Simple Finite-Dimensional Jordan Superalgebras of Prime Characteristic

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INTRODUCTION

V. G. Kac (see [K1]) classified finite-dimensional simple Jordan superalgebras over algebraically closed fields of zero characteristic.

Study of Jordan superalgebras of positive characteristics was initiated by I. Kaplansky [Kap1], [Kap2]. M. Racine and E. Zelmanov [RZ] classified finite-dimensional simple Jordan superalgebras of characteristics \( \neq 2 \) with semisimple even part.

In this paper we address the remaining case when the even part is not semisimple. Structure of these superalgebras is similar to structure of infinite-dimensional Jordan superalgebras of zero characteristic that correspond to the so-called superconformal algebras of contact brackets (see [KL], [KMZ], [NS], [R]).
DEFINITIONS AND EXAMPLES. All algebras are considered over a field $F$ of characteristic $\not= 2$.

A (linear) Jordan algebra is a vector space $J$ with a binary operation $(x, y) \to xy$ satisfying the following identities:

1. $(xy)z = x(yz)$
2. $(x^2)y = x^2(yx)$

For an element $x \in J$, let $R(x)$ denote the right multiplication $R(x) : a \to ax$ in $J$. Then linearization of (2) yields the following identity in the operator form:

\[
R((xy)z) + R(x)R(z)R(y) + R(y)R(z)R(x) = R(xy)R(z) + R(xz)R(y) + R(yz)R(x)
\]

We will refer to it as “the Jordan identity.”

For elements $x, y, z$ of a Jordan algebra $J$, by $(x, y, z)$ we denote their Jordan triple product, $(x, y, z) = xy + yz + x^2(yx).

By $U(x, y)$ we denote the operator $U(x, y) : J \to J, U(x, y) : z \to (x, z, y)$, $U(x) = U(x, x)$.

For arbitrary elements $x, y \in J$, the operator $D(x, y) = R(x)R(y) - R(y)R(x)$ is known to be a derivation of $J$ (see [J]).

We will need also the following identity:

\[
R(x)R(y)R(z) = \frac{1}{4}(-R((xz)y) + R(xy)R(z) + R(xz)R(y) + R(yz)R(x) + R(y)D(z, x) + R(x)D(z, y) - R(yD(z, x) + R(x)D(z, x)).
\]

By a superalgebra $A = A_\bar{0} + A_\bar{1}$, we mean a $\mathbb{Z}/2\mathbb{Z}$-graded algebra. Thus $A_{\bar{0}}$ is a subalgebra of $A$ and $A_{\bar{1}}$ is an $A_{\bar{0}}$-bimodule.

EXAMPLE. Let $V$ be a vector space. The Grassmann (or exterior) algebra $G(V)$ is the quotient of the tensor algebra $T(V)$ modulo the ideal generated by symmetric tensors $v \otimes w + w \otimes v; v, w \in V$. Clearly, $G(V) = G_\bar{0} + G_\bar{1}$, where $G_\bar{0}$ (resp. $G_\bar{1}$) is spanned by products of elements of $V$ of even (resp. odd) length.

DEFINITION. Let $V$ be a vector space of countable dimension. By the Grassmann envelope of a superalgebra $A = A_\bar{0} + A_\bar{1}$, we mean the subalgebra $G(A) = A_\bar{0} \otimes G_\bar{0} + A_\bar{1} \otimes G_\bar{1}$ of the tensor product $A \otimes G(V)$.

Let $\mathcal{V}$ be a homogeneous variety of algebras (see [ZSSS]).
DEFINITION. A superalgebra \( A = A_o + A_T \) is called a \( \mathcal{V} \)-superalgebra if the Grassmann envelope \( G(A) \) lies in \( \mathcal{V} \).

In particular, if \( A = A_o + A_T \) is a \( \mathcal{V} \)-superalgebra, then \( A_o \in \mathcal{V} \) and \( A_T \) is a \( \mathcal{V} \)-bimodule over \( A_o \) (see [J]).

In this way one can define superalgebras, Jordan superalgebras, etc. Associative superalgebras are just \( \mathbb{Z}/2\mathbb{Z} \)-graded associative algebras.

Let \( A = A_o + A_T \) be an associative commutative superalgebra. If \( a \in A_i \), then we denote \(|a| = i\). By a bracket on \( A \) we mean a bilinear mapping \( \{ \, , \} : A \times A \rightarrow A \).

Starting with a bracket \( \{ \, , \} \) on \( A \), consider a direct sum of vector spaces \( J = J(A, \{ \, \, \} ) = A + Ax \). We shall define a multiplication on \( J \).

For arbitrary elements \( a, b \in A \), their product in \( J \) is the product \( ab \) and \( a(ax) = (ab)x, (bx)a = (-1)^{|b|} (ba)x, (ax)(bx) = (-1)^{|a||b|} (ab) \).

The \( \mathbb{Z}/2\mathbb{Z} \)-gradation on \( A \) can be extended to a \( \mathbb{Z}/2\mathbb{Z} \)-gradation on \( J \) via \( J_o = A_o + A_T x, J_T = A_T + A_o x \). We call \( J \) the Kantor Double of \( (A, \{ \, \, \} ) \).

DEFINITION. A bracket \( \{ \, , \} \) on \( A \) is called a Jordan bracket if the Kantor Double \( J(A, \{ \, \} ) \) is a Jordan superalgebra.

EXAMPLE. A bracket \( \{ \, , \} : A \times A \rightarrow A \) is called a Poisson bracket if \( (A, \{ \, \} ) \) is a Lie superalgebra and \( \{ ab, c \} = a \{ b, c \} + (-1)^{|b||c|} \{ a, c \} b \) for arbitrary elements \( a, b, c \in A \). An arbitrary Poisson bracket is a Jordan bracket (see [K1]).

It immediately follows from the Jordan identity that for an arbitrary Jordan bracket \( \{ \, , \} \)

1. \( D: a \rightarrow \{ a, 1 \} \) is a derivation of \( A \),

2. \( \{ a, bc \} = \{ a, b \} c + (-1)^{|a||b|} b \{ a, c \} - D(a)bc \).

For other properties of Jordan brackets, see [KM]. If the superalgebra \( A \) is generated by elements \( \{ a_i \} \), then a Jordan bracket \( \{ \, , \} \) is determined by the derivation \( D \) and by values \( \{ a_i, a_j \} \).

EXAMPLES (Jordan brackets of Neveu–Schwarz and Ramon, see [NS], [R], [KMZ]). Let \( A = F[t^{-1}, t, \xi_1, \ldots, \xi_n] \), where \( t \) is the even Laurent variable; \( \xi_1, \ldots, \xi_n \) are Grassmann variables.

1. \( D = \frac{\partial}{\partial t}, \{ \xi_i, \xi_j \} = - \delta_{ij}, \{ t, \xi_i \} = 0 \).

2. \( D = \frac{\partial}{\partial t}, \{ \xi_i, \xi_j \} = - \delta_{ij}, \{ t, \xi_i \} = 0 \).

Suppose that \( \text{char } F = p > 0 \). Let \( B(m) = F[a_1, \ldots, a_m | a_i^p = 0] \) denote the algebra of truncated polynomials in \( m \) even variables. Let \( G(n) \) be the Grassmann algebra on an \( n \)-dimensional vector space, \( G(n) = \langle 1, \xi_1, \ldots, \xi_n \rangle \). Then \( B(m,n) = B(m) \otimes G(n) \) is an associative supercommutative superalgebra.

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The main result in this paper is the following.

**Theorem.** Let $J = J_0 + J_1$ be a finite-dimensional simple unital Jordan superalgebra over an algebraically closed field $F$ of characteristic $p > 2$ whose even part $J_0$ is not semisimple. Then either there exist integers $m, n$ and a Jordan bracket on $B(m, n)$ such that $J = B(m, n) + B(m, n)x$ is a Kantor Double, or $J$ is isomorphic to an exceptional Cheng–Kac Jordan superalgebra $CK(B(m, d))$ corresponding to a derivation $d: B(m) \to B(m)$ (see Section 2).

1. **GENERAL RESULTS**

1.1. **Structure of the Even Part**

Let $J = J_0 + J_1$ be a finite-dimensional simple Jordan unital superalgebra over an algebraically closed field $F$ of positive characteristic $p \neq 2$.

In what follows we will denote $A = J_0$ and $M = J_1$.

For a subspace $V \subseteq J$, by $RV$ we denote the subspace $\{R(v), v \in V\}$. Let $R^M(A)$ denote the subalgebra generated by $(R(a): M \to M, a \in A)$ in $\text{End}_F(M)$.

For arbitrary odd elements $x, y \in M$, the operator $D(x, y) = R(x)R(y) + R(y)R(x)$ is a derivation of $J$. Let $\mathcal{D} = \mathcal{D}(M, M)$ denote the linear span of all such operators. Clearly, $\mathcal{D}$ is a Lie algebra.

**Definition 1.1.1.** An $F$-algebra $B$ is called differentially simple if it does not contain proper ideals that are invariant under all derivations.

The structure of such algebras was determined by R. Block.

**Theorem 1.1.1 (see [B]).** Let $A$ be a finite-dimensional differentially simple $F$-algebra. Then either $A$ is simple or $\text{ch} F = p > 0$ and $A \cong S \otimes B(n)$, where $S$ is simple and $B(n)$ is an algebra of truncated polynomials.

Let $N(A)$ be the nilpotent radical of $A$ and let $I$ be the maximal nilpotent ideal of $A$ which is $\mathcal{D}$-invariant. Then $A/I$ is differentially semisimple; that is, it does not contain nontrivial nilpotent $\mathcal{D}$-invariant ideals.

**Lemma 1.1.1.** A finite-dimensional differentially semisimple Jordan algebra is a direct sum of differentially simple algebras.

**Proof.** Let $A$ be a finite-dimensional differentially semisimple Jordan algebra. Let $L$ be a minimal nonzero ideal of $A$ which is invariant with respect to $\text{Der}(A)$. We claim that the algebra $L$ is differentially simple. Indeed, let $K$ be a $\text{Der}(L)$-invariant ideal of $L$. We will show that $\text{id}_A(K^3) \subseteq K$. Since $K\text{Der}(A) \subseteq K$ by the identity (D), it suffices to prove that $K^3A \subseteq K$ and $(K^3A)A \subseteq K$. We have $K^2A \subseteq (KA)K + KD(K, A) \subseteq$
K. Since the restriction of $\text{Der}(A)$ to $L$ lies in $\text{Der}(L)$ and the ideal $K$ is $\text{Der}(L)$-invariant, it follows that $KD(K, A) \subseteq K$. Hence $K^2 A \subseteq A$ and $K^3 A \subseteq (K^2 A)K + K^2 D(K, A) \subseteq K^2$.

Minimality of $L$ implies that $K^3 = (0)$ or $K = L$. If $K^3 = (0)$, then $\text{id}_A(K)$ is a nilpotent (see [J], [ZSSS]) differentially invariant ideal of $A$. By differential semisimplicity of $A$, we have $K = (0)$.

We proved that $L$ is differentially simple. By Block's theorem, $L = S \otimes B(n)$, where $S$ is a simple Jordan algebra. In particular, $L$ contains an identity element $e$ which lies in the center of $L$. Then $A = L \otimes A(1 - e)$ is a direct sum of ideals. Now it remains to apply an induction assumption to $A(1 - e)$. Lemma is proved.

Denote $\overline{A} = A/I$ and let $\overline{A} = \overline{A}^{(1)} \oplus \cdots \oplus \overline{A}^{(r)}$ be a decomposition into a direct sum of differentially simple algebras.

Let $e^{(i)}$ be the identity element of $\overline{A}^{(i)}$. Clearly, $e^{(i)}$ is an idempotent element in $\overline{A}$. Let $\{e^{(i)}\}
mu$ be a maximal system of pairwise orthogonal idempotents of $\overline{A}^{(i)}$. So $\Sigma \mu e^{(i)} = \overline{A}$, and for each $\mu$ the subalgebra $\overline{A}^{(i)}(\overline{e}^{(i)})$ does not contain proper idempotents.

The system $\{e^{(i)}\}
mu$ can be lifted to a system of pairwise orthogonal idempotents $e^{(i)} \in A$, such that $\Sigma i, \mu e^{(i)} = 1$ (see [J]). Let $e^{(i)} = \Sigma \mu e^{(i)}$.

DEFINITION 1.1.2. An element $a \in J$ is said to be Peirce homogeneous if it lies in one of the subspaces $J U(e^{(i)}_\mu, e^{(j)}_\nu)$, $(i, \mu) \neq (j, \nu)$ or in $\Sigma i, \mu J U(e^{(i)}_\mu)$.

DEFINITION 1.1.3. An element $a \in J$ is said to be strongly Peirce homogeneous if it lies in one of the subspaces $J U(e^{(i)}_\mu, e^{(j)}_\nu)$ or in one of the subspaces $J U(e^{(i)}_\mu)$.

Define the trace functional $t: A \rightarrow F$ via $t(a) = \Sigma i, \mu \alpha_{i, \mu}$, where $\alpha_{i, \mu} \in F$, $\overline{a}U(e^{(i)}_\mu) = \alpha_{i, \mu}e^{(i)}_\mu$, for $a \in A$ (see [J]). In particular, $t(I) = 0$.

LEMMA 1.1.2 (see [Kap1]). Let $x$ be a Peirce homogeneous element from $M$ and let $y$ be a strongly Peirce homogeneous element from $M$. Suppose that $t([x, y]) \neq 0$. Then there exists a scalar $0 \neq \psi \in F$ and an element $a \in N(A)$ such that

$$[x, y] \cdot y = \psi y + y \cdot a$$

Proof. From $t([x, y]) \neq 0$, it follows that $[x, y] \in \Sigma i, \mu A U(e^{(i)}_\mu)$, so $[x, y] = \Sigma i, \mu \alpha_{i, \mu} e^{(i)}_\mu + a$, $a \in N(A)$, $\alpha_{i, \mu} \in F$.

If $y \in MU(e^{(i)}_\mu)$, then $[x, y] = \alpha_{i, \mu} e^{(i)}_\mu + a$, $\alpha_{i, \mu} = t([x, y]) \neq 0$. So $[x, y] y = \psi y + y \cdot a$. Take $\psi = \alpha_{i, \mu}$.

If $y \in MU(e^{(i)}_\mu, e^{(j)}_\nu)$, then $[x, y] = \alpha_{i, \mu} e^{(i)}_\mu + \alpha_{j, \nu} e^{(j)}_\nu + a$ and $t([x, y]) = \alpha_{i, \mu} + \alpha_{j, \nu} \neq 0$. Hence $[x, y] y = 1/2(\alpha_{i, \mu} + \alpha_{j, \nu})y + ya$. Take $\psi = 1/2t([x, y])$. Lemma is proved.
LEMMA 1.1.3. \([IM, M] \subseteq I.\)

Proof. It suffices to prove that \(t([IM, M]) = 0.\) Indeed, \([IM, M] + I\) is an ideal of \(A\) that contains \(I\) and is \(D\)-invariant. Since \(\tilde{A} = A/I\) is a direct sum of \(D\)-simple algebras, the ideal \([IM, M] + I/I\) has to be a sum of some of them. But each summand has an element of nonzero trace. Consequently, \([IM, M] + I/I = (0);\) that is, \([IM, M] \subseteq I.\)

Notice that the previous argument proves that \(A\) does not contain \(D\)-invariant ideals of zero trace that contain \(I\) properly.

Let \(R\langle N \rangle\) denote the ideal generated by the subspace \(R(N)\) in the multiplication algebra \(R^M(A).\) Clearly (see [I]), there exists \(r \geq 1\) such that \(MR\langle N \rangle^r = (0).\) If \([IM, M]\) does not belong to \(I,\) then there exist integers \(1 \leq i < r, 0 \leq j < r\) such that \(t([MR(I), MR\langle N \rangle^j]) \neq 0\) (for example, \((i, j) = (1, 0)).\) Let \((i, j)\) be lexicographically maximal among all pairs with this property.

If \(y \in MR\langle N \rangle^j,\) then \(yU(e_{ij}^{(i)}, e_{ij}^{(j)}), yU(e_{ij}^{(i)}, MR\langle N \rangle^j).\) So we can assume that there exists a Peirce homogeneous element \(x \in MR(I)^i\) and a strongly Peirce homogeneous element \(y \in MR\langle N \rangle^j\) such that \(t([x, y]) \neq 0.\)

By Kaplansky’s Lemma, \(y = \lambda[x, y]y + yu,\) where \(a \in N(A), 0 \neq \lambda \in F.\)

So \(t([x, ya]) \neq 0\) or \(t([x, [x, y]y]) \neq 0.\)

But \([x, ya] \subseteq [MR(I)^i, MR\langle N \rangle^{j+i}]\) and \((i, j) < (i, j + 1).\)

Let us prove that \([x, y]y = xR(y)^2 \subseteq MR(I)^i.\) Indeed, if \(x = x'R(b_1) \cdots R(b_i), b_j \in I,\) then \(xR(y)^2 = (x'R(y)^2)R(b_1) \cdots R(b_i) + x'R(b_1R(y)^2) \cdots R(b_i) + \cdots x'R(b_iR(y)^2) \subseteq MR(I)^i.\)

Now it remains to prove that \([MR(I)^i, MR(I)^j] \subseteq I + [MR(I)^{h+1}, M]\) and use that \((2i, 0) > (i, j),\) since \(i \geq 1.\)

But if \(b_1, b_2 \in I, x_1, x_2 \in M,\) then \([x_1b_1, x_2b_2] = [(x_1b_1)b_2, x_2] - b_2D(x_1b_1, x_2)\) and \(b_2D(x_1b_1, x_2) \in I.\) Lemma is proved.

Denote \(I_{ij} = \{e_{ij}^{(i)}I, e_{ij}^{(j)}, A_{ij} = \{e_{ij}^{(i)}A, e_{ij}^{(j)}, M_{ij} = \{e_{ij}^{(i)}M, e_{ij}^{(j)}\},\) where \(e_{ij}^{(i)} = \sum_{\mu}e_{ij}^{(i)}.\)

LEMMA 1.1.4. If \(i \neq j,\) then \(id_j(I_{ij}) \cap A \subseteq I.\)

Proof. Let \(u_1, \ldots, u_l \in A \cup M\) be strongly Peirce homogeneous elements, \(w = R(u_1) \cdots R(u_l).\) Suppose further that \(I_{ij}w \subseteq A,\) but \(I_{ij}w \not\subseteq I\) and \(l\) is minimal with such property. Then

(1) There are no two consecutive elements \(u_k, u_{k+1}\) of the same parity.

Indeed, if two consecutive elements \(u_k, u_{k+1}\) lie in \(A\) (resp. in \(M\)), then, using the Jordan identity, we can move them to the right end of \(w\) modulo operators of length \(< l.\) Now Lemma 1.1.3 can be used.
(2) No element \( u_k \) lies in \( I \).

Indeed, moving \( R(u_k) \) to the right via the Jordan identity, we get
\[
w = (\cdots)R(u_k)+(\cdots)R(v), \quad v \in A \cup M \quad \text{modulo operators on length } < l.
\]
If \( u_k \in I \), then \( Aw \subseteq I \) by Lemma 1.1.3.

(3) For an element \( c \in I_{ij} \), the expression \( cw + I/I \) is skew-symmetric with respect to all elements \( u_k \) lying in \( A \) and symmetric with respect to all elements \( u_k \) lying in \( M \).

Since even (odd) elements do not follow one after another, the assertion follows from the Jordan identity and minimality of \( l \).

Clearly, \( l \geq 3 \). If \( u_1 \in A \), then we can assume that \( u_1 \in AU(e^{(\alpha)}, e^{(\beta)}) \).
If \( \alpha \neq \beta \), then \( u_1 \in I \), and we have shown that this is impossible. So, the only option is \( u_1 \in AU(e^{(*)}) \) or \( u_1 \in AU(e^{(j)}) \). But then \( I_{ij}R(u_1) \subseteq I_{ij} \), contradiction by minimality of \( l \). Hence \( u_1 \in M \), and consequently, \( u_2 \in A \).
Since \( u_2 \notin I \), we can assume that \( u_2 \in AU(e^{(*)}) \).
If \( \alpha \neq i \), then \( D(I_{ij}, u_2) = 0 \) and \( I_{ij}R(u_1)R(u_2) = I_{ij}R(u_1u_2) \).
The minimality of \( l \) finishes the proof in this case.

Let \( \alpha = j \). We have
\[
I_{ij}w = I_{ij}R(e^{(i)})R(u_1)R(u_2) \cdots
\leq I_{ij}(-R(u_2)R(u_1)R(e^{(j)}) + R(e^{(j)})R(u_1u_2))
\quad + R(u_2)R(u_1e^{(j)})) \cdots + I
\]
\[
= I_{ij}R(u_2)R(u_1)R(e^{(j)}) \cdots + I.
\]

Thus, without loss of generality, we can assume that \( u_2 = e^{(j)} \).
Let \( u_1 \in \{e^{(k)}, M, e^{(*)} + e^{(j)}\} \), where \( k \neq i, k \neq j \). Then \( u_1U(I_{ij}, u_2) = (0) \), which implies \( I_{ij}R(u_1)R(u_2) \subseteq I_{ij}R(u_2)R(u_1) + I_{ij}R(u_1u_2) \). This contradicts our earlier results.

If \( u_1 \in MU(e^{(*)}) + MU(e^{(j)}) \), then \( R(u_1) \) commutes with \( R(u_2) = R(e^{(j)}) \).
The only remaining case is \( u_1 \in M_{i,j} \).
Since \( I_{ij}w + I/I \) is symmetric in \( u_1 \) and \( u_3 \), we can assume also that \( u_3 \in M_{i,j} \). But then \( I_{ij}R(u_1)R(u_2)R(u_3) \subseteq \{e^{(*)}, A, e^{(j)}\} = I_{ij} \).
Lemma is proved.

**Corollary 1.1.1.** If \( i \neq j \), then \( I_{ij} = (0) \).

**Proof.** Since the superalgebra \( J \) is simple, \( id_{j}(I_{ij}) = (0) \) or \( J \). But \( id_{j}(I_{ij}) = J \) implies that \( \tilde{A} = J \cap A \subseteq I \), the contradiction.
Lemma 1.1.5. If $I \neq (0)$, then $s = 1$; that is, $\overline{A}$ is differentially simple.

Proof. If $I \neq (0)$, then there is an index $i$ such that $I_i \neq 0$. We can assume $i = 1$.

We will show that $id_f(I_{11}) \cap A \subseteq A_{11} + I$, and consequently, $\overline{A} = \overline{A}^{(1)}$ and $s = 1$.

Indeed, suppose that the assertion is not true. Choose an operator $w = R(u_1) \cdots R(u_l), u_i \in A \cup M$, such that $I_{11}w \subseteq A$, $I_{11}w \not\subseteq A_{11} + I$ and $l$ minimal with this property. Then $l \geq 3$, since $I_{11}R(M)R(M) \subseteq I$ by Lemma 1.1.3.

Arguing as in the proof of Lemma 1.1.4, we can see that no two consecutive elements $u_k, u_{k+1}$ lie in $A$.

Suppose that two consecutive elements $u_k, u_{k+1}$ lie in $M$. If there is another odd element $u_i$ among $u_1, \ldots, u_l$, then again using the Jordan identity, we can represent $w$ as a linear combination of operators $\cdots R(u_k)R(u_{k+1})R(u_i) \cdots \cdot R(u_{k+1})R(u_k) \cdots$ of length $l$ and operators of length $< l$. The identity (D) from the Introduction implies that for odd elements $x, y, z \in M$, the operator $R(x)R(y)R(z)$ is a linear combination of operators of length $\leq 2$ and of operators of the type $R(x)D(y, z)$. We can move $D(y, z)$ to the right end and it remains to notice that $\overline{A}^{(1)}D(M, M) \subseteq \overline{A}^{(1)}$.

Suppose now that $u_k, u_{k+1}$ are the only odd elements among $u_1, \ldots, u_l$. Because of the Jordan identity, we can assume $k = 1$.

Then $I_{11}R(M)R(M) = (0)$ by Lemma 1.1.3.

The element $u_1$ clearly lies in $M$. Hence $u_1 \in A$.

If $u_2 \in A_{jj}$, $j \neq 1$, then $R(I_{11})$ and $R(u_2)$ commute, so $I_{11}R(u_1)R(u_2) = I_{11}R(u_1 \cdot u_2)$, the contradiction.

Hence we can assume that $u_2 \in A_{11}$. If $u_1 \in M_{11}$, $i \neq 1$, then $(0) = u_1U(I_{11}, u_2)$, and therefore, $I_{11}R(u_1)R(u_2) \subseteq I_{11}R(u_1u_2) + I_{11}R(u_2)R(u_1)$, the contradiction.

Hence we can assume that $u_1 \in M_{11}$. Since for an arbitrary element $c \in I_{11}$, the expression $cw + A_{11} + I/A_{11} + I$ is symmetric in odd $u_i's$ and skew-symmetric in even $u_i's$, we can assume that $u_1, \ldots, u_l \in A_{11} \cup M_{11}$, the contradiction. Lemma is proved.

The following lemma was proved in [RZ].

Lemma 1.1.6. Let $I = (0)$. Then $s \leq 2$. If $s = 2$, then $M = M_{12}$.

Proof. Let $A = A^{(1)} \oplus \cdots \oplus A^{(s)}$ with $A^{(i)}$ $\mathcal{D}$-simple. If $i \neq j$, then $[M_i, M_j] = (0)$ and $[M_i, M_{ij}] = 0$, so $[M_i, M_{ij}]M_{ij} = (0)$.

Let $Ann_{ij} = \{ a \in A^{(i)} | aM_{ij} = (0) \}$. Then $Ann_{ij}$ is an ideal of $A^{(i)}$. Since $M_{ij} = (A^{(i)}, M, A^{(j)})$ is $\mathcal{D}$-invariant, $Ann_{ij}$ is also $\mathcal{D}$-invariant. So either $Ann_{ij} = (0)$ or $Ann_{ij} = A^{(i)}$, which is possible only if $M_{ij} = 0$. 


Notice that we have proved that $[M_{hh}, M_{hh}] = (0)$ as soon as there is some other index $l \neq h$ with $M_{hl} \neq (0)$.

Suppose that $s \geq 2$. We will show that $M_{ii} = (0)$ for an arbitrary $i$. If there exists $j \neq i$ such that $M_{ij} \neq (0)$, then from the remark above it follows that $[M_{ij}, M_{ij}] = (0)$. Hence $M_{ii} \leq J$. Hence $M_{ii} = (0)$. If for an arbitrary $j \neq i$, $M_{ij} = (0)$, then $A^{(i)} + M_{ij} \leq J$, which contradicts $s \geq 2$.

Suppose that $s \geq 3$. Then there exists $k \neq i, j$ such that $\{e_i + e_j, M, e_k\} \neq (0)$. Otherwise $A^{(i)} + A^{(j)} + M_{ij} \leq J$, which contradicts simplicity of $J$.

Since $M_{ik} + M_{jk} \neq (0)$, one of them has to be nonzero. Suppose that $M_{ik} \neq (0)$. Then as we have seen above, $\text{Ann}_{ik} = (0)$.

From $\{M_{ij}, M_{ik}, M_{ij}\} = (0)$, it follows that $[M_{ij}, M_{ij}]M_{ik} = (0)$. If $[M_{ij}, M_{ij}] \not\subseteq A^{(j)}$, then $\text{Ann}_{ik} \neq (0)$. Hence $[M_{ij}, M_{ij}] \subseteq A^{(j)}$.

Suppose that there exists $l \neq i, j$ such that $M_{il} \neq (0)$. Then $[M_{ij}, M_{ij}] \subseteq A^{(i)}$ (see the argument above) and therefore, $[M_{ij}, M_{ij}] = (0)$. Hence $M_{ij} \leq J$. Hence $M_{ij} = (0)$, the contradiction.

So, for every $l \neq i, j$ we have $M_{il} = (0)$. Again this implies that $A^{(i)} + M_{ij} \leq J$, a contradiction. So $s = 2$ and $M = M_{12}$. Lemma is proved.

For elements $x, y \in M; u \in I; a, b \in A$, denote

$$t(x, y, u, a, b) = t[(x \cdot u) \cdot a, y \cdot b].$$

**Lemma 1.1.7.** The multilinear function $t: M \times M \times I \times A \times A \rightarrow F$ has the following properties:

1. $t(x, y, u, a, b) = -t(x, y, u, b, a)$.
2. $t(x, y, u, a, b) = t(x, y, u, a, b)$.
3. $t(x, y, u, a^2, b) = 2t(xa, y, u, a, b) = 2t(x, ya, u, a, b) = 2t(x, y, ua, a, b) = 2t(x, y, u, a, ab)$.

**Proof.** See Lemma 1.1.23 in [KMZ].

**Lemma 1.1.8.** For arbitrary elements $x, y \in M, u \in I, a, b, c, e \in A$, we have

1. $t(x, y, u, cD(a, b), e) = \sum_{\alpha}t(x_{\alpha}, y_{\alpha}, u, a, e) + \sum_{\beta}t(x_{\beta}, y_{\beta}, u, b, e)$ for some elements $x_{\alpha}, y_{\alpha}, x_{\beta}, y_{\beta} \in M$.
2. $t(x, y, uD(a, b), a, b) = 0$.

**Proof.** See Lemma 1.1.24 in [KMZ].

Let $FJ$ denote the free Jordan algebra on the countable set of free generators $x_1, x_2, \ldots$ (see [J], [ZSSS]).

As always, by $FJD(x_1, x_2)$ we denote the image of the derivation $D(x_1, x_2): FJ \rightarrow FJ$ and by $\langle FJD(x_1, x_2) \rangle$ we denote the subalgebra of $FJ$ generated by $FJD(x_1, x_2)$.
Let $f(x_1, x_2, x_3) = (x_1 D(x_1, x_2)^2) D(x_1, x_2)$. In [Z1] it was proved that, for any $i \geq 1$, we have $x_i \cdot f^i \in \langle FJD(x_1, x_2) \rangle$.

**Lemma 1.1.9.** Let $f = f(a, b, c); a, b, c \in A$. Then

1. $t(M, M, I, f^4, A) = (0)$.
2. For an arbitrary element $e \in A$, we have $t(M, M, I, f^{12} e, A) = (0)$, $t(M, M, I, f^{24} e, A) = (0)$.

**Proof.** See Lemma 1.1.25 in [KMZ].

Let $T$ be the ideal of the free Jordan algebra $FJ$ generated by the set $f^{24}(FJ)$. The identity (D) (see the Introduction) implies that an arbitrary multiplication operator of $FJ$ is a linear combination of operators of the type $D_1 \cdots D_r, D_1 \cdots D_r R(e) R(e)$, where $e \in FJ$, $D_i$ are inner derivations. It was proved in [Z2] that in an algebra over an infinite field, the linear span of values of an arbitrary polynomial is differentially invariant. Hence $f^{24}(FJ) D_1 \cdots D_r$ lies in the linear span of $f^{24}(FJ)$. Hence, $T$ is spanned by elements $f^{24}(a, b, c), f^{24}(a, b, c) e, (f^{24}(a, b, c) e) e$, where $a, b, c, e$ are arbitrary elements from $FJ$. Now Lemma 1.1.9 immediately implies

**Lemma 1.1.10.** $t(M, M, I, T(A), A) = 0$.

**Proof.** See Lemma 1.1.26 in [KMZ].

**Lemma 1.1.11.** If $I \neq (0)$, then $T(\overline{A}) = (0)$.

**Proof.** Since $\overline{A}$ is $\mathcal{D}$-simple and $T(\overline{A})$ is a $\mathcal{D}$-invariant ideal, then either $T(\overline{A}) = (0)$ or $T(\overline{A}) = \overline{A}$.

But $T(\overline{A}) = \overline{A}$ implies that $A = T(A) + I$. Then $t(M, M, I, A, A) = tr([M I] A, M I) A = 0$, by Lemmas 1.1.7 and 1.1.9. This proves that the ideal $I + [M I] A, M I ] A$ is $\mathcal{D}$-invariant and has zero trace. So $[M I] A, M I ] \subseteq I$. This implies that $id_I(I) \cap A \subseteq I$, and so $I = (0)$ by simplicity of $J$. Lemma is proved.

That is, we have proved that when $I \neq (0)$, then $s = 1$ and $A/I$ is $\mathcal{D}$-simple with $T(A/I) = (0)$. Consequently, either

1. $A/I = F 1 + V$ is a simple Jordan algebra of a bilinear form, or
2. $A/I = B \otimes F[a_1, \ldots, a_n \mid a^n = 0]$ and $B$ is a simple Jordan algebra of a bilinear form.

**Lemma 1.1.12.** $t([M \cdot I] A, M I ] A) = 0$.

**Proof.** Let $(i, j)$ be a lexicographically maximal pair such that $t([MD(I, A)]^j, MR(N^j)) = (0)$, where $n$ denotes the nilpotent radical of $A$. 

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Notice that \( t([(MI) A, M]) = t([MD(I, A), M]) \) since \( [(MA) I, M] \subseteq I \).
Again we can consider a homogeneous Peirce element \( x \in MD(I, A) \) and a strongly Peirce homogeneous element \( y \in MR(N)^+ \) such that \( t([x, y]) \neq 0 \). By Kaplansky’s Lemma, \( y = \psi[x, y]y + ya \), with \( 0 \neq \psi \in F \), \( a \in N(A) \). Hence, \( t([x, ya]) \neq 0 \) or \( t([x, xR(y)^2]) \neq 0 \).

But \([x, ya] \in [MD(I, A)^+, MR(N)^{+1}] \) and \( (i, j) < (i, j + 1) \).
Similarly, \([x, xR(y)^2] \in [MD(I, A)^+, MD(I, A)^+] \) \( \subseteq I + [MD(I, A)^{+1}, M] \) and \((2i, 0) > (i, j) \). This contradiction proves the lemma.

**Lemma 1.1.13.** \( [M_I, M_I] \subseteq I \), where \( M_I = (MI) A \).

**Proof.** \( M_I A \subseteq M_I \). Indeed,

\[
((xu)a)b = xR(u)R(a)R(b) \\
\quad = -xR(b)R(a)R(u) - xR((ab)a) \\
\quad + xR(a)R(b) + xR(buR(a) + xR(ab)R(u)).
\]

The subspace \( K = [M_I, M_I] A \) is an ideal of \( A \). Let us show that \( MK \subseteq M_I \). Indeed,

\[
MR(a[x_1, x_2]) = M(-R(x_1)R(a)R(x_2) + R(x_2)R(a)R(x_1) \\
\quad + R(x_1)R(ax_2) - R(x_2)R(ax_1) + R(a)R([x_1, x_2]) \\
\quad \in AM_I + M[x_1, x_2].
\]

But \( M[x_1, x_2] \subseteq x_1D(M, x_2) + x_2[M, x_1] \subseteq M_I \), since \( M_I \) is \( \mathcal{D} \)-invariant.

Consequently, \( K \) is a \( \mathcal{D} \)-invariant ideal of \( A \) and \( \tilde{A} = A/I \) is \( \mathcal{D} \)-simple, as we have seen. This implies that either \( K + I = A \) or \( K + I = I \). But \( K + I = A \) implies that \( M = MA = MK + MI \subseteq M_I \), which is a contradiction because \( I \) acts nilpotently on \( M \). So \( K + I = I \); that is, \( K \subseteq I \).

**Lemma 1.1.14.** If the algebra \( A \) contains three pairwise orthogonal idempotents, then \( I = N \), where \( N \) denotes the nilpotent radical of \( A \).

**Proof.** Since \( \tilde{A} = \oplus_{i \leq 2} \tilde{A}^{(i)} \) and each \( \tilde{A}^{(i)} \) is either simple or \( \tilde{A}^{(i)} = B_i \) \( \oplus F[a_i | a_i^2 = 0] \) and \( B_i \) is simple, it follows that \( N/I \) is generated by central elements (the elements \( a_i \)). Let \( U = (N/I) \cap Z(\tilde{A}) \). Then \( N/I = \text{id}_U(\tilde{A}) = U\tilde{A} \). Let \( I = e_1 + e_2 + e_3 \), where \( e_i \) are idempotents in \( A \). Then \( M = \oplus M_{ij}, M_{ij} = (e_i, M, e_j) \). We have \( UD(M_{ij}, M_{ij}) \subseteq Z(\tilde{A}) \cap \{ \tilde{e}_i, \tilde{e}_j \} = 0 \), and \( UD(M_{ij}, M_{iks}) \subseteq Z(\tilde{A}) \cap \{ \tilde{e}_i, \tilde{A}, \tilde{e}_k \} = 0 \) for \( i, j, k \) distinct. Hence, \( UD(M, M) = 0 \) and \( N \) is \( \mathcal{D} \)-invariant. So \( N = I \). Lemma is proved.
COROLLARY 1.1.2. If $I = (0)$, $N \neq (0)$, then $A$ is one of the following algebras:

(i) $F[a_1, \ldots, a_n | a_i^p = 0]$,
(ii) $F[a_1, \ldots, a_n | a_i^p = 0] \oplus F[b_1, \ldots, b_m | b_i^p = 0]$, $n \geq m \geq 0$ and $n \geq 1$,
(iii) $(F + V) \otimes F[a_1, \ldots, a_n | a_i^p = 0]$.

The following proposition summarizes the results proved in this section.

PROPOSITION 1.1.1. Let $J = A + M$ be a finite-dimensional simple unital Jordan superalgebra and let $I$ denote the maximal nilpotent $\mathcal{D}$-invariant ideal of $A$, $\mathcal{D} = D(M, M)$. Then one of the following statements holds:

1. $A$ is semisimple (this case was studied in [RZ]).
2. $I \neq (0)$, $A/I$ is differentially simple. Furthermore, $T(\bar{A}) = (0)$.

Hence either

2.a. $\bar{A}$ is a simple Jordan algebra of a bilinear form, or
2.b. $\bar{A} = B \otimes F[a_1, \ldots, a_n | a_i^p = 0]$ and $B$ is a simple Jordan algebra of a bilinear form.

3. $A = F[a_1, \ldots, a_n | a_i^p = 0]$.
4. $A = (F + V) \otimes F[a_1, \ldots, a_n | a_i^p = 0]$.
5. $A = F[a_1, \ldots, a_n | a_i^p = 0] \oplus F[b_1, \ldots, b_m | b_i^p = 0]$.

As it has been mentioned before, case (1) was studied in [RZ], so we will consider in this paper the other cases (2)–(5).

2. THE CASE $I = (0)$

In this section we will consider all cases that correspond to $I = (0)$.

As we have seen in Section 1, we have three possibilities:

(a) $A = F[a_1, \ldots, a_n | a_i^p = 0]$ is an algebra of truncated polynomials, or
(b) $A = F[a_1, \ldots, a_n | a_i^p = 0] \oplus F[b_1, \ldots, b_m | b_i^p = 0]$ is a direct sum of two algebras of truncated polynomials, or
(c) $A = (F \cdot 1 + V) \otimes B(m)$ is a tensor product of a finite-dimensional simple Jordan algebra of a bilinear form with the algebra of truncated polynomials in $m$ variables.

2.1. $A = F[a_1, \ldots, a_n | a_i^p = 0]$

In this section we will consider the case in which the even part $A$ of the Jordan superalgebra $J$ is an algebra of truncated polynomials, $A = F[a_1, \ldots, a_n | a_i^p = 0]$. 
Let us denote $A_0 = \{ a \in A \mid t(a) = 0 \} = \{ a = \sum_{i_1,\ldots,i_n > 0} a_{i_1} \cdots a_{i_n} \mid a_{i_1} \cdots a_{i_n} \neq 0 \}$. Denote $\deg(a) = \min\{i_1 + \cdots + i_n \mid \alpha_{i_1} \cdots \alpha_{i_n} \neq 0 \}$.

**Lemma 2.1.1.** (1) For every $a \in A$ with $\deg(a) \geq 2$ and arbitrary elements $x, y \in M$, we have

$$t([x, yR(a)]) = t([xR(a), y]).$$

(2) For arbitrary elements $a, b \in A$, $x, y \in M$, $\deg(a), \deg(b) \geq 1$, we have

$$t([x, yU(a, b)]) = -t([xU(a, b), y]).$$

(3) $t([x, yD(a, b)]) = -t([xD(a, b), y]).$

**Proof.** (1) $[xR(a), y] - [x, yR(a)] = [ax, y] + [ay, x] = aD(x, y) \in A_0$ because $\deg(a) \geq 2$.

(2) If $b \in A_0$, then $[x, yU(b)] = 4(x, b, y)b - [xU(b), y]$. Since $(x, b, y)b \in A_0$, it follows that $t([x, yU(b)] + [U(b), y]) = 0$.

(3) The assertion immediately follows from the identity: $[x, yD(a, b)] + [xD(a, b), y] = [x, y]D(a, b)$. Lemma is proved.

Let $R_i$ denote the linear span of all operators of the form $R(b_1) \cdots R(b_i)$ with $\sum_{j=1}^i \deg(b_j) \geq i$.

**Lemma 2.1.2.** For every operator $w \in R_i$, $i \geq 2$, there exist an operator $\tilde{w} \in R_2$ and operators $\omega_a \in R(A)$ such that $t([x, y\omega]) = t(\sum_a [x \tilde{w}, y \omega_a])$.

**Proof.** Let $\omega = R(b_1) \cdots R(b_i)$. If $r = 1$, then $\deg(b_1) \geq 2$ and we can use Lemma 2.1.1(1).

Suppose $r \geq 2$. Then $R(b_{r-1})R(b_r) = 1/2(D(b_{r-1}, b_r) + U(b_{r-1}, b_r) + R(b_{r-1}, b_r))$ and $t([x, y\omega]) = t(-1/2[xD(b_{r-1}, b_r), yR(b_1) \cdots R(b_{r-2})] - 1/2[xU(b_{r-1}, b_r), yR(b_1) \cdots R(b_{r-2})] + 1/2[x(b_{r-1}, b_r), yR(b_1) \cdots R(b_{r-2})])$. Lemma is proved.

Let us denote $\tilde{M} = \sum_{a, b \in A} FM(R(a)R(b) - R(ab))$. Our aim is to prove that $\tilde{M} = 0$.

Since $A$ has no nilpotent $\mathcal{D}$-invariant ideals and $A_0$ is nilpotent, then $A_0$ is not $\mathcal{D}$-invariant. So there is an element $a \in A_0$ and some element $x \in M$ such that $aR(x)^2 = 1 + b$, with $b \in A_0$. The operator $R(1 + b)$ is invertible, $R(1 + b)^{-1} = Id + \sum_{i \geq 1} (-1)^i R(b)^i$. Hence $M = MR(aR(x)^2) \subseteq (MR(x)^2)a + (MaR(x)^2) \subseteq (Ax)a + Ax$.

If $\tilde{M} = 0$, then $M = Ax$, and consequently, $J = A + Ax$ is obtained from $A = F[a_1, \ldots, a_n \mid a_n^2 = 0]$ by the Kantor-doubling process.
LEMMA 2.1.3. \( \tilde{M} = (0) \).

Proof. For arbitrary elements \( a, b, c \in A \), we have

\[
( R(a)R(b) ) - R(ab)R(c) \\
= R(a)( R(b)R(c) - R(bc) ) \\
+ ( R(a)R(bc) - R(a(bc)) ) - ( R(ab)R(c) - R((ab)c) ).
\]

Hence \( \tilde{M} \) is a submodule of \( M \). It is sufficient to prove that \( [\tilde{M}, M] = (0) \).

In such case, \( \tilde{M} \subseteq J \), and it follows from simplicity of \( J \) that \( \tilde{M} = (0) \).

To prove that \( [\tilde{M}, M] = (0) \), it is sufficient to prove that \( t([\tilde{M}, M]) = (0) \).

Indeed, the subspace \([\tilde{M}, M]\) of \( A \) is \( \mathcal{D} \)-invariant, and since \( I = (0) \), we know that there are no nonzero \( \mathcal{D} \)-invariant subspaces of \( A \) having zero trace.

Clearly, \( R_2 \) is an ideal of \( R(A) \), and for an arbitrary derivation \( d \in \mathcal{D} \), we have \([R_k, d] \subseteq R_{k-1}\) (since \([R(a), d] = R(ad)\)).

Claim. \( t([MR_i, M]) = (0) \) if \( i \geq 3 \).

Indeed, suppose the assertion is not true and let \( i \) be the maximal number with this property: \( t([MR_i, M]) \neq (0) \), but \( t([MR_{i+1}, M]) = (0) \).

If \( x, y \in M \), \( \omega \in R \) satisfy \( t([x\omega, y]) \neq 0 \) and the relevant elements are Peirce homogeneous, then by Kaplansky’s argument, \( y = \alpha x\omega R(y)^2 + \beta y\omega' \), \( \alpha, \beta \in F \), \( \omega' \in R_1 \). Hence, \( y = \alpha x\omega R(y)^2(1 - \beta\omega')^{-1} \) and \( y \in MR_{i-1} \).

Since \( i - 1 \geq 2 \), by Lemma 2.1.2 it follows that \( t([MR_i, MR_{i-1}]) = t([MR_{i+2}, M]) = (0) \). Claim is proved.

Now we can prove that \( t([\tilde{M}, M]) = (0) \). Indeed, the submodule \( \tilde{M} \) of \( M \) is \( \mathcal{D} \)-invariant. If \( x \in \tilde{M}, y \in M \) are Peirce homogeneous elements and \( t([x, y]) \neq 0 \), then again \( y = \alpha xR(y)^2 + \beta y\omega' \), \( \omega' \in R_1 \). Therefore, \( y = \alpha xR(y)^2(1 - \beta\omega')^{-1} \in \tilde{M} \subseteq MR_2 \). Now \( t([MR_2, MR_3]) \subseteq t([MR_4, M]) = (0) \). Lemma is proved.

The following proposition is due to King and McCrimmon.

PROPOSITION 2.1.1 (see [KM]). The Jordan superalgebra \( J = A + Ax \) is simple if and only if \( A \) is “bracket simple”; that is, \( A \) has no nonzero ideals \( B \) such that \( A, B ) \subseteq B \).

EXAMPLE. Let us show that there is only one (up to isomorphism) structure of a simple Jordan superalgebra on the Kantor double \( F[a_1] + F[a_1]x \). Let us denote \( a = a_1 \) and suppose that \( aR(x)^2 = 1 + b \) with \( b \in A_0 \). Since the characteristic of \( F \) is \( p \neq 2 \), the element \( 1 + b \) has a square root in \( F[a] \). Let \( h(a)^2 = 1 + b \). Let us take \( f(a) = h(a)^{-1} \).
and replace the element $x$ by $xf(a)$. Then $aR(xf(a))^2 = [xf(a)a, xf(a)] = f(a)^2 [x, a] = f(a)^2 h(a) = 1$. That is, in this case, $J = A + Ax = F[a] + Fx + \sum_{i=1}^{p} F a^i x$ and

$$a^i a^j = a^{i+j}, \quad (a^i x) a^j = a^{i+j} x, \quad [a^i x, a^j x] = (i - j) a^{i+j-1}. $$

2.2. $A = F[a_1, \ldots, a_n | a_i^p = 0] \oplus F[b_1, \ldots, b_m | b_j^p = 0]$

In this section we will consider case (5) in Proposition 1.1.1, that is, the case when the even part $A$ of the Jordan superalgebra $J$ is a direct sum of two algebras of truncated polynomials. Let us assume $n \geq m$ and denote $A' = F[a_1, \ldots, a_n]$ and $A'' = F[b_1, \ldots, b_m]$. Then (see Lemma 1.1.6) $M = \mathcal{M}$. If $e$ denotes the identity element of $A'$ and $f$ denotes the identity element $A''$, then for an arbitrary element $x \in M$, we have $xe = xf = 1/2 x$ and $1 = e + f.$ In particular, $(A', M, A') = (A'', M, A'') = (0)$.

For arbitrary elements $a, b \in A'$ (resp. $a, b \in A''$), we have $U_M(a, b) = 0$; that is, $R_M(a)R_M(b) + R_M(b)R_M(a) = R_M(ab)$.

Since $A'^2 = A'$, for an arbitrary derivation $d$ of the algebra $A$, we have $A'd \subset A'$ (resp. $A'd \subset A''$). If $a \in A'$, $b' \in A''$, then $D(a, b') = D(a, b') = D(a, e) + D(e, b') = 0$. So $R(a)R(b') = R(b')R(a)$.

The Lie algebra $\mathcal{D} = D(M, M)$ acts on $R = R^M\langle A \rangle$, $[R(a), D] = R(aD)$. Let us denote as $W$ the maximal nilpotent ideal of $R$ that is $\mathcal{D}$-invariant. As in the previous section, $A' = Fe + A'_{0}$, $A'' = Ff + A''_{0}$. If $R_{1}$ denotes the subalgebra of $R$ generated by $R(a)$, where $a \in A_{0}' + A_{0}'$, then $R = F \cdot \text{Id} + R_{1}$ and $R_{1}$ is the nilpotent radical of $R$. Since $A_{0}'$ is nilpotent, it cannot be $\mathcal{D}$-invariant. So there are elements $x \in M$, $a \in A_{0}'$, $c \in A_{0}''$ such that $aR(x)^2 = e + c$. Since the element $c$ is nilpotent, as we have seen above, the operator $R(e + c)$ is invertible and therefore $M = MR(aR(x)^2) = xR^M\langle A \rangle$.

First we will prove that $W \neq (0)$. Let us assume the contrary; that is, $R$ is $\mathcal{D}$-semisimple. The subspace $D(A, A)$ is $\mathcal{D}$-invariant and $D(A, A) \subset R_1$. Hence $D(A, A) \subset W = (0)$. So $R$ is a commutative associative algebra.

Let us note that the only element $w \in R$ that satisfies $xw = 0$ is $w = 0$. Indeed, $xw = 0$ implies that $Mw = xR^M\langle A \rangle w = xwR^M\langle A \rangle = (0)$.

Since $R$ is differentially semisimple, Block's theorem asserts that $R = \bigoplus_{i=1}^{\alpha} B(n_{i})$, where $B(n_{i}) = F[\alpha_{1}, \ldots, \alpha_{n_{i}} | \alpha_{i}^p = 0]$.

Now we will show that $s = 1$. Indeed, $s = \dim_{F} R/N(R)$. Since $N(R) = R_{1}$ and $R = F1 + R_{1}$, it follows that $s = 1$. Consequently, if we denote $r = n_{1}$, we have $R = B(r)$, and $\dim_{F} R = p^{r}$. Since $\phi: R \to M, w \to xw$ is a bijective linear mapping, we have that $\dim_{F} R = \dim_{F} M$.

Claim. $p^{n} < \dim_{F} M$.

Indeed, let us show that $\eta: A' = B(n) \to M$, $a \to xa$ is an injective linear map. We know that $xa = 0$ implies $Ma = (0)$. Let us consider
$B' = \{ a \in A' \mid Ma = (0) \}$. Let us show that $B'$ is an ideal of $A'$, so it is an ideal of $A$. For arbitrary elements $b \in A'$, $y \in M$, $a \in B'$, we have $y(ab) = 2(ya)b = 0$, since $U(a, b) = 0$ and $R(a)R(b) = R(b)R(a)$.

Since $A$ does not contain nontrivial nilpotent $D$-invariant ideals, if we assume that $\eta$ is not injective, that is, $B' \neq (0)$, then $B'$ cannot be nilpotent. This implies that $B'$ contains some element which is invertible in $A'$. Hence $B' = A'$ and $MA = (0)$, which is a contradiction, because $e \in A'$ and $ye = 1/2y$ for every $y \in M$.

Injectivity of $\eta$ implies that $\dim_F A' = p^n \leq \dim_F M$. If we assume that $p^n = \dim_F M$, then $M = \text{Im} \eta = xA'$. This implies that $[M, M] = [xA', xA'] \subseteq A'$ and so $A' + M \leq J$, which contradicts simplicity of $J$. This proves the claim.

As we have mentioned above, there is an element $a \in A'$ with $t(aR(x)^2) \neq 0$ and so $M = xA + (xA)a = xA + xA' + (xA')a = FA + xA_0 + xA'_0 + (xA'_0)a$. Hence $\dim_F M \leq 1 + (p^n - 1) + 2(p^m - 1) = p^n + 2p^m - 2$, because $\dim_F A'_0 = p^n - 1$ and $\dim_F A''_0 = p^m - 1$.

Thus, we have $p^n < \dim_F M = p^r \leq p^n + 2p^m - 2$. But $p^n < p^r$ implies that $n < r$ and therefore $p^r \geq p^{r+1} = pp^n \geq 3p^n \geq p^n + 2p^m > p^n + 2p^m - 2$. The contradiction comes from the assumption $W = (0)$. We have proved that $W \neq (0)$.

**Lemma 2.2.1.** Let us assume that $\omega = R(a) + R(b) + \sum_i R(a_i)R(b_i) \in W$, where $a, a_i \in A'_0$, $b, b_i \in A''_0$. Let $\omega^*$ be the operator $\omega^* = R(a) + R(b) - \sum_i R(a_i)R(b_i)$. Then, for arbitrary elements $y, z \in M$, we have $t(y, z \omega) = t(y \omega^*, z)$.

**Proof.** Arguing as in the proof of Lemma 2.1.1, we get $t(y, zU(a_i, b_i)) = -t(yU(a_i, b_i), z)$ for arbitrary elements $a_i \in A'_0$, $b_i \in A''_0$.

To prove that $t(y, z(a + b)) = t(y(a + b), z)$, we need to show that $t((a + b)D(y, z)) = 0$. Suppose the contrary, that is, $(a + b)D(y, z) = \alpha 1 + \beta(e - f) + c$, with $\alpha \neq 0$, $c \in A'_0 + A''_0$. Then $[\omega, D(y, z)] = R(a + b)D(y, z) + \sum_i R(a_i)R(b_i), D(y, z)]$ is an invertible operator, which contradicts nilpotency of $W$. Lemma is proved.

Now let $W_0 = [R(a) + R(b) + \sum_i R(a_i)R(b_i) \in W \mid a, a_i \in A'_0$, $b, b_i \in A''_0]$. It is clear that $W = W_0 + D(A, A)R$ and $D(A, A)$ lies in the center of $R$.

**Lemma 2.2.2.** $t([MW, M]) = 0$.

**Proof.** We will use inverse induction on $i + j$ to prove that $t([MW, A]W^i_0, M)] = (0)$, where $W^*_0 = \{ \omega^* \mid \omega \in W_0 \}$.

Let us assume that $W^q = (0)$. Then by Lemma 2.2.1, $t([MW^q_0, M]) = t([M, MW^q_0]) = (0)$.
Now let us assume that \( t([MWD(A, A)W^*, M]) = (0) \) if \( i' + j' > k \), and let \( i + j = k \). Suppose that there exist \( y_1, y_2 \in M \), \( \omega \in W \), \( d_1, \ldots, d_i \in D(A, A), \omega_1, \ldots, \omega_j \in W_0 \) such that \( t([y_1 \omega d_1 \cdots d_i \omega_j, y_2]) \neq 0 \).

Then, \( y_1 \omega d_1 \cdots d_i \omega_j, y_2 = \alpha_1 + \beta(e - f) + c \), where \( \alpha \neq 0 \), \( c \in A_0' + A_0'' \). Hence, \( y_1 \omega d_1 \cdots d_i \omega_j \), and therefore, \( y_2 = y_1 \omega d_1 \cdots d_i \omega_j, y_3 = \alpha_1 \beta(e - f) + c \), and \( \omega \in W \subseteq R \). Hence \( y_2 = y_3 \omega_{j+1} + y_4 d \), where \( \omega_{j+1} \in W_0', d \in D(A, A); y_3, y_4 \in A \).

The element
\[
[y_1 \omega d_1 \cdots d_i \omega_j, y_3] = [y_1 \omega d_1 \cdots d_i \omega_j, y_3, y_4 d]
\]
has zero trace since
\[
t([y_1 \omega d_1 \cdots d_i \omega_j, y_3]) = -t([y_1 \omega d_1 \cdots d_i \omega_j, y_3]) = 0 \quad \text{and}
\]
\[
t([y_1 \omega d_1 \cdots d_i \omega_j, y_3, y_4 d])
\]
\[
= -t([y_1 \omega d_1 \cdots d_i \omega_j, y_4])
\]
\[
+ t([y_1 \omega d_1 \cdots d_i \omega_j, y_3, y_4 d]) = 0
\]

by the induction assumption. Lemma is proved.

**Corollary 2.2.1.** \( R_M([MW, M]) \subseteq W \).

**Proof.** Indeed, by Lemma 2.2.2, \( R_M([MW, M]) \subseteq R_1 \). Since the subspace \([MW, M]\) is \( \mathcal{D} \)-invariant, it follows that \( R_M([MW, M]) \subseteq W \).

**Lemma 2.2.3.** \( AD(\tilde{M}, M) = (0) \), where \( \tilde{M} = MW \).

**Proof.** \( AD(\tilde{M}, M) \subseteq [\tilde{M}, M] \), so \( t(AD(\tilde{M}, M)) = (0) \). Since there are no nonzero \( \mathcal{D} \)-invariant nilpotent ideals of \( A' \) (resp. \( A'' \)) and \( AD(\tilde{M}, M) \) (resp. \( A'D(\tilde{M}, M) \)) has zero trace and is \( \mathcal{D} \)-invariant, it follows that \( AD(\tilde{M}, M) = (0) \) (resp. \( A'D(\tilde{M}, M) = (0) \)).

**Lemma 2.2.4.** (a) \( MWW = \tilde{M}W = (0) \). In particular, \( \tilde{M}D(A, A) = (0) \) and \( MD(A, A)D(A, A) = (0) \).

(b) \([MW, MW] = (0)\).

**Proof.** To prove assertion (a), it is sufficient to prove that \([\tilde{M}W, \tilde{M}W] \subseteq A_0' + A_0'' \). Indeed, the subspace \([\tilde{M}W, \tilde{M}W]\) is \( \mathcal{D}(M, M) \)-invariant, hence \([MW, MW] = (0) \). Hence \( \tilde{M}W \) is an ideal of \( J \). By simplicity of \( J \), we get \( MW = (0) \).
Let us prove the inclusion. Choose arbitrary elements \( \tilde{y}_1 \in \tilde{M} \), \( y_2 \in M \), \( c \in A \). By the Jordan identity and Lemma 2.2.3, we have

\[
\tilde{y}_1 R(c) R(y_2) R(e) = \tilde{y}_1 (-R(e) R(y_2) R(e) - R((ce)y_2) + R(e) R(cy_2)) + R(c) R(cy_2) + R(y_2) R(ec))
\]

\[
= -1/2[\tilde{y}_1, y_2]c - [\tilde{y}_1, (ce)y_2] + 1/2[\tilde{y}_1, cy_2] + 1/2[\tilde{y}_1 c, y_2] + [\tilde{y}_1, y_2](ec)
\]

\[
= -1/2[\tilde{y}_1, y_2]c - [\tilde{y}_1, (ce)y_2] + [\tilde{y}_1, cy_2] + [\tilde{y}_1, y_2](ec).
\]

(2.1)

If \( c \in A'_0 \), the right-hand side of (2.1) is \( \frac{1}{2} [\tilde{y}_1, y_2]c + [\tilde{y}_1, y_2]c = \frac{1}{2} [\tilde{y}_1, y_2]c \in A'_0 + A'_0 \).

If \( c \in A'_0 \), the right-hand side of (2.1) is \( \frac{1}{2} [\tilde{y}_1, y_2]c + [\tilde{y}_1, cy_2] \) and the left-hand side is \( \cdots e \in A \). So \( [\tilde{y}_1, cy_2] = ([\tilde{y}_1, cy_2] - \frac{1}{2}[\tilde{y}_1, y_2]c) + \frac{1}{2}[\tilde{y}_1, y_2]c = a + b \), where \( a \in A' \) and \( b \in A'_0 \).

By Lemma 2.2.2, \( t([\tilde{y}_1 c, y_1]) = 0 \), so \( a \in A'_0 \) and \( [\tilde{y}_1 c, y_2] \in A'_0 + A'_0 \).

(b) Since \([MW, MW]\) is \( \mathcal{D}\)-invariant and there are no nontrivial \( \mathcal{D}\)-invariant ideals in \( A'_0 + A'_0 \), it is sufficient to prove that \([MW, MW] \subseteq A'_0 + A'_0 \).

In fact, it is easy to see that we need to verify the inclusion \([MW, MR(a)] \subseteq A'_0 + A'_0 \) for an arbitrary element \( a \in A'_0 + A'_0 \), which was done above. Lemma is proved.

**Lemma 2.2.5.** For every element \( a \in A' \), there is a unique element \( \varphi(a) \in A'' \) such that \( M(a - \varphi(a)) \subseteq \tilde{M} \).

**Proof.** Since \( \tilde{M} \neq (0) \), it follows that \( [\tilde{M}, M] \neq (0) \) (otherwise \( \tilde{M} \subseteq J \)). Since \( M = xR^M(A) \), Lemma 2.2.3 implies that \([\tilde{M}, M] = [\tilde{M}, x]\).

The subspace \([\tilde{M}, M]\) is \( \mathcal{D}(M,M)\)-invariant, hence \([\tilde{M}, M] \subseteq A'_0 + A'_0 \). All elements of \([\tilde{M}, M]\) have zero trace. Therefore, there exists an element \( y \in \tilde{M} \) such that \([y, x] = (e + a_0) - (f + b_0) \), where \( a_0 \in A'_0, b_0 \in A'_0 \).

Let us consider an arbitrary element \( a \in A' \) and let \( a' = 2(e + a_0)^{-1}a \). Applying the expression (2.1) with \( y, a', \) and \( x \) instead of \( \tilde{y}_1, c, \) and \( y_2, \) we get \([ya', x]e = \frac{1}{2}[y, x]a' = [ya', x]a' - b, \) for some \( b \in A' \).

Now \( xR(a - b) = (a - b)R(x) = [ya', x]R(x) = (ya')R(x)^2 \in M \), since \( y \in \tilde{M} \) and \( \tilde{M} \) is \( \mathcal{D}\)-invariant. An arbitrary element in \( M \) is a linear combination of elements of the type \( x' = xR(c_1) \cdots R(c_r) \), with \( c_1, \ldots, c_r \in A \). We will use induction on \( r \) to prove that \( x'R(a - b) \in \tilde{M} \). If \( r = 1 \),
then \(xR(c_1)R(a - b) = xR(a - b)R(c_1) + xD(c_1, a - b) \in \tilde{M}\). In general, 
\(x'R(a - b) = xR(c_1) \cdots R(c_n)R(a - b) = xR(c_1) \cdots R(a - b)R(c_1) + xR(c_1) \cdots D(c_1, a - b) \in M\), since \(D(A, A) \subseteq W\).

Let us prove now uniqueness. If we assume that there is another \(b' \neq b\) such that \(M(a - b') \subseteq \tilde{M}\), then \(\{a' \in A' | Ma' \subseteq \tilde{M}\} \neq \{0\}\) and it is a \(\mathcal{D}\)-invariant subspace. So it contains an element \(f + a_0'\), with \(a_0' \in A_0\). But \(M = M(f + a_0') \subseteq \tilde{M}\) implies \(\tilde{M} = M = MW\), which contradicts \(MW^q = \{0\}\). Lemma is proved.

**Lemma 2.2.6.** The mapping \(\varphi : A' \rightarrow A', a \rightarrow \varphi(a)\) is an isomorphism of algebras. In particular, \(n = m\); that is, \(A\) is a direct sum of two algebras of truncated polynomials in the same number of variables.

**Proof.** Clearly, \(\varphi\) is linear. Doing as in Lemma 2.2.5, we can prove that, for every \(b \in A'\), there is a unique \(\psi(b) \in A'\) such that \(M(b - \psi(b)) \subseteq \tilde{M}\).

It is easy to see that \(\varphi\) and \(\psi\) are inverses.

Finally, let us show that, for arbitrary elements \(a, a' \in A'\), we have \(\varphi(aa') = \varphi(a)\varphi(a')\).

Indeed, 
\(R_M(aa') = R_M(a)R_M(a') + R_M(a')R_M(a) = 2R_M(a)R_M(a') + D_M(a, a')\) since \(U_M(a, a') = 0\).

Similarly, 
\(R_M(\varphi(a)\varphi(a')) = 2R_M(\varphi(a))R_M(\varphi(a')) + D_M(\varphi(a'), \varphi(a')).\) This implies 
\[
M(R(aa') - R(\varphi(a)\varphi(a'))) \\
\leq 2M(R(a)R(a') - R(\varphi(a))R(\varphi(a'))) \\
+ (MD(a', a) - MD(\varphi(a'), \varphi(a))) \\
\leq (2MR(a)R(a' - \varphi(a'))) + (2MR(a - \varphi(a))R(\varphi(a'))) \\
+ (MD(a', a) - MD(\varphi(a'), \varphi(a))) \subseteq \tilde{M},
\]

since each summand lies in \(\tilde{M}\). Lemma is proved.

**Remark.** \(\varphi\) commutes with \(\mathcal{D}(M, M)\).

Indeed, for an arbitrary element \(a \in A'\) and an arbitrary derivation \(d \in \mathcal{D}(M, M)\), we have 
\(M(a - \varphi(a))d \subseteq (Mdal(a - \varphi(a)) + M(d(a) - d(\varphi(a))).\) Therefore, 
\(M(d(a) - d(\varphi(a))) \subseteq M(a - \varphi(a))d + (Md(a - \varphi(a)) \subseteq \tilde{M}\) and so \(d(\varphi(a)) = \varphi(d(a))\).

**Notation.** Let \(S = \{a + \varphi(a) | a \in A'\}\). Then \(S\) is closed under the product \((a + \varphi(a))(a' + \varphi(a')) = aa' + \varphi(a)\varphi(a') = aa' + \varphi(aa') \in S\).

**Lemma 2.2.7.** For arbitrary elements \(s, s' \in S\), we have 
\(R_M(s)R_M(s') = R_M(ss')\).
Proof. It suffices to prove that the subspace $[\Sigma_{s, s' \in S} M(R(s)R(s') - R(ss')), M]$ lies in $A_0 + A_o'$, since it is $D(M, M)$-invariant.

Claim 1. $M(R(s)R(s') - R(ss')) \subseteq \tilde{M}$.

Let $s = a + \phi(a), s' = a' + \phi(a')$. We know that $M(a - \phi(a)) \subseteq \tilde{M}, M(a' - \phi(a')) \subseteq \tilde{M}, M(a + a')(a' - \phi(a')) \subseteq M$, and $4R(a)R(a') - 2R(a + a') = 2U(a, a') + D(a', a) = 2D(a, a')$. Hence $M(R(s)R(s') - R(ss')) \subseteq M$.

Claim 2. $MD(s, s') = (0)$.

First we will see that $[M(a - \phi(a))(a' - \phi(a'))(a - \phi(a'))] = (0)$. Indeed, by Lemma 2.2.3, we have that $[M(a - \phi(a))(a' - \phi(a'))] = [M(a - \phi(a)), M(a' - \phi(a'))] \subseteq \tilde{M}$, by Lemma 2.2.4(b).

Let us show that $M(a - \phi(a))(a' - \phi(a'))$ is an $A$-submodule of $M$. Let $u = a - \phi(a), u' = a' - \phi(a')$, and let $c$ be an arbitrary element from $A$. We have

$$MR(a - \phi(a)) R(a' - \phi(a')) R(c) = MR(u) R(u') R(c) \subseteq MR(u) R(c) R(u') + MR(u) D(u', c) = MR(u) R(c) R(u') \subseteq MD(u, c) R(u') + MR(c) R(u) R(u'),$$

since $MR(u) D(u', c), MR(u') D(u, c) \subseteq \tilde{M} W = (0)$. Now $(M(a - \phi(a))(a' - \phi(a'))) = (0)$.

In particular, $MD(a - \phi(a), a' - \phi(a')) = (0)$. Since $D(A', A') = (0)$, it follows that $[R_M(a + \phi(a)), R_M(a' + \phi(a'))] = [R_M(a - \phi(a)), R_M(a' - \phi(a'))] = (0)$, which proves Claim 2.

Now we will prove that $y_1 R(s) R(s') - R(ss'), y_2 \in A_0 + A_o'$. If $y_2 \in M(A_0 + A_o')$, then the proof of Lemma 2.2.4 implies the result (notice that $y_1 R(s) R(s') - R(ss') \subseteq M$, by Claim 1).

So we can assume that $y_2 = x$. If $y_1 = y_1' c$ with $c \in \tilde{M}$, then $c = \frac{1}{2}((c - \phi(c)) + (c + \phi(c)))$, and by Claim 1, $[y_1'(c - \phi(c))(R(s)R(s') - R(ss')), x] \subseteq [MR(A_0 + A_o'), M] \subseteq A_0 + A_o'$, as was shown in the proof of Lemma 2.2.4(a).

Similarly, by Claim 2,

$$[y_1'(c + \phi(c))(R(s)R(s') - R(ss')), x] = [y_1'(R(s)R(s') - R(ss')) R(c + \phi(c)), x] \subseteq A_0 + A_o',$$

as before.

The case $c \in A_o'$ is similar.

Finally, we have to examine the element $[y(R(s)R(s') - R(ss')), x]$. This is a linear expression in $s, s'$. The result is clearly true if one of the elements is 1. Let us assume that $s = a + \phi(a), s' = a' + \phi(a'), a, a' \in A_0'$.
Then

\[
[x(ss'), x] = (ss')R(x)^2 = (sR(x)^2)s' + (s'R(x)^2)s \in \mathcal{A}_0' + \mathcal{A}_0' \quad \text{and}
\]

\[
xR(s) R(s'), x = [x(R(a)R(a')) + R(\varphi(a))R(\varphi(a')), x]
\]

+ \[x(R(a)R(\varphi(a')) + R(\varphi(a))R(a')), x\].

Now \[xR(a)R(a') = \frac{1}{2}x(D(a, a') + R(aa'))\] implies that \([xR(a)R(a'), x] = \frac{1}{2}[xD(a, a'), x] + \frac{1}{2}[xR(aa'), x]\]. Since \((aa')R(x)^2 = a'(a'R(x)^2) + (aR(x)^2)a' \in \mathcal{A}_0' + \mathcal{A}_0'\), we have \([xR(a)R(a'), x] = \frac{1}{2}[xD(a, a'), x] \mod \mathcal{A}_0' + \mathcal{A}_0'\), and similarly, \([xR(\varphi(a))R(\varphi(a')), x] = \frac{1}{2}[xD(\varphi(a), \varphi(a')), x] \mod \mathcal{A}_0' + \mathcal{A}_0'\).

Then \([xR(a)R(a') + R(\varphi(a))R(\varphi(a'))], x] = \frac{1}{2}[x(D(a, a') + D(\varphi(a), \varphi(a')), x)] = 0\).

Finally, \(xR(a - \varphi(a))R(a' - \varphi(a')) = 0\) implies that \(0 = xR(a)R(a') + R(\varphi(a))R(\varphi(a')) - xR(a)R(\varphi(a')) + R(\varphi(a))R(a')\). Consequently, \([xR(a)R(\varphi(a')) + R(\varphi(a))R(a')], x] \in \mathcal{A}_0' + \mathcal{A}_0'\), Lemma is proved.

Our aim now is to prove that \(J\) is as a Kantor Double algebra.

Consider \(\Gamma = S + \hat{M}\), which is a commutative associative algebra.

**Lemma 2.2.8.** \(J = \Gamma + \Gamma x\).

**Proof.** We have to show that \(A = S + [\hat{M}, x]\) and \(M = \hat{M} + Sx\).

Denote \(K = \{a - \varphi(a) \mid a \in \mathcal{A}\}\). Then \(A = S + K\) and \(MK \subseteq \hat{M}\). This implies that \(M = xR^M(A) \subseteq xR^M(S) + \hat{M} = xS + \hat{M}\) by Lemma 2.2.7.

It was shown in the proof of Lemma 2.2.5 that there exists an element \(y \in \hat{M}\) such that \([y, x] = (e + a_0) - (f + b_0), a_0 \in \mathcal{A}_0, b_0 \in \mathcal{A}_0'\), and for an arbitrary element \(a \in \mathcal{A}\), we have \(2[y((e + a_0)^{-1}a), x] = a - \varphi(a)\).

This implies that \(K \subseteq [\hat{M}, x]\) and therefore \(A = S + [\hat{M}, x]\). We proved that \(J = \Gamma + \Gamma x\). Lemma is proved.

**Remark.** Since \([\hat{M}, x]x \subseteq \hat{M},\) Lemma 2.2.5 implies that \([\hat{M}, x] \subseteq K\), so \([\hat{M}, x] = K\).

**Lemma 2.2.9.** There exists an operator \(\hat{\omega} \in W\) such that \([x\hat{\omega}, x] = e - f\).

**Proof.** Since \([MW, M]\) is a \(\mathcal{D}\)-invariant subspace of \(A\), it follows that \([MW, M]\) is not contained in \(\mathcal{A}_0' + \mathcal{A}_0'\). Hence there exists an operator \(\omega \in W\) such that \([x \omega, x] = e - f + o(1)\), where \(o(1) \in \mathcal{A}_0' + \mathcal{A}_0'\). Consider the linear map \(\pi: S \rightarrow K\), \(\pi: s \rightarrow [(x \omega)s, x]\). We will show that this mapping is injective.

Let \([(x \omega)s, x] = 0\) for some element \(s \in S\).
Then

\[0 = (x \omega) R(s) R(x) R(e - f)\]
\[= (x \omega)(-R(e - f) R(x) R(s) - R((s(e - f)) x)\]
\[+ R(s) R(x(e - f)) + R(e - f) R(s x) + R(x) R(s(e - f))).\]

If \(s = c + \varphi(c), c \in A',\) then \(s(e - f) = c - \varphi(c).\) Remark that \(M(e - f) = (0)\) and \((x \omega) R((s(e - f)) x) = [(x \omega), x(c - \varphi(c))]\in [MW, MW'] = (0).\)

Hence, \([x \omega, x(c - \varphi(c))] = 0.\) Since the element \([x \omega, x] = e - f + o(1)\) is invertible in \(A,\) it follows that \(c - \varphi(c) = 0,\) \(c = 0.\)

We have \(\dim F S = \dim F K.\) This implies that \(\pi\) is a bijection. In particular, there exists an element \(s \in S\) such that \([xw, x] = e - f.\) Let \(\tilde{\omega} = \omega R(s).\) Lemma is proved.

**Lemma 2.2.10.** \(\tilde{M} = \xi S,\) where \(\xi = x \tilde{\omega}.\)

**Proof.** We have \(x \tilde{\omega} D(x, e - f) = [x \tilde{\omega}, x(e - f)] - [(x \tilde{\omega})(e - f), x] = (e - f)^2 = 1.\)

Choose an arbitrary element \(y \in \tilde{M}.\) Clearly, \([x \tilde{\omega}, y] \in [\tilde{M}, \tilde{M}] = (0).\)

Furthermore,

\[0 = [x \tilde{\omega}, y] D(x, e - f)\]
\[= - (x \tilde{\omega} D(x, e - f)) y + (x \tilde{\omega})(y D(x, e - f)) = - y + (x \tilde{\omega}) c,\]

where \(c = y D(x, e - f) \in S.\) Hence, \(y = (x \tilde{\omega}) c = \xi c \in \xi S.\) Lemma is proved.

**Lemma 2.2.11.** \((\Gamma x)(\Gamma x) \subseteq \Gamma.\)

**Proof.** Since we have already proved that \(\tilde{M} = \xi S,\) we know that \(\Gamma = S + \xi S.\) So we have to prove:

1. \([Sx, Sx] \subseteq S,\)
2. \([\xi S, x][Sx] \subseteq \tilde{M} = \xi S,\)
3. \([\xi S, x][\xi S, x] \subseteq S.\)

To check (1), let us take \(s_1 = a + \varphi(a), s_2 = b + \varphi(b), a, b \in A'.\) Then

\([x s_1, x s_2] = s_1 D(x, x s_2) - [(x s_2), x] = s_1 D(x, s_2 x) - [x(s_2 s_1), x]\) by Lemma 2.2.7.

But \(d(\varphi(a)) = \varphi(d(a))\) for every \(d \in D(M, M).\)

So \(s_1 D(x, s_2 x), (s_2 s_1) R(x)^2 \in S.\)

To prove (2), let us notice that we have proved in Lemma 2.2.9 that \(\xi R(s) R(x) R(e - f) = \xi R(x) R(s(e - f)).\)
Lemma is proved.

\[ \text{Lemma is proved.} \]

\[ \text{Lemma is proved.} \]

\[ \text{Lemma is proved.} \]

\[ \text{Lemma is proved.} \]

**Lemma 2.2.12.** For arbitrary elements \( a, b \in \Gamma \), we have \( a(bx) = (ab)x \).

**Proof.** If \( a, b \in S \), then the assertion follows from Lemma 2.2.7. If \( a, b \in \tilde{M} \), then \( a(bx) = (ab)x = 0 \), by Corollary 2.2.1 and Lemma 2.2.4.

If \( a \in \tilde{M} \), \( b \in S \), then \( a(bx) - (ab)x = bD(x, a) = 0 \) by Lemma 2.2.3.

Finally, let \( a \in S; \ b \in \tilde{M} \). We need to prove that \( bR(a)x = bR(x)R(a) \). As in the proof of Lemma 2.2.9, we have \( bR(a)x = bR(x)R(a(e - f)) \), and similarly, \( bR(x)R(a(e - f)) = bR(x)R(a(x - f)) \). Since \( e - f \) is invertible in \( A \), it follows that \( a(bx) = (bx)a = (ab)x \).

Lemma is proved.

**Lemma 2.2.13.** (a) The operator \( R(x) \) is injective on \( \Gamma \).

(b) \( J = \Gamma + \Gamma x \) is a direct sum.

**Proof.** We have \( \dim A' = \dim A'' = \dim S = p^n \). Since \( \tilde{M} = \xi S \), it follows that \( \dim \tilde{M} \leq p^n \). On the other hand, \( A = S + [M, x] \) implies that \( \dim[\tilde{M}, x] \geq \dim A - \dim S = p^n \). Hence the mapping \( \tilde{M} \to [M, x], y \to [y, x] \), is injective and \( A = S + [\tilde{M}, x] \) is a direct sum.

Let us show that \( s \in S, xs \in \tilde{M} \) implies \( s = 0 \). Indeed, if \( xs \in \tilde{M} \), then \( Ms \subseteq \tilde{M} \). The subspace \( \{ s \in S | Ms \subseteq M \} \) is \( \mathcal{D} \)-invariant. If nonzero, this subspace contains an element from \( e + f + A_0 + A_\o \), which implies \( M = \tilde{M} \), the contradiction.

Hence the mapping \( S \to xs, s \to xs \), is injective and \( M = xS + \tilde{M} \) is a direct sum. Lemma is proved.

We proved that \( J = \Gamma + \Gamma x \) is a Kantor Double with respect to the Jordan bracket \( \{, \} \) on \( \Gamma \) which is defined via

\[ [ax, bx] = (-1)^{|b|}(a, b); \quad a, b \in \Gamma. \]

The algebra \( \Gamma \) can be identified with the algebra of polynomials in \( n \) even truncated variables \( s_i = a_i + \varphi(a_i), 1 \leq i \leq n \), and one odd variable \( \xi \). Moreover, \( \{ \xi, \xi \} = (-1)(\xi, x)^2 = -1 \). For an arbitrary element \( s \in S \), we have \( \{ s, \xi \} = \{ sx \} \xi, x \} = \{ sx \} (e - f) = 0 \); that is, \( \{ S, \xi \} = \{ 0 \} \).

**Example.** Let us consider the case \( n = 1 = m \), that is,

\[ A = F[a | a^p = 0] \oplus F[b | b^p = 0]. \]

In this case we can assume that \( \varphi(b) = a \) and so \( K = \sum_{i=0}^{p-1} F(a^i - b^i) \), \( S = \sum_{i=0}^{p-1} F(a^i + b^i) = F[s] \), where \( s = a + b \). Also \( \Gamma = S + \xi S, \xi = x \omega, \)

\[ = \sum_{i=0}^{p-1} F(a^i - b^i) \oplus F[s] = F[a, b], \]

\[ = \sum_{i=0}^{p-1} F(a^i + b^i) = F[s] \]
with \([x, x] = e - f, \{x, x\} = -1, (s, x) = 0\). The derivation \(d = \{ , 1\}\) satisfies \(d(x) = -[x, x] = sR(x)^2\).

Since in this case \(D(A', A') = D(A', A^r) = 0\), it follows that \(MD(A, A) = 0\). We can choose the element \(x\) satisfying \(aR(x)^2 = e, bR(x)^2 = f\), and so \(sR(x)^2 = 1\), that is, \(d = \frac{sR(x)^2}{sR(x)^2}\).

Thus if \(n = 1 = m\), then there is only one (up to isomorphism) structure of a simple Jordan superalgebra on \(A + Ax\).

2.3. \(A = (F1 + V) \otimes F[a_1, \ldots, a_m]\)

In this section we will address the case when \(A = (F1 + V) \otimes F[a_1, \ldots, a_m]\) is a tensor product of a finite-dimensional simple Jordan algebra of a bilinear form with the algebra of truncated polynomials in \(m\) variables.

Let \(v_1, \ldots, v_n\) be an orthonormal basis in \(V\), \(v_i \cdot v_j = \delta_{ij}\).

Then we have a decomposition \(M = \oplus M(e_1, \ldots, e_n), \ e_i = \pm 1\), and \(M(e_1, \ldots, e_n) = \{x \in M \mid xU(v_i) = e_i x\}\). With every involution \(v_i\) we can associate two orthogonal idempotent elements, \(e^{(i)} = \frac{1}{2} (1 + v_i), e^{(i)} = \frac{1}{2} (1 - v_i)\).

Let us show that \(y \in M(e_1, \ldots, e_n), \ e_i = -1\), implies \(y \cdot v_i = 0\). Indeed, from \(yU(v_i) = -y\), it follows that \(yR(v_i)^2 = 0\). By the Jordan identity, \(2yR(v_i)^3 + yR(v_i) = 3yR(v_i),\) which implies \(yR(v_i) = 0\).

Let \(y \in M(e_1, \ldots, e_n), \ e_i = 1\). From \(yU(v_i) = y\), it follows that \((yw_i)v_i = y\). Hence \(D(y, v_i) = D((yw_i)v_i, v_i) = \frac{1}{2}D(yw_i, v_i^2) = 0\).

Let us note that \(MU(e^{(i)}_1 + MU(e^{(i)}_2) = \Sigma(M(e_1, \ldots, e_n) \mid e_i = 1)\) and \(MU(e^{(i)}_1, e^{(i)}_2) = \Sigma(M(e_1, \ldots, e_n) \mid e_i = -1)\).

Indeed, if \(y \in M\) is a root element, \(y \in M(e_1, \ldots, e_n), \ e_i = -1\), then \(yw_i = 0\). So \(ye^{(i)}_1 = ye^{(i)}_2 = \frac{1}{2}y\). If \(e_i = 1\), then \(D(y, v_i) = 0\) implies that \(D(y, e^{(i)}_1) = D(y, e^{(i)}_2) = 0\) and \(yU(e^{(i)}_1) + yU(e^{(i)}_2) = y(e^{(i)}_1 + e^{(i)}_2) = y\).

Now let us consider \(Z = F1 \otimes F[a_1, \ldots, a_m]\) and an arbitrary element \(a \in Z\). We will prove that if \((e_1, \ldots, e_n), (\mu_1, \ldots, \mu_n)\) are not both equal to \((-1, -1, \ldots, -1)\), then \(ad(M(e_1, \ldots, e_n), M(\mu_1, \ldots, \mu_n)) = 0\). Indeed, \(Z\) is the (associative) center of \(A\), so it is invariant under all derivations. If we assume, for instance, \(e_i = 1\), then \(M(e_1, \ldots, e_n) \subseteq MU(e^{(i)}_1) + MU(e^{(i)}_2)\).

Let \(x \in M(e_1, \ldots, e_n), x' \in M(\mu_1, \ldots, \mu_n)\). We have \(ad(xU(e^{(i)}_1), x') \in ad(MU(e^{(i)}_1), M) \cap Z\). Let us show that for an arbitrary idempotent \(e \in A\), we have \(ZD(MU(e), M) \cap Z = 0\). Clearly, \(ZD(MU(e), MU(e)) \cap Z \subseteq \{e, e, e\} \cap Z = 0\). On the other hand, if \(f = 1 - e\), then \(D(MU(e), M \cdot f) \subseteq D(A, f)\). Therefore, \(ZD(MU(e), M \cdot f) = 0\). In particular, \(AD(xU(e^{(i)}_1), x') = AD(xU(e^{(i)}_2), x') = 0\). From \(x = xU(e^{(i)}_1) + xU(e^{(i)}_2)\), it follows that \(AD(x, x') = 0\).
We have $Z = F1 + Z_0$. Since $Z_0$ is nilpotent, it generates a nilpotent ideal in $A$ that cannot be $\delta$-invariant (since $I = 0$). Hence there exists an element $a \in Z_0$ such that $\tau(aD(M(-1, \ldots, -1), M(-1, \ldots, -1))) \neq (0)$, which implies that $M = MR(aD(M(-1, \ldots, -1), M(-1, \ldots, -1))) \subseteq M(-1, \ldots, -1)A + (M(-1, \ldots, -1)A)a$.

On the other hand,

$$A = A(1, \ldots, 1) + \sum_k A(-1, \ldots, -1, \frac{1}{k}, -1, \ldots, -1).$$

Therefore, $M(\epsilon_1, \ldots, \epsilon_n) = (0)$, except if $(\epsilon_1, \ldots, \epsilon_n) = (-1, \ldots, -1)$ or $(1, \ldots, 1, -1, 1, \ldots, 1)$.

But

$$M(1, \ldots, 1, -1, 1, \ldots, 1)_{vj}$$

$$\leq M(-1, \ldots, -1, 1, \ldots, 1, -1, \ldots, -1, 1, \ldots, -1, -1, \ldots, -1) = (0)$$

if $n \neq 3$ and $i \neq j$.

For an arbitrary element

$$x \in M(1, \ldots, 1, -1, 1, \ldots, 1), \quad i \neq j,$$

we have $xU(v_i) = x$. On the other hand, if $n \neq 3$, we have

$$xU(v_j) = x\left(2R(v_j)^2 - Id\right) = -x.$$

Hence $x = 0$, and therefore, $M = M(-1, -1, \ldots, -1)$.

Since for an arbitrary element $a \in Z$,

$$M(-1, -1, \ldots, -1)(av_i) \subseteq M(1, \ldots, 1, -1, 1, \ldots, 1) = (0),$$

we deduce that

$$D(M(-1, -1, \ldots, -1), a)$$

$$= D(M(-1, -1, \ldots, -1), (av_i)v_j) \subseteq D(M(-1, \ldots, -1)(av_i), v_j)$$

$$+ D(M(-1, \ldots, -1)v_i, av_j) = (0).$$
That is, $D(M, a) = 0$. In particular, for arbitrary elements $x_1, x_2 \in M$, we have $[x_1, x_2] = [(x_1, a), x_2] = [x_1, ax_2]$.

Then $ZD(M, M) = (0)$, which contradicts our assumption that $I = (0)$.

Thus, we have proved that the only possible case is $\dim(V) = 3$. In this case,

$$A = A(1, 1, 1) + A(-1, 1, -1) + A(-1, 1, 1),$$

$$M = M(-1, 1, -1) + M(-1, 1, 1) + M(1, -1, 1) + M(1, 1, -1).$$

We have $A = Z + Zv_1 + Zv_2 + Zv_3$. There exists an element $a \in Z$ and an element $x \in M(-1, 1, -1)$ such that $aR(x^2)$ has nonzero trace. [Remember that $ZD(M(\alpha_1, \alpha_2, \alpha_3), M(\beta_1, \beta_2, \beta_3)) = (0)$ except if $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = (1, -1, -1)$, $\beta_1 = \beta_2 = \beta_3$.]

Let us assume $aR(x^2) = f$, where $f$ is an invertible element. Then $f = a + u$, $u$ nilpotent.

So $M = MR(aR(x^2)) = xA + (xA)a$. Let us denote $M' = M(-1, 1, 1, -1)$.

**Lemma 2.3.1.** \([M'(v_1Z), M'(v_1Z)] = (0)\).

**Proof.** The subspace \([M'(v_1Z), M'(v_1Z)]\) is $D(M', M')$-invariant. There are no nonzero $D(M', M')$-invariant subspaces of $Z$ with zero trace (since they would generate a $\mathcal{D}$-invariant nilpotent ideal of $A$). So it suffices to prove that the subspace \([M'(v_1Z), M'(v_1Z)]\) has zero trace.

We will use the Kaplansky argument.

Suppose that, for an even integer $r \geq 2$, we have \(\tau([M'R(v_1Z)^r, M']) \neq 0\). Choose elements $y, y_1 \in M'$ and an operator $\omega \in R(v_1Z)^r$ such that $[y_0, y_1, \alpha] = a + a$, $0 \neq \alpha \in F$, $a \in Z_0$. Applying $R(y_1)$, we get $y_1 = y_0 R(y_1)^2 (R(a_1 + a))^{-1}$.

Remark that $v_1 ZR(y_1)^2 \subset v_1 Z$ and $[R(v_1Z), R(a_1 + a)^{-1}] = 0$. Indeed, let $i \neq 1$. Then $D(v_1Z, Z) = D(v_1Z, (v_1Z)^2) \subset D((v_1Z) \cdot (v_1Z), v_1Z) = (0)$.

Hence $y_1 = y_1 \omega_1$, where $y_2 \in M'$ and $\omega_1 \in R(v_1Z)$. For an arbitrary operator $\omega_2 \in R(v_1Z)^{-2}$, we have $[y_0, y_2, \omega_2 R(v_1Z)^2] \subset [y_0 R(v_1Z)^2, y_2 \omega_1]$ since $(v_1Z)D(M'(v_1Z), M') = (0)$. We showed that \(\tau([M'R(v_1Z)^r, M']) \neq 0\) implies \(\tau([M'R(v_1Z)^r, M']') \neq 0\).

Let $z_1, \ldots, z_4 \in Z_0 \cup \{1\}$ and $M'R(v_1z_1) \cdots R(v_1z_4) \neq (0)$. If at least one $z_i = 1$, then $M'R(v_1z_1) \cdots R(v_1z_i) = (0)$ because $M'v_1 = (0)$. Hence all elements $z_i$ lie in $Z_0$. There exists $s \geq 1$ such that $MR(N)^s = (0)$ (see [J]). Hence $M'R(v_1Z)^s = (0)$, the contradiction. Lemma is proved.

**Lemma 2.3.2.** $M'R(v_1Z)R(v_1Z) = (0)$. 
Proof. Let us first prove that \([M'R(v_1Z)R(v_1Z), M] = (0)\). We know that \([M'R(v_1Z)R(v_1Z), M'] = [M'R(v_1Z), M'R(v_1Z)] = (0)\), by Lemma 2.3.1.

Let us show that \([M'R(v_1Z)R(v_1Z), (x(v_1Z))Z] = (0)\).

Since \(ZD(M', x(v_2Z)) = (0)\), it is sufficient to prove that
\[
\left[ M'R(v_1Z)R(v_1Z), x(v_2Z) \right] = (0).
\]
(Remember that \(R(Z)\) commutes with \(R(v_1Z)\).)

But \([M'R(v_1Z)R(v_1Z), x(v_2Z)] \subseteq v_2Z\), and using the Jordan identity, we get
\[
R(x(v_2Z))R(v_2) \\
\subseteq R((v_2Z)v_2)R(x) + R(xv_2)R(v_2Z) + R(x)R(v_2Z)R(v_2) \\
+ R(v_2)R(v_2Z)R(x) + R((v_2x)(v_2Z)) \\
= R(Z)R(x) + R(x)R(v_2Z)R(v_2) + R(v_2)R(v_2Z)R(x).
\]

Considering the three summands separately, we get
\[
[M'R(v_1Z)R(v_1Z), x] \subseteq [M'R(v_1Z)R(v_1Z), M'] = (0), \\
[M'R(v_1Z)R(v_1Z), x] = (0), \\
M'R(v_1Z)R(v_1Z)R(v_2) = (0).
\]

So we have proved that
\[
(M'R(v_1Z)R(v_1Z))R(x(v_2Z))R(v_2) = (0),
\]
which implies \([M'R(v_1Z)R(v_1Z), x(v_2Z)] = (0)\).

Let us show that \([M'R(v_1Z)R(v_1Z), x(v_1Z)] = (0)\). Comparing weights, we see that \([M'R(v_1Z)R(v_1Z), x(v_1Z)] \subseteq v_1Z\). By the Jordan identity,
\[
M'R(v_1Z)R(v_1Z)R(x(v_1Z))R(v_1) \\
\subseteq M'R(v_1Z)R(v_1Z)(R(xv_1)R(v_1Z)) \\
+ R(z)xR(v_1Z) + R(xv_1)R(v_1Z)R(v_1) + R(v_1)R(v_1Z)R(x) \\
+ R((xv_1)(v_1Z)) = (0).
\]

Now, taking into account that \(M = (xZ)Z + (x(v_1Z))Z + (x(v_2Z))Z + (x(v_1Z))Z\), we conclude that \([M'R(v_1Z)R(v_1Z), M] = (0)\).

Let us prove that \(M'R(v_1Z)R(v_1Z)R(A)R(M) = (0)\).

If the element in \(A\) lies in \(Z\), the assertion is clear.
Hence we only have to consider \( M'R(v,v)R(v,v)R(v,v)M \). But
\[
R(v,v)R(M) \subseteq R(M)R(A) + R(v,v)R(M) + R(Z)R(M).
\]
Now \( M'R(v,v)R(v,v)R(Z)R(M) = (0) \) and \( M'R(v,v)R(v,v)R(v,v) \subseteq M'R(v,v) = (0) \). This implies that for an arbitrary \( k \geq 1 \),
\[
M'R(v,v)R(v,v)R(A)^kR(M) = (0).
\]
Indeed, let \( k \geq 2 \). By the Jordan identity,
\[
R(A)^kR(M) \subseteq R(A)^{k-1}R(M) + R(A)^{k-2}R(M)R(A)^2.
\]
Now the \( A \)-submodule \( M'R(v,v)R(v,v)R(A) \) is an ideal of \( J \). Hence \( M'R(v,v)R(v,v) = (0) \). Lemma is proved.

**Lemma 2.3.3.** For every two elements \( z', z'' \in Z \), \( R_M(z')R_M(z'') = R_M(z'z'') \) and \( MD(A, Z) = (0) \).

**Proof.** We know that \( M = (xZ) + \sum_{i=0}^3(x(v_i))Z \). For arbitrary elements \( z', z'' \in Z \), we have
\[
0 = xR(z'v_1)R(z''v_1)
= x(-R(z''))R((z'v_1)v_1) - R(v_1)R(z''(z'v_1))
+ 2R(v_1)R(z')R(z'v_1) + R((v_1v_1z'))z''
= x(-R(z'')R(z') + R(z'z')).
\]
That is, we have proved that \( (xz'z') = x(z''z') \). In particular, \( D(Z, Z) = (0) \).

As we saw above, \( D(Z, v_1, Z) = (0) \), which implies \( MD(A, Z) = 0 \). Lemma is proved.

**Lemma 2.3.4.** Fix an element \( a \in Z \) such that \( aR(x)^2 \) is invertible. Then the mapping \( Z \to Zv_1, h \to [xh](av_1, x) \) is a bijection.

**Proof.** It is sufficient to prove that the kernel is equal to 0. We have
\[
x(aw_1)R(h)R(x)R(v_1)
= x(aw_1)(-R(v_1)R(h) - R(hv_1)x + R(x)R(hv_1))
+ R(v_1)R(hx) + R(h)R(v_1x))
= x(aw_1)R(x)R(hv_1) = [x(aw_1), x](hv_1) = a' h,
\]
where \( a' = aR(x)^2 \).
We have proved that \([(xh)(av_1)), x] = [(x(av_1))h, x] = a'hv_1.\]

By the hypothesis, the element \(a' = aR(x)^2\) is invertible. Hence \(a'hv_1 = 0\) implies that \(a'h = 0\) and \(h = 0\). Lemma is proved.

From the above lemma, it follows that there is an element \(h \in Z\) such that \([(xh)(av_1)), x] = v_1\) (it suffices to take \(h = a'^{-1}\)). Since \(h\) has nonzero constant term, \(h = g^2\) for some \(g \in Z\) and we have \(aR(xg)^2 = 1\). Let us substitute \(x\) by \(xg\).

From now on, without loss of generality, we will assume that \(a' = aR(x)^2 = 1\) and \([(x(av_1))h, x] = hv_1.\)

**Lemma 2.3.5.**

(1) \([(x(av_1))Z)R(v_1)Z) = (0).\]

(2) \(D((x(av_1))Z, Z) = (0).\]

**Proof.** Since \(MD(A, Z) = (0)\), we have

\[
((x(av_1))Z)(v_1)Z) = xR(av_1)R(v_1)ZR(Z) = (0)
\]

by Lemma 2.3.2. We have proved (1).

To prove (2), let us notice that \(D((x(av_1))Z, Z) = D((x(av_1))Z, (Zv_1)^2) = (0)\) by (1). Lemma is proved.

Our assumption \([(x(av_1))h, x] = hv_1\) and Lemma 2.3.5(2) imply that, for every \(g \in Z\), \([(x(av_1))h, xg] = [(x(av_1))h, x]g = (hv_1)g = hgv_1.\)

Similarly,

\[
x(hv_1) = ((x(av_1))h)R(x)^2
\]

\[
= (xR(x)^2)R(av_1)R(h) + x((av_1)R(x)^2)h
\]

\[
+ (x(av_1))(hR(x)^2).
\]

So, if we denote \(h' = hR(x)^2\), we have got \(x(hv_1) = (x(av_1))h'\).

The previous result together with the fact that \(M = xZ + \sum_i x(v_i)Z + \sum_i (x(v_i)Z)a\) implies that \(M = xZ + \sum_i (x(v_i)a)Z\).

We already know the product of any two elements of \(A\). Now we will derive the multiplication table for \(A \times M \to M. (xf)R(v_i g) = (x(v_i g))f = ((x(v_i a))g')f = (x(v_i g))f', (x(v_i a)Z)R(v_i g) = (0),\) by Lemma 2.3.2

Now we will determine \((x(v_i a))R(v_j g)\) for an arbitrary element \(g \in Z\) and \(i \neq j.\)

Again, from \(D(x(v_i a), g) = 0\), it follows that

\[
(x(v_i a))(v_j g) = ((x(v_i a))v_j)g.
\]
Let us show that $x$.

Clearly, $(x(v_1a)v_2 = (x(v_3a))f$ for some $f \in Z$. Applying $R(v_2)$ to both sides, we get $x(v_1a) = ((x(v_3a)v_2)f$. Now applying $D(v_3, v_1)$ to both sides, we have $-x(x(v_1a)) = ((x(v_1a)v_2)f = (x(v_3a)f)^2$. Hence $f^2 = -1$. Let us show that $f \in F$. If $f = \alpha + f_0$, where $\alpha \in F$, $f_0 \in \mathbb{Z}$, then $f^2 = \alpha^2 + 2\alpha f_0 + f_0^2 = -1$, which implies $2\alpha f_0 + f_0^2 = 0$. Therefore, $f_0 = 0$. The equation $x^2 + 1 = 0$ has two roots in $F$. Taking $-v_1$ instead of $v_1$, if necessary, we can choose either root. Thus, $(x(v_1a)v_2 = \sqrt{-1}x(v_3a)$.

Define the skew-symmetric cross product $\nu \times \nu \rightarrow F$ via $v_1 \times v_2 = v_3$, $v_1 \times v_3 = -v_2, v_2 \times v_3 = v_1$.

We have got the following multiplication table ($T_1$):

<table>
<thead>
<tr>
<th>$xf$</th>
<th>$g$</th>
<th>$v_i g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x(v_1a)f$</td>
<td>$(x(v_2a)f$</td>
<td>$(x(v_1a)(fg)$</td>
</tr>
</tbody>
</table>

It remains to determine the bracket $[\ , ] : M \times M \rightarrow A$.

We already know that $[(x(v_1a))h, xg] = hgv_i$.

Let us show that $[(x(av_1))f, (x(av_1))g] = 0$. Indeed, by Lemma 2.3.5(2), $[(x(av_1))f, (x(av_1))g] = [x(av_1), x(av_1)]fg$.

It is clear that $[x(av_1), x(av_1)] = 0$. Assume $i \neq j$. In this case,

$$[x(av_i), x(av_j)] = (av_i)D(x, x(av_j)) - [(x(ax_1))((av_1))x].$$

But $(av_i)D(x, x(av_j)) = (aD(x, x(ax_j))v_i + a(v_iD(x, x(ax_j)))$, $aD(x, x(ax_i)) = 0$, and $v_iD(x, x(ax_i)) = [(x(ax_i))v_i, x] = \sqrt{-1}[x((v_j \times v_i)a, x] = \sqrt{-1}(v_j \times v_i).$ So $[x(ax_i), x(ax_j)] = \sqrt{-1}(v_j \times v_i)a - \sqrt{-1}[x((v_j \times v_i)a, x] = 0.$

Finally,

$$[xf, xg] = xf( (gv_1)v_1)x$$

$$= xf(-R(gv_1)R(x)R(v_1) - R(v_1)R(x)R(gv_1)$$

$$+ R(gv_1)R(xv_1) + R(v_1)R(x gv_1) + R(x)R((gv_1)v_1))$$

$$= -[(xf)(gv_1), x]v_1 + [xf, x]g = -af + f'g.$$
**Remark.** Substituting $w_1 = v_1, w_2 = v_2, w_3 = \sqrt{-1}v_3$, we can rid our tables of $\sqrt{-1}$. We still have $A = Z + \sum_i Zw_i, M = xZ + \sum_i (x(w_i)a)Z$.

Let $w_1 \times w_2 = w_3, w_1 \times w_3 = w_2, w_2 \times w_3 = -w_1$.

The new multiplication tables are:

$$(T_1)$$

<table>
<thead>
<tr>
<th>$x_1 f$</th>
<th>$x_2 f$</th>
<th>$x_3 f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x(w_1)a)f$</td>
<td>$(x(w_2)a)f$</td>
<td>$(x(w_3)a)f$</td>
</tr>
</tbody>
</table>

$$(T_2)$$

<table>
<thead>
<tr>
<th>$xg$</th>
<th>$(x(w_1)a)g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1(fg)$</td>
<td>$w_2(fg)$</td>
</tr>
</tbody>
</table>

Now for an arbitrary unital associative commutative algebra $Z$ with a derivation $d: Z \to Z$, we will construct a Cheng–Kac Jordan superalgebra $CK(Z, d)$.

Let $A = Z + \sum_i Zw_i, M = xZ + \sum_i x_i Z$ be two free $Z$-modules of rank 4. The multiplication on $A$ is $Z$-linear, $w_i w_j = 0$ for $i \neq j, w_1^2 = w_2^2 = 1, w_3^2 = -1$.

Denote $x_{i\times i} = 0, x_{1\times 2} = -x_{2\times 1} = x_3, x_{1\times 3} = -x_{3\times 1} = x_2, -x_{2\times 3} = x_{3\times 2} = x_1$.

The bimodule structure $A \times M \to M$ is defined via

$$(T'_1)$$

<table>
<thead>
<tr>
<th>$x_1 f$</th>
<th>$x_2 f$</th>
<th>$x_3 f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(fg)$</td>
<td>$x(fg^d)$</td>
<td>$x_1 f x_2 f$</td>
</tr>
</tbody>
</table>

The bracket on $M$ is defined via

$$(T'_2)$$

<table>
<thead>
<tr>
<th>$xg$</th>
<th>$x_1 g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f^d g - f g^d$</td>
<td>$-w_1(fg)$</td>
</tr>
</tbody>
</table>

Now our aim is to prove that this multiplication defines a Jordan superalgebra.

**Proposition 2.3.1.** $CK(Z, d)$ is a Jordan superalgebra.
Proof. Consider the basis of $\text{CK}(Z, d)$ over $Z$: $e_1 = 1$, $e_2 = w_1$, $e_3 = w_2$, $e_4 = w_3$, $e_5 = x$, $e_6 = x_1$, $e_7 = x_2$, $e_8 = x_3$.

According to the multiplication tables $(T_1^2)$ and $(T_2^2)$, for arbitrary indices $1 \leq i, j \leq 8$, there exist integers $a_{ij}$, $b_{ij}$, $c_{ij}$ (in fact, equal to $\pm 1$ or $0$) and an index $1 \leq k \leq 8$ such that $(fe_{ij}) (ge_{ij}) = (a_{ij}fg + b_{ij}f^d g + c_{ij}fg^d)e_k$ for arbitrary $f, g \in Z$.

If we apply the Jordan identity to four elements $f_1e_{i_1}, f_2e_{i_2}, f_3e_{i_3}, f_4e_{i_4}$, $f_1, f_2, f_3, f_4 \in Z$, we get $J(f_1e_{i_1}, f_2e_{i_2}, f_3e_{i_3}, f_4e_{i_4}) = \sum S^i_{k(i_1, i_2, i_3, i_4)}e_k$, where $S^i_{k(i_1, i_2, i_3)}(f_1, f_2, f_3, f_4) = \sum \alpha_i f^i_1 f^i_2 f^i_3 f^i_4; \alpha_i$ are integers and $f^i_k \in \{f_k, f_k^d, f_k^{idd}\}$.

It was proved in [KMZ] that for $Z = \mathbb{C}[t]$, where $\mathbb{C}$ is the field of complex numbers, $d = \frac{1}{\pi t}$, the $\text{CK}(Z, d)$ is a Jordan superalgebra. This implies that each expression $S^i_{k(i_1, i_2, i_3)}(f_1, f_2, f_3, f_4)$ is identically zero. The proposition is proved.

Lemma 2.3.6. $\text{CK}(Z, d)$ is simple if and only if $Z$ does not contain proper $d$-invariant ideals.

Proof. If $I$ is a $d$-invariant ideal of $Z$, then $I + \sum_{i=1}^3 w_i I + x I + \sum_{i=1}^3 x_i I + x I$ is an ideal of $\text{CK}(Z, d)$.

Conversely, let us assume that $B$ is an ideal of $\text{CK}(Z, d)$. If $B \cap Z \neq (0)$, then $B \cap Z$ is a $d$-invariant ideal of $Z$. Hence $Z \subseteq B$, and since $1 \in Z$, it follows that $B = \text{CK}(Z, d)$.

Suppose that $B \cap Z = (0)$. Since every nonzero ideal of $A$ has a nonzero intersection with $Z$, it follows that $B \cap A = (0)$.

This implies that $[B, M] = (0)$.

Consider an element $b = xz_0 + \sum_{i=1}^3 x_i z_i$ from $B$. We have $[b, x_1] = -w_1 z_0 \in B \cap A$. Hence $z_0 = 0$. Now $[b, x] = \sum_{i=1}^3 w_i z_i \in B \cap A$, which implies that $z_i = 0$, $1 \leq i \leq 3$. Lemma is proved.

Remark. If $Z$ is an algebra of truncated polynomials in one variable, then there exists only one, up to isomorphism, simple Jordan superalgebra of the type $\text{CK}(Z, d)$.

3. THE CASE $I \neq (0)$

In this section we will consider the case when $I \neq (0)$.

As we have seen in Section 1, we have two possibilities:

(a) $A/I \simeq \overline{B}$ a simple Jordan algebra of a symmetric bilinear form, or

(b) $A/I \simeq \overline{B} \otimes B(m)$, with $\overline{B}$ as in (a) and $B(m)$ the algebra of truncated polynomials in $m$ variables.
The proof follows the general lines of the main chapter in [KMZ].

We will divide this section into two subsections. In the first subsection we will prove the existence of some element that generates $M$ as an $A$-bimodule and some general results needed in what follows. In the second subsection we will determine the structure of $M$.

### 3.1. Structure of $M$

By the theorem of E. Taft (see [Ta], [J]), in both Cases (a) and (b) there exists a subalgebra $B$ of $A$ with $1 \in B$, such that $B \cap I = (0)$ and $B = \overline{B}$. Let $B = F^1 + V$, $V \cdot V \subseteq F^1$, and choose a basis $v_1, \ldots, v_n \in V$ such that $v_iv_j = \delta_{ij}$, $n \geq 2$.

Since $v_i^2 = 1$, it follows that $U(v_i)$ is an involutive automorphism of the superalgebra $J$. Let $G$ be the group generated by $U(v_1), \ldots, U(v_n)$. Both $A$ and $M$ decompose into a direct sum of eigenspaces with respect to $G$, $J = \bigoplus J(\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_i = \pm 1$, and $J(\epsilon_1, \ldots, \epsilon_n) = A(\epsilon_1, \ldots, \epsilon_n) + M(\epsilon_1, \ldots, \epsilon_n)$.

As we have mentioned in Section 2, an element $a \in J$ belongs to the eigenvalue $-1$ with respect to $U(v_i)$ if and only if $av_i = 0$, and the element $a$ belongs to the eigenvalue $1$ if and only if $D(a, v_i) = R(a)R(v_i) - R(v_i)R(a) = 0$.

Denote $\tilde{Z} = A(1, 1, 1, \ldots, 1)$. Then $\tilde{Z} + I$ is the preimage of the center of $A/I$ under the canonical epimorphism $A \to A/I$.

In the following lemma, we will prove the existence of an element $x \in M(-1, -1, \ldots, -1)$ that generates $M$ as an $A$-bimodule in Case (b). Proving a similar result in Case (a) will take a bit more effort.

**Lemma 3.1.1.** If $A/I = \overline{B} \otimes B(m)$, then there is an element $x$ which generates $M$ as an $A$-bimodule and $x \in M(-1, 1, \ldots, -1)$.

**Proof.** Let $B(m)_0$ be the maximal (nilpotent) ideal of $B(m)$ and let $N$ be the nilpotent radical of $A$. Since $N/I = \overline{B} \otimes B(m)_0$ does not contain $D(M, M)$-invariant ideals and $B(m)D(M, M) \subseteq B(m)$, it follows that there exists an element $\overline{a} \in B(m)_0$ and an element $x \in M$ such that $\overline{a}R(x)^2$ is a polynomial with nonzero constant term. Let $a$ be a preimage of $\overline{a}$ under $A \to A/I$. The element $aR(x)^2$ can be represented as $a1 + b$ with the element $b$ nilpotent and $0 \neq a \in F$. Hence, $M = MR(aR(x)^2) \subseteq MR(x^2)R(a) + (Ma)R(x)^2 \subseteq (xA)a + xA$.

Let $x = \sum x(\epsilon_1, \ldots, \epsilon_n)$, where $x(\epsilon_1, \ldots, \epsilon_n) \in M(\epsilon_1, \ldots, \epsilon_n)$.

Arguing as in Section 2, we see that $B(m)D(M, m, \ldots, m)(\beta_1, \ldots, \beta_n) = (0)$ unless $\alpha_i = \beta_i = -1$, $1 \leq i \leq n$. Hence, $aR(x)^2 = aR(x(-1, -1, \ldots, -1))^2 \equiv (xA)a + xA$ mod $I$, and we can assume without loss of generality that $x \in M(-1, -1, \ldots, -1)$. Lemma is proved.
We will denote as $V$ the preimage of $\bar{V} \otimes B(m)$ (resp. $\bar{V}$) under $A \rightarrow A/I$ in Case (b) (resp. Case (a)). As we have done in previous sections, we will denote $M_I = (MI)A$. Clearly, $M_I$ is an $A$-sub-bimodule of $M$.

Let $\bar{Z}$ denote the center of $\bar{A}$; that is, $\bar{Z} = F$ in Case (a) and $\bar{Z} = B(m)$ in Case (b).

Let $t_Z: A \rightarrow \bar{Z}$ be the projection of $A$ onto $\bar{Z}$ and let $t: A \rightarrow F$ be the composition of $t_Z$ with the projection $\bar{Z} \rightarrow F$.

Notice that it follows from the proof of the previous lemma that, for the element $x$, we have $t(\bar{Z}R(x^2)) \neq (0)$.

It was proved in Section 1 that:

1. $[MI, M] \subseteq I$.
2. $t([MI, M]) = (0)$.
3. $[M_I, M_I] \subseteq I$.

It is not necessarily true that $[M_I, M] \subseteq I$.

**Lemma 3.1.2.** $[(MI)\bar{Z}, M] \subseteq I$.

**Proof.** Since $[(MI)\bar{Z}, M] + I$ is $D$-invariant, it suffices to prove that $[(MI)\bar{Z}, M] \subseteq N$. Let us assume that there exist elements $x, y \in M; u \in I; c \in \bar{Z}$, such that $[(xu)c, y] = \lambda 1 + v \mod N$, $\lambda 1 + v \neq 0$, $\lambda \in F$, $v \in V$.

Since $t([(xu)c, y]) = 0$, it follows that $\lambda = 0$. Let $v'$ be a vector from $V$ such that $v \cdot v' = 1$. Then $t([(xu)c, y]v') \neq 0$. Without loss of generality, we can assume that $c \in N$. By the Jordan identity,

$$[(xu)c, y]v' = (xu)R(c)R(y)R(v') = (xu)(-R(v')R(y)R(c) - R((cv')y) + R(cy)R(v') + R(v'y)R(c) + R(v'c)R(y)),$$

and each summand on the right-hand side has zero trace, which gives a contradiction. Lemma is proved.

Now, let us strengthen the assertion (2).

**Lemma 3.1.3.** $[M_I, M] \subseteq I^\Delta$.

**Proof.** Notice that the result follows immediately from (2) above in Case (a). So we will consider Case (b). Suppose that the assertion of the lemma is wrong and there exists $a \in [M_I, M]$ with $\bar{a} = \bar{z} + \bar{v}$; $0 \neq \bar{z} \in \bar{Z}$, $\bar{v} \in \bar{V} \otimes \bar{Z}$. Then $t(\bar{z}) = 0$. Since $\bar{Z}$ is differentially simple with respect to $D = D(M, M)$, there exist derivations $d_1, \ldots, d_\iota \in D$ such that $t(\bar{z}d_1 \cdots d_\iota) \neq 0$. The subspaces $\bar{Z}$ and $\bar{V} \otimes \bar{Z} = \bar{A}D(\bar{A}, \bar{A})$ are $D$-invariant. Applying $d_1 \cdots d_\iota$ to both sides of $\bar{a} = \bar{z} + \bar{v}$, we get a contradiction.
**Lemma 3.1.4.** Let \( y' \in MI; b \in A; y'' \in M; z \in \tilde{Z}. \) Then

(a) \( y'R(b)R(y''z) = y'R(b)R(y'')R(z) \mod I. \)

(b) \( y'R(v; z)R(y'') = y'R(v; z)R(y'')R(z) \mod I. \)

**Proof.** It follows immediately from the Jordan identity, the inclusion \([MI, M] \subseteq I,\) and Lemma 3.1.2.

**Lemma 3.1.5.** \( V^\delta = [M_I, M] + I. \)

**Proof.** We already know that \([M_I, M] + I \subseteq V^\delta.\) Let us prove the other inclusion. Since \( I \neq (0), \) it follows that \([M_I, M] \not\subseteq I;\) otherwise, \( I + M_I \) would be an ideal of \( J, \) contradicting simplicity of \( J. \) By Lemma 3.1.4, \([M_I, M] + I/I \) is a nonzero \( \tilde{Z}\)-submodule of \( V^\delta/I, \) which is irreducible \( D(A, A) + D(M, M). \) But \( V^\delta/I \) is an irreducible \( D(A, A) + D(M, M)\)-module. Hence \( V^\delta = [M_I, M] + I. \) Lemma is proved.

**Lemma 3.1.6.** \( [(MV^\delta)I]A, M] \subseteq I. \)

**Proof.** According to Lemmas 3.1.2 and 3.1.5, it is sufficient to prove that \([(M[M_I, M])I]A, M] \subseteq I,\) or equivalently, that

\[
\left[\left(\left(M[M_I V^\delta, M]\right)I\right)V^\delta, M\right] \subseteq I.
\]

Choose arbitrary elements \( x, y, x', y' \in M; u, u' \in I; a, b \in V^\delta.\) We have to check that \( uR(x)R(a)R(y')R(x')R(u')R(b)R(y') \in I.\)

Applying the Jordan identity to the underlined part, we have

\[
R(y) R(x') R(u') = R(u') R(x') R(y) - R([yu', x']) + R(y) R(x'u') - R(x') R(yu') + R(u') R([y, x']).
\]

Let us analyze each summand on the right-hand side separately.

(i) \( uR(x)R(a)R(u')R(y')R(b) = uR(x)R(a)R(u')R(b)R(y')R(x') - R([x'b, y]) + R([x', y])R(b) + R(x'b)R(y') - R(yb)R(x') \in [(M_I)I, M]R(M)R(M) + [(M_I)I]R(A, M] \subseteq I.\)

(ii) \( uR(x)R(a)R([yu', x])R(b)R(y') \in [(M_I, I)A, M] \subseteq I.\)

We can do similarly with other summands. Lemma is proved.

**Remark.** Notice that the inclusion of the lemma is equivalent to \([M_I, MV^\delta] \subseteq I.\)

Now, our aim is to prove the existence of an element \( x \in M(-1, \ldots, -1)\) that generates \( M \) as an \( A\)-bimodule in Case (a). So, in the rest of this
subsection we will assume that $A/I$ is a simple Jordan algebra of a bilinear form.

**Lemma 3.1.7.** If $x, y \in M; \ u \in I; \ v \in V^\sharp$ are elements such that $[(xu)v, y] \notin I$, then $M$ is generated as an $A$-bimodule by the elements $x, y$.

**Proof.** If $D \in D(V^A, V^\sharp)$, then by Lemma 3.1.5,

\[
[(xu)v, y]\ D = [((xD)u)v, y] + [(x(uD))v, y] + [(xu)(vD), y] + [(xu)v, yD] = [(x(uD))v, y] + [(xu)(vD), y] \mod I.
\]

Hence, $[(xI)v^A, y] + I/I$ is a $\text{Der}(A/I)$-invariant submodule of $V^A/I$.

Hence $V^A = [(xI)v^A, y] + I = V^\sharp D(xI, y) + I$.

Since $MD(xI, y) \subseteq [M, xI]y + [M, y](xI) \subseteq xR(A) + yR(A)$, it follows that $M(V^A D(xI, y)) \subseteq (MV^A)D(xI, y) + (MD(xI, y)V^\sharp \subseteq xR(A) + yR(A)$.

For an arbitrary $v \in V^\sharp$, there exists $u(v) \in I$ such that $v - u(v) \in V^A D(xI, y)$, and consequently, $M(v - u(v)) \subseteq xR(A) + yR(A)$.

There is $v'$ such that $vv' = 1$. Hence, for an arbitrary element $m \in M$,

\[
m = 1m = v'R(v)R(m) \in v'R\left([(xI)v^A, y]\right)R(m) + v'R(I)R(m)
\]

\[
\subseteq v'R(m)R(A) + v'D\left([(xI)v^A, y], m\right) + mI.
\]

But

\[
D\left([(xI)v^A, y], m\right) \subseteq D\left([(xI)v^A, [y, m]\right) + D\left(y, [(xI)v^A, m]\right) \subseteq D(xR(A), A) + D(y, A).
\]

So, $AD([xI)v^A, y], m) \subseteq xR(A)^+ + yR(A)$.

We have proved that $m = (mu')R(a) + mu' \mod xR(A) + yR(A)$ for some elements $a \in A, u' \in I$. But $mu' = mu(v') \mod xR(A) + yR(A)$ and the operator $R(u(v'))R(a) + R(u')$ is nilpotent. So

\[
m(1d - R(u(v'))R(a) + R(u')) \in xR(A)^+ + yR(A).
\]

We have proved that $M = xR(A) + yR(A)$.

**Lemma 3.1.8.** If $x, y \in M$ and $[(xI)v^A, y] \notin I$, then $[(xI)v^A, x] \notin I$.

**Proof.** As we have seen in the proof of Lemma 3.1.7, $V^A = [(xI)v^A, y] + I$. Hence, there are elements $w \in V$ and $v_i \in V$ such that $w^2 = 1$ and $\sum_i[(xu_i)v_i, y] = w \mod I$. 
Hence, \( \sum_i [(xu_i) v_i, y] R(w) = 1 \mod I \). By the Jordan identity,

\[
[(xu_i) v_i, y] R(w) = (xu_i) R(v_i) R(y) R(w) \\
= (xu_i) (-R(w) R(y) R(v_i) - R(y(v_i w)) \\
+ R(v_i) R(yw) + R(w) R(yv_i) + R(y) R(wv_i)) \\
= -[(xu_i) w, y] v_i \mod I.
\]

Hence, there exists \( i_0 \) such that \( [(xu_{i_0}) w, y] \not\in I \).

Now,

\[
[(xu_{i_0}) w, y] = \sum_i u_i R(x) R(v_i) R(y) R(xu_{i_0}) R(y) \\
= \sum_i u_i R(x) R(v_i) (R(x) R(yu_{i_0}) - R(u_{i_0}) R([y, x])) \\
+ R(y) R(u_{i_0}) R(x) - R(x) R(u_{i_0}) R(y) \\
+ R(u_{i_0}) [x, y]) R(y) \\
= \sum_i u_i R(x) R(v_i) R(x) R(yu_{i_0}) R(y) \mod I.
\]

This implies that \( [(xl)V^4, x] \not\in I \). Lemma is proved.

**Lemma 3.1.9.** There is an element \( x \in M(-1, -1, \ldots, -1) \) such that \( M = x R(A) \).

**Proof.** If \( x \in M(\alpha_1, \ldots, \alpha_n) \) and \( \alpha_i = 1 \), then \( x = (xv_i) v_i \in M V^4 \).

Hence, for elements \( x \in M(\alpha_1, \ldots, \alpha_n) \), \( y \in M(\beta_1, \ldots, \beta_n) \), we have \( [(xl)V^4, y] \subset I \) unless \( \alpha_i = \beta_i = -1, 1 \leq i \leq n \).

We proved that there exist elements \( x, y \in M \) such that \( [(xl)V^4, y] \not\in I \).

If \( x = \sum x(\alpha_1, \ldots, \alpha_n), y = \sum y(\alpha_1, \ldots, \alpha_n) \) are root decompositions of elements \( x, y \), then \( [(x(-1, \ldots, -1)V^4, y(-1, \ldots, -1)] \not\in I \), and therefore, \( [(x(-1, \ldots, -1)V^4, x(-1, \ldots, -1)] \not\in I \) by Lemma 3.1.8. This implies that \( M = x(-1, \ldots, -1) R(A) \) by Lemma 3.1.7. Lemma is proved.

### 3.2. Structure of \( J \)

Consider the Clifford algebra \( Cl \) on \( V \), that is, the \( F \)-algebra generated by \( v_1, \ldots, v_n \) with relations \( v_i v_j + v_j v_i = 2 \delta_{ij} \).

We say that a root element \( a \in I \) has weight \( v_{i_1} \cdots v_{i_r} \in Cl \), \( 1 \leq i_1 < \cdots < i_r \leq n \) if, for any \( 1 \leq j \leq n \), the equality \( a(U(v_{i_1}, v_{i_1}^{-1}) - \epsilon)^k = 0 \), \( \epsilon = \pm 1 \), holds for some \( k \) if and only if \( (v_{i_1} \cdots v_{i_r}) (U(v_{i_1}, v_{i_1}^{-1}) - \epsilon) = 0 \).
If \( a \in I \) is a root element and \( t_\pi(aR(x)R(v_{i_1})R(x) \cdots R(x)R(v_{i_{2r+1}})) \neq 0 \) [resp. \( t_\pi(aR(v_{i_1})R(x)R(v_{i_2})R(x) \cdots R(x)R(v_{i_{2r+1}})) \neq 0 \)], then \( a \) has weight \( v_{i_1} \cdots v_{i_{2r+1}} \) (resp. \( v_{i_1} \cdots v_{i_{2r+1}} \)).

Let \( \mathcal{W} \) be the set of all products of \( R(v_i) \)'s and \( R(x) \) which do not contain subproducts \( R(v_i)R(v_j) \) or \( R(x)^2 \).

**Lemma 3.2.1.** For an arbitrary nonzero element \( a \) from \( I \cup M_1 \), there exists an operator \( w \in \mathcal{W} \) such that \( aw \in A \setminus I \).

**Proof.** Let \( 0 \neq a \in I \cup M_1 \). Let \( r \) be the minimal integer such that there exist elements \( y_{j_1}, \ldots, y_{j_{r}} \in A \cup M \) such that \( aR(y_{j_1}) \cdots R(y_{j_r}) \in A \setminus I \). (The existence of such \( r \) follows from simplicity of \( J \).

It is easy to see that \( y_{j_r} \in M \) and there are no two consecutive multiplications of the same parity. Moreover, the expression \( aR(y_{j_1}) \cdots R(y_{j_r}) + I/I \in A/I \) is skew-symmetric in even \( y_{j_i}'s \) and symmetric in odd \( y_{j_i}'s \).

We have \( M = xR(\tilde{Z}) + M_1 + MV^A \). In view of Lemma 3.1.5 and the fact that \( [M_1, M_1] \subseteq I \), we can assume, without loss of generality, that \( y_i = xR(z_{k_i}) \cdots R(z_{k_n}) \), \( z_{k_i} \in \tilde{Z} \). By Lemma 3.1.6(a), we have \( aR(y_{j_1}) \cdots R(y_{j_{r-1}})R(x) \in A \setminus I \). Hence, we can assume that \( y_{j_r} = x \) and that all odd \( y_{j_i}'s \) are equal to \( x \).

Similarly, by Lemma 3.1.6(b), all even \( y_{j_i}'s \) can be assumed to lie in \( \{v_{i_1}, \ldots, v_{i_r}\} \). Lemma is proved.

**Corollary 3.2.1.** If \( n \) is even, then \( I(1, 1, \ldots, 1) = (0) \).

**Proof.** Suppose that \( 0 \neq a \in I(1, 1, \ldots, 1) \) and choose a shortest operator \( w \in \mathcal{W} \) such that \( aw \in A \setminus I \). It is easy to see that

\[
    w = R(v_{i_1})R(v_{i_2}) \cdots R(v_{i_{k_r}})R(x).
\]

Indeed, if \( w = R(x)R(v_{i_1}) \cdots \), then \( aR(x)R(v_{i_1}) = aR(xv_{i_1}) = 0 \). The integer \( t \) is even. Since \( aw \) has weight \( v_{i_1} \cdots v_{i_t} \) and \( n \) is even, we have got a contradiction.

Denote \( D_i = D(x, v_{i}) \).

**Lemma 3.2.2.** Let \( a \in I \cup M_1 \) be a nonzero root element. Then there are indices \( 1 \leq i_1 < \cdots < i_s \leq n \) such that \( aD_{i_1} \cdots D_{i_s} \in A \setminus I \).

**Proof.** By the previous lemma, there exists an operator \( w \in \mathcal{W} \) such that \( aw \in A \setminus I \). Let us consider all possible cases.

**Case 1.** \( a \in I, w = R(x)R(v_{i_1})R(x) \cdots R(v_{i_{2r+1}})R(x) \). Then \( aw \in [M_1, M] \subseteq V^A \). Since \( aw \) is a root element, there exist an index \( 1 \leq i \leq n \) and an element \( z \in \tilde{Z} \) such that \( aw = zv_j \mod I, \tilde{z} \neq 0 \). Now \( awR(v_j) \in A \setminus I \) and \( awR(v_j) = aD_{i_1} \cdots D_{i_2r+1}D_j \).
Case 2. If \( a \in I, w = R(v_{i_1})R(x) \cdots R(v_{i_r})R(x) \). Then \( aw = aD_{i_1} \cdots D_{i_r} \).

Case 3. If \( a \in M_I, w = R(x)R(v_{i_1}) \cdots R(v_{i_r})R(x) \). Again there exists an index \( 1 \leq j \leq n \) and an element \( z \in \hat{Z} \) such that \( aw = zw \mod I \). Then \( awR(v_{i_j}) = aD_{i_1} \cdots D_{i_r}D_j \in A \setminus I \).

Case 4. If \( a \in M_I, w = R(v_{i_1})R(x) \cdots R(v_{i_{r+1}})R(x) \). Then

\[
aw = -aD_{i_1} \cdots D_{i_{r+1}}.
\]

Lemma is proved.

**Lemma 3.2.3.** Let \( y' \in M_I, y \in M, z \in \hat{Z} \). Then

(a) \( y'R(y)R(v_{i_1}) = y'R(y)R(v_{i_1})R(z) \mod [M_I, M] + I \).

(b) \( y'R(y)R(v_{i_1}) = y'R(y)R(v_{i_1})R(z) \mod [M_I, M] + I \).

**Proof.**

(a) \( y'R(y)R(v_{i_1}) = y'(-R(v_{i_1})R(yz) - R(z)R(yv_{i_1}) + R(y)R(v_{i_1})R(z) + R(z)R(v_{i_1})R(y) + R((yz)v_{i_1})). \)

(b) \( y'R(yz)R(v_{i_1}) = y'(-R(yv_{i_1})R(z) - R(zv_{i_1})R(y) + R(y)R(v_{i_1})R(z) + R(z)R(v_{i_1})R(y) + R((yz)v_{i_1})). \)

Now we only need to use the fact that \( y'R(yv_{i_1}) \in [M_I, M\hat{N}] \subseteq I \).

**Lemma 3.2.4.** For arbitrary indices \( 1 \leq i \neq j \leq n \), there exists an element \( a_{ij} \in I \) such that \( t(a_{ij}D_jD_i) = 1 \).

**Proof.** We have seen in Section 1 that \( I \neq (0) \) implies

\[
t(R(M)R(A)R(M)R(A)) = (0).
\]

Hence there exists elements \( b \in I; y_1, y_2 \in M; a_1, a_2 \in A \), such that \( t(bR(y_1)R(a_1)R(y_2)R(a_2)) \neq 0 \).

It is easy to see that the expression \( t(\cdots) \) is skew-symmetric in \( a_1, a_2 \) and symmetric in \( y_1, y_2 \).

By Lemma 3.2.3(a), we can assume that \( a_1, a_2 \in \{v_{i_1}, \ldots, v_n\} \); and by Lemma 3.2.3(b) we can assume that \( y_1 = y_2 = x \).

The element \( b \) can be assumed to be a root element. If \( a_1 = v_k, a_2 = v_i, \) then \( bR(x)R(v_k)R(x)R(v_i) = bD_kD_i \).

If \( \{i, j\} \cap \{k, l\} = \emptyset \), then \( t(bD(v_k, v_i)D(v_j, v_l)D_kD_j) \neq 0 \). If, say \( i = k \), then \( t(bD(v_k, v_i)D_jD_l) \neq 0 \). Lemma is proved.

**Remark.** Let \( A(1) \) denote the subalgebra of all elements of weight 1. Then the restriction of the trace to \( A(1) \) is a homomorphism.

**Lemma 3.2.5.** Let \( n \geq 6 \). Let \( 1 \leq i_1, \ldots, i_{2k} \leq n \) be distinct integers. Let

\[
b = a_{i_{12}}R(a_{i_{34}}) \cdots R(a_{i_{2k-1, 2k}}).
\]

Then \( t(bD_{i_{12}}D_{i_{34}} \cdots D_{i_{2k-1, 2k}}) = 1 \).
Proof. Without loss of generality, we can assume \( k \geq 2 \). Denote \( b' = a_{i_1i_2}R(a_{i_3i_4}) \cdots R(a_{i_{2k-1}i_{2k-2}}) \). We have
\[
bD_{i_1} \cdots D_{i_{2k}} = \sum_{\mu + \nu = 2k} \pm (b'D_{j_1} \cdots D_{j_\nu})(a_{i_{2k-\nu}i_{2k}}D_{i_1} \cdots D_{i_\mu}).
\]

Suppose that \( \nu \) is even. The element \( a_{i_{2k-\nu}i_{2k}}D_{i_1} \cdots D_{i_\mu} \) has the weight \( v_{i_{2k-\nu}}v_{i_{2k}} \cdots v_{i_\nu} \). If \( n \) is even, then \( a_{i_{2k-\nu}i_{2k}}D_{i_1} \cdots D_{i_\mu} \in I \) unless \( \nu = 2 \), \( \{t_1, t_2\} = \{i_{2k-1}, i_{2k}\} \). Now we apply the induction assumption and the Remark above.

If \( n \) is odd, then \( n \geq 7 \). In this case, either again \( \nu = 2 \), \( \{t_1, t_2\} = \{i_{2k-1}, i_{2k}\} \), or \( \nu = n - 3 \) and \( \{i_{2k-1}, i_{2k}\} \cap \{t_1, \ldots, t_\nu\} = \emptyset \). In the latter case, \( \mu = 2 \) or \( \mu = 0 \).

If \( \mu = 0 \), then the summand clearly lies in \( I \).

Let \( \mu = 2 \). Then \( 2k = \mu + \nu = n - 1 \geq 6 \); hence, \( k \geq 3 \). In this case, \( b'D_{j_1}D_{j_2} \in I \).

Now suppose that \( \nu \) is odd. Then \( a_{i_{2k-\nu}i_{2k}}D_{i_1} \cdots D_{i_\mu} \in M_I \) unless \( \{t_1, \ldots, t_\nu\} = \{1, \ldots, n\} \setminus \{i_{2k-1}, i_{2k}\} \). In this case, \( \mu = 1 \) and \( b'D_{j_1} \in M_I \). In any case, the summand has zero trace. Lemma is proved.

**Lemma 3.2.6.** Let \( z \in \tilde{Z} \), and \( b \) is a root element from \( I \cup M_I \). Then
\[
t_\mathcal{Z}( (bz)D_{i_1} \cdots D_{i_k} ) = t_\mathcal{Z}( bD_{i_1} \cdots D_{i_k} ) z.
\]

**Proof.** We have \( (bz)D_{i_1} \cdots D_{i_k} = \sum (bD_{i_1} \cdots D_{i_k})(zD_{i_1} \cdots D_{i_k}) \).

If \( \nu \neq 0 \), then the element \( zD_{i_1} \cdots D_{i_k} \) lies in \( I \cup M_I \) unless \( \nu \geq n - 1 \), in which case \( \mu = 0 \) or \( \mu = 1 \). In both cases, \( b \) and \( bD_{i_1} \) lie in \( I \cup M_I \). Lemma is proved.

**Lemma 3.2.7.** Let \( n \neq 5 \). Then for arbitrary indices \( 1 \leq i \neq j \leq n \), there is a root element \( a_{i_j} \in I \) such that \( t(a_{i_i}D_{i_j}) = 1 \), and for any subset \( \{i_1 < \cdots < i_{2\nu}\} \neq \{i, j\} \), we have \( a_{i_{i_1}}D_{i_{i_2}} \cdots D_{i_{i_{2\nu}}} \in I \).

**Proof.** If \( n \) is even, then the old \( a'_{i_j}s \) of Lemma 3.2.4 will do because of weight considerations.

Let \( n \) be odd. For \( n = 3 \), let us show that \( a_{i_1}D_{i_1}D_{i_3} \in I \). It is easy to see that \( a_{i_1}D_{i_1}D_{i_3} = -a_{i_1}R(x)R(v_1)R(v_3)R(x) \). By the Jordan identity, \( R(x)R(v_1)R(v_3) = -R(v_3)R(v_1)R(x) \).

Hence
\[
a_{i_1}D_{i_1}D_{i_3} = a_{i_1}R(v_3)R(v_1)R(x)^2 \in IR(x^2) \subseteq I.
\]

Similarly, \( a_{i_2}D_{i_2}D_{i_3} \in I \).
Now let $n \geq 7$. Without loss of generality, we can assume that $\{i, j\} = \{1, 2\}$. 

If $a_{i_1}D_{i_1} \cdots D_{i_r} \in A \setminus I$ and $\{i_1, \ldots, i_2\} \neq \{1, 2\}$, then $\{i_1, \ldots, i_2\} \cap \{1, 2\} = \emptyset$ and $2r = n - 3$, $r \geq 2$.

Suppose that $a_{i_1}D_{i_1} \cdots D_{i_r} = zu_k$ mod $I$, $k = \{1, \ldots, n\} \setminus \{1, 2, i_1, \ldots, i_2\}$, $z \in \mathbb{Z}$.

Let $b = a_{i_1} \cdots R(a_{i_{2r-1}, i_{2r}})$. By Lemma 3.2.5, $bD_{i_1} \cdots D_{i_{2r}} = z_0$ mod $I$, $t(z_0) = 1$, hence $z_0$ is invertible.

Let $a_{i_2} := a_{i_2} - (z_0^{-1}z)(bv_k)$.

From $r \geq 2$, it follows that $((z_0^{-1}z)(bv_k))D_1D_2 \in I$.

By Lemma 3.2.6, $a_{i_2}D_{i_1} \cdots D_{i_{2r}} \in I$ as well. This finishes the case $n \geq 7$.

Lemma is proved.

Now let us consider the case $n = 5$.

We will establish at first the existence of such an element $0 \neq b_{i_2} \in I$ of weight $v_i v_2$ that $b_{i_2}D_3D_4 \in I$.

For an element $v_{i_1} \cdots v_{i_r}$, $1 \leq i_1 < \cdots < i_r \leq n$, of the Clifford algebra, let $I(v_{i_1} \cdots v_{i_r})$ denote the subspace of $I$ consisting of elements of weight $v_{i_1} \cdots v_{i_r}$.

**Lemma 3.2.8.** Let $n = 5$. There exists a nonzero element $b_{i_2} \in I(v_i v_2)$ such that $b_{i_2}R(v_j)R(x)R(v_k)R(v_r)R(v_3) \in I$.

**Proof.** First we remark that if $1 \leq j \leq 5$, $\{k, l, q\} = \{1, 2, \ldots, 5\} \setminus \{i, j\}$, and $b_{i_2} \in I(v_i v_j)$, then the following four inclusions are equivalent:

$$b_{i_j}R(v_k)R(x)R(v_1)R(x)R(v_q) \in I; \quad b_{i_j}D_kD_l \in I;$$

$$b_{i_j}D_kD_q \in I; \quad b_{i_j}D_lD_q \in I.$$

Suppose that, contrary to the assertion of the lemma, $0 \neq b_{i_2} \in I(v_i v_2)$ implies that $b_{i_2}D_3D_4 \notin I$.

We will finish the proof in several steps.

Step 1. For arbitrary $1 \leq i \neq j \leq 5$, $0 \neq b_{i_j} \in I(v_i v_j)$ implies that $b_{i_j}R(v_k)R(x)R(v_1)R(x)R(v_q) \notin I$ as long as $i, j, k, l, q$ are distinct integers. Indeed, if $\{1, 2\} \cap \{i, j\} = \emptyset$, then $0 \neq b_{i_j}D_{i_j}D_{i_l}D_{i_q} \in I(v_i v_j)$, and therefore, $b_{i_j}D_{i_j}D_{i_l}D_{i_q}D_{i_k}D_{i_r} = \pm b_{i_j}R(v_k)R(x)R(v_l)R(x)R(v_q)R(x)R(v_r) \notin I$. If $i = 1$, $j \neq 2$, then $b_{i_j}D_{i_j}D_{i_q} \in I(v_i v_j)$ and we argue as above.

Step 2. $0 \neq b_{i_j} \in I(v_i v_j)$ implies $b_{i_j}D_i D_j \notin I$. 

Indeed, if \( b_{ij}D_iD_j \in I \) and \( k \not\in \{i, j\} \), then

\[
b_{ij}D_iD_jR(v_k) = b_{ij}R(v_k)D_iD_j \in I.
\]

But \( b_{ij}v_k \) is a nonzero element from \( I(v_i,v_j,v_k) = I(v_i,v_q) \), where \( \{l, q\} = \{1, 2, \ldots, 5\}\setminus\{i, j, k\} \).

This contradicts the assertion of Step 1.

**Step 3.** For arbitrary nonzero elements \( a, b \in I(v_i,v_j) \), there exist \( f, g \in \hat{Z} \) such that \( fg \not\in I \) and \( af = bg \).

Indeed, let \( \{k, l, q\} = \{1, 2, 3, 4, 5\}\setminus\{i, j\} \). Then \( aD_kD_l = gv_q \neq 0 \mod I \) and \( bD_kD_l = fv_q \neq 0 \mod I \) for some elements \( f, g \in \hat{Z} \).

By Lemma 3.2.6, \( (af)D_kD_l = (fg)v_q \mod I \) and \( (bg)D_kD_l = (fg)v_q \mod I \). Hence, \( (af - bg)D_kD_l \in I \). By our assumption, \( af = bg \).

**Step 4.** For arbitrary distinct integers \( 1 \leq i, j, k \leq 5 \), we have

\[
I(v_i,v_j)I(v_j,v_k) = I(v_i)I(v_j,v_k) = (0).
\]

Indeed, let \( a \in I(v_i,v_j), b \in I(v_j,v_k) \). Then \( c = ab \in I(v_i,v_k) \). From Lemma 3.2.4, it follows that there exists an element \( a_{ik} \in I(v_i,v_k) \) such that \( t(a_{ik}D_iD_k) = 1 \).

By Step 3, there exist \( f, g \in \hat{Z} \) such that \( fg \not\in I \) and \( a_{ik}f = cg \).

If two elements of weight 1 have equal traces \( t \), then they are equal modulo \( I \). Hence by Lemma 3.2.6, \( (cg)D_iD_k = ((ab)D_iD_k)g \mod I \). Furthermore, \( (ab)D_iD_k \subseteq [M_i, M_j] + I \) by Lemma 1.1.13.

Hence, \( t_\tau(a_{ik}D_iD_k) = t_\tau(a_{ik}D_iD_k) \not\equiv 0, \) which is impossible, because \( \hat{f} \neq 0 \) and \( t_\tau(a_{ik}D_iD_k) \) is invertible.

The equality \( I(v_i)R(v_j,v_k) = (0) \) is proved similarly.

**Step 5.** \( xI(v_i,v_j)R(v_j,v_k) = (0) \).

As above, choose arbitrary elements \( a \in I(v_i,v_j), b \in I(v_j,v_k) \).

We will first show that \( c = xR(a)R(b)R(x) = 0 \). The element \( c \) lies in \( I(v_i,v_k) \). That is why, to show that \( c = 0 \), it is sufficient to show that \( cD_iD_k = 0 \). Since \( cv_i = 0 \), it follows that \( cD_iD_k = cR(x)R(v_j)D_k = xR(a)R(b)R(v_j)D_k R(x)^2 = xR(a)R(b)D_k R(v_j)R(x)^2 \). We have

\[
xR(a)R(b)R(x)R(v_k) = 0.
\]

Therefore,

\[
xR(a)R(b)D_k = -xR(a)R(b)R(v_k)R(x) \in I,
\]

because \( [(MI)A, M] \subseteq I \).
We have proved that \([x_{ij}]_{jk}, x] = (0)\). It is easy to see that for an arbitrary \(1 \leq l \leq 5\), we have \(\langle x_{ij} \rangle_{jk} D_l \subseteq I\).

By Lemma 3.2.2, there exist distinct integers \(1 \leq l_1, \ldots, l_r \leq 5\), \(r\) is odd, such that \(\langle x_{ij} \rangle_{jk} D_{l_1} \cdots D_{l_r} \subseteq A \setminus I\). If at least one integer \(l_q\) is not equal to \(i\) or \(k\), then we can assume that \(l_1 \notin \{i, k\}\).

Then \(\langle x_{ij} \rangle_{jk} R(v_{l_1}) = 0\), and therefore, \(\langle x_{ij} \rangle_{jk} D_{l_1} = 0\). Hence \(l_1, \ldots, l_r \in \{i, k\}, r \leq 2\). This contradicts \(r\) being odd and greater than \(1\). We proved that \(\langle x_{ij} \rangle_{jk} = 0\).

**Step 6.** Let us show that for arbitrary elements \(a \in I(v_{i_2} v_{j_2}), b \in I(v_{i_3} v_{j_3})\), we have \(aR(x)R(v_{i_2})R(b) = 0\). Indeed, by the Jordan identity,

\[ R(v_1)R(x)R(b) = -R(b)R(x)R(v_1) + R(xb)R(v_1). \]

In Step 5 it was proved that \(aR(x)R(b) = 0\). Now, \(aR(x)R(xb)R(v_1) = aR(x)R(xb)R(v_1) + R(v_1)R(b)(R(xb)) = (I(v_{i_2}v_{j_2})R(x_{i_2}v_{j_2}))v_1 = 0\) by Step 4.

The element \(xR(a)R(v_1)R(b)\) belongs to the eigenvalue \(1\) with respect to \(U(v_2)\). Hence \(xR(a)R(v_1)R(b) \neq 0\) implies \(xR(a)R(v_1)R(b)R(v_4) \neq 0\).

Again by the Jordan identity, \(R(v_1)R(b)R(v_4) = R(v_1)R(b)R(v_4) + R(bv_4)R(v_1)\). We have \(xR(a)R(v_4) = xR(v_4)R(a) = 0\). As for the second summand, the element \(bv_4\) lies in \(I(v_{i_2}v_{j_2}) = I(v_2v_3)\). Hence, \(xR(a)R(bv_4) = 0\) by Step 5. This finishes the proof of Step 6.

Now we are ready to finish the proof of the lemma.

By Lemma 3.2.4, there exists an element \(a_{12} \in I(v_{i_2}v_{j_2})\) such that \(t(a_{12})D_1 D_2 = 1\). This implies that \(a_{12} R(x)R(v_1)R(x) = f v_2 \mod I(v_2)\), where \(f \in Z, t(f) = 1\).

From Steps 4 and 6, it follows that \((fv_2)I(v_{i_2}v_{j_2}) = (0)\).

Now \((fv_2)I(v_{i_2}v_{j_2}) = (fI(v_{i_2}v_{j_2}))v_2\) and \((fI(v_{i_2}v_{j_2}))v_2 = fI(v_{i_2}v_{j_2}) = (0)\).

Since \(t(f) = 1\), it follows that \(1 - f\) is a nilpotent element. Hence the multiplication \(R(f): J \to J\) is an invertible operator. Hence \(I(v_{i_2}v_{j_2}) = (0)\), the contradiction. Lemma is proved.

**Lemma 3.2.9.** For arbitrary integers \(1 \leq i \neq j \leq n\), there exists an element \(a_{ij} \in I(v_{i_2}v_{j_2})\) such that \(t(a_{ij})D_1 .. D_5 = 1\), and for any subset \(\{i_1 < \cdots < i_2\} \subseteq \{1, 2, \ldots, n\}\), which does not coincide with \(\{i, j\}\), we have \(a_{ij}D_1 \cdots D_{2r} \subseteq I\).

**Proof.** The lemma has been proved for all \(n\) except 5.

Let \(n = 5\). By Lemma 3.2.8, there exists a nonzero element \(b_{12} \in I(v_{i_2}v_{j_2})\) such that \(b_{12} D_1 D_2 \in I\), \(b_{12} D_3 D_5 \in I\), \(b_{12} D_4 D_5 \in I\). Then, by Lemma 3.2.1, \(b_{12} D_1 D_2 \in A \setminus I\).

Clearly, \(b_{12} D_1 D_2 = f \in Z, f \neq 0\). Since \(Z\) does not contain nilpotent ideals which are invariant with respect to \(D(M(-1, \ldots, -1), M(-1, \ldots, -1))\), there exist derivations \(d_1, \ldots, d_r) \in D(M(-1, \ldots, -1), M(-1, \ldots, -1)) \subseteq I\) such that \(t(fd_1 \cdots d_r) \neq 0\).
We have
\[ b_{12} D_1 D_2 d_1 \cdots d_j \]
\[ = \sum (b_{12} d_{i_1} \cdots d_{i_k}) D(x d_{j_1} \cdots d_{j_{\mu}}, v_1) D(x d_{\mu_1} \cdots d_{\mu_{\mu}}, v_2). \]

Hence, for some summand
\[ t\left( (b_{12} d_{i_1} \cdots d_{i_k}) D(x d_{j_1} \cdots d_{j_{\mu}}, v_1) D(x d_{\mu_1} \cdots d_{\mu_{\mu}}, v_2) \right) \]
\[ = t\left( (b_{12} d_{i_1} \cdots d_{i_k}) R(x d_{j_1} \cdots d_{j_{\mu}}) R(v_1) D(x d_{\mu_1} \cdots d_{\mu_{\mu}}, v_2) \right) \neq 0. \]

Denote \( b'_{12} = b_{12} d_{i_1} \cdots d_{i_k} \).

We have already mentioned that \( M = x R \langle \tilde{Z} \rangle + M_I + MV^A \).

From \( t(b'_{12} R(M) R(v_1) R(M) R(v_2)) \neq (0) \), it follows that
\[ t\left( b'_{12} R(x R \langle \tilde{Z} \rangle) R(v_1) R(x R \langle \tilde{Z} \rangle) R(v_2) \right) \neq (0). \]

By Lemma 3.2.3(a), \( t(b'_{12} R(x) R(v_1) R(x) R(v_2)) \neq 0 \), and therefore, we can assume that \( t(b'_{12} D_1 D_2) = 1 \).

Let us show that \( b'_{12} R(v_3) R(v_4) R(x) R(v_5) \in I \). We have
\[ b'_{12} R(v_3) R(x) R(v_4) R(x) R(v_5) \]
\[ = \sum (b_{12} R(v_3) R(x w_1) R(v_4) R(x w_2) R(v_5) w_3), \]

where \( w_1, w_2, w_3 \) are products (may be empty) of derivations from \( D(M(-1, \ldots, -1), M(-1, \ldots, -1)) \).

To show that each summand lies in \( I \), it is sufficient to show that \( b_{12} R(v_3) R(x R \langle \tilde{Z} \rangle) R(v_4) R(x R \langle \tilde{Z} \rangle) R(v_5) \subseteq I \), which boils down (see Lemma 3.2.3(a)) to \( b_{12} R(v_3) R(x) R(v_4) R(x) R(v_5) \in I \). Lemma is proved.

**Remark 1.** Without loss of generality, we can assume in Lemma 3.2.9 that \( a_{ij} D_1 D_2 = 1 \bmod I \). Indeed, by Lemma 3.2.9, \( a_{ij} D_1 D_2 = f_0 \bmod I \), where \( f_0 \in \tilde{Z} \) is an invertible element. Then, by Lemma 3.2.6,
\[ \left( a_{ij} f_0^{-1} \right) D_1 D_2 = 1 \bmod I. \]

**Remark 2.** The element \( a_{ij} \) with this property is unique in \( I(v_i, v_j) \). If \( a'_{ij} \) is another element with the same property, then \( a'_{ij} - a_{ij} \) fails Lemma 3.2.1. From this, it follows that \( a_{ij} D(v_i, v_k) = a_{ik} \).

**Lemma 3.2.10.** \( MV^A \subseteq M_I \).
Proof. We have $V^d = I + \sum_{k=1}^n \tilde{Z}v_k$. Choose $f \in \tilde{Z}$ and $y \in M$ and let us show that $y(fv_j) \in M_I$. We have $a_{ij}R(x)R(v_j)R(x) = v_j \mod I$.
We have $(a_{ij}f)R(x)R(v_j)R(x) = (a_{ij}f)D_iD_jR(v_j)R(f) = a_{ij}R(x)R(v_j)R(x)R(f)$ by Lemma 3.2.6 (we used here the fact that the elements $(a_{ij}f)D_iD_j$ have weight 1).

Hence $(a_{ij}f)R(x)R(v_j)R(x) = fv_j \mod I$.

By the Jordan identity,
\[
R(v_j)R(x)R(y) = R(y)R(x)R(v_j) - R([x, yv_j]) - R(v_jy)R(x) + R([x, y])R(v_j).
\]

We have
\[
(a_{ij}f)R(x)R(y)R(x), (a_{ij}f)R(x)R(v_jy)R(x) \in IR(M)R(M)R(M) \subseteq IM,
\]
while $(a_{ij}f)R(x)R([x, yv_j])$ and $(a_{ij}f)R(x)R([x, y]R(v_j))$ lie in $M_I$. Lemma is proved.

**Corollary 3.2.2.** $M = xR(\tilde{Z}) + M_I$.

**Lemma 3.2.11.** Let $1 \leq q \leq n$ be an odd integer and let $1 \leq i_1 < \cdots < i_q \leq n$. Then $a_{i_1}D_{i_1} \cdots D_{i_q-1} \in \tilde{Z} + I$. Clearly, $ID_{i_q} \subseteq M_I$.

Furthermore, $\tilde{Z}D(x, v_{i_q}) = (Zv_{i_q})x \in M_I$ by Lemma 3.2.10.

For an ordered subset $\pi = \{i_1, \ldots, i_{2r}\}$ of $\{1, \ldots, n\}$, let
\[
a_\pi = a_{i_1i_2}R(a_{i_3i_4}) \cdots R(a_{i_{2r-1}i_{2r}}).
\]

**Lemma 3.2.12.** For an arbitrary element $f \in \tilde{Z}$, we have $((a_\pi f)D_{i_1} \cdots D_{i_{2r}}) = f \mod I$.

For any subset $\{j_1, \ldots, j_k\}$ of $\{1, \ldots, n\}$ other than $\{i_1, \ldots, i_{2r}\}$, we have
\[
(a_\pi f)D_{i_1} \cdots D_{i_{2r}} \in I \cup M_I.
\]

**Proof.** By Lemma 3.2.6, $t_{\pi}(a_\pi f)D_{i_1} \cdots D_{i_{2r}} = t_{\pi}(aD_{i_1} \cdots D_{i_{2r}})\tilde{f}$; hence, to prove the first part of the assertion it is sufficient to prove that $a_\pi D_{i_1} \cdots D_{i_{2r}} = 1 \mod I$.

If $r = 1$, then see Remark 1. Let $r > 1$ and let $\pi' = \{i_1, \ldots, i_{2r-2}\}$. Then $a_\pi = a_\pi a_{i_{2r-1}i_{2r}}$ and
\[
a_\pi D_{i_1} \cdots D_{i_{2r}} = \sum_{s+q=2r} (a_{\pi'}D_{a_1} \cdots D_{a_s})(a_{i_{2r-1}i_{2r}}D_{p_1} \cdots D_{p_q}),
\]
where $\pi'$ is an ordered subset of $\{1, \ldots, n\}$ other than $\{i_1, \ldots, i_{2r-2}\}$.
Consider the element \( a_{i_{2r-1},i_{2r}} D_{\beta_1} \cdots D_{\beta_j} \). If \( q \) is odd, then this element lies in \( M_I \). Hence \( t_\Sigma(a_{\pi} D_{\alpha_1} \cdots D_{\alpha_r} a_{i_{2r-1},i_{2r}} D_{\beta_1} \cdots D_{\beta_j}) = 0 \).

Now let \( q \) be even. Unless \( q = 2 \), \( \beta_1 = i_{2r-1} \), \( \beta_2 = i_{2r} \), the element \( a_{i_{2r-1},i_{2r}} D_{\beta_1} \cdots D_{\beta_j} \in I \). Hence \( t_\Sigma(a_{\pi} D_{i_1} \cdots D_{i_{2r}}) = t_\Sigma(a_{\pi} D_{i_1} \cdots D_{i_{2r-2}}) \) and we can use the induction assumption.

Now let \( \{j_1, \ldots, j_k\} \neq \{i_1, \ldots, i_{2r}\} \). We have to show that \( (a_{\pi} f) D_{j_1} \cdots D_{j_k} \in I \cup M_I \). Assume the contrary. Let \( k \) be even. By weight considerations, \( n \) is odd and \( \{i_1, \ldots, i_{2r}\} \cap \{j_1, \ldots, j_k\} = \emptyset \), \( k = n - 2r - 1 \). Distributing \( D_i \)'s to the factors of \( a_{\pi} \) and to \( f \), we see that \( (a_{\pi} f) D_{j_1} \cdots D_{j_k} \) lies in the subsuperalgebra of \( J \) generated by \( \tilde{Z} + I + M_I \), that is, \( \tilde{Z} + I + M_I \). Comparing weights, we see that \( (a_{\pi} f) D_{j_1} \cdots D_{j_k} \in I \).

Now let \( k \) be odd. If \( (a_{\pi} f) D_{j_1} \cdots D_{j_{k-1}} \in I \), then \( (a_{\pi} f) D_{j_1} \cdots D_{j_k} \in M_I \).

If \( (a_{\pi} f) D_{j_1} \cdots D_{j_{k-1}} \in \tilde{Z} \), then \( (a_{\pi} f) D_{j_1} \cdots D_{j_k} \in \tilde{Z} D_{j_k} \subseteq M_I \) by Lemma 3.2.10. Lemma is proved.

Let \( \pi = \{i_1, \ldots, i_{2r+1}\} \) be an ordered subset of \( \{1, \ldots, n\} \). Denote \( a_\pi = a_{i_1 \cdots i_2 \cdots i_{2r+1}} \).

**Lemma 3.2.13.** Let \( 1 \leq i_1, \ldots, i_k \leq n \) be distinct integers, \( \pi = \{i_1, \ldots, i_k\} \) is an ordered subset, \( \sigma \in S_k \), and the ordered subset \( \pi' \) is obtained from \( \pi \) by the permutation \( \sigma \). Then \( a_{\pi'} = (-1)^{\sigma} a_\pi \).

**Proof.** If \( k \) is even, then the element \( a_{\pi'} = (-1)^{\sigma} a_\pi \), if nonzero, fails Lemma 3.2.1. Let \( k \) be odd. It is sufficient to prove that \( a_{i_1 \cdots i_k} = -a_{i_1 \cdots i_k} \) for any distinct \( 1 \leq i, j, k \leq n \). It is easy to see that \( a_{i_1 \cdots i_k} = a_{i_1 \cdots i_k} D(v_j, v_k) \). Hence, \( a_{i_1 \cdots i_k} R(v_j) = a_{i_1 \cdots i_k} (R(v_j) R(v_k) - R(v_k) R(v_j)) R(v_j) \).

From the Jordan identity, it follows that \( R(v_j) R(v_k) R(v_j) = 0 \). Hence, \( a_{i_1 \cdots i_k} R(v_j) = a_{i_1 \cdots i_k} R(v_k) R(v_j) = -a_{i_1 \cdots i_k} R(v_k) \). Lemma is proved.

**Lemma 3.2.14.** Let \( \pi = \{i_1, \ldots, i_{2r+1}\} \) be an ordered subset of \( \{1, \ldots, n\} \), \( f \in \tilde{Z} \). For arbitrary distinct integers \( 1 \leq j_1, \ldots, j_k \leq n \), we have \( (a_{\pi} f) D_{i_1} \cdots D_{i_k} \in I \cup M_I \) unless

(a) \( k = 2r \) and \( j_1, \ldots, j_k \in \pi \). In this case, \( (a_{\pi} f) D_{i_1} \cdots D_{i_k} = \pm f \) mod \( I \), where \( \{l\} = \pi \setminus \{j_1, \ldots, j_k\} \); or

(b) \( k = 2r + 2 \) and \( \pi \subseteq \{j_1, \ldots, j_k\} \). In this case, \( (a_{\pi} f) D_{i_1} \cdots D_{i_k} = \pm (R(f x)) \) mod \( I \), where \( \{l\} = \{j_1, \ldots, j_k\} \setminus \pi \); or

(c) \( k = 2r + 1, j_1, \ldots, j_k \) is a permutation of \( i_1, \ldots, i_{2r+1} \). In this case, \( (a_{\pi} f) D_{i_1} \cdots D_{i_k} = \pm f \) mod \( M_I \).

**Proof.** Suppose that \( (a_{\pi} f) D_{i_1} \cdots D_{i_k} \in I \cup M_I \). Let \( k \) be even. Comparing weights, we see that either \( k = 2r \) and \( j_1, \ldots, j_k \in \pi \) or \( k = 2r + 2 \) and \( \pi \subseteq \{j_1, \ldots, j_k\} \) or \( n \) is odd and \( \{j_1, \ldots, j_k\} = \{1, \ldots, n\} \setminus \pi \). Let us
exclude the last possibility. Indeed, since \(i_{2r+1} \notin \{j_1, \ldots, j_k\}\), it follows that 
\[(a_\pi f)D_{j_1} \cdots D_{j_k} = (a_{i_1 \ldots i_{2r}} f)D_{j_1} \cdots D_{j_k} R(v_{i_{2r+1}}),\]
and it is sufficient to refer to Lemma 3.2.12.

Let \(k = 2r\); \(j_1, \ldots, j_k \in \pi\), \(l = \pi \setminus \{j_1, \ldots, j_k\}\). By Lemma 3.2.13, without loss of generality, we can assume that \(i_{2r+1} = \bar{l}\). Then again \((a_\pi f)D_{j_1} \cdots D_{j_k} = (a_{i_1 \ldots i_{2r}} f)D_{j_1} \cdots D_{j_k} R(v_{\bar{l}})\), and the assertion follows from Lemma 3.2.12.

Let \(k = 2r + 2\). Without loss of generality, we will assume that \(i_{2r+1} = j_{2r+2}, l = j_{2r+1}\).

Then
\[
(a_\pi f)D_{j_1} \cdots D_{j_k} = (a_{i_1 \ldots i_{2r}} f)D_{j_1} \cdots D_{j_k} R(v_{j_{2r+1}})D_{j_{2r+1}} = - (a_{i_1 \ldots i_{2r}} f)D_{j_1} \cdots D_{j_k} R(x),
\]
because \(R(v_{j_{2r+1}})R(x)R(v_{j_{2r+1}}) = 0\). Since \(j_1, \ldots, j_{2r}\) is a permutation of \(i_1, \ldots, i_{2r}\), it follows from Lemma 3.2.12 that \((a_{i_1 \ldots i_{2r}} f)D_{j_1} \cdots D_{j_{2r}} = \pm f + a, a \in I(1, 1, \ldots, 1)\).

Now \((\pm f + a)D_{j_1} R(x) = - (\pm f + a)R(v_{\bar{l}}) R(x)^2 = \pm (R(x)^2)v_{\bar{l}} \mod I\).

Now let \(k\) be odd. Since \((a_\pi f)D_{j_1} \cdots D_{j_{k-1}} \notin I\), it follows that every \((k - 1)\)-element subset of \(\{j_1, \ldots, j_k\}\) satisfies (a) or (b). This is possible only if \(j_1, \ldots, j_k\) is a permutation of \(\pi\). We can assume that \(i_{2r+1} = j_{2r+1}\). Then \((a_\pi f)D_{j_1} \cdots D_{j_k} = (a_{i_1 \ldots i_{2r}} f)D_{j_1} \cdots D_{j_{2r}} R(v_{\bar{l}}) = \pm f x \mod M_{\bar{l}}\) by Lemma 3.2.12. Lemma is proved.

If \(n\) is even, then by Corollary 5.2.1, we have \(\bar{Z} = \bar{Z}\).

Let \(n\) be odd. Denote \(w_0 = R(v_{\bar{l}}) \cdots R(x) R(v_{\bar{n}})\).

**Lemma 3.2.15.** For an arbitrary element \(\tilde{f} \in \bar{Z}\), there exists an element \(f \in \bar{Z}\) such that \(\tilde{f} - f \in I\) and \(fw_0 \in I\).

**Proof.** Let \(\tilde{f} w_0 = g \in \bar{Z}\). Let \(\pi = \{1, 2, \ldots, n\}\), \(\pi' = \{1, \ldots, n - 1\}\). By Lemma 3.2.6,
\[
(a_\pi g)w_0 = a_\pi R(v_{\bar{n}}) R(g)w_0 = (a_\pi g) D_1 \cdots D_{n-1} R(v_{\bar{n}}) R(v_{\bar{n}}) = (a_\pi g) D_1 \cdots D_{n-1} = g \mod I.
\]

Hence \((\tilde{f} - a_\pi g)w_0 \in I\) and it is sufficient to let \(f = \tilde{f} - a_\pi g\). Lemma is proved.

**Lemma 3.2.16.** Let \(a, b \in \bar{Z}\), \(aw_0 \in I\), \(bw_0 \in I\). Then \((ab)w_0 \in I\).

**Proof.** It is sufficient to prove that \((ab)D_{j_1} \cdots D_{n-1} \in I\). We have \((ab)D_{j_1} \cdots D_{n-1} = \sum \pm (aD_{j_1} \cdots D_{k}) (bD_{j_1} \cdots D_{l})\).
If both $k$ and $q$ are nonzero, then comparing weights, we see that $aD_{i_1} \cdots D_{i_k}$ and $bD_{i_1} \cdots D_{i_q} \in I$.

If $q = 0$, then $aD_{i_1} \cdots D_{i_k} = aD_1 \cdots D_{n-1} \in I$. If $k = 0$, then $bD_{i_1} \cdots D_{i_q} \in I$. Lemma is proved.

Recall that in Case (b), $\bar{Z} = F[\bar{a}_1, \bar{a}_m]$ the algebra of truncated polynomials in $m$ variables $\bar{a}_1, \ldots, \bar{a}_m$. By Lemma 3.2.15, for each $\bar{a}$, there exists a preimage $a_1$ in $\bar{Z}$ such that $a_1w_0 \in I$. Let $Z$ be the subalgebra of $A$ generated by $a_1, \ldots, a_m$. By Lemma 3.2.16, $aw_0 \subseteq I$. This implies that $Z \cap I = \{0\}$ and $Z = \bar{Z}, \bar{Z} = Z + I(1, 1, \ldots, 1)$.

The same result in Case (a) is obvious.

**Lemma 3.2.17.** For an arbitrary element $f \in Z$, an arbitrary nonempty subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, we have $fD_{i_1} \cdots D_{i_k} \in I \cup M_I$.

**Proof.** Let $k$ be even. If $fD_{i_1} \cdots D_{i_k} \notin I$, then $n$ is odd and $k = n - 1$, which contradicts the definition of $Z$. If $k = 1$, then the assertion follows from Lemma 3.2.10. If $k$ is odd and $k > 1$, then $fD_{i_1} \cdots D_{i_{k-1}} \in I$, which implies the assertion. Lemma is proved.

**Lemma 3.2.18.** $ZR(x)^2 \subseteq Z$.

**Proof.** Clearly, $ZR(x)^2 \subseteq \bar{Z}$. If $n$ is even, then $\bar{Z} = Z$. If $n$ is odd, then $\bar{Z} = Z + Za_1, \ldots, a_m$.

If $f \in Z$ and $fR(x)^2 = g + ha_1 \cdots a_m$, where $g, h \in Z$, $h \neq 0$, then $fR(x)^2w_0 = h \mod I$. But $fR(x)^2w_0 = fw_0R(x)^2 \in I$, the contradiction. Lemma is proved.

**Lemma 3.2.19.** Let $f \in Z$ and let $\{i_1, \ldots, i_k\}$ be a nonempty subset of $\{1, \ldots, n\}$. Then

$$\langle xf \rangle D_{i_1} \cdots D_{i_k} \in I \cup M_I,$$

unless $k = 1$, $(xf)D_i = (fR(x)^2)v_i$.

**Proof.** Since $(xf)R(v_i) = 0$, it follows that $(xf)D_i = (xf)R(x)R(v_i) = (fR(x)^2)v_i$.

Let $k$ be odd and $k > 1$. Then $(xf)D_{i_1} \cdots D_{i_k} = fR(x)^2R(v_{i_1})R(x)R(v_{i_2}) \cdots R(x)R(v_{i_k}) = (fR(x)^2)D_{i_1} \cdots D_{i_{k-1}}R(v_{i_k}) \in I$ by Lemma 3.2.17.

Let $k$ be even. If $k = 2$, then $(xf)D_{i_1}D_{i_2} = (fR(x)^2)v_{i_1}R(x)R(v_{i_2}) \in M_I$ by Lemma 3.2.10.

If $k > 2$, then $(xf)D_{i_1} \cdots D_{i_{k-1}} \in I$, which implies the assertion. Lemma is proved.

**Lemma 3.2.20.** Let $\pi = \{i_1, \ldots, i_k\}$ be a nonempty subset of $\{1, \ldots, n\}$, and let $f \in Z$. For any distinct integers $1 \leq j_1, \ldots, j_k \leq n$, the element
If \( (x_\alpha) f D_{j_1} \cdots D_{j_k} \neq 0 \) lies in \( I \cup M_f \) unless

(a) \( k = 2r - 1, \{j_1, \ldots, j_k\} \subseteq \{i_1, \ldots, i_{2r}\} \). Then \( (x_\alpha) f D_{j_1} \cdots D_{j_k} \in Zv_l \mod I, \) where \( l = \pi \setminus \{j_1, \ldots, j_k\} \), or

(b) \( k = 2r, \{j_1, \ldots, j_k\} = \pi \), in which case \( (x_\alpha) f D_{j_1} \cdots D_{j_k} \in Zx \mod M_f \), or

(c) \( k = 2r + 1, \pi \subseteq \{j_1, \ldots, j_k\} \), in which case \( (x_\alpha) f D_{j_1} \cdots D_{j_k} \in Zv_l \mod I, l = (j_1, \ldots, j_k) \setminus \pi \).

Proof. Let us at first prove the assertion when \( f = 1 \). Suppose that

\( (x_\alpha) D_{j_1} \cdots D_{j_k} \neq I \cup M_f \). Let \( k \) be odd. If \( j_1 \notin \pi \), then \( (x_\alpha) D_{j_1} = a_\alpha R(x)^2 R(v_j) \). In this case, \( (x_\alpha) D_{j_1} \cdots D_{j_k} = a_\alpha D_{j_1} \cdots D_{j_k} R(x)^2 R(v_{j_1}) \). By Lemma 3.2.12, \( \{j_2, \ldots, j_k\} = \pi \) and \( a_\alpha D_{j_1} \cdots D_{j_{k-1}} = \pm 1 \mod I \), so \( a_\alpha D_{j_1} \cdots D_{j_{k-1}} R(x)^2 \in I \). Hence \( j_1, \ldots, j_k \in \pi \).

The element \( (x_\alpha) D_{j_1} \cdots D_{j_k} \) has weight \( v_{i_1} \cdots v_{i_{2r}} \cdots v_{j_k} \). It can lie outside of \( I \) only if \( k = 2r - 1 \). Comparing weights, we see that \( (x_\alpha) D_{j_1} \cdots D_{j_k} \in Zv_{j_{2r-1}} \), where \( l = \pi \setminus \{j_1, \ldots, j_{2r-1}\} \).

Now let \( k \) be even. Then \( (x_\alpha) D_{j_1} \cdots D_{j_{k-1}} \neq I \); hence, every \( (k - 1) \)-element subset of \( \{j_2, \ldots, j_k\} \) lies in \( \pi \); hence, \( k = 2r \) and \( \{j_1, \ldots, j_k\} = \pi \). We have \( (x_\alpha) D_{j_1} \cdots D_{j_k} \in Zv_{j_{2r}} \), hence, \( (x_\alpha) D_{j_1} \cdots D_{j_k} \in Zv_{j_{2r}} \in Zx \).

Now we will drop the assumption that \( f = 1 \).

We have \( ((x_\alpha) f) D_{j_1} \cdots D_{j_k} = \sum \pm (x_\alpha) D_{a_1} \cdots D_{a_s} (f D_{\beta_1} \cdots D_{\beta_s}) \). Suppose that \( (x_\alpha) D_{a_1} \cdots D_{a_s} (f D_{\beta_1} \cdots D_{\beta_s}) \neq I \cup M_f \). The case of \( q = 0 \) has already been considered above. If \( q > 0 \), then \( f D_{\beta_1} \cdots D_{\beta_s} \in I \cup M_f \). Then \( (x_\alpha) D_{a_1} \cdots D_{a_s} \notin I \cup M_f \), \( q \) is odd, and \( s \) is even. From what we proved above, it follows that \( \{a_1, \ldots, a_s\} = \pi \).

The element \( (x_\alpha) D_{a_1} \cdots D_{a_s} (f D_{\beta_1} \cdots D_{\beta_s}) \) has weight \( v_{i_1} \cdots v_{i_s} \) and lies in \( A \). This implies that \( q = 1 \) (notice that \( s \geq 2 \)). Now \( k = 2r + 1, \{j_1, \ldots, j_k\} \supset \pi \). Comparing weights, we see that \( (x_\alpha) f D_{j_1} \cdots D_{j_k} \in Zv_{j_1} \mod I, l = (j_1, \ldots, j_k) \setminus \pi \). Lemma is proved.

Lemma 3.2.21. Let \( \pi = \{i_1, \ldots, i_{2r}\} \) be a subset of \( \{1, \ldots, n\} \), let \( i \in \pi \), and let \( f \in Z \). For any distinct integers \( 1 \leq j_1, \ldots, j_k \leq n \), the element \( (a_\alpha D_i) f D_{j_1} \cdots D_{j_k} \) lies in \( I \cup M_f \), unless \( k = 2r - 1, \{j_1, \ldots, j_k\} = \pi \setminus \{i\} \), in which case \( (a_\alpha D_i) f D_{j_1} \cdots D_{j_k} \in Z \mod I \).

Proof. As in the previous lemma, let us at first assume that \( f = 1 \).

If \( i \notin \{j_1, \ldots, j_k\} \), then the assertion follows from Lemma 3.2.14. Suppose that \( i = j_1 \). Then \( a_\alpha D_i D_{j_1} = -a_\alpha R(x)^2 \) and therefore \( a_\alpha D_i D_{j_1} \cdots D_{j_k} = -a_\alpha D_{j_2} \cdots D_{j_k} R(x)^2 \in I \) by Lemma 3.2.14.
Now let us drop the assumption that $f = 1$. We have $((a_i D_i) f) D_i \cdots D_i = \sum (a_i D_i D_{a_i} \cdots D_{a_i}(f D_{\beta_i} \cdots D_{\beta_i}))$. Let $(a_i D_i D_{a_i} \cdots D_{a_i}(f D_{\beta_i} \cdots D_{\beta_i})) \notin I \cup M_i$. The case $q = 0$ has been considered above. If $q \geq 1$, then

$$ f D_{\beta_i} \cdots D_{\beta_i} \in I \cup M_i. $$

If $a_i D_i D_{a_i} \cdots D_{a_i} \notin I \cup M_i$, then from what we proved above, it follows that $s = 2r - 1$, and therefore, $a_i D_i D_{a_i} \cdots D_{a_i} \in A$. Now

$$ (a_i D_i D_{a_i} \cdots D_{a_i})(f D_{\beta_i} \cdots D_{\beta_i}) \in A(I \cup M_i) \subseteq I \cup M_i. $$

Lemma is proved.

**Lemma 3.2.22.** For arbitrary elements $f, g \in Z$, $x(R(f) R(g) - R(fg)) \in M_i$.

**Proof.** We have

$$ xR(f) R(g) = xR(f) R((g v_1) v_1) $$

$$ = x(-R(g v_1) R(f v_1) - R(v_1) R(f g v_1)) $$

$$ + R(g v_1) R(f) R(v_1) + R(v_1) R(f) R(g v_1) + R(fg)) $$

$$ = xR(fg) \mod M_i $$

by Lemma 3.2.10. Lemma is proved.

**Lemma 3.2.23.** $A = Z + \Sigma v_i + \Sigma a_i Z + \Sigma_{i \in \pi}(a_i v_i) Z; M = xZ + \Sigma(a_i D_i) Z$, where $\pi$ runs over all nonempty subsets of the set \{1, \ldots, n\} containing even number of elements.

**Proof.** Clearly, $A = Z + \Sigma v_i + I$. From the previous lemma and Lemma 3.2.10, it follows that $M = xZ + M_i$.

Recall that by $\mathcal{W}$ we denoted the set of (nonempty) products of $R(v_i)$'s and $R(x)$'s which do not contain subproducts $R(v_i)R(v_j)$ and $(x)^2$. Consider the subset

$$ \mathcal{W}_0 = \{ R(x) R(v_{i_1}) R(x) \cdots R(x) R(v_{i_k}) R(v_{i_1}) R(x) R(v_{i_2}) \cdots R(x) R(v_{i_k}), 1 \leq i_1 < \cdots < i_k \leq n \}. $$

If $n$ is odd, then we exclude from $\mathcal{W}_0$ the operator $R(x) R(v_{i_1}) R(x) \cdots R(v_{i_k})$. From Lemma 3.2.1, it follows that for an arbitrary nonzero element $a \in I \cup M_i$, there exists an operator $w \in \mathcal{W}_0$ such that $t^2 aw \neq 0$. In-
every operator will find an element $R$ an element $af$. This implies that if $aDg$ then from what we proved above, it follows that $af$ has zero kernel. This will imply the existence of $af$.

Let $w = R(x)R(v_{i_1}) \cdots R(x)R(v_{i_k})$, $k$ is even. Let $a_w(f) = a_{i_1, \ldots, i_k}$. From Lemma 3.2.12, it follows that $a_w(f)w = f \mod I$. Comparing weights, we see that the only other candidate $\varphi \in W_0$ for $a_w(f)\varphi \neq 0$ is $\varphi = R(v_{i_{k+1}})R(x) \cdots R(x)R(v_{i_1})$, where $i_1, \ldots, i_n$ is a permutation of $1, 2, \ldots, n$. However, $a_w(f)\varphi = a_w(f)D_{i_{k+1}} \cdots D_{i_1}R(v_{i_1}) \in I$ by Lemma 3.2.12.

Let $w = R(x)R(v_{i_1}) \cdots R(x)R(v_{i_k})$, $k$ is odd. Choose an integer $i \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$ and let $\sigma = (i_1, \ldots, i_k, i)$.

For an arbitrary element $g \in Z$ we have $(a_x D_i)gw = (a_x D_i)gD_i \cdots D_{i_k} \in Z \mod I$ by Lemma 3.2.14. Let $\varphi \in W_0$, $\varphi \neq w$, and $t_\varphi((a_x D_i)g\varphi) \neq 0$. Comparing weights, we see that $n$ should be odd and $\varphi = R(v_{i_{k+1}})R(x) \cdots R(x)R(v_{i_1})$, where $i_1, \ldots, i_n$ is a permutation of $1, 2, \ldots, n$.

Then $(a_x D_i)g\varphi = -(a_x D_i)gD_{i_{k+1}} \cdots D_{i_1}R(v_{i_1}) \in I$ by Lemma 3.2.14. This implies that if $(a_x D_i)gw \neq 0$, then, by Lemma 3.2.1, $t_\varphi(a_x D_i)gw \neq 0$.

Let us show that the linear mapping $\theta: Z \to Z$, $\theta(g) = (a_x D_i)gw + I/1$ has zero kernel. This will imply the existence of $a_w(f)$. If $(a_x D_i)gw \in I$, then from what we proved above, it follows that $(a_x D_i)g = 0$. Hence $(a_x g) = (a_x g)R(v_{i_1}) = 0$. Consider the element $0 = (a_x g)D_{i_{k+1}} \cdots D_{i_1} \in Z \mod I$. By Lemma 3.2.17, $(a_x g)D_{i_{k+1}} \cdots D_{i_1} \in I \cup M_f$.

Hence $(a_x g)D_{i_{k+1}} \cdots D_{i_1} \in I$. We have $(a_x D_{i_{k+1}} \cdots D_{i_1} = -a_x D_{i_{k+1}} \cdots D_{i_1}R(x)$. Moreover, $(a_x D_{i_{k+1}} \cdots D_{i_1}R(v_{i_1}) = -a_x D_{i_{k+1}} \cdots D_{i_1}D_{i_1} = -1 \mod I$ by Lemma 3.2.12. Hence $(a_x D_{i_{k+1}} \cdots D_{i_1}g)R(v_{i_1}) = -g \mod I$, hence $g = 0$.

We proved that the linear mapping $\theta: Z \to Z$ is bijective and therefore an element $a_w(f)$ exists.

Let

$$w = R(v_{i_1})R(x) \cdots R(x)R(v_{i_k})$$

and let

$$w' = R(x)R(v_{i_1}) \cdots R(x)R(v_{i_k}).$$
Let us show that the element $a_n(f) = a_w(f) v_i$, satisfies our requirements. Clearly, $a_w(f) R(v_i) w = f \mod I$. Let $\varphi \in W_0$, $\varphi \neq w$, and $t_\varphi (a_w(f) R(v_i) \varphi) \neq 0$. Comparing weights, we see that $n$ is odd and

$$\varphi = R(x) R(v_{i_1}) \cdots R(x) R(v_{i_q}) \quad \text{or} \quad \varphi = R(v_{i_1}) R(x) R(v_{i_2}) \cdots R(x) R(v_{i_q}),$$

where $\{j_1, \ldots, j_q\} = \{1, 2, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$. In the first case, $R(v_{i_1}) \varphi \in W_0 \cup (-W_0)$, which contradicts the choice of $a_w(f)$. In the second case, applying the Jordan identity $q - 1$ times, we get $R(v_{i_1}) \varphi = (\cdots R(v_{i_1}) R(v_{i_2})$, and it remains to notice that $t_\varphi (AR(v_{i_1}) R(v_{i_2})) = (0)$. Thus, the existence of $a_n(f)$ is proved for all operators $w \in W_0$ and elements $f \in Z$. For an element $\tilde{f} \in Z$, we find a preimage $f \in Z$ and denote $a_n(f) = a_n(\tilde{f})$.

Now, for an arbitrary element $a \in I \cup M_1$, for an arbitrary operator $w \in W_0$, we have $t_\varphi ((a - \sum_{i=1}^n a_i (t_\varphi (aw) w) = 0$ for every $w \in W_0$. Hence $a = \sum_{i=1}^n a_i (t_\varphi (aw))$. It remains to notice that all elements $a_n(f)$ were chosen from the subspace $\sum_{i=1}^n a_i Z + \sum_{i=1}^n (a_{-n} v_i) Z + \sum_{i=1}^n (x a_{-n}) Z \in \sum_{i \in \pi} (a_{-n} D_i) Z$. Lemma is proved.

Denote

$$\Gamma = Z + \sum_{\pi} a_{-n} Z + \sum_{i \in \pi} (a_{-n} D_i) Z,$$

$$\Gamma' = \sum_{i=1}^n Z v_i + x Z + \sum_{i \notin \pi} (a_{-n} v_i) Z + \sum_{\pi} (x a_{-n}) Z,$$

where $\pi$ runs over all nonempty subsets of $\{1, 2, \ldots, n\}$ containing even number of elements.

For an arbitrary element $a \in \Gamma$, arbitrary (may be empty) subset $1 \leq i_1 < \cdots < i_k \leq n$, we have $a D_{i_1} \cdots D_{i_k} \in Z + I + M_1$ (see Lemmas 3.2.12, 3.2.17, 3.2.21).

For an arbitrary element $a \in \Gamma'$, arbitrary (may be empty) subset $1 \leq i_1 < \cdots < i_k \leq n$, we have $a D_{i_1} \cdots D_{i_k} \in \sum_{i=1}^n Z + I + M_1$ (see Lemmas 3.2.14, 3.2.19, 3.2.20).

Hence $J = \Gamma + \Gamma'$ is a direct sum of subspaces.

LEMMA 3.2.24. (a) $\Gamma \Gamma' \subseteq \Gamma$.

(b) $\Gamma' \Gamma' \subseteq \Gamma'$.

(c) $\Gamma' \Gamma \subseteq \Gamma$.

Proof. (a) Let $a, b \in \Gamma$ be root elements. If $ab \notin \Gamma$, then there exists a subset $1 \leq i_1 < \cdots < i_k \leq n$ such that the projection of $(ab) D_{i_1} \cdots D_{i_k}$
on $\sum_{i=1}^n v_i Z$ is not zero. However,
\[(ab) D_{i_1} \cdots D_{i_k} = \sum (aD_{i_1} \cdots D_{i_n})(bD_{i_1} \cdots D_{i_n}) \leq (Z + I + M_I)(Z + I + M_I) \subseteq Z + I + M_I,
\]
the contradiction.

(b) Let $a \in \Gamma$, $b \in \Gamma'$. If $ab \notin \Gamma'$, then there exists a subset $1 \leq i_1 < \cdots < i_k \leq n$ such that the projection of $(ab) D_{i_1} \cdots D_{i_k}$ on $Z$ is not zero. However, $(ab) D_{i_1} \cdots D_{i_k} = \sum (aD_{i_1} \cdots D_{i_n})(bD_{i_1} \cdots D_{i_n}) \leq (Z + I + M_I)(\sum v_i Z + I + M_I) \subseteq Z + I + M_I$, the contradiction.

(c) First we will show that, for an arbitrary $1 \leq i \leq n$, we have $\Gamma' D_i \subseteq \Gamma'$. Clearly, $t_{\mathcal{Q}}(\Gamma' D_i) = (0)$. If $\Gamma' D_i \subseteq \Gamma'$, then there exists a subset $1 \leq i_1 < \cdots < i_k \leq n$ such that $t_{\mathcal{Q}}(\Gamma' D_i D_{i_1} \cdots D_{i_k}) \neq (0)$. If $i \notin \{i_1, \ldots, i_k\}$, then this contradicts what we already know about $\Gamma'$. If $i \in \{i_1, \ldots, i_k\}$, then $\Gamma' D_i D_{i_1} \cdots D_{i_k} = \Gamma' D_{i_1} \cdots D_{i_{k-1}} D_{i_k}$, where $\{j_1, \ldots, j_{k-1}\} = \{i_1, \ldots, i_k\} \setminus \{i\}$. It is easy to see that $\Gamma' D_{j_1} \cdots D_{j_{k-1}} D_{j_k} \subseteq \Gamma' D_i \cdots D_{i_{k-1}} D_{i_k}$. Hence $t_{\mathcal{Q}}(\Gamma' D_{j_1} \cdots D_{j_{k-1}} D_{j_k}) = (0)$ follows from $t_{\mathcal{Q}}(\Gamma' D_{j_1} \cdots D_{j_{k-1}} D_{j_k}) = (0)$.

We have $(\sum_{i=1}^n v_i Z + \sum_{i=1}^n (xa_i Z) J \subseteq I + M_I$. Hence
\[\Gamma' \Gamma' = \left(\sum_{i=1}^n Z v_i + xZ\right) \left(\sum_{i=1}^n Z v_i + xZ\right) \text{ mod } I + M_I.\]

Hence $\Gamma' \Gamma'$ has zero projection on $\sum_{i=1}^n Z v_i$. If $\Gamma' \Gamma' \not\subseteq \Gamma$, then there exists a subset $1 \leq i_1 < \cdots < i_k \leq n$ such that $\Gamma' \Gamma' D_{i_1} \cdots D_{i_k}$ has a nonzero projection on $\sum_{i=1}^n Z v_i$. But $\Gamma' \Gamma' D_{i_1} \cdots D_{i_k} \subseteq \Gamma' \Gamma' D_{i_1} \cdots D_{i_k} \subseteq \Gamma' \Gamma'$, the contradiction. Lemma is proved.

**Lemma 3.2.25.** For arbitrary elements $a, b \in \Gamma$, we have $R(ab) = R(a)R(b)$.

**Proof.** Let us first check that for arbitrary elements $a, b \in Z + I + M_I$, we have $JR(a)R(b) \subseteq I + M_I$. We can assume that $a, b \in Z \cup I \cup M_I$.

If $a \in I$ or $b \in I$, then both $JR(a)R(b)$ and $JR(ab)$ lie in $(IJ)J \subseteq I + M_I$.

Assume therefore that $a, b \in Z \cup M_I$. If $a, b \in Z$, then the assertion follows from Lemma 3.2.12. If $a, b \in M_I$, then $ab \in I$, $AR(M_I)R(M_I) \subseteq [M_I, M_I] \subseteq I$, and $MR(M_I)R(M_I) \subseteq AM_I \subseteq M_I$. Let $a \in M_I$, $b \in Z$. Clearly, $AR(a)R(b)$ and $AR(ab)$ lie in $M_I$. We have to check that $M(R(a)R(b) - R(ab)) \in I$. Since $M_I = (MI)A$, we can assume that $a = ac$, where $a' \in MI$, $c \in A$. Then $R(a'c)R(b) - R((a'c)b) = -R(ab)R(c)$.
\(- R(bc)R(a') + R(a')R(b)R(c) + R(c)R(b)R(a')\). In Section 1 it was
proved that \([M,(MI)Z] \subseteq I\); hence, \(MR(a'b) \subseteq I\). Other summands lie in
\(I\) for similar reasons.

If \(a \in Z, b \in M_f\), the argument is similar. Let \(b = b'c\), where \(b' \in
MI, c \in A\). Then \(R(a)R(b'c) - R(a(b'c)) = -R(b')R(ac) - R(c)R(b'a)
+ R(b')R(a)R(c) + R(c)R(a)R(b')\), which implies the result.

In particular, since \(\Gamma \subseteq Z + I + M_f\), it follows that for arbitrary ele-
ments, \(u \in A \cup M, a, b \in \Gamma\), we have \(u(R(a)R(b) - R(ab)) \subseteq I + M_f\). If
\(u(R(a)R(b) - R(ab)) \neq 0\), then there exists a subset \(1 \leq i_1 < \cdots < i_k \leq n\)
such that \(u(R(a)R(b) - R(ab))D_{i_1} \cdots D_{i_k} \in A \setminus I\). But

\[
u(R(a)R(b) - R(ab))D_{i_1} \cdots D_{i_k} = \sum \nu' \left( R(aD_{n_1} \cdots D_{n_s})R(bD_{\beta_1} \cdots D_{\beta_t}) \right)
- R\left( (aD_{n_1} \cdots D_{n_s})(bD_{\beta_1} \cdots D_{\beta_t}) \right) \]

From \(a, b \in \Gamma\), it follows that \(aD_{n_1} \cdots D_{n_s}, bD_{\beta_1} \cdots D_{\beta_t} \in Z + I + M_f\).

Hence, every summand of the sum above lies in \(I + M_f\), the contradiction.

Lemma is proved.

**Lemma 3.2.26.** \(\Gamma' = \Gamma x\).

**Proof.** We have \(\Gamma' R(x)^2 = (\Gamma x)x \subseteq \Gamma x\). Suppose that \(N \neq I\). Then
there exists an element \(a \in Z\) such that the right multiplication \(R(aR(x)^2)\)
is invertible. Hence \(\Gamma' = \Gamma'(aR(x)^2) \subseteq (\Gamma'a)R(x)^2 + (\Gamma'R(x)^2)a \subseteq \Gamma x\).

Now let \(A/I\) be a simple Jordan algebra of a bilinear form. We have
\(a_i R(x)R(v_j)R(x) = v_j \bmod I(v_j)\). Hence \(v_j \in \Gamma x + I(v_j)\). If \(n\) is even, by
Corollary 3.2.1, \(I(v_i) = \langle 0 \rangle\). If \(n\) is odd, then \(I(v_j)\) is a product of \(a_i\)'s and
\(Z\), and therefore, \(I(v_j) \subseteq \Gamma\). Hence \(v_j \in \Gamma x + \Gamma\). From Lemma 3.2.23, it
follows that \(Z, a_i, v_j, \ldots, v_n, x\) generate \(J\). Since \(\Gamma + \Gamma x\) is a subsuperal-
gebra of \(J\), we conclude that \(J = \Gamma + \Gamma x, \Gamma' = \Gamma x\). Lemma is proved.

To finish the proof of the theorem, we need only to check that \(0 \neq a \in \Gamma\)
implies that \(ax \neq 0\). If \(K = \{a \in \Gamma | ax = 0\}\), then \(K\) is an ideal of \(J\).
Hence \(K = \langle 0 \rangle\).

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