{when } \tau \in \left(-\infty, \frac{1}{4}\right], \quad C_4 \quad \text{when } \tau \in \left[\frac{3}{2}, \infty\right).$$

Then our switching process contains the cases: turning off (on): $C_2 = 0$ ($C_1 = 0$) and tuning up (down): $C_1 \leqslant C_2$ ($C_2 \leqslant C_1$). As long as $\eta_1(\tau) \in C^1$ and satisfies Condition 2, $\eta_1$ can behave as one likes inside $[\frac{1}{2}, 1]$. Same comment applies to $\eta_2$ too. Furthermore, Theorem 1 addresses the time intervals $(-\infty, 0]$ and $[\frac{3}{2}, \infty)$ as well. Since the equation is time translation invariant in these two intervals, we can use Theorem 1 separately in each sufficiently small time intervals.

**Remark 1.** Technically, one should interpret Conditions 1 and 2 in the following way. Due to Condition 1, we have a $C^1$ even extension of $\eta_l$ i.e. we define $\eta_l(\tau) = \eta_l(-\tau)$ for $\tau < 0$. The fast switching Condition 2 in fact ensures that $\beta_l$ defined via Eq. (4.1) is nonzero in $[0, T_0]$ which is crucial in this paper. See Claim 1 for the proof.

**Remark 2.** We assume $\beta \in (0, \frac{3}{4})$ to match Kirkpatrick–Schlein–Staffilani [23] in which the authors studied the $\eta_l = 0$ case. $\beta = 0$ will yield a Hartree equation instead of the cubic NLS.

The approach with $\gamma_N^{(k)}$ has been proven to be successful in the $\eta_l = 0$ and $n = 3$ case, which corresponds to the evolution after the removal of the traps, in the fundamental papers [11–17] by Elgart, Erdös, Schlein, and Yau. Their program, outlined by Spohn [28], consists of two principal parts: on the one hand, they prove that an appropriate limit of the sequence $\{\gamma_N^{(k)}\}_{k=1}^N$ as $N \to \infty$ solves the Gross–Pitaevskii hierarchy

$$\left(i\partial_t + \frac{1}{2}\triangle_{\vec{\mathbf{x}}_k} - \frac{1}{2}\triangle_{\vec{\mathbf{x}}'_k}\right)\gamma^{(k)} = b_0 \sum_{j=1}^{k} B_{j,k+1}\left(\gamma^{(k+1)}\right), \quad k = 1, \dots, n, \tag{1.4}$$

where $B_{j,k+1}$ are in formula (1.7); on the other hand, they show that hierarchy (1.4) has a unique solution which is therefore a completely factored state. However, the uniqueness theory for hierarchy (1.4) is surprisingly delicate due to the fact that it is a system of infinitely many coupled equations over an unbounded number of variables. In [25], by assuming a space–time bound, Klainerman and Machedon gave another proof of the uniqueness in [14] through a collapsing estimate and a board game argument. We call the space–time estimates of the solution of Schrödinger equations restricted to a subspace of $\mathbb{R}^n$ "collapsing estimates". We can interpret them as local smoothing estimates for which integrating in time results in a gain of one hidden derivative in the sense of the trace theorem. To be specific, the collapsing estimate of [25] reads: Suppose $u^{(k+1)}$ solves

$$\left(i\partial_t + \frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}_{k+1}}} - \frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}'_{k+1}}}\right)u^{(k+1)} = 0,$$

there is $C > 0$, independent of $j, k$ or $u^{(k+1)}(0, \overrightarrow{\mathbf{x}_{k+1}}; \overrightarrow{\mathbf{x}'_{k+1}})$ s.t.

$$\begin{aligned}&\left\| \left(\prod_{j=1}^{k}(\nabla_{\mathbf{x}_j}\nabla_{\mathbf{x}'_j})\right) u^{(k+1)}\left(t, \vec{\mathbf{x}}_k, \mathbf{x}_j; \vec{\mathbf{x}}'_k, \mathbf{x}_j\right) \right\|_{L^2(\mathbb{R}\times\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \\ &\quad \leqslant C \left\| \left(\prod_{j=1}^{k+1}(\nabla_{\mathbf{x}_j}\nabla_{\mathbf{x}'_j})\right) u^{(k+1)}\left(0, \overrightarrow{\mathbf{x}_{k+1}}; \overrightarrow{\mathbf{x}'_{k+1}}\right) \right\|_{L^2(\mathbb{R}^{3(k+1)}\times\mathbb{R}^{3(k+1)})}.\end{aligned} \tag{1.5}$$

Later, the method in Klainerman and Machedon [25] was taken up by Kirkpatrick, Schlein, and Staffilani in [23], where they studied the corresponding problem in 2d, and Chen, Pavlović and Tzirakis [4–6], in which they considered the 1d and 2d 3-body interaction problem and the general existence theory of hierarchy (1.4).

We are interested in the case $\eta_l \neq 0$. So we study the Gross–Pitaevskii hierarchy with anisotropic switchable quadratic traps. That is a sequence of functions $\{\gamma^{(k)}(\tau, \vec{\mathbf{y}}_k; \vec{\mathbf{y}}'_k)\}_{k=1}^{\infty}$, where $\tau \in \mathbb{R}$, $\vec{\mathbf{y}}_k, \vec{\mathbf{y}}'_k \in \mathbb{R}^{nk}$, which are symmetric, in the sense that

$$\gamma^{(k)}\left(\tau, \vec{\mathbf{y}}_k; \vec{\mathbf{y}}'_k\right) = \overline{\gamma^{(k)}\left(\tau, \vec{\mathbf{y}}'_k; \vec{\mathbf{y}}_k\right)}$$

and

$$\gamma^{(k)}\left(\tau, \mathbf{y}_{\sigma(1)}, \mathbf{y}_{\sigma(2)}, \dots, \mathbf{y}_{\sigma(k)}; \mathbf{y}'_{\sigma(1)}, \mathbf{y}'_{\sigma(2)}, \dots, \mathbf{y}'_{\sigma(k)}\right) = \gamma^{(k)}\left(\tau, \mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k; \mathbf{y}'_1, \mathbf{y}'_2, \dots, \mathbf{y}'_k\right)$$

for any permutation $\sigma$, since we are considering Bosons, and satisfy the anisotropic switchable quadratic traps Gross–Pitaevskii infinite hierarchy of equations:

$$\left(i\partial_\tau - \frac{1}{2}H_{\vec{\mathbf{y}}_k}(\tau) + \frac{1}{2}H_{\vec{\mathbf{y}}'_k}(\tau)\right)\gamma^{(k)} = b_0 \sum_{j=1}^{k} B_{j,k+1}\left(\gamma^{(k+1)}\right). \tag{1.6}$$

In the above, $B_{j,k+1} = B^1_{j,k+1} - B^2_{j,k+1}$ are defined as

$$\begin{aligned}B^1_{j,k+1}\left(\gamma^{(k+1)}\right)\left(\tau, \vec{\mathbf{y}}_k; \vec{\mathbf{y}}'_k\right) &= \int\int \delta(\mathbf{y}_j - \mathbf{y}_{k+1})\delta\left(\mathbf{y}_j - \mathbf{y}'_{k+1}\right)\gamma^{(k+1)}\left(\tau, \overrightarrow{\mathbf{y}_{k+1}}; \overrightarrow{\mathbf{y}'_{k+1}}\right)d\mathbf{y}_{k+1}\, d\mathbf{y}'_{k+1} \\ B^2_{j,k+1}\left(\gamma^{(k+1)}\right)\left(\tau, \vec{\mathbf{y}}_k; \vec{\mathbf{y}}'_k\right) &= \int\int \delta\left(\mathbf{y}'_j - \mathbf{y}_{k+1}\right)\delta\left(\mathbf{y}'_j - \mathbf{y}'_{k+1}\right)\gamma^{(k+1)}\left(\tau, \overrightarrow{\mathbf{y}_{k+1}}; \overrightarrow{\mathbf{y}'_{k+1}}\right)d\mathbf{y}_{k+1}\, d\mathbf{y}'_{k+1}.\end{aligned} \tag{1.7}$$

These Dirac delta functions in $B_{j,k+1}$ are the reason we consider the collapsing estimates like estimate (1.5).

When the initial data is a BEC state (factorized state)

$$\gamma^{(k)}\big(0,\overrightarrow{\mathbf{y}_k};\overrightarrow{\mathbf{y}_k'}\big)=\prod_{j=1}^{k}\phi_0(\mathbf{y}_j)\overline{\phi_0\big(\mathbf{y}_j'\big)},$$

hierarchy (1.6) admits one solution

$$\gamma^{(k)}\big(\tau,\overrightarrow{\mathbf{y}_k};\overrightarrow{\mathbf{y}_k'}\big)=\prod_{j=1}^{k}\phi(\tau,\mathbf{y}_j)\overline{\phi\big(\tau,\mathbf{y}_j'\big)},$$

which is also a BEC state, provided $\phi$ solves the $n$d Gross–Pitaevskii equation

$$\begin{aligned} i\partial_\tau\phi-\frac{1}{2}H_{\mathbf{y}}(\tau)\phi&=b_0|\phi|^2\phi,\\ \phi(0,\mathbf{y})&=\phi_0(\mathbf{y}).\end{aligned}$$

Hence we would like to have uniqueness theorems of hierarchy (1.6).

### 1.1. Main auxiliary theorems

To obtain Theorem 1, we need the auxiliary theorems in this subsection which are of independent interest. We show them in 3d as well. On the one hand, the general idea for the 2d case is derived from the higher dimensional case. On the other hand, the 2d and 3d cases are dramatically different when they are viewed in the context of Theorem 1. We will explain this difference between the 2d and 3d cases in Section 7. For the moment, notice that the uniqueness theorems in 2d and 3d address two different Gross–Pitaevskii hierarchies which stand for the two sides of the lens transform. Also, we currently do not have a 3d version of the 2d convergence (Theorem 1). We state our auxiliary theorems regarding different dimensions separately for comparison.

First, we have the following collapsing estimates which generalizes estimate (1.5).

**Theorem 2** *($3*nd$ optimal collapsing estimate). Let $n=2$ or $3$, write*

$$L_{\mathbf{x}}(t)=\sum_{l=1}^{n}a_l(t)\frac{\partial^2}{\partial x_l^2},$$

*where the $L^1_{loc}$ functions $a_l$ satisfy*

$$a_l\geqslant c_0>0\quad a.e.$$

*Assume $u(t,\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_2')$ solves the Schrödinger equation*

$$\begin{aligned} iu_t+L_{\mathbf{x}_1}(t)u+L_{\mathbf{x}_2}(t)u\pm L_{\mathbf{x}_2'}(t)u&=0\quad in\ \mathbb{R}^{3n+1},\\ u\big(0,\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_2'\big)&=f\big(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_2'\big),\end{aligned}\tag{1.8}$$

*then*

$$\int_{\mathbb{R}^{n+1}}\big||\nabla_{\mathbf{x}}|^{\frac{n-1}{2}}u(t,\mathbf{x},\mathbf{x},\mathbf{x})\big|^2\,d\mathbf{x}\,dt\leqslant C\big\||\nabla_{\mathbf{x}_1}|^{\frac{n-1}{2}}|\nabla_{\mathbf{x}_2}|^{\frac{n-1}{2}}|\nabla_{\mathbf{x}_2'}|^{\frac{n-1}{2}}f\big\|_2^2.$$

Theorem 2 is a scale invariant estimate when $a_l=1$ hence it is optimal. In fact, it holds for all $n\geqslant 2$. The proof is different for $n=2$ and $n\geqslant 3$. We name the third spatial variables $\mathbf{x}_2'$ to match the uniqueness theorems. We point out that Kirkpatrick, Schlein and Staffilani proved the almost optimal result for the 2d constant coefficient case in [23]. Some other collapsing estimates were attained in [7,19].

*1.1.1. 2d auxiliary theorems*

Theorem 2 is the key to show the following uniqueness theorem.

**Theorem 3** *(Uniqueness of 2d GP with time-dependent coefficients). Let $L_{\mathbf{x}_k}$ be as in Theorem 2 and $B_{j,k+1}$ be defined via formula (1.7). Say $\{u^{(k)}(\tau, \overrightarrow{\mathbf{y}_k}; \overrightarrow{\mathbf{y}'_k})\}_{k=1}^{\infty}$ solves the Gross–Pitaevskii hierarchy with variable coefficients*

$$\left(i\partial_t + L_{\overrightarrow{\mathbf{x}_{k+1}}}(t) - L_{\overrightarrow{\mathbf{x}'_{k+1}}}(t)\right)u^{(k)} = b_0 \sum_{j=1}^{k} B_{j,k+1}\left(u^{(k+1)}\right),$$

*subject to zero initial data and the space–time bound*

$$\int_0^T \left\| \left( \prod_{j=1}^{k} |\nabla_{\mathbf{x}_j}|^{\frac{1}{2}} |\nabla_{\mathbf{x}'_j}|^{\frac{1}{2}} \right) B_{j,k+1} u^{(k+1)}(t, \cdot; \cdot) \right\|_{L^2(\mathbb{R}^{2k}\times\mathbb{R}^{2k})} dt \leqslant C^k$$

*for some $C > 0$ and all $1 \leqslant j \leqslant k$. Then $\forall k, t \in [0, T]$,*

$$\left\| \prod_{j=1}^{k} \left( |\nabla_{\mathbf{x}_j}|^{\frac{1}{2}} |\nabla_{\mathbf{x}'_j}|^{\frac{1}{2}} \right) u^{(k)}(t, \cdot; \cdot) \right\|_{L^2(\mathbb{R}^{2k}\times\mathbb{R}^{2k})} = 0.$$

In contrast to the standard Elgart–Erdös–Schlein–Yau program, we do not need a uniqueness theorem regarding the Gross–Pitaevskii hierarchy with anisotropic switchable quadratic traps (hierarchy (1.6)) to establish Theorem 1. It is enough to have Theorem 3 which has no quadratic potential inside. At a glance, the analysis of the above hierarchy based on the Laplacian is unrelated to the hierarchy (1.6) based on a Hermite like operator $H_{\mathbf{y}}(\tau)$. However, Carles' generalized lens transform [3] links them together. In fact, the generalized lens transform preserves $L^2$ critical NLS and thus the 2d Gross–Pitaevskii hierarchies. The specific version of the lens transform we need is in Section 4.

*1.1.2. 3d auxiliary theorems*

As mentioned before, the uniqueness theorem here addresses a different hierarchy from Theorem 3. Of course we can prove a 3d version of Theorem 3. However, the disparity between the 2d and 3d case renders such a theorem of little value because the lens transform does not preserve the 3d cubic NLS. See Section 7 for details.

We consider the norm

$$\left\| R_\tau^{(k)} \gamma^{(k)}(\tau, \cdot; \cdot) \right\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \tag{1.9}$$

in which

$$R_\tau^{(k)} = \left( \prod_{j=1}^{k} P_{\mathbf{y}_j}(\tau) P_{\mathbf{y}'_j}(-\tau) \right),$$

$$P_{\mathbf{y}}(\tau) = \begin{pmatrix} i\beta_1(\tau)\frac{\partial}{\partial y_1} + \dot{\beta}_1(\tau) y_1 \\ i\beta_2(\tau)\frac{\partial}{\partial y_2} + \dot{\beta}_2(\tau) y_2 \\ i\beta_3(\tau)\frac{\partial}{\partial y_3} + \dot{\beta}_3(\tau) y_3 \end{pmatrix},$$

where $\beta_l$ solves

$$\ddot{\beta}_l(\tau) + \eta_l(\tau)\beta_l(\tau) = 0, \qquad \beta_l(0) = 1, \qquad \dot{\beta}_l(0) = 0. \tag{1.10}$$

The operator $i\beta_l(\tau)\frac{\partial}{\partial y_l} + \dot{\beta}_l(\tau) y_l$ was introduced by Carles in [3]. Lemma 3 and relation (5.2) indicate that the norm (1.9) is natural. That is because this operator is in fact the evolution of the momentum operator $-i\nabla$. We will compute it in Appendix A.

Through a specific generalized lens transform (Proposition 3) we produce the collapsing estimate which is the key estimate to our 3d uniqueness theorem regarding hierarchy (1.6) when $n = 3$.

**Theorem 4.** *Let $[s, T] \subset [0, T_0]$ and $\beta_l$ be defined through Eq. (1.10), assume $\gamma^{(k+1)}(\tau, \mathbf{y}_{k+1}; \mathbf{y}'_{k+1})$ satisfies the homogeneous equation*

$$\left(i\partial_\tau - \frac{1}{2}H_{\overrightarrow{\mathbf{y}_{k+1}}}(\tau) + \frac{1}{2}H_{\overrightarrow{\mathbf{y}'_{k+1}}}(\tau)\right)\gamma^{(k+1)} = 0,$$
$$\gamma^{(k+1)}\big(0, \overrightarrow{\mathbf{y}_{k+1}}; \overrightarrow{\mathbf{y}'_{k+1}}\big) = \gamma_0^{(k+1)}\big(\overrightarrow{\mathbf{y}_{k+1}}; \overrightarrow{\mathbf{y}'_{k+1}}\big). \tag{1.11}$$

*Then there exists a $C > 0$ independent of $\gamma_0^{(k+1)}$, $j$, $k$, $s$, and $T$ s.t.*

$$\big\| R_\tau^{(k)} B_{j,k+1}\big(\gamma^{(k+1)}\big)\big\|^2_{L^2([s,T]\times\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \leqslant C\left(\inf_{\tau\in[0,T_0]}\prod_{l=2}^{3}\beta_l^2(\tau)\right)^{-1}\big\| R_\tau^{(k+1)}\gamma^{(k+1)}\big\|^2_{L^2(\mathbb{R}^{3(k+1)}\times\mathbb{R}^{3(k+1)})},$$

*where the $\tau$ on the RHS of the above estimate can be chosen freely in $[s, T]$.*

From Theorem 4, it follows

**Theorem 5** *(Uniqueness of 3d GP with anisotropic switchable quadratic traps). Let $\{\gamma^{(k)}(\tau, \overrightarrow{\mathbf{y}_k}; \overrightarrow{\mathbf{y}'_k})\}_{k=1}^{\infty}$ solve the 3d Gross–Pitaevskii hierarchy with anisotropic switchable quadratic traps (hierarchy (1.6) when $n = 3$) subject to zero initial data and the space–time bound*

$$\int_0^{T_0}\big\| R_\tau^{(k)} B_{j,k+1}\gamma^{(k+1)}(\tau, \cdot; \cdot)\big\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})}\,d\tau \leqslant C^k \tag{1.12}$$

*for some $C > 0$ and all $1 \leqslant j \leqslant k$. Then $\forall k, \tau \in [0, T_0]$,*

$$\big\| R_\tau^{(k)}\gamma^{(k)}(\tau, \cdot; \cdot)\big\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})} = 0.$$

**Remark 3.** It is currently unknown how to show directly that the limit of $\gamma_N^{(k)}$ in 3d satisfies the space–time bound (1.12).

### *1.2. Organization of the paper*

We show Theorem 2 for $n = 3$ first in Section 2. Utilizing the same scheme, we prove Theorem 2 for $n = 2$ in Section 3. Compared to [25] which uses the approach in the Klainerman–Machedon null form paper [24], the proofs of Theorem 2 here are closer to Beals and Bezard [2] which is a simplification of [24] in the sense that duality takes the place of convolution with surface measures.

In Section 4, we lay down the tools, a generalized lens transform and its related properties, involved in establishing Theorems 4 and 5 whose proofs are in Sections 5 and 6. Theorem 3 follows from the same procedure.

In Section 7, we put together the generalized lens transform, Theorem 3, and the estimates in Kirkpatrick, Schlein and Staffilani [23] to establish Theorem 1. We also explain the differences between the 2d and 3d cases there.

In Appendix A, we present an algebraic explanation of the generalized lens transform, one of the vital tools in this paper.

## 2. Proof of Theorem 2 when $n = 3$ ($3 * 3$d collapsing estimate)

We will make use of the lemma.

**Lemma 1.** *(See [25].) Let $\boldsymbol{\xi} \in \mathbb{R}^3$ and $P$ be a 2d plane or sphere in $\mathbb{R}^3$ with the usual induced surface measure $dS$.*

(1) *Say $0 < a,\ b < 2,\ a + b > 2$, then*

$$\int\limits_P \frac{dS(\boldsymbol{\eta})}{|\boldsymbol{\xi} - \boldsymbol{\eta}|^a |\boldsymbol{\eta}|^b} \leqslant \frac{C}{|\boldsymbol{\xi}|^{a+b-2}}.$$

(2) *Say $\varepsilon = \frac{1}{10}$, then*

$$\int\limits_P \frac{dS(\boldsymbol{\eta})}{|\frac{\boldsymbol{\xi}}{2} - \boldsymbol{\eta}||\boldsymbol{\xi} - \boldsymbol{\eta}|^{2-\varepsilon} |\boldsymbol{\eta}|^{2-\varepsilon}} \leqslant \frac{C}{|\boldsymbol{\xi}|^{3-2\varepsilon}}.$$

*Both the constants in the above estimates are independent of $P$.*

**Proof.** See pages 174–175 of [25]. □

By duality, to gain Theorem 2 when $n = 3$, it suffices to prove

$$\left| \int\limits_{\mathbb{R}^{3+1}} |\nabla_{\mathbf{x}}| u(t, \mathbf{x}, \mathbf{x}, \mathbf{x}) h(t, \mathbf{x})\, d\mathbf{x}\, dt \right| \leqslant C \|h\|_2 \|\nabla_{\mathbf{x}_1} \nabla_{\mathbf{x}_2} \nabla_{\mathbf{x}_2'} f\|_2.$$

Let

$$A_t = \begin{pmatrix} \int_0^t a_1(s)\, ds & 0 & 0 \\ 0 & \int_0^t a_2(s)\, ds & 0 \\ 0 & 0 & \int_0^t a_3(s)\, ds \end{pmatrix},$$

then it brings the solution of Eq. (1.8)

$$u(t, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2') = \int e^{i(\boldsymbol{\xi}_1^T A_t \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2 \pm (\boldsymbol{\xi}_2')^T A_t \boldsymbol{\xi}_2')} e^{i\mathbf{x}_1 \boldsymbol{\xi}_1} e^{i\mathbf{x}_2 \boldsymbol{\xi}_2} e^{i\mathbf{x}_2' \boldsymbol{\xi}_2'} \hat{f}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_2')\, d\boldsymbol{\xi}_1\, d\boldsymbol{\xi}_2\, d\boldsymbol{\xi}_2'.$$

Accordingly, the spatial Fourier transform of $|\nabla_{\mathbf{x}}| u(t, \mathbf{x}, \mathbf{x}, \mathbf{x})$ is

$$|\boldsymbol{\xi}_1| \int e^{i((\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2')^T A_t (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2') + \boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2 \pm (\boldsymbol{\xi}_2')^T A_t \boldsymbol{\xi}_2')} \hat{f}(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2', \boldsymbol{\xi}_2, \boldsymbol{\xi}_2')\, d\boldsymbol{\xi}_2\, d\boldsymbol{\xi}_2',$$

which allows us to compute that

$$\begin{aligned}
&\left| \int |\nabla_{\mathbf{x}}| u(t, \mathbf{x}, \mathbf{x}, \mathbf{x}) h(t, \mathbf{x})\, d\mathbf{x}\, dt \right|^2 \\
&= \left| \int |\boldsymbol{\xi}_1| e^{i((\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2')^T A_t (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2') + \boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2 \pm (\boldsymbol{\xi}_2')^T A_t \boldsymbol{\xi}_2')} \hat{f}(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2', \boldsymbol{\xi}_2, \boldsymbol{\xi}_2') \hat{h}(t, \boldsymbol{\xi}_1)\, dt\, d\boldsymbol{\xi}_1\, d\boldsymbol{\xi}_2\, d\boldsymbol{\xi}_2' \right|^2 \\
&\quad \text{(spatial Fourier transform on } h) \\
&= \left| \int \left( \int |\boldsymbol{\xi}_1| e^{i((\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2')^T A_t (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2') + \boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2 \pm (\boldsymbol{\xi}_2')^T A_t \boldsymbol{\xi}_2')} \hat{h}(t, \boldsymbol{\xi}_1)\, dt \right) \hat{f}(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2', \boldsymbol{\xi}_2, \boldsymbol{\xi}_2')\, d\boldsymbol{\xi}_1\, d\boldsymbol{\xi}_2\, d\boldsymbol{\xi}_2' \right|^2 \\
&\leqslant I(h) \|\nabla_{\mathbf{x}_1} \nabla_{\mathbf{x}_2} \nabla_{\mathbf{x}_2'} f\|_{L^2}^2 \quad \text{(Cauchy–Schwarz)},
\end{aligned}$$

where

$$I(h) = \int \frac{|\boldsymbol{\xi}_1|^2 |\int e^{i((\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2')^T A_t (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2') + \boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2 \pm (\boldsymbol{\xi}_2')^T A_t \boldsymbol{\xi}_2')} \hat{h}(t, \boldsymbol{\xi}_1)\, dt|^2}{|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2'|^2 |\boldsymbol{\xi}_2|^2 |\boldsymbol{\xi}_2'|^2}\, d\boldsymbol{\xi}_1\, d\boldsymbol{\xi}_2\, d\boldsymbol{\xi}_2'.$$

So the target of the rest of this section is to show

$$I(h) \leqslant C \|h\|_{L^2}^2.$$

Noticing that the integral $I(h)$ is symmetric in $|\xi_1 - \xi_2 - \xi_2'|$ and $|\xi_2|$, we deal with the region: $|\xi_1 - \xi_2 - \xi_2'| > |\xi_2|$ only. We separate this region into two parts, Cases I and II.

When the "$\pm$" in Eq. (1.8) is "$+$", Case I is sufficient. To show the estimate for "$-$", we need both Cases I and II.

Away from $|\xi_1 - \xi_2 - \xi_2'| > |\xi_2|$, there are other restrictions on the integration regions in Cases I and II. We state the restrictions in the beginning of both Cases I and II. Due to the limited space near "$\int$", we omit the actual region. Please keep this in mind during reading.

### 2.1. *Case I: $I(h)$ restricted to the region $|\xi_2'| < |\xi_2|$ with integration order $d\xi_2$ prior to $d\xi_2'$*

Write the phase function of the $dt$ integral inside $I(h)$ as

$$\begin{aligned}&\left(\xi_1 - \xi_2 - \xi_2'\right)^T A_t\left(\xi_1 - \xi_2 - \xi_2'\right) + \xi_2^T A_t \xi_2 \pm \left(\xi_2'\right)^T A_t \xi_2' \\ &\quad = \frac{(\xi_1 - \xi_2')^T A_t (\xi_1 - \xi_2')}{2} + 2\left(\xi_2 - \frac{\xi_1 - \xi_2'}{2}\right)^T A_t \left(\xi_2 - \frac{\xi_1 - \xi_2'}{2}\right) \pm \left(\xi_2'\right)^T A_t \xi_2'.\end{aligned}$$

The change of variable

$$\xi_{2,new} = \xi_{2,old} - \frac{\xi_1 - \xi_2'}{2} \tag{2.1}$$

leads to

$$\begin{aligned}I(h) &= \int \frac{|\xi_1|^2 |\int e^{i(\frac{(\xi_1-\xi_2')^T A_t(\xi_1-\xi_2')}{2} + 2\xi_2^T A_t \xi_2 \pm (\xi_2')^T A_t \xi_2')} \hat{h}(t,\xi_1)\,dt|^2}{|\xi_2 - \frac{\xi_1-\xi_2'}{2}|^2 |\xi_2 + \frac{\xi_1-\xi_2'}{2}|^2 |\xi_2'|^2}\,d\xi_1\,d\xi_2\,d\xi_2' \\ &= \int \frac{|\xi_1|^2}{|\xi_2 - \frac{\xi_1-\xi_2'}{2}|^2 |\xi_2 + \frac{\xi_1-\xi_2'}{2}|^2 |\xi_2'|^2} e^{i(2\frac{(\xi_1-\xi_2')^T A_t(\xi_1-\xi_2')}{2} + 2\xi_2^T A_t \xi_2 \pm (\xi_2')^T A_t \xi_2')} \\ &\quad \times e^{-i(\frac{(\xi_1-\xi_2')^T A_{t'}(\xi_1-\xi_2')}{2} + 2\xi_2^T A_{t'} \xi_2 \pm (\xi_2')^T A_{t'} \xi_2')} \hat{h}(t,\xi_1)\overline{\hat{h}(t',\xi_1)}\,dt\,dt'\,d\xi_1\,d\xi_2\,d\xi_2' \\ &= \int d\xi_1 \int J(\overline{\hat{h}})(t,\xi_1)\hat{h}(t,\xi_1)\,dt,\end{aligned}$$

where

$$J(\overline{\hat{h}})(t,\xi_1) = \int \frac{|\xi_1|^2 e^{i2\xi_2^T A_t \xi_2} e^{-i2\xi_2^T A_{t'} \xi_2}}{|\xi_2 - \frac{\xi_1-\xi_2'}{2}|^2 |\xi_2 + \frac{\xi_1-\xi_2'}{2}|^2 |\xi_2'|^2} e^{i(\frac{(\xi_1-\xi_2')^T (A_t - A_{t'})(\xi_1-\xi_2')}{2} \pm (\xi_2')^T (A_t - A_{t'})\xi_2')} \overline{\hat{h}(t',\xi_1)}\,dt'\,d\xi_2\,d\xi_2'.$$

Assume for the moment that

$$\int \left|J(\overline{\hat{h}})(t,\xi_1)\right|^2 dt \leqslant C \left\|\hat{h}(\cdot,\xi_1)\right\|_{L_t^2}^2$$

with $C$ independent of $h$ or $\xi_1$, then

$$I(h) \leqslant C \int d\xi_1 \left\|\hat{h}(\cdot,\xi_1)\right\|_{L_t^2}^2.$$

Hence we end Case I by this proposition.

**Proposition 1.**

$$\int \left|J(f)(t,\xi_1)\right|^2 dt \leqslant C \left\|f(\cdot,\xi_1)\right\|_{L_t^2}^2,$$

*where $C$ is independent of $f$ or $\xi_1$.*

**Remark 4.** To avoid confusing notation in the proof of the proposition, we use $f(t',\xi_1)$ to replace $\overline{\hat{h}(t',\xi_1)}$.

**Proof of Proposition 1.** Again, by duality, we just need to prove

$$\left|\int J(f)(t,\boldsymbol{\xi}_1)\overline{g(t)}\,dt\right| \leqslant C\left\|f(\cdot,\boldsymbol{\xi}_1)\right\|_{L_t^2}\|g\|_{L_t^2}.$$

For convenience, let

$$\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2') = \frac{(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')^T A_t(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')}{2} \pm (\boldsymbol{\xi}_2')^T A_t\boldsymbol{\xi}_2'.$$

Then

$$\begin{aligned}
&\left|\int J(f)(t,\boldsymbol{\xi}_1)\overline{g(t)}\,dt\right| \\
&=\left|\int \frac{|\boldsymbol{\xi}_1|^2 e^{i2\boldsymbol{\xi}_2^T A_t\boldsymbol{\xi}_2}e^{-i2\boldsymbol{\xi}_2^T A_{t'}\boldsymbol{\xi}_2}}{|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}|^2|\boldsymbol{\xi}_2+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}|^2|\boldsymbol{\xi}_2'|^2}\big(e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}f(t',\boldsymbol{\xi}_1)\big)\big(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g(t)}\big)\,dt\,dt'\,d\boldsymbol{\xi}_2\,d\boldsymbol{\xi}_2'\right| \\
&=\left|\int \frac{(\int e^{2i\boldsymbol{\xi}_2^T A_t\boldsymbol{\xi}_2}(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g(t)})\,dt)(\int e^{-2i\boldsymbol{\xi}_2^T A_{t'}\boldsymbol{\xi}_2}(e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}f(t',\boldsymbol{\xi}_1))\,dt')|\boldsymbol{\xi}_1|^2\,d\boldsymbol{\xi}_2\,d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}|^2|\boldsymbol{\xi}_2+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}|^2|\boldsymbol{\xi}_2'|^2}\right| \\
&\leqslant \int \frac{|\boldsymbol{\xi}_1|^2\,d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_2'|^2}\int \frac{|\int e^{2i\boldsymbol{\xi}_2^T A_t\boldsymbol{\xi}_2}(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g(t)})\,dt||\int e^{-2i\boldsymbol{\xi}_2^T A_{t'}\boldsymbol{\xi}_2}(e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}f(t',\boldsymbol{\xi}_1))\,dt'|}{|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}|^2|\boldsymbol{\xi}_2+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}|^2}\,d\boldsymbol{\xi}_2.
\end{aligned}$$

To deal with the $dt$ and $dt'$ integrals, for every fixed $\boldsymbol{\xi}_2$, let

$$u(t) = 2\frac{\boldsymbol{\xi}_2^T A_t\boldsymbol{\xi}_2}{|\boldsymbol{\xi}_2|^2},$$

then

$$\frac{du}{dt} = 2\frac{a_1(t)\xi_{2,1}^2+a_2(t)\xi_{2,2}^2+a_3(t)\xi_{2,3}^2}{|\boldsymbol{\xi}_2|^2} \geqslant 2c_0 > 0$$

which provides a well-defined inverse $t(u)$.

Consequently, the integral

$$\int e^{2i\boldsymbol{\xi}_2^T A_t\boldsymbol{\xi}_2}\big(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g(t)}\big)\,dt = \overline{\int e^{-iu|\boldsymbol{\xi}_2|^2}\left(e^{-i\phi(t(u),\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g\big(t(u)\big)\left|\frac{dt}{du}\right|\right)du}$$

is indeed the Fourier transform of

$$G(u) = e^{-i\phi(t(u),\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g\big(t(u)\big)\left|\frac{dt}{du}\right|.$$

This is well-defined since

$$\int_{\mathbb{R}}\left|G(u)\right|^2du = \int_{\mathbb{R}}\left|\overline{e^{-i\phi(t(u),\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)}g\big(t(u)\big)}\left|\frac{dt}{du}\right|\right|^2du = \int_{\mathbb{R}}\left|g(t)\right|^2\left|\frac{dt}{du}\right|dt \leqslant \frac{1}{2c_0}\left\|g(\cdot)\right\|_{L_t^2}^2.$$

Hence

$$\begin{aligned}
&\left|\int J(f)(t,\boldsymbol{\xi}_1)\overline{g(t)}\,dt\right| \\
&\leqslant \int \frac{|\boldsymbol{\xi}_1|^2\,d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_2'|^2}\int \frac{|\int e^{2i\boldsymbol{\xi}_2^T A_t\boldsymbol{\xi}_2}(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)}g(t)})dt||\int e^{-2i\boldsymbol{\xi}_2^T A_{t'}\boldsymbol{\xi}_2}(e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)}f(t',\boldsymbol{\xi}_1))\,dt'|}{|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}|^2|\boldsymbol{\xi}_2+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}|^2}\,d\boldsymbol{\xi}_2 \\
&= \int \frac{|\boldsymbol{\xi}_1|^2\,d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_2'|^2}\int \frac{|\overline{\hat{G}(|\boldsymbol{\xi}_2|^2)}\hat{F}(|\boldsymbol{\xi}_2|^2,\boldsymbol{\xi}_1)|}{|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}|^2|\boldsymbol{\xi}_2+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}|^2}\,d\boldsymbol{\xi}_2
\end{aligned}$$

$$= \int \frac{|\xi_1|^2\,d\xi_2'}{|\xi_2'|^2} \int \frac{|\hat F(\rho^2,\xi_1)\overline{\hat G(\rho^2)}|}{|\xi_2 - \frac{\xi_1-\xi_2'}{2}|^2|\xi_2+\frac{\xi_1-\xi_2'}{2}|^2}\rho^2\,d\rho\,d\sigma \quad \text{(spherical coordinate in } \xi_2)$$

$$\leqslant \int \frac{|\xi_1|^2\,d\xi_2'}{|\xi_2'|^2} \sup_\rho \left( \int \frac{\rho^2\,d\sigma}{\rho|\xi_2 - \frac{\xi_1-\xi_2'}{2}|^2|\xi_2+\frac{\xi_1-\xi_2'}{2}|^2} \right) \left( \int |\hat F(\rho^2,\xi_1)|^2 \rho\,d\rho \right)^{\frac12} \left( \int \left|\hat G(\rho^2)\right|^2 \rho\,d\rho \right)^{\frac12}$$

(Hölder in $\rho$)

$$\leqslant C \| f(\cdot,\xi_1)\|_{L_t^2}\|g\|_{L_t^2} \left\{ \int \frac{|\xi_1|^2}{|\xi_2'|^2} \sup_\rho \left( \int \frac{\rho^2\,d\sigma}{\rho|\xi_2 - \frac{\xi_1-\xi_2'}{2}|^2|\xi_2+\frac{\xi_1-\xi_2'}{2}|^2} \right) d\xi_2' \right\}.$$

However,

$$\int \frac{|\xi_1|^2}{|\xi_2'|^2} \sup_\rho \left( \int \frac{\rho^2\,d\sigma}{\rho|\xi_2 - \frac{\xi_1-\xi_2'}{2}|^2|\xi_2+\frac{\xi_1-\xi_2'}{2}|^2} \right) d\xi_2'$$

$$= \int \frac{|\xi_1|^2}{|\xi_2'|^2} \sup_\rho \left( \int \frac{|\xi_2 - \frac{\xi_1-\xi_2'}{2}|^2\,d\sigma}{|\xi_2 - \frac{\xi_1-\xi_2'}{2}||\xi_1-\xi_2-\xi_2'|^2|\xi_2|^2} \right) d\xi_2' \quad \text{(reverse the change of variable in formula (2.1))}$$

$$= |\xi_1|^2 \int \frac{d\xi_2'}{|\xi_2'|^{2+2\varepsilon}} \sup_\rho \left( \int \frac{|\xi_2 - \frac{\xi_1-\xi_2'}{2}|^2\,d\sigma}{|\xi_2 - \frac{\xi_1-\xi_2'}{2}||\xi_1-\xi_2-\xi_2'|^{2-\varepsilon}|\xi_2|^{2-\varepsilon}} \right)$$

$$\leqslant C|\xi_1|^2 \int \frac{d\xi_2'}{|\xi_2'|^{2+2\varepsilon}|\xi_1-\xi_2'|^{3-2\varepsilon}} \quad \text{(second part of Lemma 1)}$$

$$\leqslant C.$$

In the above calculation, the $\sigma$ in the first line lives on the unit sphere centered at the origin while the $\sigma$ in the second line is on a unit sphere centered at $\frac{\xi_1-\xi_2'}{2}$. We use the same symbol because Lebesgue measure is translation invariant.

Thus,

$$\left| \int J(f)(t,\xi_1)\overline{g(t)}\,dt \right| \leqslant C \| f(\cdot,\xi_1)\|_{L_t^2}\|g\|_{L_t^2}. \qquad \square$$

**Remark 5.** Because the integral $I(h)$ is also symmetric in $\xi_2$ and $\xi_2'$ when the "±" in Eq. (1.8) is "+", we have acquired the estimate in that case. In Case II, we will assume that "±" is "−".

*2.2. Case II: $I(h)$ restricted to the region $|\xi_2'| > |\xi_2|$ with integration order $d\xi_2'$ prior to $d\xi_2$*

This time we write the phase function to be

$$\begin{aligned}
&(\xi_1-\xi_2-\xi_2')^T A_t (\xi_1-\xi_2-\xi_2') + \xi_2^T A_t \xi_2 - (\xi_2')^T A_t \xi_2' \\
&\quad = (\xi_1-\xi_2)^T A_t(\xi_1-\xi_2) - 2(\xi_1-\xi_2)^T A_t \xi_2' + \xi_2^T A_t \xi_2 \\
&\quad = \phi(t,\xi_1,\xi_2) - 2(\xi_1-\xi_2)^T A_t \xi_2',
\end{aligned}$$

and let

$$J(\bar{\hat h})(t,\xi_1) = \int \frac{|\xi_1|^2 e^{-2i(\xi_1-\xi_2)^T A_t \xi_2'} e^{2i(\xi_1-\xi_2)^T A_{t'}\xi_2'}}{|\xi_1-\xi_2-\xi_2'|^2|\xi_2|^2|\xi_2'|^2} e^{-i\phi(t',\xi_1,\xi_2)} \overline{e^{-i\phi(t,\xi_1,\xi_2')}\hat h(t',\xi_1)}\,dt'\,d\xi_2'\,d\xi_2.$$

Again, we want to prove

**Proposition 2.**

$$\int \left|J(f)(t,\boldsymbol{\xi}_1)\right|^2 dt \leqslant C \left\| f(\cdot,\boldsymbol{\xi}_1)\right\|_{L_t^2}^2,$$

*where $C$ is independent of $f$ or $\boldsymbol{\xi}_1$.*

**Proof.** We calculate

$$\begin{aligned}
&\left|\int J(f)(t,\boldsymbol{\xi}_1)\overline{g(t)}\,dt\right| \\
&\quad = \left|\int \frac{|\boldsymbol{\xi}_1|^2 e^{-2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'} e^{2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_{t'}\boldsymbol{\xi}_2'}}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^2|\boldsymbol{\xi}_2|^2|\boldsymbol{\xi}_2'|^2}\big(e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} f\big(t',\boldsymbol{\xi}_1\big)\big)\big(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} g(t)}\big)\,dt\,dt'\,d\boldsymbol{\xi}_2'\,d\boldsymbol{\xi}_2\right| \\
&\quad = \left|\int \frac{(\int e^{-2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'}(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} g(t)})\,dt)(\int e^{2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_{t'}\boldsymbol{\xi}_2}(e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} f(t',\boldsymbol{\xi}_1))\,dt')|\boldsymbol{\xi}_1|^2\,d\boldsymbol{\xi}_2\,d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^2|\boldsymbol{\xi}_2|^2|\boldsymbol{\xi}_2'|^2}\right| \\
&\quad \leqslant \int \frac{|\boldsymbol{\xi}_1|^2\,d\boldsymbol{\xi}_2}{|\boldsymbol{\xi}_2|^2}\int \frac{d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^2|\boldsymbol{\xi}_2'|^2} \\
&\qquad \times \left|\int e^{-2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'}\big(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} g(t)}\big)\,dt\right|\left|\int e^{2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_{t'}\boldsymbol{\xi}_2'}\big(e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} f\big(t',\boldsymbol{\xi}_1\big)\big)\,dt'\right|.
\end{aligned}$$

Fix $\boldsymbol{\xi}_1-\boldsymbol{\xi}_2$ and $\boldsymbol{\xi}_2'$, write

$$\int e^{-2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'}\big(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g(t)}\big)\,dt = \int e^{-2i|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\omega^T A_t \boldsymbol{\xi}_2'}\big(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g(t)}\big)\,dt,$$

where $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)$ is a unit vector in $\mathbb{R}^3$. Without loss of generality, we assume

$$\max\big\{|\omega_1|,|\omega_2|,|\omega_3|\big\}=|\omega_1|$$

which implies

$$\frac{1}{\sqrt{3}} \leqslant |\omega_1| \leqslant 1.$$

Let us further assume that $\omega_1>0$ (the proof works exactly the same for the $\omega_1<0$ case), then we can write

$$\begin{aligned}
\boldsymbol{\xi}_2' &= (x,0,0)+(0,y_1,y_2), \\
u(t) &= 2\omega_1\int_0^t a_1(s)\,ds.
\end{aligned}$$

Again $u$ is invertible with

$$\frac{du}{dt} \geqslant \frac{2c_0}{\sqrt{3}} > 0.$$

So we have

$$\begin{aligned}
&\int e^{-2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'}\big(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g(t)}\big)\,dt \\
&\quad = \int e^{-2i|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\boldsymbol{\omega}^T A_{t'}\boldsymbol{\xi}_2'}\big(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g(t)}\big)\,dt \\
&\quad = \int e^{-iu(\omega_1|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|x)}\left(e^{-2i|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|(0,\omega_2,\omega_3)^T A_{t(u)}(0,y_1,y_2)}\overline{e^{-i\phi(t(u),\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g\big(t(u)\big)}\left|\frac{dt}{du}\right|\right)du \\
&\quad = \overline{\hat{G}\big(-\omega_1|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|x\big)},
\end{aligned}$$

where

$$G(u)=\overline{e^{-2i|\xi_1-\xi_2|(0,\omega_2,\omega_3)^T A_{t(u)}(0,y_1,y_2)}}e^{-i\phi(t(u),\xi_1,\xi_2')}g\bigl(t(u)\bigr)\left|\frac{dt}{du}\right|$$

which still has the property that

$$\int\bigl|G(u)\bigr|^2\,du\leqslant\frac{\sqrt{3}}{2c_0}\int\bigl|g(t)\bigr|^2\,dt.$$

Just as in Case I, this procedure hands us

$$\begin{aligned}
&\left|\int J(f)(t,\xi_1)\overline{g(t)}\,dt\right|\\
&\quad\leqslant\int\frac{|\xi_1|^2\,d\xi_2}{|\xi_2|^2}\int\frac{d\xi_2'}{|\xi_1-\xi_2-\xi_2'|^2|\xi_2'|^2}\\
&\qquad\times\left|\int e^{-2i(\xi_1-\xi_2)^T A_t\xi_2'}\bigl(\overline{e^{-i\phi(t,\xi_1,\xi_2)}g(t)}\bigr)\,dt\right|\left|\int e^{2i(\xi_1-\xi_2)^T A_{t'}\xi_2'}\bigl(e^{-i\phi(t',\xi_1,\xi_2)}f\bigl(t',\xi_1\bigr)\bigr)\,dt'\right|\\
&\quad=\int\left(\int\frac{dx\,dy_1\,dy_2}{|\xi_1-\xi_2-\xi_2'|^2|\xi_2'|^2}\left|\overline{\hat{G}\bigl(-\omega_1|\xi_1-\xi_2|x\bigr)}\hat{F}\bigl(-\omega_1|\xi_1-\xi_2|x,\xi_1\bigr)\right|\right)\frac{|\xi_1|^2}{|\xi_2|^2}\,d\xi_2\\
&\quad=\int\left(\int\frac{dx\,dy_1\,dy_2}{|\xi_1-\xi_2-\xi_2'|^2|\xi_2'|^2}\left|\overline{\hat{G}(x)}\hat{F}(x,\xi_1)\right|\right)\frac{|\xi_1|^2}{|\omega_1||\xi_1-\xi_2||\xi_2|^2}\,d\xi_2\\
&\quad\leqslant C\int\frac{|\xi_1|^2}{|\xi_1-\xi_2||\xi_2|^2}\left(\sup_x\int\frac{dy_1\,dy_2}{|\xi_1-\xi_2-\xi_2'|^2|\xi_2'|^2}\right)\left(\int\bigl|\hat{F}(x,\xi_1)\bigr|^2\,dx\right)^{\frac{1}{2}}\left(\int\bigl|\hat{G}(x)\bigr|^2\,dx\right)^{\frac{1}{2}}d\xi_2\quad\text{(Hölder in }x\text{)}\\
&\quad\leqslant C\bigl\|f(\cdot,\xi_1)\bigr\|_{L_t^2}\|g\|_{L_t^2}\int\frac{|\xi_1|^2}{2|\xi_1-\xi_2||\xi_2|^2}\left(\sup_x\int\frac{dy_1\,dy_2}{|\xi_1-\xi_2-\xi_2'|^2|\xi_2'|^2}\right)d\xi_2.
\end{aligned}$$

The first part of Lemma 1 and the restrictions that $|\xi_1-\xi_2-\xi_2'|>|\xi_2|$ and $|\xi_2'|<|\xi_2|$ show

$$\begin{aligned}
&\int\frac{|\xi_1|^2}{2|\xi_1-\xi_2||\xi_2|^2}\left(\sup_x\int\frac{dy_1\,dy_2}{|\xi_1-\xi_2-\xi_2'|^2|\xi_2'|^2}\right)d\xi_2\\
&\quad\leqslant\int\frac{|\xi_1|^2}{2|\xi_1-\xi_2||\xi_2|^{2+2\varepsilon}}\left(\sup_x\int\frac{dy_1\,dy_2}{|\xi_1-\xi_2-\xi_2'|^{2-\varepsilon}|\xi_2'|^{2-\varepsilon}}\right)d\xi_2\\
&\quad\leqslant C\int\frac{|\xi_1|^2\,d\xi_2}{2|\xi_1-\xi_2|^{3-2\varepsilon}|\xi_2|^{2+2\varepsilon}}\\
&\quad\leqslant C,
\end{aligned}$$

which finishes the proposition. □

## 3. Proof of Theorem 2 when $n=2$ ($3*2$d collapsing estimate)

By the proof of the $n=3$ case in Section 2, we only need to show these two estimates:

Case I. Under the restrictions $|\xi_1-\xi_{2,old}-\xi_2'|>|\xi_{2,old}|$ and $|\xi_2'|<|\xi_{2,old}|$, we have

$$\int\frac{|\xi_1|}{|\xi_2'|}\sup_\rho\left(\int\frac{d\sigma(\xi_{2,new})}{|\xi_{2,new}-\frac{\xi_1-\xi_2'}{2}||\xi_{2,new}+\frac{\xi_1-\xi_2'}{2}|}\right)d\xi_2'\leqslant C,$$

where $\xi_{2,new}$ and $\xi_{2,old}$ are related by formula (2.1) and we write

$$\xi_{2,new}=\rho\sigma\quad\text{with }\sigma\in\mathbb{S}^1.$$

Case II. Under the restrictions $|\xi_1 - \xi_2 - \xi_2'| > |\xi_2|$ and $|\xi_2'| > |\xi_2|$, we have

$$\int \frac{|\xi_1|}{|\xi_1 - \xi_2||\xi_2|}\left(\sup_x \int \frac{dy}{|\xi_1 - \xi_2 - \xi_2'||\xi_2'|}\right) d\xi_2 \leqslant C,$$

where $\xi_2' = (x, y)$.

Lemma 1 plays an important role in giving the corresponding estimates in Section 2. In the 2d case, the subsequent lemma provides its replacement.

**Lemma 2.** *Let $\xi \in \mathbb{R}^2$ and $L$ be a 1d line or circle in $\mathbb{R}^2$ with the usual induced line element $dS$.*

(1) *Say $0 < a, b < 1$, $a + b > 1$, then there exists a $C$ independent of $L$ s.t.*

$$\int_L \frac{dS(\eta)}{|\xi - \eta|^a |\eta|^b} \leqslant \frac{C}{|\xi|^{a+b-1}}.$$

(2) *Let $\varepsilon = \frac{1}{80}$, then*

$$\sup_{|\eta|}\left(\int_{\mathbb{S}^1} \frac{d\sigma(\eta)}{|\xi - \eta|^{1-\varepsilon}|\xi + \eta|^{1-\varepsilon}}\right) \leqslant \frac{C}{|\xi|^{2-2\varepsilon}}.$$

**Proof.** We will show the second part in the end of this section. The first part shares exactly the same proof with Lemma 2.2 in [25]. □

*3.1. Proof of Case I*

The change of variable (2.1) turns the restrictions into

$$\left|\xi_{2,new} - \frac{\xi_1 - \xi_2'}{2}\right| = \left|\xi_1 - \xi_{2,old} - \xi_2'\right| > |\xi_{2,old}| > \left|\xi_2'\right|,$$
$$\left|\xi_{2,new} + \frac{\xi_1 - \xi_2'}{2}\right| = |\xi_{2,old}| > \left|\xi_2'\right|.$$

Noticing that $\xi_{2,new} = \rho\sigma$, we in fact have

$$\begin{aligned}
&\int \frac{|\xi_1|}{|\xi_2'|} \sup_\rho \left(\int \frac{d\sigma(\xi_{2,new})}{|\xi_{2,new} - \frac{\xi_1-\xi_2'}{2}||\xi_{2,new} + \frac{\xi_1-\xi_2'}{2}|}\right) d\xi_2' \\
&\leqslant \int \frac{|\xi_1|}{|\xi_2'|^{1+2\varepsilon}} \sup_\rho \left(\int_{\mathbb{S}^1} \frac{d\sigma(\xi_{2,new})}{|\xi_{2,new} - \frac{\xi_1-\xi_2'}{2}|^{1-\varepsilon}|\xi_{2,new} + \frac{\xi_1-\xi_2'}{2}|^{1-\varepsilon}}\right) d\xi_2' \\
&\leqslant C|\xi_1| \int \frac{1}{|\xi_2'|^{1+2\varepsilon}} \frac{1}{|\xi_1 - \xi_2'|^{2-2\varepsilon}}\, d\xi_2' \quad \text{(second part of Lemma 2)} \\
&\leqslant C.
\end{aligned}$$

*3.2. Proof of Case II*

Recall that $\xi_2' = (x, y)$, we estimate

$$\begin{aligned}
&\int \frac{|\xi_1|}{|\xi_1 - \xi_2||\xi_2|}\left(\sup_x \int \frac{dy}{|\xi_1 - \xi_2 - \xi_2'||\xi_2'|}\right) d\xi_2 \\
&\leqslant \int \frac{|\xi_1|}{|\xi_1 - \xi_2||\xi_2|^{1+2\varepsilon}}\left(\sup_x \int \frac{dy}{|\xi_1 - \xi_2 - \xi_2'|^{1-\varepsilon}|\xi_2'|^{1-\varepsilon}}\right) d\xi_2
\end{aligned}$$

$$\leqslant C|\xi_1| \int \frac{1}{|\xi_1 - \xi_2|^{2-2\varepsilon}|\xi_2|^{1+2\varepsilon}}\, d\xi_2 \quad \text{(first part of Lemma 2)}$$
$$\leqslant C.$$

### 3.3. *Proof of the second part of Lemma 2*

Due to

$$|\xi| \leqslant |\xi - \eta| + |\xi + \eta|,$$

we can separate the integral as

$$\sup_{|\eta|}\left(\int_{\mathbb{S}^1} \frac{d\sigma(\eta)}{|\xi-\eta|^{1-\varepsilon}|\xi+\eta|^{1-\varepsilon}}\right) \leqslant \sup_{|\eta|}\left(\int_{\mathbb{S}^1 \text{ and } |\xi-\eta|\geqslant\frac{|\xi|}{2}}\right) + \sup_{|\eta|}\left(\int_{\mathbb{S}^1 \text{ and } |\xi+\eta|\geqslant\frac{|\xi|}{2}}\right).$$

We will only show

$$\sup_{|\eta|}\left(\int_{\mathbb{S}^1 \text{ and } |\xi+\eta|\geqslant\frac{|\xi|}{2}} \frac{d\sigma(\eta)}{|\xi-\eta|^{1-\varepsilon}|\xi+\eta|^{1-\varepsilon}}\right) \leqslant \frac{C}{|\xi|^{2-2\varepsilon}}$$

since the other part is similar. It is clear that

$$\sup_{|\eta|}\left(\int_{\mathbb{S}^1 \text{ and } |\xi+\eta|\geqslant\frac{|\xi|}{2}} \frac{d\sigma(\eta)}{|\xi-\eta|^{1-\varepsilon}|\xi+\eta|^{1-\varepsilon}}\right) \leqslant \frac{C}{|\xi|^{1-\varepsilon}} \sup_{|\eta|}\left(\int_{\mathbb{S}^1} \frac{d\sigma(\eta)}{|\xi-\eta|^{1-\varepsilon}}\right). \tag{3.1}$$

Rotate $\mathbb{S}^1$ such that $\xi$ is on the positive $x$-axis, then write $\eta = \rho e^{i\theta}$ for $(\rho\cos\theta, \rho\sin\theta)$ and observe:

- When $\theta \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]$,
$$\left|\rho e^{i\theta} - \left(|\xi|, 0\right)\right| \geqslant |\xi||\sin\theta|$$
because $|\xi||\sin\theta|$ is the distance between the point $(|\xi|, 0)$ and the line ($angle = \theta$).
- When $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$,
$$\left|\rho e^{i\theta} - \left(|\xi|, 0\right)\right| \geqslant |\xi|$$
because $\rho e^{i\theta} - (|\xi|, 0)$ is the longest edge in the obtuse triangle which consists of $\rho e^{i\theta}$, $(|\xi|, 0)$ and $\rho e^{i\theta} - (|\xi|, 0)$.

Inserting these two elementary observations into estimate (3.1), we have

$$\begin{aligned}
&\sup_{|\eta|}\left(\int_{\mathbb{S}^1 \text{ and } |\xi+\eta|\geqslant\frac{|\xi|}{2}} \frac{d\sigma(\eta)}{|\xi-\eta|^{1-\varepsilon}|\xi+\eta|^{1-\varepsilon}}\right) \\
&\quad\leqslant \frac{C}{|\xi|^{1-\varepsilon}} \sup_{|\eta|}\left(\int_{\mathbb{S}^1} \frac{d\sigma(\eta)}{|\xi-\eta|^{1-\varepsilon}}\right) \\
&\quad\leqslant \frac{C}{|\xi|^{1-\varepsilon}}\left[\sup_{\rho}\left(\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{d\theta}{|\rho e^{i\theta} - (|\xi|, 0)|^{1-\varepsilon}}\right) + 2\sup_{\rho}\left(\int_0^{\frac{\pi}{2}} \frac{d\theta}{|\rho e^{i\theta} - (|\xi|, 0)|^{1-\varepsilon}}\right)\right] \\
&\quad\leqslant \frac{C}{|\xi|^{1-\varepsilon}}\left[\left(\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{d\theta}{|\xi|^{1-\varepsilon}}\right) + 2\left(\int_0^{\frac{\pi}{2}} \frac{d\theta}{||\xi|\sin\theta|^{1-\varepsilon}}\right)\right] \\
&\quad\leqslant \frac{C}{|\xi|^{2-2\varepsilon}}.
\end{aligned}$$

To show the other part, namely

$$\sup_{|\boldsymbol{\eta}|}\left(\int_{\mathbb{S}^1 \text{ and } |\boldsymbol{\xi}-\boldsymbol{\eta}|\geqslant\frac{|\boldsymbol{\xi}|}{2}} \frac{d\boldsymbol{\sigma}(\boldsymbol{\eta})}{|\boldsymbol{\xi}-\boldsymbol{\eta}|^{1-\varepsilon}|\boldsymbol{\xi}+\boldsymbol{\eta}|^{1-\varepsilon}}\right)\leqslant\frac{C}{|\boldsymbol{\xi}|^{2-2\varepsilon}},$$

one just needs to notice

$$|\boldsymbol{\xi}+\boldsymbol{\eta}| = \left|\big(|\boldsymbol{\xi}|,0\big)-\rho e^{i(\theta+\pi)}\right|,$$

then one can proceed as above. Therefore we conclude the proof of the second part of Lemma 2.

## 4. The lens transform (preparation for Theorem 4)

From now on, we enter the proof of Theorems 4 and 5. We set $n=3$ until Section 7. In this section, we set up the tools involved in the proof of Theorem 4. We build the lens transform we need and state the related properties. For simplicity of notations, we write $U^{(k+1)}(\tau;s)$ to be the solution operator of Eq. (1.11) and $U_{\mathbf{y}}(\tau;s)$ to be the solution operator of

$$\left(i\partial_\tau-\frac{1}{2}H_{\mathbf{y}}(\tau)\right)u=0,$$
$$u(s,\mathbf{y})=u_s(\mathbf{y}),$$

i.e. $U^{(k+1)}(\tau;s)\gamma_0^{(k+1)}$ solves Eq. (1.11). By definition,

$$U^{(k)}(\tau;s)=\prod_{j=1}^{k}\big(U_{\mathbf{y}_j}(\tau;s)U_{\mathbf{y}'_j}(-\tau;-s)\big).$$

To be specific, we need this version of the generalized lens transform:

**Proposition 3.** *There is an operator $L_{\mathbf{x}}(t)$ which satisfies the hypothesis in Theorem 2 such that*

$$\begin{aligned}
&U^{(k+1)}(\tau;0)\gamma_0^{(k+1)}\\
&=\prod_{j=1}^{k+1}\left(\prod_{l=1}^{3}\frac{e^{i\frac{\dot\beta_l(\tau)}{\beta_l(\tau)}\frac{(|y_{j,l}|^2-|y'_{j,l}|^2)}{2}}}{\beta_l(\tau)}\right)\\
&\quad\times u^{(k+1)}\left(\frac{\alpha_1(\tau)}{\beta_1(\tau)},\frac{y_{1,1}}{\beta_1(\tau)},\frac{y_{1,2}}{\beta_2(\tau)},\frac{y_{1,3}}{\beta_3(\tau)},\dots,\frac{y_{k+1,1}}{\beta_1(\tau)},\frac{y_{k+1,2}}{\beta_2(\tau)},\frac{y_{k+1,3}}{\beta_3(\tau)};\right.\\
&\qquad\left.\frac{y'_{1,1}}{\beta_1(\tau)},\frac{y'_{1,2}}{\beta_2(\tau)},\frac{y'_{1,3}}{\beta_3(\tau)},\dots,\frac{y'_{k+1,1}}{\beta_1(\tau)},\frac{y'_{k+1,2}}{\beta_2(\tau)},\frac{y'_{k+1,3}}{\beta_3(\tau)}\right)
\end{aligned}$$

*in $[-T_0,T_0]$, where $\alpha_l$ and $\beta_l$ are defined as in Claim 1, and $u^{(k+1)}(t,\overrightarrow{\mathbf{x}_{k+1}};\overrightarrow{\mathbf{x}'_{k+1}})$ is the solution of*

$$\big(i\partial_t+L_{\overrightarrow{\mathbf{x}_{k+1}}}(t)-L_{\overrightarrow{\mathbf{x}'_{k+1}}}(t)\big)u^{(k+1)}=0\quad \text{in } \mathbb{R}^{(6k+6)+1},$$
$$u^{(k+1)}\big(0,\overrightarrow{\mathbf{x}_{k+1}};\overrightarrow{\mathbf{x}'_{k+1}}\big)=\gamma_0^{(k+1)}.$$

The proposition will be a corollary of a sequence of claims.

**Claim 1.** *Assuming Conditions 1 and 2, for $l=1,2,3$, the system*

$$\begin{aligned}
&\ddot\alpha_l(\tau)+\eta_l(\tau)\alpha_l(\tau)=0, &&\alpha_l(0)=0, &&\dot\alpha_l(0)=1,\\
&\ddot\beta_l(\tau)+\eta_l(\tau)\beta_l(\tau)=0, &&\beta_l(0)=1, &&\dot\beta_l(0)=0
\end{aligned}\tag{4.1}$$

*defines an odd $\alpha_l$ and an even $\beta_l\in C^2(\mathbb{R})$ with the following properties*:

(1) *$\beta_l$ is nonzero in* $[-T_0, T_0]$;
(2) *The Wronskian of $\alpha_l$ and $\beta_l$ is constant* 1 *i.e.*

$$\dot{\alpha}_l(\tau)\beta_l(\tau) - \alpha_l(\tau)\dot{\beta}_l(\tau) = 1;$$

(3) *The odd function*

$$\upsilon_l(\tau) = \frac{\alpha_l(\tau)}{\beta_l(\tau)}$$

*is invertible in* $[-T_0, T_0]$ *because*

$$\dot{\upsilon}_l(\tau) = \frac{1}{(\beta_l(\tau))^2} > 0 \quad \textit{in } [-T_0, T_0].$$

**Proof.** We show (1) only since all other statements are fairly trivial.

Suppose $\beta_l(\tau_0) = 0$ for some $\tau_0$ in $[-T_0, T_0]$ then $\beta_l(-\tau_0) = 0$ via $\beta_l$ is even. Of course $\tau_0 \neq 0$ because $\beta_l(0) = 1$. Notice that $\cos(\tau\sqrt{\sup_\tau |\eta_l(\tau)|})$ is a nontrivial solution of

$$\ddot{v}(\tau) + \sup_\tau \left|\eta_l(\tau)\right| v(\tau) = 0.$$

Since $\cos(\tau\sqrt{\sup_\tau |\eta_l(\tau)|})$ is not a multiple of $\beta_l$, $\cos(\tau\sqrt{\sup_\tau |\eta_l(\tau)|})$ must have at least one zero in $[-\tau_0, \tau_0]$ due to the Sturm–Picone comparison theorem. But this creates a contradiction. □

Though Claim 1 is elementary, its consequences lying below make our procedure well-defined.

**Definition 1** *(A reminder of the norm).* Let $\beta_l$ be defined via Eq. (4.1). We define

$$P_{\mathbf{y}}(\tau) = \begin{pmatrix} i\beta_1(\tau)\frac{\partial}{\partial y_1} + \dot{\beta}_1(\tau)y_1 \\ i\beta_2(\tau)\frac{\partial}{\partial y_2} + \dot{\beta}_2(\tau)y_2 \\ i\beta_3(\tau)\frac{\partial}{\partial y_3} + \dot{\beta}_3(\tau)y_3 \end{pmatrix}$$

and

$$R_\tau^k = \prod_{j=1}^k P_{\mathbf{y}_j}(\tau)P_{\mathbf{y}'_j}(-\tau).$$

**Lemma 3.** *$P_{\mathbf{y}}(\tau)$ commutes with the linear operator*

$$i\partial_\tau - \frac{1}{2}\left(-\triangle_{\mathbf{y}_k} + \eta(\tau)|\mathbf{y}_k|^2\right).$$

*Moreover,*

$$P_{\mathbf{y}}(\tau)U_{\mathbf{y}}(\tau; s)f = U_{\mathbf{y}}(\tau; s)P_{\mathbf{y}}(s)f.$$

**Lemma 4.** *Say $K_1(t, x_0, y_0)$ is the Green's function of the* 1*d free Schrödinger equation*

$$\left(i\partial_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)v = 0,$$

*then*

$$U_{\mathbf{y}}(\tau; 0)u_0 = \left(\prod_{l=1}^3 \frac{e^{i\frac{\dot{\beta}_l(\tau)}{\beta_l(\tau)}\frac{y_l^2}{2}}}{(\beta_l(\tau))^{\frac{1}{2}}}\right)\int\left(\prod_{l=1}^3 K_1\left(\frac{\alpha_l(\tau)}{\beta_l(\tau)}, \frac{y_l}{\beta_l(\tau)}, y_{0l}\right)\right)u_0(y_{01}, y_{02}, y_{03})\, dy_{01}\, dy_{02}\, dy_{03} \tag{4.2}$$

*is valid in the interval $[-T, T]$ in which $\eta_l$ are Lipschitzian and $\beta_l(\tau) \neq 0$.*

**Proof.** Carles computed the isotropic case of formula (4.2) in [3]. We include a proof of Lemmas 3 and 4 using the metaplectic representation in Appendix A. □

We can now prove Proposition 3. On the one hand, via Claim 1, we can invert

$$t(\tau)=\upsilon_1(\tau)=\frac{\alpha_1(\tau)}{\beta_1(\tau)}\quad \text{in } [-T_0,T_0].$$

Therefore, the integral part of formula (4.2)

$$\phi(t,\mathbf{x})=\int \big(K_1(t,x_1,y_{01})K_1\big(\upsilon_2\big(\upsilon_1^{-1}(t)\big),x_2,y_{02}\big)K_1\big(\upsilon_3\big(\upsilon_1^{-1}(t)\big),x_3,y_{03}\big)\big)u_0(y_{01},y_{02},y_{03})\,dy_{01}\,dy_{02}\,dy_{03}$$

in fact solves

$$\begin{aligned}\big(i\partial_t+\widetilde{L_{\mathbf{x}}}(t)\big)\phi=0\quad &\text{in } \mathbb{R}^3\times\big[-\upsilon_1^{-1}(T_0),\upsilon_1^{-1}(T_0)\big],\\ \phi(0,\mathbf{x})=u_0,&\end{aligned}$$

where

$$\widetilde{L_{\mathbf{x}}}(t)=\frac{1}{2}\frac{\partial^2}{\partial x_1^2}+\frac{1}{2}\frac{\beta_1^2(\upsilon_1^{-1}(t))}{\beta_2^2(\upsilon_1^{-1}(t))}\frac{\partial^2}{\partial x_2^2}+\frac{1}{2}\frac{\beta_1^2(\upsilon_1^{-1}(t))}{\beta_3^2(\upsilon_1^{-1}(t))}\frac{\partial^2}{\partial x_3^2}.$$

On the other hand, plugging $-\tau$ into formula (4.2) yields

$$U_{\mathbf{y}}(-\tau;0)u_0=\left(\prod_{l=1}^{3}\frac{e^{-i\frac{\dot\beta_l(\tau)}{\beta_l(\tau)}\frac{y_l^2}{2}}}{(\beta_l(\tau))^{\frac12}}\right)\int\left(\prod_{l=1}^{3}K_1\left(-\frac{\alpha_l(\tau)}{\beta_l(\tau)},\frac{y_l}{\beta_l(\tau)},y_{0l}\right)\right)u_0(y_{01},y_{02},y_{03})\,dy_{01}\,dy_{02}\,dy_{03}$$

because $\alpha_l$ and $\dot\beta_l$ are odd while $\beta_l$ are even.

Whence in $[-T_0,T_0]$

$$\begin{aligned}U^{(k+1)}(\tau;0)\gamma_0^{(k+1)}&=\prod_{j=1}^{k+1}\big(U_{\mathbf{y}_j}(\tau;0)U_{\mathbf{y}'_j}(-\tau;0)\big)\gamma_0^{(k+1)}\\&=\prod_{j=1}^{k+1}\left(\prod_{l=1}^{3}\frac{e^{i\frac{\dot\beta_l(\tau)}{\beta_l(\tau)}\frac{(|y_{j,l}|^2-|y'_{j,l}|^2)}{2}}}{\beta_l(\tau)}\right)\\&\quad\times u^{(k+1)}\left(\frac{\alpha_1(\tau)}{\beta_1(\tau)},\frac{y_{1,1}}{\beta_1(\tau)},\frac{y_{1,2}}{\beta_2(\tau)},\frac{y_{1,3}}{\beta_3(\tau)},\dots,\frac{y_{k+1,1}}{\beta_1(\tau)},\frac{y_{k+1,2}}{\beta_2(\tau)},\frac{y_{k+1,3}}{\beta_3(\tau)};\right.\\&\quad\left.\frac{y'_{1,1}}{\beta_1(\tau)},\frac{y'_{1,2}}{\beta_2(\tau)},\frac{y'_{1,3}}{\beta_3(\tau)},\dots,\frac{y'_{k+1,1}}{\beta_1(\tau)},\frac{y'_{k+1,2}}{\beta_2(\tau)},\frac{y'_{k+1,3}}{\beta_3(\tau)}\right)\end{aligned}$$

if $u^{(k+1)}(t,\overrightarrow{\mathbf{x}_{k+1}};\overrightarrow{\mathbf{x}'_{k+1}})$ solves

$$\begin{aligned}\big(i\partial_t+\widetilde{L_{\overrightarrow{\mathbf{x}_{k+1}}}}(t)-\widetilde{L_{\overrightarrow{\mathbf{x}'_{k+1}}}}(t)\big)u^{(k+1)}=0\quad &\text{in } \mathbb{R}^{6k+6}\times\big[-\upsilon_1^{-1}(T_0),\upsilon_1^{-1}(T_0)\big],\\ u^{(k+1)}\big(0,\overrightarrow{\mathbf{x}_{k+1}};\overrightarrow{\mathbf{x}'_{k+1}}\big)=\gamma_0^{(k+1)}.&\end{aligned}$$

At long last, define

$$L_{\mathbf{x}}(t)=\begin{cases}\widetilde{L_{\mathbf{x}}}(t), & \text{when } t\in[-\upsilon_1^{-1}(T_0),\upsilon_1^{-1}(T_0)],\\ \widetilde{L_{\mathbf{x}}}(\upsilon_1^{-1}(T_0)), & \text{when } t\geqslant\upsilon_1^{-1}(T_0)\text{ or } t\leqslant-\upsilon_1^{-1}(T_0),\end{cases}$$

then we obtain the desired variant of the generalized lens transform i.e. Proposition 3.

## 5. Proof of Theorem 4

Without loss of generality, we show Theorem 4 for $B^1_{j,k+1}$ in $B_{j,k+1}$ when $j$ is taken to be 1. This corresponds to the estimate:

$$\int_s^T d\tau \int_{\mathbb{R}^{3k}\times\mathbb{R}^{3k}} \left|R_\tau^{(k)}\gamma^{(k+1)}\left(\tau,\overrightarrow{\mathbf{y}_k},\mathbf{y}_1;\overrightarrow{\mathbf{y}'_k},\mathbf{y}_1\right)\right|^2 d\overrightarrow{\mathbf{y}_k}\,d\overrightarrow{\mathbf{y}'_k}$$
$$\leqslant C\left(\inf_{\tau\in[0,T_0]}\prod_{l=2}^3\beta_l^2(\tau)\right)^{-1}\int_{\mathbb{R}^{3(k+1)}\times\mathbb{R}^{3(k+1)}}\left|R_\tau^{(k+1)}\gamma^{(k+1)}\left(\tau,\overrightarrow{\mathbf{y}_{k+1}};\overrightarrow{\mathbf{y}'_{k+1}}\right)\right|^2 d\overrightarrow{\mathbf{y}_{k+1}}\,d\overrightarrow{\mathbf{y}'_{k+1}}, \tag{5.1}$$

$\forall\tau\in[s,T]$, if $\gamma^{(k+1)}$ satisfies Eq. (1.11).

By Proposition 3, we compute

$$R_\tau^{(k)}\gamma^{(k+1)}\left(\tau,\overrightarrow{\mathbf{y}_k},\mathbf{y}_1;\overrightarrow{\mathbf{y}'_k},\mathbf{y}_1\right)$$
$$=\left(\prod_{l=1}^3\frac{1}{\beta_l(\tau)}\right)\prod_{j=1}^k\left(\prod_{l=1}^3\frac{e^{i\frac{\dot\beta_l(\tau)}{\beta_l(\tau)}\frac{(|y_{j,l}|^2-|y'_{j,l}|^2)}{2}}}{\beta_l(\tau)}\right)\left(\left(\prod_{j=1}^k(\nabla_{\mathbf{x}_j}\nabla_{\mathbf{x}'_j})\right)u^{(k+1)}\left(\frac{\alpha_1(\tau)}{\beta_1(\tau)},\overrightarrow{\mathbf{x}_k},\mathbf{x}_1;\overrightarrow{\mathbf{x}'_k},\mathbf{x}_1\right)\right), \tag{5.2}$$

if we let

$$x_{j,l}=\frac{y_{j,l}}{\beta_l(\tau)}\quad\text{and}\quad x'_{j,l}=\frac{y'_{j,l}}{\beta_l(\tau)},$$

because of the relations

$$i\beta_l(\tau)\frac{\partial}{\partial y_{j,l}}\left(e^{i\frac{\dot\beta_l(\tau)}{\beta_l(\tau)}\frac{|y_{j,l}|^2}{2}}\right)+\dot\beta_l(\tau)y_{j,l}\left(e^{i\frac{\dot\beta_l(\tau)}{\beta_l(\tau)}\frac{|y_{j,l}|^2}{2}}\right)=0,$$
$$\beta_l(\tau)\frac{\partial}{\partial y_{j,l}}=\frac{\partial}{\partial x_{j,l}}.$$

Consequently,

$$\int_s^T d\tau\int_{\mathbb{R}^{3k}\times\mathbb{R}^{3k}}\left|R_\tau^{(k)}\gamma^{(k+1)}\left(\tau,\overrightarrow{\mathbf{y}_k},\mathbf{y}_1;\overrightarrow{\mathbf{y}'_k},\mathbf{y}_1\right)\right|^2 d\overrightarrow{\mathbf{y}_k}\,d\overrightarrow{\mathbf{y}'_k}$$
$$=\int_s^T d\tau\int_{\mathbb{R}^{6k}}\left|\left(\prod_{l=1}^3\frac{1}{\beta_l(\tau)}\right)^{k+1}\left(\prod_{j=1}^k(\nabla_{\mathbf{x}_j}\nabla_{\mathbf{x}'_j})\right)u^{(k+1)}\left(\frac{\alpha_1(\tau)}{\beta_1(\tau)},\overrightarrow{\mathbf{x}_k},\mathbf{x}_1;\overrightarrow{\mathbf{x}'_k},\mathbf{x}_1\right)\right|^2 d\overrightarrow{\mathbf{y}_k}\,d\overrightarrow{\mathbf{y}'_k}$$
$$=\int_s^T\frac{d\tau}{(\beta_1(\tau))^2}\int_{\mathbb{R}^{6k}}\left(\prod_{l=2}^3\frac{1}{\beta_l(\tau)}\right)^2\left|\left(\prod_{j=1}^k(\nabla_{\mathbf{x}_j}\nabla_{\mathbf{x}'_j})\right)u^{(k+1)}\left(\frac{\alpha_1(\tau)}{\beta_1(\tau)},\overrightarrow{\mathbf{x}_k},\mathbf{x}_1;\overrightarrow{\mathbf{x}'_k},\mathbf{x}_1\right)\right|^2 d\overrightarrow{\mathbf{x}_k}\,d\overrightarrow{\mathbf{x}'_k}$$
$$\leqslant\left(\inf_{\tau\in[0,T_0]}\prod_{l=2}^3\beta_l^2(\tau)\right)^{-1}\int_s^T\frac{d\tau}{(\beta_1(\tau))^2}\int_{\mathbb{R}^{6k}}\left|\left(\prod_{j=1}^k(\nabla_{\mathbf{x}_j}\nabla_{\mathbf{x}'_j})\right)u^{(k+1)}\left(\frac{\alpha_1(\tau)}{\beta_1(\tau)},\overrightarrow{\mathbf{x}_k},\mathbf{x}_1;\overrightarrow{\mathbf{x}'_k},\mathbf{x}_1\right)\right|^2 d\overrightarrow{\mathbf{x}_k}\,d\overrightarrow{\mathbf{x}'_k}$$
$$\leqslant\left(\inf_{\tau\in[0,T_0]}\prod_{l=2}^3\beta_l^2(\tau)\right)^{-1}\int_{-\infty}^\infty dt\int_{\mathbb{R}^{3k}\times\mathbb{R}^{3k}}\left|\left(\prod_{j=1}^k(\nabla_{\mathbf{x}_j}\nabla_{\mathbf{x}'_j})\right)u^{(k+1)}\left(t,\overrightarrow{\mathbf{x}_k},\mathbf{x}_1;\overrightarrow{\mathbf{x}'_k},\mathbf{x}_1\right)\right|^2 d\overrightarrow{\mathbf{x}_k}\,d\overrightarrow{\mathbf{x}'_k},$$

where we used the fact that the Wronskian of $\alpha_l$ and $\beta_l$ is constant 1, i.e.

$$\frac{dt}{d\tau} = \frac{\dot{\alpha}_1(\tau)\beta_1(\tau) - \alpha_1(\tau)\dot{\beta}_1(\tau)}{(\beta_1(\tau))^2} = \frac{1}{(\beta_1(\tau))^2}$$

as shown in Claim 1.

A corollary of Theorem 2 tells us that

**Corollary 1.** *Let $L_{\mathbf{x}}(t)$ be the same as in Theorem 2 and $u^{(k+1)}$ verify*

$$\left(i\partial_t + L_{\overrightarrow{\mathbf{x}_{k+1}}}(t) - L_{\overrightarrow{\mathbf{x}'_{k+1}}}(t)\right)u^{(k+1)} = 0.$$

*Then there is a $C > 0$, independent of $j$, $k$, and $u^{(k+1)}$ s.t.*

$$\begin{aligned}
&\left\| \left( \prod_{j=1}^{k} (\nabla_{\mathbf{x}_j} \nabla_{\mathbf{x}'_j}) \right) \left(B^1_{j,k+1} u^{(k+1)}\right)\left(t, \overrightarrow{\mathbf{x}_k}; \overrightarrow{\mathbf{x}'_k}\right) \right\|_{L^2(\mathbb{R}\times\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \\
&= \left\| \left( \prod_{j=1}^{k} (\nabla_{\mathbf{x}_j} \nabla_{\mathbf{x}'_j}) \right) u^{(k+1)}\left(t, \overrightarrow{\mathbf{x}_k}, \mathbf{x}_j; \overrightarrow{\mathbf{x}'_k}, \mathbf{x}_j\right) \right\|_{L^2(\mathbb{R}\times\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \\
&\leqslant C \left\| \left( \prod_{j=1}^{k+1} (\nabla_{\mathbf{x}_j} \nabla_{\mathbf{x}'_j}) \right) u^{(k+1)}\left(0, \overrightarrow{\mathbf{x}_{k+1}}; \overrightarrow{\mathbf{x}'_{k+1}}\right) \right\|_{L^2(\mathbb{R}^{3(k+1)}\times\mathbb{R}^{3(k+1)})}.
\end{aligned}$$

Whence inequality (5.1) follows.

## 6. The uniqueness of hierarchy (1.6)

To get Theorem 5, we of course use the Klainerman–Machedon board game argument to group the terms. For convenience, we assume $b_0 = 1$ here.

**Lemma 5.** *One can express $\gamma^{(1)}(\tau_1, \cdot; \cdot)$ in the Gross–Pitaevskii hierarchy (1.6) as a sum of at most $4^n$ terms of the form*

$$\int_D J(\underline{\tau}_{n+1}, \mu_m)\, d\underline{\tau}_{n+1},$$

*or in other words,*

$$\gamma^{(1)}(\tau_1, \cdot; \cdot) = \sum_m \int_D J(\underline{\tau}_{n+1}, \mu_m)\, d\underline{\tau}_{n+1}. \tag{6.1}$$

*Here $\underline{\tau}_{n+1} = (\tau_2, \tau_3, \ldots, \tau_{n+1})$, $D \subset [s, \tau_1]^n$, $\mu_m$ are a set of maps from $\{2, \ldots, n+1\}$ to $\{1, \ldots, n\}$ satisfying $\mu_m(2) = 1$ and $\mu_m(j) < j$ for all $j$, and*

$$J(\underline{\tau}_{n+1}, \mu_m) = U^{(1)}(\tau_1; \tau_2) B_{1,2} U^{(2)}(\tau_2; \tau_3) B_{\mu_m(3),2} \cdots U^{(n)}(\tau_n; \tau_{n+1}) B_{\mu_m(n+1),n+1}\left(\gamma^{(n+1)}(\tau_{n+1}, \cdot; \cdot)\right).$$

**Proof.** The RHS of formula (6.1) is in fact a Duhamel principle. This lemma follows from the proof of Theorem 3.4 in [25] which uses a board game inspired by the Feynman graph argument in [14]. One just needs to replace $e^{i(t_1-t_2)\triangle_y}$ by $U_{\mathbf{y}}(t_1; t_2)$, and $e^{i(t_1-t_2)\triangle^{(k)}}$ by $U^{(k)}(t_1; t_2)$. □

Let $D_{\tau_2} = \{(\tau_3, \ldots, \tau_{n+1}) \mid (\tau_2, \tau_3, \ldots, \tau_{n+1}) \in D\}$ where $D$ is as in Lemma 5. Assuming that we have already verified

$$\left\| R_s^{(1)} \gamma^{(1)}(s, \cdot) \right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} = 0,$$

applying Lemma 5 to $[s,\tau_1]\subset[0,T_0]$, we have

$$\begin{aligned}
&\left\|R^{(1)}_{\tau_1}\gamma^{(1)}(\tau_1,\cdot)\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}\\
&\quad=\left\|R^{(1)}_{\tau_1}\int_D U^{(1)}(\tau_1;\tau_2)B_{1,2}U^{(2)}(\tau_2;\tau_3)B_{\mu_m(3),2}\cdots d\tau_2\cdots d\tau_{n+1}\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}\\
&\quad=\left\|\int_s^{\tau_1} U^{(1)}(\tau_1;\tau_2)\left(\int_{D_{\tau_2}} R^{(1)}_{\tau_2}B_{1,2}U^{(2)}(\tau_2;\tau_3)B_{\mu_m(3),2}\cdots d\tau_3\cdots d\tau_{n+1}\right)d\tau_2\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \quad \text{(Lemma 3)}\\
&\quad\leqslant\int_s^{\tau_1}\left\|\int_{D_{\tau_2}} R^{(1)}_{\tau_2}B_{1,2}U^{(2)}(\tau_2;\tau_3)B_{\mu_m(3),2}\cdots d\tau_3\cdots d\tau_{n+1}\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}d\tau_2\\
&\quad\leqslant\int_{[s,\tau_1]^n}\left\|R^{(1)}_{\tau_2}B_{1,2}U^{(2)}(\tau_2;\tau_3)B_{\mu_m(3),2}\cdots\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}d\tau_2\,d\tau_3\cdots d\tau_{n+1}\\
&\quad\leqslant(\tau_1-s)^{\frac12}\int_{[s,\tau_1]^{n-1}}\left\|R^{(1)}_{\tau_2}B_{1,2}U^{(2)}(\tau_2;\tau_3)B_{\mu_m(3),2}\cdots\right\|_{L^2(\tau_2\in[s,\tau_1]\times\mathbb{R}^3\times\mathbb{R}^3)}d\tau_3\cdots d\tau_{n+1}\\
&\quad\leqslant C(\tau_1-s)^{\frac12}\int_{[s,\tau_1]^{n-1}}\left\|R^{(2)}_{\tau_2}U^{(2)}(\tau_2;\tau_3)B_{\mu_m(3),2}\cdots\right\|_{L^2(\mathbb{R}^6\times\mathbb{R}^6)}d\tau_3\cdots d\tau_{n+1} \quad \text{(Theorem 4)}\\
&\qquad\text{(same procedure } n-2 \text{ times)}\\
&\quad\leqslant C\big(C(\tau_1-s)\big)^{\frac{n-1}{2}}\int_s^{\tau_1}\left\|R^{(n)}_{\tau_{n+1}}B_{\mu_m(n+1),n+1}\gamma^{(n+1)}(\tau_{n+1},\cdot)\right\|_{L^2(\mathbb{R}^{3n}\times\mathbb{R}^{3n})}d\tau_{n+1}\\
&\quad\leqslant C\big(C(\tau_1-s)\big)^{\frac{n-1}{2}}.
\end{aligned}$$

Let $(\tau_1-s)$ be sufficiently small, and $n\to\infty$, then we infer that

$$\left\|R^{(1)}_{\tau_1}\gamma^{(1)}(\tau_1,\cdot)\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}=0 \quad \text{in } [s,\tau_1].$$

Similar arguments show that $\|R^{(k)}_\tau\gamma^{(k)}(\tau,\cdot)\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}=0$, $\forall k$, $\tau\in[0,T_0]$. Hence we have attained Theorem 5.

## 7. Derivation of the 2d cubic NLS with anisotropic switchable quadratic traps (proof of Theorem 1)

For a more comprehensible presentation, let us suppose

$$H_{\mathbf{y}}(\tau)=\sum_{l=1}^n\left(-\frac{\partial^2}{\partial y_{j,l}^2}+\eta_l(\tau)y_{j,l}^2\right)$$

is the ordinary Hermite operator

$$H_{\mathbf{y}}=-\triangle_{\mathbf{y}}+|\mathbf{y}|^2$$

in this section to make formulas shorter and more explicit. We will add two remarks in the proof to address the small modifications needed for the general case.

We start by reviewing the standard Elgart–Erdös–Schlein–Yau program in this setting.

Step A. Observe that, by definition, $\{\gamma_N^{(k)}\}$ solves the quadratic trap Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy

$$\begin{aligned}&\left(i\partial_\tau-\frac{1}{2}\left(-\triangle_{\overrightarrow{\mathbf{y}_k}}+|\overrightarrow{\mathbf{y}_k}|^2\right)+\frac{1}{2}\left(-\triangle_{\overrightarrow{\mathbf{y}'_k}}+\left|\overrightarrow{\mathbf{y}'_k}\right|^2\right)\right)\gamma_N^{(k)}\\&\quad=\frac{1}{N}\sum_{1\leqslant i<j\leqslant k}\left(V_N(\mathbf{y}_i-\mathbf{y}_j)-V_N\left(\mathbf{y}'_i-\mathbf{y}'_j\right)\right)\gamma_N^{(k)}\\&\qquad+\frac{N-k}{N}\sum_{j=1}^{k}\int dy_{k+1}\left[\left(V_N(\mathbf{y}_i-\mathbf{y}_{k+1})-V_N\left(\mathbf{y}'_i-\mathbf{y}_{k+1}\right)\right)\gamma_N^{(k+1)}\left(\tau,\overrightarrow{\mathbf{y}_k},\mathbf{y}_{k+1};\overrightarrow{\mathbf{y}'_k},\mathbf{y}_{k+1}\right)\right],\end{aligned}\tag{7.1}$$

where $V_N(\mathbf{x})=N^{n\beta}V(N^\beta\mathbf{x})$. It converges (at least formally) to the quadratic trap Gross–Pitaevskii infinite hierarchy

$$\left(i\partial_\tau-\frac{1}{2}\left(-\triangle_{\overrightarrow{\mathbf{y}_k}}+|\overrightarrow{\mathbf{y}_k}|^2\right)+\frac{1}{2}\left(-\triangle_{\overrightarrow{\mathbf{y}'_k}}+\left|\overrightarrow{\mathbf{y}'_k}\right|^2\right)\right)\gamma^{(k)}=b_0\sum_{j=1}^{k}B_{j,k+1}\left(\gamma^{(k+1)}\right).\tag{7.2}$$

Prove rigorously that the sequence $\{\gamma_N^{(k)}\}$ is compact with respect to the weak* topology on the trace class operators and every limit point $\{\gamma^{(k)}\}$ satisfies hierarchy (7.2).

Step B. Utilize a suitable uniqueness theorem of hierarchy (7.2) to conclude that

$$\gamma^{(k)}\left(\tau,\overrightarrow{\mathbf{y}_k};\overrightarrow{\mathbf{y}'_k}\right)=\prod_{j=1}^{k}\phi(\tau,\mathbf{y}_j)\overline{\phi\left(\tau,\mathbf{y}'_j\right)},$$

where $\phi$ solves the 2d quadratic trap cubic NLS

$$i\partial_\tau\phi=\frac{1}{2}\left(-\triangle+|\mathbf{y}|^2\right)\phi+b_0\phi|\phi|^2.$$

So the compact sequence $\{\gamma_N^{(k)}\}$ has only one limit point, i.e.

$$\gamma_N^{(k)}\to\prod_{j=1}^{k}\phi(\tau,\mathbf{y}_j)\overline{\phi\left(\tau,\mathbf{y}'_j\right)}$$

in the weak* topology. Since $\gamma^{(k)}$ is an orthogonal projection, the convergence in the weak* topology is equivalent to the convergence in the trace norm topology.

We modify this procedure to show Theorem 1. We remark that the main additional tool is the lens transform. When $H_{\mathbf{y}}(\tau)$ is the Hermite operator, $\alpha_l=\sin\tau$, $\beta_l=\cos\tau$ and $T_0<\frac{\pi}{2}$ i.e. the lens transform and its inverse reads as follows.

**Definition 2.** We define the lens transform $T_l:L^2(d\overrightarrow{\mathbf{x}_k}\,d\overrightarrow{\mathbf{x}'_k})\to L^2(d\overrightarrow{\mathbf{y}_k}\,d\overrightarrow{\mathbf{y}'_k})$ and its inverse by

$$\begin{aligned}\left(T_lu^{(k)}\right)\left(\tau,\overrightarrow{\mathbf{y}_k};\overrightarrow{\mathbf{y}'_k}\right)&=\frac{1}{(\cos\tau)^{nk}}u^{(k)}\left(\tan\tau,\frac{\overrightarrow{\mathbf{y}_k}}{\cos\tau};\frac{\overrightarrow{\mathbf{y}'_k}}{\cos\tau}\right)e^{-i\frac{\tan\tau}{2}(|\overrightarrow{\mathbf{y}_k}|^2-|\overrightarrow{\mathbf{y}'_k}|^2)},\\\left(T_l^{-1}\gamma^{(k)}\right)\left(t,\overrightarrow{\mathbf{x}_k};\overrightarrow{\mathbf{x}'_k}\right)&=\frac{1}{(1+t^2)^{\frac{nk}{2}}}\gamma^{(k)}\left(\arctan t,\frac{\overrightarrow{\mathbf{x}_k}}{\sqrt{1+t^2}};\frac{\overrightarrow{\mathbf{x}'_k}}{\sqrt{1+t^2}}\right)e^{\frac{it}{2(1+t^2)}(|\overrightarrow{\mathbf{x}_k}|^2-|\overrightarrow{\mathbf{x}'_k}|^2)}.\end{aligned}$$

$T_l$ is unitary by definition and the variables are related by

$$\tau=\arctan t,\qquad\mathbf{y}_k=\frac{\mathbf{x}_k}{\sqrt{1+t^2}}\quad\text{and}\quad\mathbf{y}'_k=\frac{\mathbf{x}'_k}{\sqrt{1+t^2}}.$$

**Remark 6.** For the general anisotropic case, we still need the 2d version of Proposition 3.

Let us write

$$\left(T_l^{-1}\gamma^{(k)}\right)\left(t,\overrightarrow{\mathbf{x}_k};\overrightarrow{\mathbf{x}'_k}\right)=\gamma^{(k)}\left(\tau,\overrightarrow{\mathbf{y}_k};\overrightarrow{\mathbf{y}'_k}\right)\frac{e^{\frac{it}{2(1+t^2)}(|\mathbf{x}_k|^2-|\mathbf{x}'_k|^2)}}{(1+t^2)^{\frac{nk}{2}}}:=\gamma^{(k)}\left(\tau,\overrightarrow{\mathbf{y}_k};\overrightarrow{\mathbf{y}'_k}\right)h_n^{(k)}\left(t,\overrightarrow{\mathbf{x}_k};\overrightarrow{\mathbf{x}'_k}\right),$$

then we have a more explicit version of Proposition 3.

**Proposition 4.**

$$\left(i\partial_t + \frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}_k}} - \frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}'_k}}\right)\left(T_l^{-1}\gamma^{(k)}\right)\left(t, \overrightarrow{\mathbf{x}_k}; \overrightarrow{\mathbf{x}'_k}\right)$$
$$= \frac{h_n^{(k)}}{1+t^2}\left[\left(i\partial_\tau - \frac{1}{2}\left(-\triangle_{\overrightarrow{\mathbf{y}_k}} + |\overrightarrow{\mathbf{y}_k}|^2\right) + \frac{1}{2}\left(-\triangle_{\overrightarrow{\mathbf{y}'_k}} + |\overrightarrow{\mathbf{y}'_k}|^2\right)\right)\gamma^{(k)}\left(\tau, \overrightarrow{\mathbf{y}_k}; \overrightarrow{\mathbf{y}'_k}\right)\right].$$

**Proof.** This is a direct computation. □

Via this proposition, we understand how the lens transform acts on hierarchies (7.1) and (7.2).

**Lemma 6** *(Gross–Pitaevskii hierarchy under the lens transform). $\{\gamma^{(k)}\}$ solves the quadratic trap Gross–Pitaevskii hierarchy (7.2) if and only if $\{u^{(k)} = T_l^{-1}\gamma^{(k)}\}$ solves the infinite hierarchy*

$$\left(i\partial_t + \frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}_k}} - \frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}'_k}}\right)u^{(k)} = \frac{(1+t^2)^{\frac{n}{2}}}{1+t^2}b_0\sum_{j=1}^{k}B_{j,k+1}\left(u^{(k+1)}\right). \tag{7.3}$$

*In particular, when $n = 2$, the lens transform preserves the Gross–Pitaevskii hierarchy.*

**Lemma 7** *(BBGKY hierarchy under the lens transform). $\{\gamma_N^{(k)}\}$ solves the quadratic trap BBGKY hierarchy (7.1) if and only if $\{u_N^{(k)} = T_l^{-1}\gamma_N^{(k)}\}$ solves the hierarchy*

$$\left(i\partial_t + \frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}_k}} - \frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}'_k}}\right)u_N^{(k)}$$
$$= \frac{1}{N}\frac{1}{1+t^2}\sum_{1\leqslant i<j\leqslant k}\left(V_N\left(\frac{\mathbf{x}_i - \mathbf{x}_j}{\sqrt{1+t^2}}\right) - V_N\left(\frac{\mathbf{x}'_i - \mathbf{x}'_j}{\sqrt{1+t^2}}\right)\right)u_N^{(k)}$$
$$+ \frac{N-k}{N}\frac{1}{1+t^2}\sum_{j=1}^{k}\int d\mathbf{x}_{k+1}\left[\left(V_N\left(\frac{\mathbf{x}_i - \mathbf{x}_{k+1}}{\sqrt{1+t^2}}\right) - V_N\left(\frac{\mathbf{x}'_i - \mathbf{x}_{k+1}}{\sqrt{1+t^2}}\right)\right)u_N^{(k+1)}\left(t, \overrightarrow{\mathbf{x}_k}, \mathbf{x}_{k+1}; \overrightarrow{\mathbf{x}'_k}, \mathbf{x}_{k+1}\right)\right]. \tag{7.4}$$

We can now prove Theorem 1.

## *7.1. Proof of Theorem 1*

Step 1. Let $n = 2$, consider $\{u_N^{(k)} = T_l^{-1}\gamma_N^{(k)}\}$ which solves hierarchy (7.4).
Step 2. Write

$$\widetilde{V}(\mathbf{x}) = \frac{1}{1+t^2}V\left(\frac{\mathbf{x}}{\sqrt{1+t^2}}\right),$$

then

$$\frac{1}{(1+T^2)^{1-\frac{1}{p}}}\|V\|_p \leqslant \|\widetilde{V}\|_p \leqslant \|V\|_p \quad \text{when } T < \infty \text{ and } p \geqslant 1.$$

Therefore we can employ the proof in Kirkpatrick, Schlein and Staffilani [23] to show that the sequence $\{u_N^{(k)}\}$ is compact with respect to the weak* topology on the trace class operators and every limit point $\{u^{(k)}\}$ satisfies the Gross–Pitaevskii hierarchy (7.3). Moreover, based on a fixed time trace theorem argument as in [23], for $\alpha < 1$, we have

$$\int_0^T dt\left\|\prod_{j=1}^{k}\left(\langle\nabla_{\mathbf{x}_j}\rangle^\alpha\langle\nabla_{\mathbf{x}_j}\rangle^\alpha\right)B_{j,k+1}\left(u^{(k+1)}\right)\right\|_{L^2(\mathbb{R}^{2k}\times\mathbb{R}^{2k})} \leqslant C^k,$$

for every limit point $\{u^{(k)}\}$. To be more precise, the proof in [23] involves a smooth approximation. We omit this detail here.

**Remark 7.** The auxiliary Hamiltonian

$$\widetilde{H_N}(t) = \frac{1}{2}\sum_{j=1}^{N} L_{\mathbf{X}_j}(t) + \frac{1}{N}\sum_{i<j} N^{2\beta}\tilde{V}\big(N^{\beta}(\mathbf{x}_i - \mathbf{x}_j)\big),$$

which corresponds to the anisotropic quadratic potential case, does not lead to the conservation of the quantity

$$\big\langle \psi_N, \big(\widetilde{H_N}(t)\big)^k \psi_N\big\rangle.$$

On the other hand, the following estimate controls the energy:

$$\begin{aligned}\frac{d}{dt}\big\langle \psi_N, \big(\widetilde{H_N}(t)\big)^k \psi_N\big\rangle &= \left\langle \psi_N, \left[\frac{d}{dt}, \big(\widetilde{H_N}(t)\big)\right]\big(\widetilde{H_N}(t)\big)^{k-1}\psi_N\right\rangle + \cdots + \left\langle \psi_N, \big(\widetilde{H_N}(t)\big)^{k-1}\left[\frac{d}{dt}, \big(\widetilde{H_N}(t)\big)\right]\psi_N\right\rangle \\ &\leqslant Ck\big\langle \psi_N, \big(\widetilde{H_N}(t)\big)^k \psi_N\big\rangle\end{aligned}$$

since $a_1$ and $a_2$, the coefficients of $L_{\mathbf{X}}$, are $C^1$ in the context of Theorem 1. Thus Gronwall's inequality takes care of the problem for us as long as we are considering finite time.

Step 3. By Theorem 3 (2d uniqueness) or Theorem 7.1 in [23], we deduce that

$$u^{(k)}\big(t, \overrightarrow{\mathbf{x}_k}; \overrightarrow{\mathbf{x}'_k}\big) = \prod_{j=1}^{k} \tilde{\phi}(t, \mathbf{x}_j)\overline{\tilde{\phi}\big(t, \mathbf{x}'_j\big)},$$

where $\tilde{\phi}$ solves the 2d cubic NLS

$$i\partial_t\tilde{\phi} = -\frac{1}{2}\triangle\tilde{\phi} + b_0\tilde{\phi}|\tilde{\phi}|^2.$$

Hence the compact sequence $\{u_N^{(k)}\}$ has only one limit point, so

$$u_N^{(k)} \to \prod_{j=1}^{k} \tilde{\phi}(t, \mathbf{x}_j)\overline{\tilde{\phi}\big(t, \mathbf{x}'_j\big)}$$

in the weak* topology. Since $u^{(k)}$ is an orthogonal projection, the convergence in the weak* topology is equivalent to the convergence in the trace norm topology.

**Remark 8.** It is necessary to use Theorem 3 in this paper for the general anisotropic quadratic traps case.

Step 4. Let $\phi$ solve the 2d quadratic trap cubic NLS

$$i\partial_\tau\phi = \frac{1}{2}\big(-\triangle + |\mathbf{y}|^2\big)\phi + b_0\phi|\phi|^2,$$

then the lens transform of $u^{(k)}$ is

$$\gamma^{(k)}\big(\tau, \overrightarrow{\mathbf{y}_k}; \overrightarrow{\mathbf{y}'_k}\big) = \prod_{j=1}^{k} \phi(\tau, \mathbf{y}_j)\overline{\phi\big(\tau, \mathbf{y}'_j\big)},$$

due to the fact that the lens transform preserves mass critical NLS, which is the cubic NLS in 2d.

Step 5. The convergence

$$u_N^{(k)} \to u^{(k)}$$

in the trace norm indicates the convergence in the Hilbert–Schmidt norm. But the lens transform

$$T_l : L^2\big(d\overrightarrow{\mathbf{x}}\, d\overrightarrow{\mathbf{x}}'\big) \to L^2\big(d\overrightarrow{\mathbf{y}}\, d\overrightarrow{\mathbf{y}}'\big)$$

is unitary (so preserves the norm) and thus

$$\gamma_N^{(k)} = T_l u_N^{(k)} \to T_l u^{(k)} = \gamma^{(k)}.$$

Thence we conclude that $\gamma_N^{(k)}$ converges to

$$\gamma^{(k)}\big(\tau, \overrightarrow{\mathbf{y}}_k; \overrightarrow{\mathbf{y}}_k'\big) = \prod_{j=1}^{k} \phi(\tau, \mathbf{y}_j)\overline{\phi\big(\tau, \mathbf{y}_j'\big)},$$

in the Hilbert–Schmidt norm, which is Theorem 1.

### 7.2. Comments about the 3d case

It is natural to wonder what we can say about the 3d case using the above method. Visiting Lemma 6 again yields the hierarchy

$$\left(i\partial_t + \frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}}_k} - \frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}}_k'}\right)u^{(k)} = \big(1+t^2\big)^{\frac{1}{2}} b_0 \sum_{j=1}^{k} B_{j,k+1}\big(u^{(k+1)}\big). \tag{7.5}$$

Due to the factor $(1+t^2)^{\frac{1}{2}}$, it is difficult to see of what use a 3d version of Theorem 3 might be. We can certainly give a uniqueness theorem regarding hierarchy (7.5) with the techniques in this paper. But it is unknown how to verify the space–time bound when $n = 3$ as stated earlier.

Another possibility to attack the 3d case is the standard Elgart–Erdos–Schlein–Yau procedure, but we presently know very little about the analysis of the Hermite like operator $H_{\mathbf{y}}(\tau)$.

Finally, we remark that it is not clear whether the Feynman diagrams argument, the key to the uniqueness theorem in [14] on which [13–17] are based, leads to a 3d uniqueness theorem of hierarchy (1.6) or (7.5), which represent the two sides of the lens transform.

## 8. Conclusion

In this paper, we have derived rigorously the 2d cubic NLS with anisotropic switchable quadratic traps through a modified Elgart–Erdös–Schlein–Yau procedure. We have attained partial results in 3d as well. Unfortunately, when $n = 3$, we still have unsolved problems as stated in Section 7.2.

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## Appendix A. The generalized lens transform and the metaplectic representation

In this appendix, we prove Lemmas 3 and 4 via the metaplectic representation. The 3d anisotropic case drops out once we show the 1d case. Before we delve into the proof, we remark that we currently do not have an explanation away from direct computations for Proposition 4 or for the fact that the generalized lens transform preserves $L^2$ critical NLS. The group theory proof presented in this appendix only shows the linear case: Lemmas 3 and 4.

Throughout this appendix, we consider the metaplectic representation

$$\mu : Sp(2,\mathbb{R}) \to \text{unitary operators on } L^2(\mathbb{R}),$$

which has the property:

$$d\mu\left(\begin{pmatrix} 0 & 1 \\ -\eta(\tau) & 0 \end{pmatrix}\right) = i\left(-\frac{1}{2}\partial_y^2 + \eta(\tau)\frac{y^2}{2}\right).$$

For more information regarding $\mu$ and $d\mu$, we refer the readers to Folland's monograph [18]. We comment that $\mu$ is not a well-defined group homomorphism on all of $Sp(2,\mathbb{R})$, but the fact that it is well-defined in a neighborhood of the identity of $Sp(2,\mathbb{R})$ is good enough for our purpose here.

*A.1. The generalized lens transform (proof of Lemma 4)*

**Proposition 5.** *Define $\alpha$ and $\beta$ through the system*

$$\begin{aligned} &\ddot{\alpha}(\tau) + \eta(\tau)\alpha(\tau) = 0, \qquad \alpha(0) = 0, \qquad \dot{\alpha}(0) = 1, \\ &\ddot{\beta}(\tau) + \eta(\tau)\beta(\tau) = 0, \qquad \beta(0) = 1, \qquad \dot{\beta}(0) = 0, \end{aligned}$$

*and let*

$$B(\tau) = \begin{pmatrix} \beta(\tau) & -\alpha(\tau) \\ -\dot{\beta}(\tau) & \dot{\alpha}(\tau) \end{pmatrix}.$$

*Assume $\beta$ is nonzero in some time interval $[0,T]$, then $\mu(B(\tau))f$ solves the Schrödinger equation with switchable quadratic trap*:

$$\begin{aligned} &i\partial_\tau u = \left(-\frac{1}{2}\partial_y^2 + \eta(\tau)\frac{y^2}{2}\right)u \quad \text{in } \mathbb{R}\times[0,T], \\ &u(0,y) = f(y) \in L^2(\mathbb{R}). \end{aligned} \tag{A.1}$$

**Proof.** We calculate

$$\begin{aligned} \partial_\tau|_{\tau=0}\mu\big(B(\tau_0+\tau)\big)f &= \big(\partial_\tau|_{\tau=0}\mu\big(B(\tau_0+\tau)\big)\big)f \\ &= \big(\partial_\tau|_{\tau=0}\mu\big(B(\tau_0+\tau)B^{-1}(\tau_0)B(\tau_0)\big)\big)f \\ &= \big(\partial_\tau|_{\tau=0}\mu\big(B(\tau_0+\tau)B^{-1}(\tau_0)\big)\big)\mu\big(B(\tau_0)\big)f \\ &= d\mu\big(B'(\tau_0)B^{-1}(\tau_0)\big)\mu\big(B(\tau_0)\big)f, \end{aligned}$$

where

$$\begin{aligned} B'(\tau_0)B^{-1}(\tau_0) &= \begin{pmatrix} \dot{\beta}(\tau_0) & -\dot{\alpha}(\tau_0) \\ -\ddot{\beta}(\tau_0) & \ddot{\alpha}(\tau_0) \end{pmatrix}\begin{pmatrix} \dot{\alpha}(\tau_0) & \alpha(\tau_0) \\ \dot{\beta}(\tau_0) & \beta(\tau_0) \end{pmatrix} \\ &= \begin{pmatrix} \dot{\beta}(\tau_0) & -\dot{\alpha}(\tau_0) \\ \eta(\tau_0)\beta(\tau_0) & -\eta(\tau_0)\alpha(\tau_0) \end{pmatrix}\begin{pmatrix} \dot{\alpha}(\tau_0) & \alpha(\tau_0) \\ \dot{\beta}(\tau_0) & \beta(\tau_0) \end{pmatrix} \\ &= \begin{pmatrix} 0 & \dot{\beta}(\tau_0)\alpha(\tau_0) - \dot{\alpha}(\tau_0)\beta(\tau_0) \\ \eta(\tau_0)(\dot{\alpha}(\tau_0)\beta(\tau_0) - \dot{\beta}(\tau_0)\alpha(\tau_0)) & 0 \end{pmatrix}. \end{aligned}$$

Notice that the Wronskian of $\alpha$ and $\beta$ is constant 1 i.e.

$$\dot{\alpha}(\tau)\beta(\tau) - \alpha(\tau)\dot{\beta}(\tau) = 1.$$

So

$$\begin{aligned} d\mu\big(B'(\tau_0)B^{-1}(\tau_0)\big) &= d\mu\left(\begin{pmatrix} 0 & -1 \\ \eta(\tau_0) & 0 \end{pmatrix}\right) \\ &= -\frac{i}{2}\big(-\partial_y^2 + \eta(\tau_0)y^2\big). \end{aligned}$$

In other words,

$$\partial_\tau\big(\mu\big(B(\tau)\big)f\big) = -\frac{i}{2}\big(-\partial_y^2 + \eta(\tau)y^2\big)\big(\mu\big(B(\tau)\big)f\big).$$

Before we end the proof, we remark that $\beta \neq 0$ is required for the metaplectic representation to be well-defined. □

Through the LDU decomposition of the matrix $B$, we derive the generalized lens transform. The LDU decomposition of the matrix $B$ is

$$\begin{aligned} B(\tau) &= \begin{pmatrix} \beta(\tau) & -\alpha(\tau) \\ -\dot\beta(\tau) & \dot\alpha(\tau) \end{pmatrix} \\ &= \begin{pmatrix} \beta(\tau) & -\alpha(\tau) \\ -\dot\beta(\tau) & \alpha(\tau)\frac{\dot\beta(\tau)}{\beta(\tau)} + \frac{1}{\beta(\tau)} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -\frac{\dot\beta(\tau)}{\beta(\tau)} & 1 \end{pmatrix}\begin{pmatrix} \beta(\tau) & 0 \\ 0 & \frac{1}{\beta(\tau)} \end{pmatrix}\begin{pmatrix} 1 & -\frac{\alpha(\tau)}{\beta(\tau)} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence we have

$$\mu\big(B(\tau)\big)f = \mu\left(\begin{pmatrix} 1 & 0 \\ -\frac{\dot\beta(\tau)}{\beta(\tau)} & 1 \end{pmatrix}\right)\mu\left(\begin{pmatrix} \beta(\tau) & 0 \\ 0 & \frac{1}{\beta(\tau)} \end{pmatrix}\right)\mu\left(\begin{pmatrix} 1 & -\frac{\alpha(\tau)}{\beta(\tau)} \\ 0 & 1 \end{pmatrix}\right)f, \tag{A.2}$$

where

$$\begin{aligned} \mu\left(\begin{pmatrix} 1 & 0 \\ -\frac{\dot\beta(\tau)}{\beta(\tau)} & 1 \end{pmatrix}\right)f(y) &= e^{i\frac{\dot\beta(\tau)}{\beta(\tau)}\frac{y^2}{2}}f(y) \quad \text{by (4.25) in [18]}, \\ \mu\left(\begin{pmatrix} \beta(\tau) & 0 \\ 0 & \frac{1}{\beta(\tau)} \end{pmatrix}\right)f(y) &= \frac{1}{(\beta(\tau))^{\frac12}}f\left(\frac{y}{\beta(\tau)}\right) \quad \text{by (4.24) in [18]}, \\ \mu\left(\begin{pmatrix} 1 & -\frac{\alpha(\tau)}{\beta(\tau)} \\ 0 & 1 \end{pmatrix}\right)f(y) &= e^{i\frac{\alpha(\tau)}{\beta(\tau)}\frac{\partial_y^2}{2}}f \quad \text{by (4.54) in [18]}. \end{aligned}$$

Due to the definition of $\mu$, equality (A.2) in fact holds up to a "±" sign which depends on the time interval. However, the LHS and the RHS of equality (A.2) agree for sufficiently small $\tau$. By continuity, they must agree on the time interval $[0, T]$ where $\beta \neq 0$. So we conclude the following lemma concerning the generalized lens transform.

**Lemma 8.** *(See [3].) Assume $\beta$ is nonzero in the time interval $[0, T]$, then the solution of the Schrödinger equation with switchable quadratic trap (Eq. (A.1)) in $[0, T]$ is given by*

$$u(\tau, y) = \frac{e^{i\frac{\dot\beta(\tau)}{\beta(\tau)}\frac{y^2}{2}}}{(\beta(\tau))^{\frac12}}v\left(\frac{\alpha(\tau)}{\beta(\tau)}, \frac{y}{\beta(\tau)}\right),$$

*if $v(t, x)$ solves the free Schrördinger equation*

$$\begin{aligned} i\partial_t v &= -\frac12\partial_x^2 v \quad \text{in } \mathbb{R}^{1+1}, \\ v(0, x) &= f(x) \in L^2(\mathbb{R}). \end{aligned}$$

The anisotropic case, Lemma 4, follows from the above lemma.

### *A.2. Evolution of momentum (proof of Lemma 3)*

Using the metaplectic representation, we can also compute the evolution of momentum and position.

**Lemma 9.** *The evolution of momentum and position is given by*

$$\begin{aligned} P(\tau) &= \mu\big(B(\tau)\big)\circ(-i\partial_y)\circ\big(\mu\big(B(\tau)\big)\big)^{-1} = -i\beta(\tau)\partial_y - \dot\beta(\tau)y, \\ Y(\tau) &= \mu\big(B(\tau)\big)\circ y\circ\big(\mu\big(B(\tau)\big)\big)^{-1} = i\alpha(\tau)\partial_y + \dot\alpha(\tau)y. \end{aligned}$$

**Proof.** Let us only compute the momentum, position can be obtained similarly

$$\begin{aligned}\mu\big(B(\tau)\big)(-i\partial_y)\big(\mu\big(B(\tau)\big)\big)^{-1} &= \mu\big(B(\tau)\big)\,(1\quad 0)\begin{pmatrix}-i\partial_y\\ y\end{pmatrix}\big(\mu\big(B(\tau)\big)\big)^{-1}\\ &= (1\quad 0)\,\big(B(\tau)\big)^T\begin{pmatrix}-i\partial_y\\ y\end{pmatrix}\quad \big(\text{Theorem 2.15 in [18]}\big)\\ &= (1\quad 0)\begin{pmatrix}\beta(\tau) & -\dot\beta(\tau)\\ -\alpha(\tau) & \dot\alpha(\tau)\end{pmatrix}\begin{pmatrix}-i\partial_y\\ y\end{pmatrix}\\ &= -i\beta(\tau)\partial_y-\dot\beta(\tau)y. \qquad \square\end{aligned}$$

**Remark 9.** We select $-i\partial_y$ to be the momentum to match the canonical commutation relations in Folland [18] which is

$$[-i\partial_y, y] = -iI.$$

The above lemma reproduces the following result in Carles [3].

**Lemma 10.** *(See [3].) The operators $P(\tau)$ and $Y(\tau)$ commute with the linear operator*

$$i\partial_\tau + \frac{1}{2}\partial_y^2 - \eta(\tau)\frac{y^2}{2}.$$

*Moreover,*

$$\begin{aligned}P(\tau)U(\tau;s) &= U(\tau;s)P(s),\\ Y(\tau)U(\tau;s) &= U(\tau;s)Y(s)\end{aligned}$$

*if we let $U_y(\tau;s)$ be the solution operator of*

$$\begin{aligned}i\partial_\tau u &= \left(-\frac{1}{2}\partial_y^2+\eta(\tau)\frac{y^2}{2}\right)u \quad \text{in } \mathbb{R}^{1+1},\\ u(s,y) &= u_s(y)\in L^2(\mathbb{R}),\end{aligned}$$

*or in other words*

$$U_y(\tau;s) = \mu\big(B(\tau)\big)\mu\big(B(s)\big)^{-1}.$$

Thence we have shown Lemma 3.

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