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Phase Shift of Solutions of Second Order Linear Ordinary Differential Equations and Related Problems*

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I. FORMULATION OF THE PROBLEMS

We study here two questions concerning solutions of the differential equation

$$Lx = x'' + f(t)\, x = 0, \qquad f \in C_p[0, \infty). \tag{1.1}$$

The restriction to piecewise continuous f's is not essential here; it insures the existence of a fundamental set of solutions of (1.1) on $[0, \infty)$, but it can be weakened somewhat without complicating anything in what follows.

First, let $f(t) = 1 + q(t)$, and assume that $\int^{\infty} | q(\tau) | \, d\tau < \infty$. One can show then that (1.1) has solutions y_1 and y_2 such that

$$\begin{aligned} y_1(t) &= \cos t + o(1), \\ y_2(t) &= \sin t + o(1), \end{aligned} \tag{1.2}$$

as $t \to \infty$. See [2, Chapter 5] and [1, Theorem (2-10), pp. 60-62, p. 135]. From (1.2) it follows that any solution y of (1.1) can be expressed in the form

$$y(t) = k \cos (t - \psi_{\infty}) + o(1) \tag{1.3}$$

as $t \to \infty$, where k and ψ_{∞} are constants, $k \geq 0$ and ψ_{∞} is determined up to an additive multiple of 2π. Consider, in particular, the solution y of (1.1) with initial data $(\cos \theta, \sin \theta)$, $-\pi/2 < \theta \leq \pi/2$. Then as $t \to 0$, $y(t) = \cos (t - \theta) + o(t)$, and there exist continuous functions v and ψ such that for every $\tau \in [0, \infty)$ and $t \in [0, \infty)$,

$$y(t) = v(\tau) \cos [(t - \tau) - (\psi(\tau) - \tau)] + o(t - \tau) \tag{1.4}$$

as $t \to \tau$.

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This follows since $y^2 + y'^2 > 0$; by using the Taylor formula for y at $t = \tau$ one finds that v and ψ are uniquely determined by

$$y(\tau) = v(\tau) \cos(\psi(\tau) - \tau), \qquad y'(\tau) = v(\tau) \sin(\psi(\tau) - \tau),$$
$$\psi(0) = \theta, \qquad \psi \in C^1[0, \infty), \qquad v(0) = 1. \tag{1.5}$$

Comparing (1.4) with (1.3) one can conclude that $\psi(\infty) = \lim_{t\to\infty} \psi(t)$ exists. The number

$$\varphi = \psi(\infty) - \theta \tag{1.6}$$

is called the *phase shift* and our objective is to compute it.

Next, let $f(t) = -1 + q(t)$ and assume again that $\int^\infty | q(\tau) | \, d\tau < \infty$. In this case the DE (1.1) has two solutions z_1 and z_2 such that as $t \to \infty$

$$z_1(t) = e^{-t}[1 + o(1)],$$
$$z_2(t) = e^{t}[1 + o(1)]. \tag{1.7}$$

See, e.g., [1, pp. 125-6]. Any solution y of the DE will behave like z_2 as $t \to \infty$ unless it is a constant multiple of z_1. On the other hand, a solution y of the DE is uniquely determined by its initial data $y(0)$ and $y'(0)$. It will be a multiple of z_1 if and only if

$$\frac{y(0)}{y'(0)} = \frac{z_1(0)}{z_1'(0)}, \qquad \text{if} \qquad z_1'(0) \neq 0,$$
$$y'(0) = 0, \qquad \text{if} \qquad z_1'(0) = 0. \tag{1.8}$$

Our second question concerns the determination of the ratio $z_1(0)/z_1'(0)$. This also amounts to the characterization through their initial data of the class of solutions of the DE which vanish as $t \to \infty$.

The two problems are treated by exploiting a theory of polar representations of solution of second order linear, homogeneous, differential equations, developed in Section II. Using the results of Section II we obtain in Section III the formula (3.10) for the phase shift, and in Section 4 the formula (4.3) for the ratio $z_1(0)/z_2'(0)$ mentioned above. Sections V and VI contain suggestions concerning the determination of functions u and $\bar{u}$ appearing in formulas (3.10) and (4.3).

II. Polar Representation of Solutions

Our study exploits the existence of an *amplitude* $u \in C^1(I) \cap C_p^2(I)$ and a *phase* $g \in C^2(I) \cap C_p^3(I)$ such that on I,

(a) $u > 0$,
(b) $g' > 0$,

(c) if y is a solution of (1.1) then there exist constants a and θ such that

$$y(t) = au(t) \cos (g(t) - \theta). \tag{2.1}$$

Here I is the interval of definition of f, not necessarily restricted to the interval $[0, \infty)$. Without loss of generality we assume that $0 \in I$ and that $g(0) = 0$. Clearly, if a and θ are such constants, $(-1)^n a$ and $\theta + n\pi$ will do likewise for every integer n.

The representation (2.1) of y will be called *polar representation* (u, g), and (u, g) will be referred to as a *polar pair*. The study of such representations is based solely on the existence theory for solutions of (1.1), and, as noted already in Section I, the restriction to $f \in C_p(I)$ is sufficient (though not necessary) for such a theory.

Polar representations apparently were first studied by E'lsin [4-6] and independently used by Courant and Snyder [3]. Although they seem to play an important role in the study of solutions of second order linear differential equations, there is no mention of them in the current standard texts on the subject. A deterrent to their use may lie in that formula (2.1) allegedly predicts an infinite number of zeros, while solutions of homogeneous linear differential equations sometimes may have only a finite number of zeros. This would preclude the possibility of such representations. The paradox is resolved, however, if one notes that the range of g may be bounded.

THEOREM 2.1. *If there exists a polar pair (u, g), then there exist solutions y_1 and y_2 of $Lx = 0$ with positive Wronskian $W(y_1, y_2)$ such that*

$$u(t) = \sqrt{y_1^2(t) + y_2^2(t)},$$

$$g(t) = \int_0^t \frac{W(y_1, y_2)(\tau)}{u^2(\tau)} \, d\tau. \tag{2.2}$$

PROOF. Let y_1 and y_2 be the solutions of $Lx = 0$ with initial data $(u(0), u'(0))$ and $(0, u(0)\, g'(0))$ respectively. Since the coefficient of x' in Lx is zero, their Wronskian is constant and

$$W(y_1, y_2)(t) = W(y_1, y_2)(0) = u^2(0)\, g'(0) > 0.$$

Using the assumed existence of a polar pair one concludes that

$$y_1(t) = u(t) \cos g(t)$$

$$y_2(t) = u(t) \sin g(t).$$

Squaring and adding these two, one gets the first of the formulas (2.2).
Computing the Wronskian, one gets

$$W(y_1, y_2)(t) = u^2(t) g'(t),$$

and the second of formulas (2.2) follows.

Theorem (2.1) associates with each polar pair (u, g) a pair (y_1, y_2) of solutions of $Lx = 0$. Of course, such association need not be unique. However, it conjectures formulae for determination of polar pairs and facilitates the proof of an existence theorem for polar pairs.

THEOREM 2.2. *Let (y_1, y_2) be a fundamental set of solutions of $Lx = 0$ with positive Wronskian $W(y_1, y_2)$. Then the pair (u, g) defined by formulas (2.2) is a polar pair.*

PROOF. We verify that (u, g) has all the properties of polar pairs. Conditions (a) and (b) are satisfied since y_1 and y_2 do not vanish simultaneously and $W(y_1, y_2) > 0$. (u, g) have the required smoothness properties since $y_1, y_2 \in C^1(I) \cap C^2_p(I)$, and $W(y_1, y_2) =$ constant. By direct computation, one verifies that $W(u \cos g, u \sin g) = W(y_1, y_2) > 0$, and $L(u\, e^{ig}) = 0$. This means that $(u \cos g, u \sin g)$ is a fundamental set of solutions of $Lx = 0$. Thus, if y is any solution of $Lx = 0$,

$$y(t) = \alpha\, u(t) \cos g(t) + \beta\, u(t) \sin g(t),$$

and the representation (2.1) for y follows by choosing $a = \sqrt{\alpha^2 + \beta^2}$, and θ such that $a \cos \theta = \alpha$, $a \sin \theta = \beta$.

Theorem (2.2) not only asserts the existence of a polar pair, but associates a polar pair with each fundamental set of solutions of $Lx = 0$ with positive Wronskian. Theorem (2.1) asserts that each polar pair can be so associated. We shall now discuss the set of polar pairs and establish relations between them. Observe first that if (u, g) is a polar pair associated with the fundamental set (y_1, y_2), then $(|\, k \,|\, u, g)$ is a pair associated with the fundamental set (ky_1, ky_2), where $k \neq 0$. (This follows from $W(ky_1, ky_2) = k^2 W(y_1, y_2)$.) Two pairs (u, g) and $(\bar{u}, \bar{g})$ will be called *distinct* if $\bar{u} \not\equiv ku$ for some constant k. To study distinct pairs it will suffice to consider only those associated with fundamental sets (y_1, y_2) such that $W(y_1, y_2) = 1$. Such pairs will be called *normal*, and from now on, a polar pair will mean a *normal polar pair*, unless stated otherwise. In view of Theorem (2.1), a polar pair is normal if and only if $u^2 g' \equiv 1$.

Polar pairs form a two parameter family. To show this we take a fixed fundamental set (z_1, z_2) (with $W(z_1, z_2) = 1$). Then any other fundamental set (y_1, y_2) is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

where a, b, c, and d are real, and $ad - bc = 1$. The amplitude u associated with (y_1, y_2) is then given by the formula

$$u^2(t) = y_1^2(t) + y_2^2(t) = (a^2 + c^2)\, z_1^2(t) + 2(ab + cd)\, z_1(t)\, z_2(t$$
$$+ (b^2 + d^2)\, z_2^2(t).$$

The identity

$$(ab + cd)^2 + (ad - cb)^2 = (a^2 + c^2)(b^2 + d^2)$$

implies now that u depends on values of $a^2 + c^2$ and $b^2 + d^2$ restricted by the condition $ad - bc = 1$, and not on a, b, c and d individually. This result is also included in the following theorem establishing explicit relations between polar pairs.

THEOREM 2.3. *If (u, g) and $(\bar{u}, \bar{g})$ are two polar pairs then there exist numbers $\alpha \geq 1$ and σ, $-\pi/2 < \sigma \leq \pi/2$, such that*

$$\bar{u}^2(t) = [\alpha - \sqrt{\alpha^2 - 1} \cos 2(g(t) + \sigma)]\, u^2(t),$$
$$\sin \bar{g}(t) = \frac{\sin g(t)}{\sqrt{\alpha - \sqrt{\alpha^2 - 1} \cos 2\sigma}\sqrt{\alpha - \sqrt{\alpha^2 - 1} \cos 2(g(t) + \sigma)}}. \tag{2.3}$$

Conversely, if (u, g) is a polar pair and α, σ are two numbers such that $\alpha \geq 1$ and $-\pi/2 < \sigma \leq \pi/2$, then the pair $(\bar{u}, \bar{g})$ defined by the formulas (2.3) with the additional requirements $\bar{u} > 0$, $\bar{g}(0) = 0$, $\bar{g}' > 0$, is a polar pair. The two pairs are distinct if $\alpha > 1$.

PROOF. We begin with the identities

$$\bar{u}(t) \sin \bar{g}(t) = \frac{u(0)\, u(t)}{\bar{u}(0)} \sin g(t), \tag{1}$$

$$\bar{u}(t) \cos \bar{g}(t) = \frac{\bar{u}(0)\, u(t)}{u(0)} \cos g(t) + W(u, \bar{u})(0)\, u(t) \sin g(t), \tag{2}$$

which follow from that both sides in each represent the same solution of the DE $Lx = 0$. Squaring (1) and (2) and adding one gets the first of formulas (2.3) on setting

$$\alpha = \tfrac{1}{2}\left[\left(\frac{u(0)}{\bar{u}(0)}\right)^2 + \left(\frac{\bar{u}(0)}{u(0)}\right)^2 + [W(u, \bar{u})(0)]^2\right],$$

$$\cos 2\sigma = \frac{\alpha - [\bar{u}(0)/u(0)]^2}{\sqrt{\alpha^2 - 1}}, \qquad \sin 2\sigma = \frac{[\bar{u}(0)/u(0)]\, W(u, \bar{u})(0)}{\sqrt{\alpha^2 - 1}}.$$

If either $u(0) \neq \bar{u}(0)$ or $u'(0) \neq \bar{u}'(0)$, $\alpha > 1$, and σ is defined. If $u(0) = \bar{u}(0)$ and $u'(0) = \bar{u}'(0)$, $\alpha = 1$ and the choice of σ is immaterial since formulas (2.3) then reduce to the statement $(u, g) = (\bar{u}, \bar{g})$. From (1) one next gets $\sin \bar{g}(t) = [u(0)\, u(t)/\bar{u}(0)\, \bar{u}(t)] \sin g(t)$, and the second formula (2.3) follows from the first.

To prove the converse, one verifies that the equations (2.3) together with the conditions $\bar{u} > 0, \bar{g}(0) = 0, \bar{g}' > 0$ determine a pair $(\bar{u}, \bar{g})$, and imply that

$$\bar{u}(t) \sin \bar{g}(t) = \frac{u(t) \sin g(t)}{\sqrt{\alpha - \sqrt{\alpha^2 - 1}\cos 2\sigma}}, \tag{1}$$

$$\bar{u}(t) \cos \bar{g}(t) = \frac{(\sqrt{\alpha^2 - 1}\sin 2\sigma)\, u(t) \sin g(t)}{\sqrt{\alpha - \sqrt{\alpha^2 - 1}\cos 2\sigma}} + \sqrt{\alpha - \sqrt{\alpha^2 - 1}\cos 2\sigma}\; u(t) \cos g(t). \tag{2}$$

The conclusion follows by observing that

$$L(\bar{u}\, e^{i\bar{g}}) = 0, \; W(\bar{u} \sin \bar{g}, \bar{u} \cos \bar{g}) = W(u \sin g, u \cos g) \equiv 1,$$

and $\bar{u} \neq u$ unless $\alpha = 1$.

The first of the formulas (2.3) yields an important characterization of polar pairs stated in the following theorem.

THEOREM 2.4. *Assume that* $I \supset [0, \infty)$ *and that* $\lim_{t\to\infty} u(t)$ *exists. Let* $\lim_{t\to\infty} u(t) = k$. *Then:*

(a) *if* $k = \infty$, $\lim_{t\to\infty} \bar{u}(t) = \infty$ *for all amplitudes* $\bar{u}$;
(b) *if* $k = 0$, $\lim_{t\to\infty} \bar{u}(t) = 0$ *for all amplitudes* $\bar{u}$;
(c) *if* $0 < k < \infty$, *then* $\lim_{t\to\infty} \bar{u}(t)$ *does not exist unless* $\bar{u} = u$.

PROOF. Parts (a) and (b) follow from

$$u(t)\sqrt{\alpha - \sqrt{\alpha^2 - 1}} \leq \bar{u}(t) \leq u(t)\sqrt{\alpha + \sqrt{\alpha^2 - 1}}.$$

To prove part (c) one observes that since u is bounded, $g \to \infty$ as $t \to \infty$. Thus $\cos 2(g + \sigma)$ has no limit as $t \to \infty$, and the same is true of $\alpha - \sqrt{\alpha^2 - 1}$ $\cos 2(g + \sigma)$ unless $\alpha = 1$. Thus $\lim_{t\to\infty} \bar{u}(t)$ exists only if $\alpha = 1$, i.e., when $\bar{u} = u$.

Theorem (2.4) shows then that if there exists an amplitude possessing a bounded nonzero limit as $t \to \infty$, this amplitude is unique. We shall refer to it as the *preferred* amplitude.

So far polar pairs were characterized in terms of solution of $Lx = 0$ via Theorems (2.1) and (2.2). In applications one should prefer to discuss solutions of $Lx = 0$ by using their polar representation. Thus one needs an intrinsic characterization of polar pairs which follows from the next theorem.

THEOREM 2.5. *If (u, g) is a polar pair, then $Lu = u^{-3}$. Conversely, for any $t_0 \in I$, $a > 0$ and any b, the differential equation*

$$Lx = x^{-3} \tag{2.4}$$

has exactly one solution $u \in C^1(I) \cap C_p^2(I)$ such that $u(t_0) = a$, $u'(t_0) = b$ and (u, g), where $g(t) = \int_0^t d\tau/u^2(\tau)$ is a polar pair.

PROOF. If u is an amplitude, there exists a fundamental set (y_1, y_2) of solutions of $Lx = 0$ such that $W(y_1, y_2) = 1$ and $u = \sqrt{y_1^2 + y_2^2}$. By direct substitution one verifies then that $Lu = u^{-3}$.

To prove the converse, we first observe that the DE (2.4) satisfies standard conditions for existence and uniqueness of solutions on $I \times (0, \infty) \times (-\infty, \infty)$, see e.g., [2]. There exists then a unique solution u defined on a maximal interval $J \subset I$ satisfying the required initial and regularity conditions. Now let z_1 and z_2 be solutions of $Lx = 0$ with initial data $z_1(t_0) = a$, $z_1'(t_0) = b$, $z_2(t_0) = 0$, $z_2'(t_0) = 1/a$. Consider $v \in C^1(I) \cap C_p^2(I)$ defined by

$$v(t) = \sqrt{z_1^2(t) + z_2^2(t)}.$$

Since $W(z_1, z_2) = 1$, v is an amplitude and $Lv = v^{-3}$. Since $v(t_0) = a$, $v'(t_0) = b$, it follows by uniqueness that $u = v$ on J. Since v is defined on I, it is a continuation of u to I; this shows however that $J = I$, since J is maximal interval for u. Thus $u = v$ on I, u is an amplitude, and by Theorem (2.1), $g(t) = \int_0^t d\tau/u^2(\tau)$ is the associated phase.

Remarks.

1. The equation (2.4) for the amplitudes is easily derived as follows. Let $\tau = g(t)$, $z(\tau) = a \cos(\tau - \theta)$ in the polar representatjon of y, Eq. (2.1). Then

$$\ddot{z} + z = 0, \qquad \left(\dot{z} = \frac{d}{d\tau}\right). \tag{2.5}$$

Since $Ly = 0$, one obtains the requirement

$$L(uz) = (Lu)\, z + (2u'g' + ug'')\, \dot{z} + u(g')^2\, \ddot{z} = 0. \tag{2.6}$$

For normal polar pairs $u^2 g' \equiv 1$ so that the coefficient of $\dot{z}$ is zero, and that of $\ddot{z}$ is u^{-3}. Equation (2.4) now follows from (2.6) by using (2.5), on the set on which $z \neq 0$.

2. The formulas of Theorem (2.3) can be obtained from the preceding remark. Since z is a solution of a second order linear differential equation, it has a polar representation

$$z(\tau) = \bar{a}\, w(\tau) \cos [h(\tau) - \bar{\theta}].$$

Here w and h have the properties

$$\dot{h}(\tau) = w^{-2}(\tau), \qquad \ddot{w} + w = w^{-3}, \tag{2.7}$$

the second of these being the amplitude equation associated with (2.5). Thus,

$$y(t) = a\, u(t) \cos [g(t) - \theta] = \bar{a}\, u(t)\, w(g(t)) \cos [h(g(t)) - \bar{\theta}]$$

Now, since

$$\frac{d}{dt}[h(g(t))] = \dot{h}(g(t))\, g'(t) = [u(t)\, w(g(t))]^{-2},$$

one concludes that $(u \cdot w(g), h(g))$ is a polar pair for solutions of $Lx = 0$. Thus if u is an amplitude of solutions of $Lx = 0$, and w is an amplitude of solutions of (2.5), then $\bar{u} = u \cdot w(g)$ is also an amplitude of solutions of $Lx = 0$. (This also follows by direct substitution:

$$\begin{aligned} L\bar{u} &= L(u \cdot w(g)) = (Lu)\, w(g) + (2u'g' + ug'')\, \dot{w}(g) + u(g')^2\, \ddot{w}(g) \\ &= (Lu)\, w(g) + u^{-3}\, \ddot{w}(g); \end{aligned}$$

since $Lu = u^{-3}$, and $w + \ddot{w} = w^{-3}$, one gets $L\bar{u} = \bar{u}^{-3}$.) By considering the initial conditions on w one can show that every amplitude $\bar{u}$ of solutions of $Lx = 0$ can be represented this way with fixed u. The first of formulas (2.3) now follows by using the general solution of (2.7) which can be obtained in closed form, namely,

$$w(\tau) = \sqrt{\alpha - \sqrt{\alpha^2 - 1} \cos 2(\tau + \sigma)}, \qquad \alpha \geq 1.$$

The formula for the corresponding phase $\bar{g}$ is obtained by integrating $(h(g))'$. Thus

$$\bar{g}(t) = \int_0^t \dot{h}(g(\sigma))\, g'(\sigma)\, d\sigma = \int_0^{g(t)} \frac{d\tau}{\alpha - \sqrt{\alpha^2 - 1}\cos 2(\tau + \sigma)}$$

$$= n\pi + \operatorname{arc\,cot}\,[\sqrt{\alpha^2 - 1}\sin 2\sigma + (\alpha - \sqrt{\alpha^2 - 1}\cos 2\sigma)\cot g(t)] \quad (2.8)$$

for t such that $n\pi \leq g(t) \leq (n+1)\pi$. This explicit formula for $\bar{g}$ is equivalent to the implicit formula (2.3).

3. From formula (2.8) one easily concludes that $\bar{g} - g$ is a periodic function of g with period π, which vanishes at $k\pi$ and whose range is included in an interval of length π. (It is, of course, defined only on the range of g.) Furthermore if θ and $\bar{\theta}$ are such that

$$\cot \bar{\theta} = (\alpha - \sqrt{\alpha^2 - 1}\cos 2\sigma)\cot\theta + \sqrt{\alpha^2 - 1}\sin 2\sigma \qquad (2.9)$$

then

$$\cot(\bar{g} - \bar{\theta}) = [\alpha - \sqrt{\alpha^2 - 1}\cos 2(g + \sigma)]\cot(g - \theta) - \sqrt{\alpha^2 - 1}\sin 2(g + \sigma),$$

showing that $\bar{g} - \bar{\theta} = \bar{k}\pi$ if and only if $g - \theta = k\pi$. This is in agreement with the fact that if a solution of $Lx = 0$ has p zeros in some interval, then any other solution of $Lx = 0$ must have at least $p - 1$ and at most $p + 1$ zeros; and that if $au(t)\sin[g(t) - \theta]$ and $\bar{a}\bar{u}(t)\sin[\bar{g}(t) - \bar{\theta}]$ are two polar representations of the same solution then $\sin(\bar{g}(t) - \bar{\theta})$ must vanish whenever $\sin(g(t) - \theta)$ does. $\bar{\theta}$ is determined by formula (2.9) up to an integral multiple of π.

III. The Phase Shift Problem

We now consider again the DE (1.1) and assume that $f(t) = 1 + q(t)$, $\int^\infty |q(\tau)|\, d\tau < \infty$. We first show that there exists a preferred amplitude u, that $\lim_{t\to\infty} u(t) = 1$ and that $\lim_{t\to\infty}(g(t) - t)$ exists.

Referring back to solutions y_1 and y_2 of (1.1) having the property (1.2), we observe that $z = y_1 + iy_2$ is the unique solution of the integral equation

$$x(t) = e^{it} + \int_t^\infty q(\tau)\sin(t - \tau)\, x(\tau)\, d\tau. \qquad (3.1)$$

Let w be the solution of the integral equation

$$x(t) = 1 + \int_t^\infty q(\tau) \sin (t - \tau) e^{-i(t-\tau)} x(\tau) d\tau \tag{3.2}$$

obtained from (3.1) through the substitution $x(t) \to x(t) e^{it}$.[1] Since, as is easily verified, $w(t) = 1 + o(1)$, $w'(t) = o(1)$, and $w \neq 0$ for t sufficiently large, a continuous argument arg w can be defined by

$$\arg w(t) = \operatorname{Im} \log w(t), \qquad \operatorname{Im} \log 1 = 0, \tag{3.3}$$

for t sufficiently large. Since $z(t) = w(t)e^{it}$ one obtains for y_1 and y_2 the representation

$$\begin{aligned} y_1(t) &= | w(t) | \cos [t + \arg w(t)] \\ y_2(t) &= | w(t) | \sin [t + \arg w(t)] \end{aligned} \tag{3.4}$$

valid at least for t sufficiently large, and $\lim_{t\to\infty} \arg w(t) = 0$.

From (3.4) we conclude that the pair $(| w |, g)$ where

$$g(t) = t + \arg w(t) + \sigma,$$

for some σ, is polar for t sufficiently large, but it is not necessarily a normal polar pair. Since an amplitude cannot vanish on $[0, \infty)$, it follows that $w \neq 0$ on $[0, \infty)$ so that (3.3) defines arg w on $[0, \infty)$, and the representation (3.4) is valid on $[0, \infty)$. Thus the polar pair $(| w |, g)$ is also defined on $[0, \infty)$. Since $\lim_{t\to\infty} | w(t) | = 1$, it follows from Theorem (2.4) that a preferred amplitude u exists. Also,

$$\lim_{t\to\infty} (g(t) - t) = \lim_{t\to\infty} (\arg w(t) + \sigma) = \sigma.$$

We still have to show that $\lim_{t\to\infty} u(t) = 1$. Now,

$$W(y_1, y_2)(t) = | w(t) |^2 g'(t) = | w(t) |^2 [1 + (\arg w(t))'] = \text{constant}.$$

This shows that $\lim_{t\to\infty} (\arg w(t))'$ exists. It must be zero for otherwise $\lim_{t\to\infty} \arg w(t)$ would not exist. One now gets:

$$W(y_1, y_2)(t) = \text{constant} = \lim_{t\to\infty} W(y_1, y_2)(t) = 1.$$

[1] The existence of a unique solution of (3.2) is easily established. One considers the Banach Space $\{C[t_0, \infty), \| u \| = \sup_{t_0 \leq t < \infty} | u(t) |\}$ where t_0 is such that $\int_{t_0}^\infty | q(\tau) | d\tau = k < 1$, and one shows that the transformation defined by the right side of (3.2) maps the sphere $\{u \mid \| u \| \leq 1/(1 - k)\}$ into itself and is contracting. This establishes the solution on $[t_0, \infty)$. A continuation to $[0, \infty)$ is next achieved by showing that the interval on which (3.2) defines a solution is nonempty and both closed and open in $[0, \infty)$.

Thus $(|w|, g)$ is, after all, a normal polar pair. Since it is a preferred pair, it follows, by uniqueness of preferred pairs, that $u = |w|$. Hence,

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} |w(t)| = 1.$$

From the above and Theorems (2.4) and (2.5) we see that the preferred amplitude is fully determined by the requirements

$$u'' + (1 + q)\, u = u^{-3}, \qquad u(\infty) = 1. \tag{3.5}$$

In Section V we shall consider some questions concerning the numerical determination of u. Here, we assume that u is known on $[0, \infty)$, and, in particular, that the numbers $a = u(0)$, $b = u'(0)$ are known and that $g(t) = \int_0^t d\tau/u^2(\tau)$ is the associated phase. The phase shift problem will be now solved in terms of these.

Consider the solution y of (1.1) with initial data $(\cos\theta, \sin\theta)$, $-\pi/2 < \theta \leq \pi/2$. By Theorems (2.1) and (2.2), there exist constants $k > 0$ and $-\pi/2 < X \leq \pi/2$ such that

$$y(t) = k\, u(t) \cos(g(t) - X), \qquad t \in [0, \infty). \tag{3.6}$$

From the identification at $t = 0$ one gets

$$X = \arctan(a^2 \tan\theta - ab). \tag{3.7}$$

Comparing (3.6) with (1.5) we see that

$$k\, u(t) \cos[g(t) - X] = v(t) \cos[\psi(t) - t],$$

$$k\, u'(t) \cos[g(t) - X] - \frac{k}{u(t)} \sin[g(t) - X] = v(t) \sin[\psi(t) - t]. \tag{3.8}$$

We will show now that as $t \to \infty$

$$\psi(t) - t = X - g(t) + o(1). \tag{3.9}$$

Consider the set $S = \{t \mid t - \psi(t) = n\pi/2\}$ of zeros of y and y'. Let t_1 and t_2 be two consecutive zeros of y, n_1, and n_2—the corresponding values of n; they are both odd. Since between two consecutive zeros of y there is exactly one zero t_1' of y', and $t - \psi(t)$ is continuous, we conclude that $n_2 = n_1 + 2\epsilon$, $n_1' = n_1 + \epsilon$, where $n_1' = (2/\pi)(t_1' - \psi(t_1'))$ and $\epsilon = 1$ or -1. Using induction one now concludes that on S, $(t - \psi(t))$ is a monotone function into a set of consecutive multiples of $\pi/2$. Since $-\psi(0) = -\theta < \pi/2$ and $t - \psi(t) \to \infty$ as $t \to \infty$, it follows now that $(t - \psi(t))$ is increasing on S onto all positive multiples of $\pi/2$.

Since $g(t)$ is increasing on $[0, \infty)$ and $t - \psi(t)$ is increasing on S, it follows from (3.8) that $g(t) - X = (n + \frac{1}{2})\pi$ whenever $t - \psi(t) = (n + \frac{1}{2})\pi$. For $t_1 < t < t_2$, $n_1\pi/2 < t - \psi(t) < n_2\pi/2$ and $n_1\pi/2 < g(t) - X < n_2\pi/2$. Thus

$$\psi(t) - t = X - g(t) + \delta, \qquad | \delta | < \pi. \tag{*}$$

Assume next that $X - g(t) \neq (n + \frac{1}{2})\pi$. Dividing the second of the identities (3.8) by the first, one gets

$$\tan [\psi(t) - t] = \frac{1}{u^2(t)} \tan [X - g(t)] + \frac{u'(t)}{u(t)}. \tag{**}$$

Now,

$$\frac{u'}{u} = \frac{(u^2)'}{2u^2} = \frac{(w\bar{w})'}{2 \mid w \mid^2} = \operatorname{Re} \frac{w'}{w} = o(1),$$

and $1/u^2 = 1 + o(1)$ as $t — \infty$. Thus (**) implies that

$$\psi(t) - t = X - g(t) + o(1), \qquad (\operatorname{mod} \pi), \tag{***}$$

as $t \to \infty$. The truth of (3.9) now follows from (*) and (***).

From (3.9) we obtain

$$\lim_{t\to\infty} \psi(t) = \psi(\infty) = X + \lim_{t\to\infty} (t - g(t)) = X + \int_0^\infty \left[1 - \frac{1}{u^2(\tau)}\right] d\tau.$$

This now yields the final formula for the phase shift (1.6).

$$\varphi = \psi(\infty) - \theta = \int_0^\infty \left[1 - \frac{1}{u^2(\tau)}\right] d\tau - \theta + \operatorname{arc\,tan} (a^2 \tan \theta - ab). \tag{3.10}$$

IV. Determination of Solutions Behaving Like e^{-t} as $t \to \infty$

We now consider the second problem, Section I, p. 2 which makes sense when $f(t) = -1 + q(t)$, and $\int^\infty | q(\tau) | \, d\tau < \infty$. Let $\bar{u}$ be the amplitude with initial data $(1, 0)$; it is defined in view of Theorem (2.5). Let $\bar{g}$ be the corresponding phase. From (1.7) and Theorem (2.1), $\bar{u}(t) \sim ke^t$ for some $k > 0$ as $t \to \infty$. Consequently,

$$\lim_{t\to\infty} \bar{g}(t) = \int_0^\infty \frac{d\tau}{\bar{u}^2(\tau)} = \bar{g}(\infty)$$

exists.

The solution

$$y(t) = a\, \bar{u}(t) \sin [\bar{g}(t) - \bar{g}(\infty) + \theta], \qquad a \neq 0, \tag{4.1}$$

of (1.1) is unbounded unless $\theta = n\pi$. Consequently, if $z_1(t) = e^{-t}[1 + o(1)]$ as $t \to \infty$, then

$$z_1(t) = a\,\bar{u}(t) \sin [\bar{g}(t) - \bar{g}(\infty)] \tag{4.2}$$

for some $a \neq 0$. The desired ratio $z_1(0)/z_1'(0)$ characterizing z_1 through its initial data is now easily computed in terms of initial data of $\bar{u}$. One gets

$$\frac{z_1'(0)}{z_1(0)} = -\cot [\bar{g}(\infty)], \qquad \text{if} \qquad \bar{g}(\infty) \neq n\pi. \tag{4.3}$$

If $\bar{g}(\infty) = n\pi$, (4.2) implies that $z_1(0) = 0$.

If an arbitrary amplitude u is used in place of $\bar{u}$, one obtains in place of (4.3) a slightly more complicated formula,

$$\frac{z_1'(0)}{z_1(0)} = \frac{u'(0)}{u(0)} - \frac{1}{u^2(0)} \cot [g(\infty)], \tag{4.4}$$

where $g(\infty) = \int_0^\infty d\tau/u^2(\tau)$. That both (4.4) and (4.3) are the same follows by using the formula (2.8) together with definitions of α and σ given in Section II.

V. Determination of the Preferred Amplitude

In order to compute the phase shift of solutions of the DE (1.1) through formula (3.10) we must know the preferred amplitude u on $[0, \infty)$ and in particular, the values $a = u(0)$ and $b = u'(0)$. The amplitude is determined by the conditions (3.5) and generally must be found by a suitable numerical integration of the nonlinear second order differential equation. Since the domain of integration is infinite, important errors may result.

We shall show here that $v = u^2$ is determined as a solution of a *linear* equation; furthermore, that if q has an asymptotic expansion in powers of $1/t$ as $t \to \infty$—the expansion then necessarily begins with a term in $1/t^2$—then v and v' have also easily computable asymptotic expansions. The asymptotic expansion for v may supply its values with sufficient accuracy on its domain of definition except for a finite interval terminating at zero, thus alleviating the computational problem.

We first derive the equation for v. Multiply the DE (3.5) by u' and integrate from t to ∞, obtaining

$$u'^2 + u^2 + u^{-2} - 2 \int_t^\infty q(\tau)\, u(\tau)\, u'(\tau)\, d\tau = 2. \tag{5.1}$$

Next multiply (5.1) by u^2, set $u^2 = v$, differentiate and divide by v', obtaining the desired equation for v,

$$v'' + 4v = 4 + 2\left[\int_t^\infty q(\tau)\, v'(\tau)\, d\tau - qv\right]. \tag{5.2}$$

The last step is permissible unless $v' = 0$ on some open set. This will occur only if $q = 0$ for $t \geq t_0$ with some t_0; in that case, however, $u = 1$ for $t \geq t_0$ and one can verify directly that (5.2) is a consequence of the DE (3.5). Conversely, one can easily verify that a unique solution of (5.2) with the terminal condition $v(\infty) = 1$ exists and that $v > 0$, $\sqrt{v} = u$.

Assume now that

$$q(t) \sim \sum_{n=2}^\infty q_n t^{-n}, \tag{5.3}$$

and consider again $z = y_1 + iy_2$ where y_1 and y_2 are the solutions of (1.1) having the property (1.2). As is well known, see e.g. [2, Chapter 5], z has an asymptotic expansion,

$$z(t) \sim e^{it} \sum_{n=0}^\infty z_n t^{-n}.^{2} \tag{5.4}$$

Since, as shown in Section III, $u = | w | = | z |$ so that $v = | z |^2$, it follows from (5.4) that v has an asymptotic expansion

$$v(t) \sim \sum_{n=0}^\infty v_n t^{-n}, \qquad v_0 = 1. \tag{5.5}$$

The coefficients v_n may be obtained by using (5.4), but they are much more simply obtained by a formal substitution of (5.3) and (5.5) in (5.2). This yields, on identifying the coefficients of the t^{-n}, the *linear* recurrence relation

$$v_0 = 1, \qquad v_1 = 0$$

$$v_n = -\tfrac{1}{4}\left[(n-1)(n-2)\, v_{n-2} + 2\sum_{\nu=0}^{n-2}\left(1 + \frac{\nu}{n}\right) q_{n-\nu} v_\nu\right], \qquad n \geq 2. \tag{5.6}$$

As an example consider the Bessel Equation

$$t^2 y'' + ty' + (t^2 - m^2)\, y = 0, \qquad m - \text{real}. \tag{5.7}$$

[2] The z_n are determined by $z_0 = 1$, $z_n = -\frac{i}{2}\left[(n-1)\, z_{n-1} + 1/n \sum_{\nu=0}^{n-1} q_{n+1-\nu} z_\nu\right]$.

This equation is not of form (1.1), but on setting $x = y\sqrt{t}$ one finds that

$$x'' + \left(1 - \frac{m^2 - \frac{1}{4}}{t^2}\right) x = 0. \tag{5.7}$$

(Here, of course, the previous theory is applicable only on $I = [a, \infty)$ with $a > 0$.) Here $q = -\alpha/t^2$, $\alpha = m^2 - \frac{1}{4}$, $q_2 = -\alpha$, $q_n = 0$ for $n \geq 2$. The recurrence relation (5.6) now yields

$$v_{2n} = \frac{(2n)!}{16^n (n!)^2} \prod_{\sigma=0}^{n-1} (4m^2 - (2\sigma + 1)^2), \qquad n \geq 1,$$

$$v_{2n+1} = 0. \tag{5.8}$$

Under certain restrictions one can determine the phase shift of a solution even when the differential equation has a regular singularity at $t = 0$. Specifically assume that f in (1.1) is analytic on $(0, \infty)$ but has a pole of order at most two at $t = 0$, and that $\lim_{t\to 0} t^2 f(t) = k \leq \frac{1}{4}$. Then the eigen-exponents of the solutions of $Lx = 0$ are $\lambda_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - k}$ (see, e.g. [2, Chapter 4]). The differential equation has now a fundamental set of solutions y_1 and y_2 such that as $t \to 0$,

$$y_1(t) = O(\sqrt{t}), \qquad y_2 = O(\sqrt{t}\,\log t), \qquad \text{if} \quad k = \tfrac{1}{4},$$

$$y_1(t) = O(t^{\lambda_-}), \qquad y_2 = O(t^{\lambda_+}), \qquad \text{if} \quad k < \tfrac{1}{4}.$$

It follows that any amplitude $\bar{u}$ behaves near $t = 0$ as $\sqrt{t}\,\log t$ if $k = \frac{1}{4}$ and as t^{λ_-} if $k < \frac{1}{4}$. Since $2\lambda_- = 1 - \sqrt{1 - 4k} < 1$, we conclude that

$$\lim_{\epsilon \to 0_+} \int_\epsilon^t \frac{d\tau}{\bar{u}^2(\tau)}$$

exists, so that the phase g can be defined at 0, and the solutions of $Lx = 0$ still have a polar representation at 0 with $\lim_{t\to 0} \bar{u}(t) = 0$ or $+\infty$.

The initial phase of a solution y cannot be determined anymore by the formula (1.4). If however we agree to call the number θ appearing in the polar representation of y the initial phase, then the phase shift φ will be given by the formula

$$\varphi = \lim_{t\to\infty} [t - g(t)] = \int_0^\infty \left[1 - \frac{1}{u^2(\tau)}\right] d\tau, \tag{5.9}$$

where again u is the preferred amplitude. Observe that in the singular case φ is the same for all solutions of $Lx = 0$.

If $k > \frac{1}{4}$ the eigenexponents $\lambda_{\mp}$ are complex and $\bar{u}(t) = 0(\sqrt{t})$ as $t \to 0$. But then $\lim_{\epsilon \to 0_+} \int_\epsilon^t d\tau/\bar{u}^2(\tau)$ does not exist and one cannot speak of the phase of a solution at $t = 0$.

For the Bessel Equation $k = \frac{1}{4} - m^2 \leq \frac{1}{4}$ and the above considerations apply. If $m = p + \frac{1}{2}$, where p is an integer, the asymptotic series for u^2 terminate. Thus they yield exact solutions on $(0, \infty)$ which can be used for the computation of φ. We get in particular,

$$u(t) = 1, \qquad \varphi = \int_0^\infty \left[1 - \frac{1}{u^2(\tau)}\right] d\tau = 0, \qquad \text{if} \qquad p = 0,$$

$$u(t) = 1 + \frac{1}{t^2}, \qquad \varphi = \int_0^\infty \left[1 - \frac{1}{u^2(\tau)}\right] d\tau = \frac{\pi}{2}, \qquad \text{if} \qquad p = 1,$$

$$u(t) = 1 + \frac{3}{t^2} + \frac{9}{t^4}, \qquad \varphi = \int_0^\infty \left[1 - \frac{1}{u^2(\tau)}\right] d\tau = \pi, \qquad \text{if} \qquad p = 2.$$

This is in agreement with the formula $\varphi = (-\frac{1}{4} + m/2)\,\pi$ for the phase shift of Bessel functions obtained by using their conventional asymptotic expansions. See e.g. [7, vol. II, p. 85].

VI. Determination of the Amplitude for the Second Problem

In this problem the amplitude $\bar{u}$ is determined by the conditions

$$\begin{gathered} \bar{u}'' + [-1 + q(t)]\,\bar{u} = \bar{u}^{-3}, \\ \bar{u}(0) = 1, \qquad \bar{u}'(0) = 0. \end{gathered} \tag{6.1}$$

See Section IV. Here again recourse must be made to a suitable numerical procedure. If however q is analytic at zero, and q has an asymptotic expansion in powers of t^{-1} as $t \to \infty$, one can use a power series for $\bar{u}$ near $t = 0$ and an asymptotic expansion for $\bar{u}$ as $t \to \infty$.

We shall show here that a power series for $v = \bar{u}^2$ can be determined by means of a *linear* recurrence relation. One proceeds as in Section V with minor modifications. Since $\bar{u} \not\equiv$ constant, $\bar{u}' \not\equiv 0$. Multiplying (6.1) by $\bar{u}'$ and integrating from 0 to t one gets

$$\bar{u}'^2 - \bar{u}^2 + \bar{u}^{-2} + 2\int_0^t q(\tau)\,\bar{u}(\tau)\,\bar{u}'(\tau)\,d\tau = 0. \tag{6.2}$$

On multiplying (6.2) by $v = \bar{u}^2$, differentiating through and dividing by v' one gets

$$v'' - 4v + 2\left[\int_0^t q(\tau)\,v'(\tau)\,d\tau + q(t)\,v\right] = 0. \tag{6.3}$$

Thus v is a solution of a linear equation with initial data $v(0) = 1$, $v'(0) = 0$.

Assume that q is analytic for $| t | < \rho$, i.e.,

$$q(t) = \sum_{n=0}^{\infty} q_n t^n. \tag{6.4}$$

Then every solution of (6.3) is analytic on $| t | < \rho$, and one gets for v

$$v(t) = \sum_{n=0}^{\infty} v_n t^n, \tag{6.5}$$

where $v_0 = 1$, $v_1 = 0$ and

$$v_{n+2} = \frac{1}{(n+1)(n+2)} \left[4v_n - 2 \sum_{\nu=0}^{n} \left(1 + \frac{\nu}{n}\right) q_{n-\nu} v_\nu \right].^3 \tag{6.6}$$

If as $t \to \infty$, q has an asymptotic expansion

$$q(t) \sim \sum_{\nu=2}^{\infty} q_\nu t^{-\nu}, \tag{6.7}$$

then v has an asymptotic expansion of the form

$$v(t) \sim e^{2t} \sum_{n=0}^{\infty} v_n t^{-n}. \tag{6.8}$$

Indeed, let z_1 and z_2 be the solutions of $Lx = 0$ mentioned in (1.7), i.e.,

$$\begin{aligned} z_1(t) &= O(e^{-t}), \\ z_2(t) &= e^t[1 + o(1)]. \end{aligned} \tag{6.9}$$

It now follows from the formula preceding Theorem (2.3) that

$$v \sim k z_2^2, \tag{6.10}$$

where k is a positive constant. The assertion (6.8) is now a consequence of that z_2/e^t has an asymptotic power series.

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[3] The series (6.5) has a remarkable feature: it converges to a positive sum (on $[0, \rho)$) no matter what are the coefficients q_n.

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