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Weak Krull–Schmidt for Infinite Direct Sums of Uniserial Modules

Nguyen Viet Dung

Institute of Mathematics, P.O. Box 631, Bo Ho, Hanoi, Vietnam

and

Alberto Facchini

Dipartimento di Matematica e Informatica, Università di Udine, 33100 Udine, Italy

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1. INTRODUCTION

The well-known Krull–Schmidt–Azumaya theorem [2] says that if $M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j$ are two indecomposable decompositions of a module M, where M_i has local endomorphism ring for each $i \in I$, then there is a bijection $\sigma\colon I \to J$ such that $M_i \cong N_{\sigma(i)}$ for every $i \in I$. In this theorem the local endomorphism ring hypothesis is crucial to ensure the uniqueness of the decomposition of M into indecomposable summands, and it is always tempting to ask whether under some weaker hypotheses the indecomposable decomposition of a module is unique. This was done, for example, by Anderson and Fuller [1, Theorem 12.4], who showed that the conclusion of the Krull–Schmidt–Azumaya theorem still remains valid for modules with an indecomposable decomposition that complements maximal direct summands. In general, however, there exist various examples in the literature which show that Krull–Schmidt may fail even under "rather good" hypotheses on the rings or on the indecomposable summands (see, e.g., [5], [7], and [8]).

In 1975 Warfield proved that every finitely presented module over a serial ring is a finite direct sum of uniserial modules, and asked whether

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the direct decomposition of a finitely presented module into uniserial summands is unique up to isomorphism [11]. Recently the second author has answered Warfield's question completely, showing that although there exist serial rings for which the Krull–Schmidt theorem does not hold for finitely presented modules, it is possible to prove a *weak* Krull–Schmidt theorem for *finite* direct sums of uniserial modules [4]. In order to recall this main result of [4] we need the concepts of monogeny class and epigeny class of a module. Two modules A and B are said to *belong to the same monogeny class*, written $[A]_m = [B]_m$, if there are a monomorphism $A \to B$ and a monomorphism $B \to A$. Similarly, the modules A and B are said to *belong to the same epigeny class*, written $[A]_e = [B]_e$, if there are an epimorphism $A \to B$ and an epimorphism $B \to A$. The weak Krull–Schmidt theorem for finite direct sums of uniserial modules proved in [4] says that if $U_1, \dots, U_n, V_1, \dots, V_t$ are nonzero uniserial modules over an arbitrary ring R, then $U_1 \oplus \cdots \oplus U_n \cong V_1 \oplus \cdots \oplus V_t$ if and only if $n = t$ and there are two permutations σ, τ of $\{1, 2, \dots, n\}$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i = 1, 2, \dots, n$.

The purpose of this paper is to determine necessary and sufficient conditions for two *arbitrary* families of uniserial modules $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ in order to have the property that $\bigoplus_{i \in I} U_i \cong \bigoplus_{j \in J} V_j$. Our main results are Theorems 3.1, 3.3, 4.7, 4.8, and 5.4, which show that most of the results proved in [4] for finite families can be extended to arbitrary (infinite) families as well, and during this extension new concepts, like the concept of *quasismall* uniserial modules, appear in a natural way (see Sect. 3 for the definition of quasismall modules).

The weak Krull–Schmidt theorem for finite direct sums of uniserial modules consists of *two* logical implications (the necessary condition and the sufficient condition). In Theorems 3.1 and 3.3 we prove that "one and a half" of these two logical implications still hold for infinite direct sums of uniserial modules. This generalizes the results in [4] considerably. In Theorem 4.8 we show that if $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ are two families of quasismall uniserial modules over an arbitrary ring R, then $\bigoplus_{i \in I} U_i \cong \bigoplus_{j \in J} V_j$ if and only if there are two bijections $\sigma, \tau \colon I \to J$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i \in I$. The class of quasismall uniserial modules is large enough to contain all finitely generated and all uncountably generated uniserial modules, and also uniserial modules with local endomorphism rings. Moreover, we show that, in some sense, this is the largest possible class of uniserial modules for which the above weak Krull–Schmidt theorem holds (Theorem 4.9). Finally, we establish a weak Krull–Schmidt theorem for locally semi-T-nilpotent families of uniserial modules (Theorem 5.4). This also is a generalization of the finite direct sum case studied in [4], but in a different direction.

2. PRELIMINARY LEMMAS

Throughout this paper we consider unitary right modules over an associative ring R with identity. If $f\colon A \to B$ is a mapping and C is a subset of A, we will write $f|_C$ for the restriction of f to C.

Recall that a module is *uniserial* if its lattice of submodules is linearly ordered under inclusion, and is *serial* if it is a direct sum of uniserial modules. Two modules A and B are said to belong to *the same monogeny class*, written $[A]_m = [B]_m$, if there are a monomorphism $f\colon A \to B$ and a monomorphism $g\colon B \to A$. Similarly, A and B are in *the same epigeny class*, written $[A]_e = [B]_e$, if there are an epimorphism $h\colon A \to B$ and an epimorphism $l\colon B \to A$. Obviously, these are two equivalence relations. Roughly speaking, the modules which belong to the same monogeny or epigeny classes share many common properties, but they need not be isomorphic. The significance of these concepts for uniserial modules is highlighted by the fact that any uniserial module is uniquely determined by its monogeny and epigeny classes.

LEMMA 2.1. *Let U and V be uniserial modules such that $[U]_m = [V]_m$ and $[U]_e = [V]_e$. Then $U \cong V$.*

Proof. See [4, Proposition 1.6]. ∎

In the rest of this section, for the reader's convenience, we collect some results that will be used repeatedly in the sequel. The proof of Lemma 2.2 follows from [4, Proposition 1.7 and Lemma 1.8], Lemmas 2.3 and 2.4 are [4, Proposition 1.6 and Lemma 1.1], and Lemma 2.5 is a special case of [4, Lemma 1.4].

LEMMA 2.2. *Let V, V' be uniserial modules. Suppose that there exists a uniserial module U such that $[U]_m = [V]_m$ and $[U]_e = [V']_e$. Then there are uniserial submodules X and Y of $V \oplus V'$ such that $V \oplus V' = X \oplus Y$ and $X \cong U$. Moreover, $[Y]_m = [V']_m$ and $[Y]_e = [V]_e$.*

LEMMA 2.3 (cancellation property). *Let A, B be right modules over an arbitrary ring R, and let U be a uniserial right R-module such that $A \oplus U \cong B \oplus U$. Then $A \cong B$.*

LEMMA 2.4. *Let A, C be nonzero right modules over an arbitrary ring R, let B be a uniserial right R-module, and let $\alpha\colon A \to B$, $\beta\colon B \to C$ be homomorphisms. Then*

(a) *$\beta\alpha$ is a monomorphism if and only if β and α are both monomorphisms;*

(b) *$\beta\alpha$ is an epimorphism if and only if β and α are both epimorphisms.*

LEMMA 2.5. *Let U be a uniserial module over an arbitrary ring R.*

(a) *If f and g are two endomorphisms of U, f is injective and nonsurjective, and g is surjective and noninjective, then $f+g$ is an automorphism.*

(b) *Conversely, suppose that $f_1, \dots, f_n$ are n endomorphisms of U, none of which is an automorphism. If $f_1 + \cdots + f_n$ is an automorphism, then there exist two distinct indices $i, j = 1, 2, \dots, n$ such that f_i is injective and nonsurjective, and f_j is surjective and noninjective.*

The proof of the next result is essentially based on Lemmas 2.4 and 2.5(b); see [4, Proposition 1.5 and 1.7] for details.

LEMMA 2.6. *Suppose U is a uniserial module such that $U \oplus B = C_1 \oplus \cdots \oplus C_n$ for arbitrary modules $B, C_1, \dots, C_n$ and $n \geq 2$. Then there are two distinct indices i and j and a direct decomposition $C_i \oplus C_j = U' \oplus C'$ such that $U' \cong U$ and $B \cong C' \oplus (\bigoplus_{k \neq i, j} C_k)$. In particular, if $C_1, \dots, C_n$ are uniserial modules, there are two indices i, j, possibly equal, such that $[U]_m = [C_i]_m$ and $[U]_e = [C_j]_e$.*

Finally, we record the following elementary lemma which is very useful in the study of direct sum decompositions of modules.

LEMMA 2.7. *Let M be a module and A, B, C submodules of M. Suppose $M = A \oplus B$. Let $\pi_B \colon M = A \oplus B \to B$ denote the canonical projection. Then $M = A \oplus C$ if and only if $\pi_B|_C \colon C \to B$ is an isomorphism.*

Proof. See [1, Proposition 5.5]. ∎

3. ARBITRARY FAMILIES OF UNISERIAL MODULES

In this section we consider two arbitrary families of uniserial modules $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$. Our first result says that if the families $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ have the same monogeny and epigeny classes, then $\bigoplus_{i \in I} U_i \cong \bigoplus_{j \in J} V_j$. This fact was proved in [4] for finite families by induction on the number of uniserial summands. In the case in which the index sets I and J are infinite, such induction argument does not seem to be applicable, and a new method of proof is required. The idea of our proof below is reminiscent of the proof of the classical Cantor–Schroeder–Bernstein theorem in set theory.

THEOREM 3.1. *Let $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ be two families of uniserial modules over an arbitrary ring R. Suppose that there are two bijections $\sigma, \tau \colon I \to J$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i \in I$. Then $\bigoplus_{i \in I} U_i \cong \bigoplus_{j \in J} V_j$.*

Proof. Suppose that there are two bijections $\sigma, \tau\colon I \to J$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i \in I$. We have to show that $\bigoplus_{i \in I} U_i \cong \bigoplus_{j \in J} V_j$.

The symmetric group S_I consisting of all bijection $I \to I$ acts on the set I in a natural way. Let C be the cyclic subgroup of S_I generated by $\tau^{-1}\sigma \in S_I$. Then C acts on the set I. For every element $i \in I$ let

$$[i] = \left\{ (\tau^{-1}\sigma)^z (i) \mid z \in \mathbf{Z} \right\}$$

denote the C-orbit of i. Let $\sigma([i]) = \{\sigma(x) \mid x \in [i]\}$ be the image of the orbit $[i]$ via the bijection σ.

Fix an element $i \in I$. We claim that $\bigoplus_{k \in [i]} U_k \cong \bigoplus_{l \in \sigma([i])} V_l$.

For simplicity of notation set $i_z = (\tau^{-1}\sigma)^z(i)$, $j_z = \sigma(i_z)$, $U_z = U_{i_z}$, and $V_z = V_{j_z}$ for every $z \in \mathbf{Z}$. Hence if the orbit $[i]$ is infinite, then $\sigma([i])$ is infinite, and $U_n = U_m$ if and only if $n = m$. If the orbit $[i]$ is finite of order q, then $\sigma([i])$ is finite of order q, and $U_n = U_m$ if and only if $n \equiv m \pmod q$. Note that $\tau(i_z) = \tau(\tau^{-1}\sigma)^z(i) = \sigma(\tau^{-1}\sigma)^{z-1}(i) = \sigma(i_{z-1}) = j_{z-1}$. In this notation the equality $[U_i]_m = [V_{\sigma(i)}]_m$ for every $i \in I$ implies that

$$[U_z]_m = [V_z]_m \tag{1}$$

for all $z \in \mathbf{Z}$, and similarly the equality $[U_i]_e = [V_{\tau(i)}]_e$ implies that

$$[U_z]_e = [V_{z-1}]_e \tag{2}$$

for all $z \in \mathbf{Z}$.

For every integer $n \geq 0$ define two direct sum decompositions $X_n \oplus Y_n = V_n \oplus V_{-n-1}$ of $V_n \oplus V_{-n-1}$ and $X'_n \oplus Y'_n = U_{n+1} \oplus U_{-n-1}$ of $U_{n+1} \oplus U_{-n-1}$ by induction on n with the following properties:

(a) $[X_n]_m = [U_n]_m$ and $[X_n]_e = [U_{-n}]_e$ for every $n \geq 0$;

(b) $X_{n+1} \cong X'_n$ for every $n \geq 0$;

(c) $Y_n \cong Y'_n$ for every $n \geq 0$.

Since $[U_0]_m = [V_0]_m$ and $[U_0]_e = [V_{-1}]_e$, there are submodules X_0 and Y_0 of $V_0 \oplus V_{-1}$ such that $V_0 \oplus V_{-1} = X_0 \oplus Y_0$, $X_0 \cong U_0$, $[Y_0]_m = [V_{-1}]_m$, and $[Y_0]_e = [V_0]_e$ (Lemma 2.2). Hence $[U_1]_e = [Y_0]_e$. Similarly, we get that $[U_{-1}]_m = [V_{-1}]_m = [Y_0]_m$. Hence, again by Lemma 2.2, there are submodules X'_0 and Y'_0 of $U_1 \oplus U_{-1}$ such that $U_1 \oplus U_{-1} = X'_0 \oplus Y'_0$ and $Y'_0 \cong Y_0$. Thus X_0, Y_0, X'_0, Y'_0 have the required properties.

Suppose now $n \geq 1$ and that X_t, Y_t, X'_t, Y'_t satisfying the required properties have already been defined for every $t = 0, 1, \dots, n-1$. Then $X_{n-1} \oplus Y_{n-1} = V_{n-1} \oplus V_{-n}$. Since $[X_{n-1}]_m = [U_{n-1}]_m = [V_{n-1}]_m$ by (a) and (1), it follows from Lemma 2.2 that $[Y_{n-1}]_m = [V_{-n}]_m = [U_{-n}]_m$. Similarly,

$[X_{n-1}]_e = [U_{-n+1}]_e = [V_{-n}]_e$ by (a) and (2), so that $[Y_{n-1}]_e = [V_{n-1}]_e = [U_n]_e$. By (c) it follows that $[Y'_{n-1}]_m = [U_{-n}]_m$ and $[Y'_{n-1}]_e = [U_n]_e$. These last two equalities and $X'_{n-1} \oplus Y'_{n-1} = U_n \oplus U_{-n}$ imply that

$$[X'_{n-1}]_m = [U_n]_m \quad \text{and} \quad [X'_{n-1}]_e = [U_{-n}]_e. \tag{3}$$

From (1) and (2) we get that $[X'_{n-1}]_m = [V_n]_m$ and $[X'_{n-1}]_e = [V_{-n-1}]_e$. By Lemma 2.2 there exist X_n, Y_n such that $X_n \oplus Y_n = V_n \oplus V_{-n-1}$ and $X_n \cong X'_{n-1}$. Then property (a) holds for X_n and Y_n by (3). And property (b) holds as well.

From (3) it follows that $[X_n]_m = [X'_{n-1}]_m = [U_n]_m = [V_n]_m$ and $[X_n]_e = [X'_{n-1}]_e = [U_{-n}]_e = [V_{-n-1}]_e$. These equalities and $X_n \oplus Y_n = V_n \oplus V_{-n-1}$ imply that $[Y_n]_m = [V_{-n-1}]_m$ and $[Y_n]_e = [V_n]_e$. Hence $[Y_n]_m = [U_{-n-1}]_m$ and $[Y_n]_e = [U_{n+1}]_e$ by (1) and (2). Lemma 2.2 yields a decomposition $X'_n \oplus Y'_n = U_{n+1} \oplus U_{-n-1}$ of $U_{n+1} \oplus U_{-n-1}$, where X'_n and Y'_n are suitable uniserial submodules of $U_{n+1} \oplus U_{-n-1}$ and $Y'_n \cong Y_n$. This completes the induction. Note that (a) implies that $X_0 \cong U_0$.

Suppose that the orbit $[i]$ is an infinite set. Then

$$\begin{aligned} \bigoplus_{k \in [i]} U_k &= \bigoplus_{n \in \mathbf{Z}} U_n = U_0 \oplus \left(\bigoplus_{n \geq 0} (X'_n \oplus Y'_n) \right) \\ &\cong X_0 \oplus \left(\bigoplus_{n \geq 0} (X_{n+1} \oplus Y_n) \right) = \bigoplus_{n \geq 0} (V_n \oplus V_{-n-1}) = \bigoplus_{l \in \sigma([i])} V_l . \end{aligned}$$

This proves the claim for the case of an infinite orbit $[i]$.

Now suppose that the orbit $[i]$ is a finite set with q elements. Consider the case that $q = 2n + 1$ for some $n \geq 0$. Then $V_n = V_{-n-1}$, and by the equality $X_n \oplus Y_n = V_n \oplus V_{-n-1}$, it follows by Lemma 2.2 that $[X_n]_m = [V_n]_m$ and $[X_n]_e = [V_n]_e$, hence by Lemma 2.1 we get $X_n \cong V_n$. Therefore

$$\begin{aligned} \bigoplus_{k \in [i]} U_k &= U_0 \oplus \left(\bigoplus_{k=1}^{n} (U_k \oplus U_{-k}) \right) = U_0 \oplus \left(\bigoplus_{k=1}^{n} (X'_{k-1} \oplus Y'_{k-1}) \right) \\ &\cong X_0 \oplus \left(\bigoplus_{k=1}^{n} (X_k \oplus Y_{k-1}) \right) = \bigoplus_{k=0}^{n-1} (X_k \oplus Y_k) \oplus X_n \\ &\cong \bigoplus_{k=0}^{n-1} (V_k \oplus V_{-k-1}) \oplus V_n = \bigoplus_{l \in \sigma([i])} V_l , \end{aligned}$$

which proves our claim for the case of a finite orbit $[i]$ with an odd number q of elements. Similarly, for the case $q = 2n$ with $n \geq 1$, we have $U_n = U_{-n}$, hence the equality $X'_{n-1} \oplus Y'_{n-1} = U_n \oplus U_{-n}$ implies that $Y_{n-1} \cong$

$Y'_{n-1} \cong U_n$. Then we have

$$\begin{aligned}\bigoplus_{k\in[i]} U_k &= U_0 \oplus \left(\bigoplus_{k=1}^{n-1} (U_k \oplus U_{-k})\right) \oplus U_n \\ &= U_0 \oplus \left(\bigoplus_{k=1}^{n-1} (X'_{k-1} \oplus Y'_{k-1})\right) \oplus U_n \\ &\cong X_0 \oplus \left(\bigoplus_{k=1}^{n-1} (X_k \oplus Y_{k-1})\right) \oplus Y_{n-1} \\ &= \bigoplus_{k=0}^{n-1} (V_k \oplus V_{-k-1}) = \bigoplus_{l\in\sigma([i])} V_l.\end{aligned}$$

This concludes the proof of the claim.

When the index i runs over all the indices in I, we get that the orbits $[i]$ form a partition of I into disjoint countable subsets $I = \bigcup_{i\in I}[i]$ and their images $\sigma([i])$ form a partition of J into disjoint countable subsets $J = \bigcup_{i\in I}\sigma([i])$. By the claim $\bigoplus_{k\in[i]}U_k \cong \bigoplus_{l\in\sigma([i])}V_l$ for every orbit $[i]$, so that $\bigoplus_{i\in I}U_i \cong \bigoplus_{j\in J}V_j$. ∎

In view of Theorem 3.1 it is natural to ask whether the existence of two bijections between the monogeny and epigeny classes of $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ is a necessary condition for the isomorphism $\bigoplus_{i\in I}U_i \cong \bigoplus_{j\in J}V_j$. Now we establish a partial converse of Theorem 3.1, namely that if $\bigoplus_{i\in I}U_i \cong \bigoplus_{j\in J}V_j$, then the families $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ have the same monogeny classes.

First we prove an auxiliary lemma.

LEMMA 3.2. *Let $M = \bigoplus_{j\in J}A_j$ be a direct sum of arbitrary modules A_j. Suppose that $M = U \oplus B$, where U and B are submodules of M, and U is nonzero uniserial. Let $\pi_U\colon M = U \oplus B \to U$ and $\pi_i\colon M = \bigoplus_{j\in J}A_j \to A_i$ be the canonical projections. Then there exists an index $k \in J$ such that $\pi_U|_{A_k} \circ \pi_k|_U$ is an injective endomorphism of U.*

Proof. Take a nonzero element $x \in U$. Then $x \in A_{j_1} \oplus \cdots \oplus A_{j_n}$ for some $j_1, \ldots, j_n \in J$. Set $C = \bigoplus_{j\neq j_1,\ldots,j_n}A_j$, so that $M = A_{j_1} \oplus \cdots \oplus A_{j_n} \oplus C$. If $\varepsilon_U, \pi_U, \varepsilon_B, \pi_B, \varepsilon_{j_t}, \pi_{j_t}$ $(t = 1,\ldots,n)$ and ε_C, π_C are the injections and the projections associated with the two direct decompositions $M = U \oplus B = A_{j_1} \oplus \cdots \oplus A_{j_n} \oplus C$, then

$$\begin{aligned}1_U = \pi_U\varepsilon_U &= \pi_U(\varepsilon_{j_1}\pi_{j_1} + \cdots + \varepsilon_{j_n}\pi_{j_n} + \varepsilon_C\pi_C)\varepsilon_U \\ &= \pi_U\varepsilon_{j_1}\pi_{j_1}\varepsilon_U + \cdots + \pi_U\varepsilon_{j_n}\pi_{j_n}\varepsilon_U + \pi_U\varepsilon_C\pi_C\varepsilon_U.\end{aligned}$$

By Lemma 2.5(b) at least one of these summands must be a monomorphism. Since $\pi_U \varepsilon_C \pi_C \varepsilon_U(x) = 0$, the last summand cannot be a monomorphism. Therefore there is an index $t = 1, \dots, n$ such that $\pi_U \varepsilon_{j_t} \pi_{j_t} \varepsilon_U$ is a monomorphism. Set $k = j_t$. ∎

THEOREM 3.3. *Let $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ be two families of nonzero uniserial modules over an arbitrary ring R and suppose that $\bigoplus_{i \in I} U_i \cong \bigoplus_{j \in J} V_j$. Then there is a bijection $\sigma\colon I \to J$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ for every $i \in I$.*

Proof. We may suppose that $M = \bigoplus_{i \in I} U_i = \bigoplus_{j \in J} V_j$ with U_i, V_j nonzero uniserial modules for every i and j.

Fix an index $k \in I$ and consider the two subsets $I(k) = \{i \in I \mid [U_i]_m = \{U_k]_m\}$ of I and $J(k) = \{j \in J \mid [V_j]_m = [U_k]_m\}$ of J. It is obvious that the $I(k)$, $k \in I$, form a partition of I. Note that the $J(k)$, $k \in I$, also form a partition of J, because for every $j \in J$ there is a $k \in I$ with $[V_j]_m = [U_k]_m$ and for every $k \in I$ there is a $j \in J$ with $[V_j]_m = [U_k]_m$ (Lemmas 2.4(a) and 3.2).

In order to establish the existence of the bijection between the monogeny classes of $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$, it is sufficient to prove that the cardinalities $|I(k)|$ and $|J(k)|$ are equal for every $k \in I$.

Suppose first that either $I(k)$ or $J(k)$ is a finite set. Without loss of generality we may assume $|I(k)| \le |J(k)|$. Suppose that $|I(k)| < |J(k)|$. If $I(k) = \{i_1, \dots, i_n\}$, let $\{j_1, \dots, j_{n+1}\}$ be a subset of $J(k)$ of cardinality $n+1$. Write

$$M = U_{i_1} \oplus \cdots \oplus U_{i_n} \oplus B = V_{j_1} \oplus \cdots \oplus V_{j_{n+1}} \oplus C,$$

where

$$B = \bigoplus_{i \neq i_1, \dots, i_n} U_i \quad \text{and} \quad C = \bigoplus_{j \neq j_1, \dots, j_{n+1}} V_j.$$

We shall show by induction on n that the serial module B has a uniserial direct summand V isomorphic to V_{j_t} for some $t = 1, \dots, n+1$, which gives us a contradiction because by Lemmas 2.4(a) and 3.2 there exists $i \in I \setminus \{i_1, \dots, i_n\}$ with $[U_i]_m = [V]_m = [V_{j_t}]_m$.

Apply Lemma 2.6 to U_{i_1} and the finite direct sum $M = V_{j_1} \oplus \cdots \oplus V_{j_{n+1}} \oplus C$. One of the following two cases must hold.

(1) There are distinct indices, say j_1 and j_2, and a direct decomposition $V_{j_1} \oplus V_{j_2} = W_1 \oplus W_2$ of $V_{j_1} \oplus V_{j_2}$ such that $W_1 \cong U_{i_1}$. From Lemma 2.6 it follows that either $[U_{i_1}]_e = [V_{j_1}]_e$ or $[U_{i_i}]_e = [V_{j_2}]_e$. Since $[U_{i_1}]_m = [V_{j_1}]_m = [V_{j_2}]_m$, it follows that either $U_{i_1} \cong V_{j_1}$ or $U_{i_1} \cong V_{j_2}$ (Lemma 2.1). If, for instance, $U_{i_1} \cong V_{j_1}$, then

$$U_{i_2} \oplus \cdots \oplus U_{i_n} \oplus B \cong V_{j_2} \oplus \cdots \oplus V_{j_{n+1}} \oplus C$$

because the cancellation property holds for uniserial modules (Lemma 2.3).

(2) There is an index, say j_1, and a direct decomposition $V_{j_1} \oplus C = W_1 \oplus D$ of $V_{j_1} \oplus C$ such that $W_1 \cong U_{i_1}$. Then again by Lemma 2.3

$$U_{i_2} \oplus \cdots \oplus U_{i_n} \oplus B \cong V_{j_2} \oplus \cdots \oplus V_{j_{n+1}} \oplus D.$$

An easy induction shows that after n steps we get the required contradiction.

Now suppose that $I(k)$ and $J(k)$ are both infinite. By symmetry it is sufficient to prove that $|J(k)| \leq |I(k)|$. Let $\pi_k\colon \bigoplus_{i \in I} U_i \to U_k$ and $p_l\colon \bigoplus_{j \in J} V_j \to V_l$ be the canonical projections. For every $t \in I$ define a subset $A(t)$ of J as follows: $A(t) = \{ l \in J \mid \pi_t|_{V_l}\colon V_l \to U_t$ and $p_l|_{U_t}\colon U_t \to V_l$ are both monomorphisms$\}$.

Note that $A(t)$ is a finite set because there is a finite subset F of J with $U_t \cap (\bigoplus_{j \in F} V_j) \neq 0$, so that $p_l|_{U_t}$ is not a monomorphism for every $l \in J \setminus F$.

We claim that

$$J(k) \subseteq \bigcup_{t \in I(k)} A(t).$$

In order to prove the claim take an index $j \in J(k)$. By Lemma 3.2 applied to the direct summand V_j of $M = \bigoplus_{i \in I} U_i$, there exists an index $t \in I$ such that $\pi_t|_{V_j} \circ p_j|_{U_t}$ is a monomorphism. Hence both $\pi_t|_{V_j}$ and $p_j|_{U_t}$ are monomorphisms by Lemma 2.4(a), i.e., $j \in A(t)$. Since $[U_t]_m = [V_j]_m = [U_k]_m$, it follows that $t \in I(k)$. This proves the claim.

It follows that $|J(k)| \leq \aleph_0 |I(k)| = |I(k)|$. Hence $|J(k)| = |I(k)|$ if $I(k)$ and $J(k)$ are both infinite. ∎

Remark. A careful analysis of our arguments shows that the results of this section hold for a class of modules more general than the class of uniserial modules, namely, the class of modules of both Goldie dimension 1 and dual Goldie dimension 1 (i.e., modules A with the property that, for any submodules B, C of A, $B \cap C = 0$ implies $B = 0$ or $C = 0$ and $B + C = A$ implies $B = A$ or $C = A$). However, the condition that the modules are uniserial will be essential in the next section.

4. QUASISMALL UNISERIAL MODULES

Suppose $M = \bigoplus_{i \in I} U_i = \bigoplus_{j \in J} V_j$ with U_i, V_j uniserial modules for every $i \in I$ and $j \in J$. In this section we discuss the existence of a bijection $\tau\colon I \to J$ preserving the epigeny classes. In [4] it was shown that such a bijection always exists if M has finite Goldie dimension. However, when M has infinite Goldie dimension, the situation is more complicated. We will show that there exists a bijection between the epigeny classes of only certain subfamilies of $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$, and these subfamilies are in some sense the largest possible subfamilies for which our weak Krull–Schmidt theorem holds.

DEFINITION 4.1. Let U be a module over an arbitrary ring R. We say that U is *quasismall* if whenever U is isomorphic to a direct summand of a direct sum $\bigoplus_{\lambda \in \Lambda} M_\lambda$ of arbitrary modules M_λ, there is a finite subset $F \subseteq \Lambda$ such that U is isomorphic to a direct summand of $\bigoplus_{\lambda \in F} M_\lambda$.

The reason for the terminology just introduced is the following. A module U is *small* if for any direct sum $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ with projection π_λ and any monomorphism $f\colon U \to M$ we have $\pi_\lambda \circ f = 0$ for all but a finite number of indices λ (see [6] or [9]). Obviously every small module is quasismall. In particular, every finitely generated module is quasismall.

The following result is due to Fuchs and Salce [6, Lemma 24]. Since their lemma was considered in the setting of commutative valuation rings, we include its proof for the reader's convenience.

LEMMA 4.2. *Every uniserial module is either countably generated or small.*

Proof. Let U be a uniserial module and suppose that U is not small. Then U is a submodule of a direct sum $\bigoplus_{i \in I} A_i$ of suitable modules A_i such that $U \not\subseteq \bigoplus_{i \in F} A_i$ for every finite subset F of I.

For every element $x = (x_i)_{i \in I} \in \bigoplus_{i \in I} A_i$ set $\operatorname{supp}(x) = \{i \in I \mid x_i \neq 0\}$. Define a sequence of elements $u_n \in U$ by induction on the positive integer n as follows. Let u_1 be an arbitrary nonzero element of U. Suppose $u_n \in U$ has been defined. Then $U \not\subseteq \bigoplus_{i \in \operatorname{supp}(u_n)} A_i$, so that there is an element $u_{n+1} \in U$ with $\operatorname{supp}(u_{n+1}) \not\subseteq \operatorname{supp}(u_n)$. This defines the sequence u_n.

Since U is uniserial, $u_{n+1}R \supseteq u_n R$, so that $\operatorname{supp}(u_{n+1}) \supset \operatorname{supp}(u_n)$. Suppose that $\bigcup_{n \geq 0} u_n R \subset U$. If $v \in U$ is an element such that $v \notin \bigcup_{n \geq 0} u_n R$, then $\operatorname{supp}(v)$ is a finite subset of I that contains all the sets of the strictly ascending chain $\operatorname{supp}(u_1) \subset \operatorname{supp}(u_2) \subset \operatorname{supp}(u_3) \subset \cdots$. This contradiction shows that $\bigcup_{n \geq 0} u_n R = U$, so that U is countably generated. ∎

In particular, it follows from the lemma above that every uncountably generated uniserial module is quasismall. We do not know an example of a uniserial module that is not quasismall.

Another important class of quasismall modules is the class of modules with local endomorphism rings. To show this, we recall a definition.

DEFINITION 4.3. Following Crawley and Jónsson [3], a module M is said to have the *exchange property* if whenever M is a direct summand of a direct sum $A = \oplus_{i \in I} A_i$, there are submodules B_i of A_i such that $A = M \oplus (\oplus_{i \in I} B_i)$. If M satisfies this property for any finite index set I, we say that M has the *finite exchange property*.

A module M with $\mathrm{End}(M)$ local has the exchange property by Warfield's theorem [10]. Thus, if M is a direct summand of a direct sum $A = \oplus_{i \in I} A_i$, for each $i \in I$ there are submodules B_i and C_i of A_i such that $A_i = B_i \oplus C_i$ and $A = M \oplus (\oplus_{i \in I} B_i)$. It follows that $M \cong \oplus_{i \in I} C_i$, and since M is indecomposable, $M \cong C_j$ for some index $j \in I$. Hence M is quasismall.

In the next lemma we give an internal characterization of quasismall uniserial modules in terms of summable families of endomorphisms. Let A and B be arbitrary modules. We say that a family $\{f_i \mid i \in I\}$ of homomorphisms from A into B is a *summable family* if for every $x \in A$ we have $f_i(x) = 0$ for all but a finite set of indices $i \in I$ (e.g., see [12]). Note that if $\{f_i \mid i \in I\}$ is a summable family of homomorphisms from A into B, it is possible to define their sum $\sum_{i \in I} f_i \colon A \to B$ in the obvious way.

LEMMA 4.4. *The following conditions are equivalent for a uniserial module U:*

(a) *U is quasismall.*

(b) *If $\{f_i \mid i \in I\}$ is a summable family of endomorphisms of U such that $\sum_{i \in I} f_i = 1_U$, then at least one of the f_i is an epimorphism.*

Proof. (a) ⇒ (b) Suppose that (a) holds and $\{f_i \mid i \in I\}$ is a summable family of endomorphisms of U such that $\sum_{i \in I} f_i = 1_U$ and no f_i is an epimorphism. Then for every $i \in I$ there exists a cyclic module C_i with $f_i(U) \subseteq C_i \subseteq U$. Consider the homomorphisms $f \colon U \to \oplus_{i \in I} C_i$ defined by $f(x) = (f_i(x))_{i \in I}$ for every $x \in U$ and $g \colon \oplus_{i \in I} C_i \to U$ defined by $g((x_i)_{i \in I}) = \sum_{i \in I} x_i$ for every $(x_i)_{i \in I} \in \oplus_{i \in I} C_i$. The composite mapping gf is equal to 1_U, so that U is isomorphic to a direct summand of $\oplus_{i \in I} C_i$. Since U is quasismall, there is a finite subset F of I such that U is isomorphic to a direct summand of $\oplus_{i \in F} C_i$. In particular, U is finitely generated, hence cyclic. It follows that in the summable family $\{f_i \mid i \in I\}$ all but a finite number of f_i are zero. Since U is uniserial, the equality $\sum_{i \in I} f_i(U) = U$ implies that at least one of the f_i is an onto mapping. This is a contradiction.

(b) ⇒ (a) Suppose that (b) holds and that $M = \oplus_{i \in I} A_i = U \oplus C$. Let $\varepsilon_i \colon A_i \to M$, $\pi_i \colon M \to A_i$, $\varepsilon_U \colon U \to M$, and $\pi_U \colon M \to U$ be the canonical embeddings and projections relative to these decompositions. Then $\{\pi_U \varepsilon_i \pi_i \varepsilon_U \mid i \in I\}$ is a summable family of endomorphisms of U whose sum is 1_U. One of these endomorphisms, say $\pi_U \varepsilon_j \pi_j \varepsilon_U$, must be onto by hypothesis (b). By Lemma 3.2 there is an index k such that $\pi_U \varepsilon_k \pi_k \varepsilon_U$ is an injective endomorphism. If $\pi_U \varepsilon_j \pi_j \varepsilon_U$ is an automorphism, the composite mapping of $\pi_j \varepsilon_U \colon U \to A_j$ and $(\pi_U \varepsilon_j \pi_j \varepsilon_U)^{-1} \pi_U \varepsilon_j \colon A_j \to U$ is the identity of U, hence U is isomorphic to a direct summand of A_j, and we are done. Similarly, if $\pi_U \varepsilon_k \pi_k \varepsilon_U$ is an automorphism, U must be isomorphic to a direct summand of A_k. Otherwise, $\pi_U \varepsilon_j \pi_j \varepsilon_U + \pi_U \varepsilon_k \pi_k \varepsilon_U$ is an automorphism by Lemma 2.5(a). Now it follows easily that U is isomorphic to a direct summand of $A_i \oplus A_j$, which shows that U is quasismall. ∎

We will need the following characterization of the uniserial modules that are not quasismall. By Lemma 4.2, such a uniserial module must be countably generated.

LEMMA 4.5. *Let U be a countably generated uniserial module. The following conditions are equivalent*:

(a) *U is not quasismall.*

(b) *For any element $x \in U$ there is an endomorphism f of U such that $f(x) = x$, but f is not an automorphism of U.*

Furthermore, if U is not quasismall, then any nonzero factor module of U is not quasismall either.

Proof. (a) ⇒ (b) Suppose (b) false, i.e., that there exists an element u of U such that for every endomorphism f of U, $f(u) = u$ implies that f is an automorphism. Let $\{f_i | i \in I\}$ be a summable family of endomorphisms of U such that $\sum_{i \in I} f_i = 1_U$. Let J be a finite subset of I such that $f_i(u) = 0$ for every $i \in I \setminus J$. Then $\sum_{i \in J} f_i(u) = u$, hence $\sum_{i \in J} f_i$ is an automorphism. It follows from Lemma 2.5(b) that one of the mappings f_i is onto. Hence U is quasismall by Lemma 4.4.

(b) ⇒ (a) Suppose that (b) holds. Let x_n, $n \geq 0$, be a countable set of generators of U with $0 \subset x_0 R \subset x_1 R \subset x_2 R \cdots$. Since (b) holds, for every n there is an endomorphism f_n of U such that $f_n(x_n) = x_n$ but f_n is not an automorphism. Since U is uniserial and $f_n(x_n) = x_n$ with $x_n \neq 0$, it follows that f_n is monic for every n. Therefore the mappings f_n are not onto. Define a family of endomorphisms as follows: $g_0 = f_0$, and $g_n = f_n - f_{n-1}$ for every $n \geq 1$. It is easily seen that $\{g_n \mid n \geq 0\}$ is a summable family of endomorphisms of U, $\sum_{n \geq 0} g_n = 1_U$, and each g_n is not onto. Hence U is not quasismall by Lemma 4.4.

Now suppose that U is a uniserial module, U not quasismall, and U/A is a nonzero factor of U. For every element $x \in U$, there is an endomorphism f of U such that $f(x) = x$ and f is not an epimorphism. Hence for the nonzero element $x + A$ of U/A, the endomorphism f of U is the identity on A, so that f induces an endomorphism f' of U/A. Moreover $f'(x+A) = x+A$ and f' is not an epimorphism. Therefore U/A is not quasismall by the first part of the statement. ∎

COROLLARY 4.6. *Let U and V be uniserial modules such that $[U]_e = [V]_e$. Then U is quasismall if and only if V is quasismall.*

Proof. It follows immediately from Lemma 4.5, because there are epimorphisms $f\colon U \to V$ and $g\colon V \to U$. ∎

THEOREM 4.7. *Let $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ be two families of nonzero uniserial modules over an arbitrary ring R and suppose that $\bigoplus_{i\in I} U_i \cong \bigoplus_{j\in J} V_j$. Set $I' = \{i \in I \mid U_i$ is quasismall$\}$ and $J' = \{j \in J \mid V_j$ is quasismall$\}$. Then there is a bijection $\tau\colon I' \to J'$ such that $[U_i]_e = [V_{\tau(i)}]_e$ for every $i \in I'$.*

Proof. Without loss of generality we may suppose that $M = \bigoplus_{i\in I} U_i = \bigoplus_{j\in J} V_j$. Let $\varepsilon_k\colon U_k \to \bigoplus_{i\in I} U_i$ and $e_l\colon V_l \to \bigoplus_{j\in J} V_j$ be the embeddings, and let $\pi_k\colon \bigoplus_{i\in I} U_i \to U_k$ and $p_l\colon \bigoplus_{j\in J} V_j \to V_l$ be the canonical projections.

Observe the following fact, which will be crucial for the proof. Fix an index $i \in I'$. Then the family $\{\pi_i e_j p_j \varepsilon_i \mid j \in J\}$ is a summable family of endomorphisms of the quasismall module U_i and its sum is the identity 1_{U_i}, hence there exists an index $j \in J$ such that $\pi_i e_j p_j \varepsilon_i$ is an epimorphism (Lemma 4.4). It follows from Lemma 2.4(b) that both $\pi_i e_j\colon V_j \to U_i$ and $p_j \varepsilon_i\colon U_i \to V_j$ are epimorphisms, hence $[V_j]_e = [U_i]_e$. In particular, V_j is quasismall by Corollary 4.6. Therefore $j \in J'$.

Similarly, for every $j \in J'$ there exists an $i \in I'$ such that both $\pi_i e_j\colon V_j \to U_i$ and $p_j \varepsilon_i\colon U_i \to V_j$ are epimorphisms.

Fix an index $k \in I'$ and consider the two subsets $I(k) = \{i \in I' \mid [U_i]_e = [U_k]_e\}$ of I' and $J(k) = \{j \in J' \mid [V_j]_e = [U_k]_e\}$ of J'. The $I(k)$, $k \in I'$, form a partition of I', and from the fact that we have just observed it follows that the $J(k)$, $k \in I'$, also form a partition of J'.

As in the proof of Theorem 3.3, in order to establish the existence of the bijection τ it is sufficient to prove that the cardinalities $|I(k)|$ and $|J(k)|$ are equal for every $k \in I'$.

In the case in which either $I(k)$ or $J(k)$ is a finite set, we can use the fact observed above and argue exactly as in the proof of Theorem 3.3.

If $I(k)$ and $J(k)$ are both infinite, it is sufficient to prove that $|J(k)| \le |I(k)|$. For every $t \in I$ define a subset $A(t)$ of J' as follows: $A(t) = \{j \in J' \mid \pi_t e_j\colon V_j \to U_t$ and $p_j \varepsilon_t\colon U_t \to V_j$ are both epimorphisms$\}$.

By Lemma 4.2 there is a countable subset $H(t)$ of J such that $U_t \subseteq \bigoplus_{j\in H(t)} V_j$, hence clearly $A(t) \subseteq H(t)$ is a countable set.

Now the same argument of the proof of Theorem 3.3 allows us to conclude that $|J(k)| \le |I(k)|$. ∎

We are now in the position to establish a weak Krull–Schmidt theorem for two families of quasismall uniserial modules. The proof follows from Theorems 3.1, 3.3, and 4.7.

THEOREM 4.8. *Let* $\{U_i \mid i \in I\}$ *and* $\{V_j \mid j \in J\}$ *be two families of nonzero quasismall uniserial modules over an arbitrary ring* R. *Then* $\bigoplus_{i \in I} U_i \cong \bigoplus_{j \in J} V_j$ *if and only if there are two bijections* $\sigma, \tau\colon I \to J$ *such that* $[U_i]_m = [V_{\sigma(i)}]_m$ *and* $[U_i]_e = [V_{\tau(i)}]_e$ *for every* $i \in I$.

The natural question that arises here is: What happens to decompositions of a serial module if a uniserial summand that is not quasismall appears? Our next result shows that given any non-quasismall uniserial module U, we can construct a serial module with two different decompositions into uniserial summands for which the above weak Krull–Schmidt theorem does not hold.

THEOREM 4.9. *Let* U *be a uniserial* R*-module that is not quasismall. Then* $U \oplus (\bigoplus_{i \in I} A_i) \cong \bigoplus_{i \in I} A_i$ *for any family* A_i, $i \in I$, *of proper submodules of* U *with* $\bigcup_{i \in I} A_i = U$. *In particular, there exists a countable family* C_n, $n \geq 0$, *of cyclic submodules of* U *such that* $U \oplus (\bigoplus_{n \geq 0} C_n) \cong \bigoplus_{n \geq 0} C_n$.

Proof. By Lemma 4.2 the module U is countably generated and not finitely generated. Let x_n, $n \geq 0$, be a countable family of generators of U such that $x_0 R \subset x_1 R \subset x_2 R \subset \cdots$.

For every $n \geq 0$ we define an index $i(n) \in I$ and an endomorphism f_n of U with the following properties: (1) $x_n \in A_{i(n)}$; (2) $f_n(x) = x$ for any $x \in A_{i(n-1)}$ and $n \geq 1$; (3) $f_n(U) \subseteq A_{i(n)}$. We proceed by induction on n. Since U is not quasismall, there is an endomorphism f_0 of U such that $f_0(x_0) = x_0$ and f_0 is not onto (Lemma 4.5). Let $i(0) \in I$ be an index such that $f_0(U) \subseteq A_{i(0)}$.

Suppose that $i(n) \in I$ and f_n with the required properties have been defined. If y is any element of $U \setminus A_{i(n)}$, by Lemma 4.5 there is an endomorphism f_{n+1} of U that is not onto and such that $f_{n+1}(y) = y$. In particular $f_{n+1}(x) = x$ for every $x \in A_{i(n)}$. Let $i(n+1) \in I$ be an index such that $A_{i(n)} \subseteq A_{i(n+1)}$, $x_{n+1} \in A_{i(n+1)}$, and $f_{n+1}(U) \subseteq A_{i(n+1)}$. Then $i(n+1)$ and f_{n+1} satisfy properties (1)–(3), which completes the construction by induction. Note that $U = \bigcup_{n \geq 0} A_{i(n)}$.

Now set $g_0 = f_0$ and $g_n = f_n - f_{n-1}$ for every $n \geq 1$. Note that $g_n(x) = 0$ for every $x \in A_{i(n-2)}$, so that for every element $x \in U$ one has that $g_n(x) = 0$ for almost all n. This shows that the family $\{g_n \mid n \geq 0\}$ is a summable family of endomorphisms of U. Moreover $g_n(U) \subseteq A_{i(n)}$, so that we can define a homomorphism $g\colon U \to \bigoplus_{n \geq 0} A_{i(n)}$ via $g(x) = (g_n(x))_{n \geq 0}$. Consider the homomorphism $h\colon \bigoplus_{n \geq 0} A_{i(n)} \to U$, $h((x_n)_{n \geq 0}) = \sum_{n \geq 0} x_n$.

If $x \in U$, we have that $x \in A_{i(n)}$ for some n, so that $g_m(x) = 0$ for every $m \geq n+2$, and thus $hg(x) = \sum_{i=0}^{n+1} g_i(x) = f_{n+1}(x) = x$. This proves that $hg = 1_U$, hence $U \oplus \ker h \cong \bigoplus_{n \geq 0} A_{i(n)}$.

For every $n \geq 0$ let K_n be the submodule of $A_{i(n)} \oplus A_{i(n+1)}$ defined by $K_n = \{(x, -x) \mid x \in A_{i(n)}\}$ (note that $A_{i(n)} \subseteq A_{i(n+1)}$). Clearly, $\{K_n \mid n \geq 0\}$ is an independent family of submodules of $M = \bigoplus_{n \geq 0} A_{i(n)}$, and $K_n \cong A_{i(n)}$, so that $K = \bigoplus_{n \geq 0} K_n \cong \bigoplus_{n \geq 0} A_{i(n)}$. If $x = (x_1, x_2, \dots, x_n, 0, 0, \dots) \in \ker h$, then $x_1 + x_2 + \cdots + x_n = 0$, so $x_n = -x_1 - \cdots - x_{n-1}$. Thus we can write x as

$$x = (x_1, -x_1, 0, 0, \dots) + (0, x_1 + x_2, -x_1 - x_2, 0, 0, \dots) + \cdots + \left(0, \dots, 0, \sum_{i=1}^{n-1} x_i, -\sum_{i=1}^{n-1} x_i, 0, 0, \dots\right).$$

It follows that $K = \ker h$. Thus $U \oplus (\bigoplus_{n \geq 0} A_{i(n)}) \cong (\bigoplus_{n \geq 0} A_{i(n)})$, which implies that $U \oplus (\bigoplus_{i \in I} A_i) \cong (\bigoplus_{i \in I} A_i)$. This concludes the proof of the first part of the statement.

The second part follows immediately from the first and the fact that the uniserial module U is not quasismall, and thus it is the union of a countable chain of cyclic submodules. ∎

Note that if two modules belong to the same epigeny class, they must have the same number of generators. Therefore the above theorem shows that, in Theorem 4.7, there may not exist a bijection between the epigeny classes of $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ if $I \neq I'$ or $J \neq J'$, that is, if at least one of the modules U_i or V_j is not quasismall.

Remark. For uniserial modules the property of being quasismall has a topological interpretation as well. If E is the endomorphism ring of a uniserial right R-module $U \neq 0$, then E is endowed with the *finite topology* having as a basis of neighborhoods of zero the left ideals of the form

$$W(x) = \{f \in E \mid f(x) = 0\},$$

where x is an element of U. In this topology E is a Hausdorff and complete left linearly topologized ring. If the uniserial module U is cyclic, the topology on E is the discrete topology. As was observed in [4], E has (at most) two maximal right ideals, namely, I, the (two-sided) ideal of E consisting of all the endomorphisms of U_R that are not monic, and J, the (two-sided) ideal of consisting of all the endomorphisms of U_R that are not epic. Note that the ideal I is always an open ideal in E, because $I = \bigcup_{x \neq 0} W(x)$. In particular I is always closed. It is possible to prove that a countably generated uniserial module U is not quasismall if and only if the element $1_U \in E$ belongs to the closure of J in the finite topology.

5. LOCALLY SEMI-T-NILPOTENT FAMILIES

In this section we do not impose any conditions on the uniserial summands, but instead we require certain additional hypotheses on the families of uniserials. As the main result of this section, we establish a weak Krull–Schmidt theorem for two locally semi-T-nilpotent families of uniserial modules, which may be regarded as a generalization of the finite direct sum case considered in [4].

First we recall the following definition.

DEFINITION 5.1. A family of modules $\{M_i \mid i \in I\}$ is called *locally semi-T-nilpotent* if, for any countably infinite set of nonisomorphisms $\{f_n: M_{i_n} \to M_{i_{n+1}}\}$ with all the i_n distinct in I, and for any $x \in M_{i_1}$, there exists a positive integer k (depending on x) such that $f_k \cdots f_1(x) = 0$.

It is obvious from the definition that every finite family of modules is locally semi-T-nilpotent. And it is well known that any family of indecomposable right modules over a right pure semisimple ring (i.e., a ring over which every right module is pure-injective), in particular over a ring of finite representation type, is locally semi-T-nilpotent.

We quote the following important result, due to Zimmermann-Huisgen and Zimmermann [12], which provides the connection between the locally semi-T-nilpotency of a family of modules with local endomorphism rings and the exchange property of their direct sum.

PROPOSITION 5.2. *Let M_i, $i \in I$, be modules with* $\mathrm{End}(M_i)$ *local for all $i \in I$. Then $M = \bigoplus_{i \in I} M_i$ has the (finite) exchange property if and only if the family $\{M_i \mid i \in I\}$ is locally semi-T-nilpotent.*

Proof. See [12, Corollary 6]. ∎

The next proposition is crucial for establishing our weak Krull–Schmidt theorem for two locally semi-T-nilpotent families of uniserial modules.

PROPOSITION 5.3. *Let $M = \bigoplus_{i \in I} U_i$ be a direct sum of uniserial modules U_i, and let V be a nonzero uniserial direct summand of M. Assume that the family $\{U_i \mid i \in I\}$ is locally semi-T-nilpotent. Then there exists $j \in I$ such that $[V]_e = [U_j]_e$.*

Proof. Suppose that for some $i \in I$ there is an infinite number of $k \in I$ such that $U_i \cong U_k$. Then it follows from the condition of locally semi-T-nilpotency that any monomorphism $f: U_i \to U_i$ must be an automorphism, hence $\mathrm{End}(U_i)$ is local. Now set $A = \bigoplus_{i \in T} U_i$, where $T = \{i \in I \mid U_i \cong U_k$ for an infinite number of $k \in I\}$. Then A has the (finite) exchange property by Proposition 5.2.

Set $B = \bigoplus_{i \in I \setminus T} U_i$, so that $M = A \oplus B$. Let D be a complement of V in M, hence $M = V \oplus D$. By the finite exchange property of A, there is a submodule D' of D such that either $M = A \oplus D'$ or $M = A \oplus V \oplus D'$. The direct summand D' of M is contained in D, hence $D = D' \oplus D''$ for some submodule D''. Thus $M = V \oplus D' \oplus D''$. If $M = A \oplus D'$, we can factorize M modulo D' and get that $V \oplus D'' \cong A$. Since V is indecomposable and $A = \bigoplus_{i \in T} U_i$ is a direct sum of modules with local endomorphism rings, we can apply Azumaya's theorem [1, Theorem 12.6] to deduce that V is isomorphic to U_i for some $i \in T$, and we are done. If $M = A \oplus V \oplus D'$, we get $V \oplus D' \cong B$. Therefore, without loss of generality, from now on we may assume that $M = \bigoplus_{i \in I} U_i = V \oplus D$, where for each $i \in I$ there exists only a finite number of $k \in I$ such that $U_i \cong U_k$.

By Lemma 4.2 we may assume I countable. If I is finite, the result follows from Lemma 2.6. Hence we can assume $I = \mathbf{N}$, the set of natural numbers.

There is $n \in \mathbf{N}$ such that $V \cap (\bigoplus_{i \le n} U_i) \ne 0$. If we set $C = \bigoplus_{i \ge n+1} U_i$, then we have $M = (\bigoplus_{i \le n} U_i) \oplus C = V \oplus D$.

Let π_C: $M = (\bigoplus_{i \le n} U_i) \oplus C \to C$, p_V: $M = V \oplus D \to V$, and p_D: $M = V \oplus D \to D$ denote the canonical projections relative to these decompositions. Since $V \cap (\bigoplus_{i \le n} U_i) \ne 0$ and V is uniserial, it follows that $V \cap C = 0$. Hence $p_D|_C$: $C \to D$ is a monomorphism. Set $U_i' = p_D(U_i)$ for all $i > n$. Then $\{U_i' \mid i > n\}$ is an independent family of submodules of D. Consider the submodule $D' = \bigoplus_{i > n} U_i'$ of D. If $[V]_e = [U_k]_e$ for some $k > n$, we are done. Thus, from now on, we assume that $[V]_e \ne [U_k]_e$ for all $k > n$. Our aim is to show that under this assumption $\pi_C|_{D'}$: $D' \to C$ is an isomorphism. We shall first prove that

$$M = \left(\bigoplus_{i \le n} U_i \right) \oplus \left(\bigoplus_{n < i \le m} U_i' \right) \oplus \left(\bigoplus_{i > m} U_i \right) \tag{4}$$

for every $m \ge n$, by induction on m. For $m = n$ there is nothing to prove. Suppose that (4) holds for $m - 1$, i.e., that

$$M = \left(\bigoplus_{i \le n} U_i \right) \oplus \left(\bigoplus_{n < i \le m-1} U_i' \right) \oplus \left(\bigoplus_{i \ge m} U_i \right). \tag{5}$$

Let π_m: $M \to U_m$ denote the canonical projection with kernel $K = (\bigoplus_{i \le n} U_i) \oplus (\bigoplus_{n < i \le m-1} U_i') \oplus (\bigoplus_{i > m} U_i)$. Then $\pi_m|_V \circ p_V|_{U_m} + \pi_m|_D \circ p_D|_{U_m} = 1_{U_m}$.

Note that $\pi_m|_V$ is not injective, because $V \cap (\bigoplus_{i \le n} U_i) \ne 0$, and that either $\pi_m|_V$ or $p_V|_{U_m}$ is not surjective, because otherwise $[V]_e = [U_m]_e$, contrary to our assumption. Hence $\pi_m|_V \circ p_V|_{U_m}$ is neither injective nor surjective by Lemma 2.4. Since U_m is uniserial, it follows from Lemma

2.5(b) that $\pi_m|_D \circ p_D|_{U_m}$ is an automorphism. Therefore the restriction of the projection π_m to $p_D(U_m) = U'_m$ is an isomorphism. By Lemma 2.7 we get that (4) holds.

From (4) it follows that $(\oplus_{i \leq n} U_i) \cap (\oplus_{i > n} U'_i) = 0$, hence $\pi_C|_{D'}: D' \to C$ is a monomorphism.

Now we shall use the locally semi-T-nilpotency of the family $\{U_i \mid i \in I\}$ to show that $\pi_C|_{D'}: D' \to C$ is an epimorphism. Our argument is inspired by a technique used in the proof of [12, Theorem 5].

Suppose, on the contrary, that $\pi_C(D') \neq C$. This means that there exist a positive integer $m > n$ and an element $x \in U_m$ such that $x \notin \pi_C(D')$. Since

$$M = \left(\bigoplus_{i \leq n} U_i \right) \oplus \left(\bigoplus_{n < i \leq m} U_i \right) \oplus \left(\bigoplus_{i \geq m+1} U_i \right),$$

we obtain from (4) and Lemma 2.7 that the canonical projection of M onto $\oplus_{n < i \leq m} U_i$ with kernel $(\oplus_{i \leq n} U_i) \oplus (\oplus_{i > m} U_i)$, restricted to $\oplus_{n < i \leq m} U'_i$, is an isomorphism. We denote this isomorphism by

$$f_m\colon \bigoplus_{n<i\leq m} U'_i \to \bigoplus_{n<i\leq m} U_i .$$

Set $x' = f_m^{-1}(x)$. Then we have $x' = a + x + b$, where $a \in (\oplus_{i \leq n} U_i)$ and $b \in \oplus_{i > m} U_i$. Clearly $\pi_C(x') = x + b$. From $x \notin \pi_C(D')$ it follows that $b = (\pi_C(x') - x) \notin \pi_C(D')$. Therefore, there is a positive integer $m_1 \geq m + 1$ such that the m_1-component of b in $\oplus_{i > m} U_i$ does not belong to $\pi_C(D')$. Let $\pi_{m_1}: M = (\oplus_{i \geq 1} U_i) \to U_{m_1}$ be the canonical projection. Set $x_1 = \pi_{m_1} f_m^{-1}(x) \in U_{m_1}$, then $x_1 \notin \pi_C(D')$. Denote by g_0 the homomorphism $\pi_{m_1} f_m^{-1}|_{U_m}: U_m \to U_{m_1}$, then $x_1 = g_0(x)$.

Now we repeat the same argument for x_1 instead of x. By an obvious induction, we get an infinite sequence of positive integers $m < m_1 < m_2 < \cdots < m_k < \cdots$, and a countable family of homomorphisms $\{g_k: U_{m_k} \to U_{m_{k+1}}\}$, with $k = 1, 2, 3, \ldots$, so that $g_k g_{k-1} \cdots g_0(x) \neq 0$ for all $k \geq 1$. Since for any U_{m_k} there exists only a finite number of m_l such that $U_{m_l} \cong U_{m_k}$, this readily implies that the family $\{U_n \mid n \in \mathbf{N}\}$ is not locally semi-T-nilpotent (for instance, if $g_0, g_1, \ldots, g_n$ are isomorphisms, but g_{n+1} is not an isomorphism, then the composition map $g_{n+1} \cdots g_1 g_0: U_m \to U_{n+2}$ is not an isomorphism, and so on). Thus we obtain the contradiction which shows that $\pi_C|_{D'}: D' \to C$ must be an isomorphism.

Now, again by Lemma 2.7, it follows that $M = (\oplus_{i \leq n} U_i) \oplus D'$. Then D' is also a direct summand of D, hence $D = D' \oplus D''$ for some submodule D'' of D. Thus we can write $M = V \oplus D = V \oplus D' \oplus D''$, which implies that $V \oplus D'' \cong (\oplus_{i \leq n} U_i)$. Thus the uniserial module V is isomorphic to a direct summand of a finite direct sum $(\oplus_{i \leq n} U_i)$, so, by Lemma 2.6, $[V]_e = [U_i]_e$ for some $1 \leq i \leq n$, which completes the proof. ∎

THEOREM 5.4. *Let* $\{U_i \mid i \in I\}$ *and* $\{V_j \mid j \in J\}$ *be two locally semi-T-nilpotent families of nonzero uniserial modules over an arbitrary ring* R. *Then* $\oplus_{i \in I} U_i \cong \oplus_{j \in J} V_j$ *if and only if there are two bijections* $\sigma, \tau \colon I \to J$ *such that* $[U_i]_m = [V_{\sigma(i)}]_m$ *and* $[U_i]_e = [V_{\tau(i)}]_e$ *for every* $i \in I$.

Proof. By Theorems 3.1 and 3.3, it is sufficient to prove that if the two modules $\oplus_{i \in I} U_i$ and $\oplus_{j \in J} V_j$ are isomorphic, then a bijection τ with the required properties exists. This can be done by modifying the proof of Theorem 3.3. By Lemma 4.2 and [9, Lemma 5] we may assume that I and J are countable sets. With the notation of Theorem 3.3, if $I(k)$ and $J(k)$ are both infinite, they are countable, hence $|I(k)| = |J(k)|$. If at least one of them is finite, the proof is entirely similar to the proof of Theorem 3.3 using Proposition 5.3. ∎

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