Properties of the telegrapher’s random process with or without a trap

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Abstract

The properties of the telegrapher random process which is a Poissonian random walk on a straight line are studied in detail in probabilistic terms. The paper contains, besides the details of a rapid communication (Foong, 1992) by one of the authors, a number of new results. The distributions of the first passage time subject to an arbitrary number of reversals in the walk are obtained explicitly for both the starting directions. These distributions are then used to obtain, again explicitly, the corresponding distributions of the maximum of the walk, proving the conjecture by Orsingher (1990) for the one started moving right. The densities of the displacements from the origin in the presence of a trap are also given in detail. The relationship between this density and (1) the first passage time and (2) the maximum are given.

Key words: Poissonian walk; First passage; Maximum of walk; Trap; Telegrapher equation; Wave equation

1. Introduction

There have been continuing interests in the path integral solution of the telegrapher equation and the underlying Poissonian random walk, since the work of Goldstein (1951) and Kac (1956). Recently, DeWitt-Morette and Foong (1989) and Foong (1990) gave the solution in terms of an ordinary integral. An example solution in Foong (1991) was very recently applied by Mugnai et al. (1992) to semi-classical analysis of tunneling time. There are also other applications and extensions, for example, application to polymer by Gaveau and Schulman (1990), and extension to the case when the reversal intensity of the random walk is time-dependent (Kaplan, 1964; Foong and van Kolck, 1992). Orsingher (1990) studied the distributions of the maximum of the walk, and gave an elegant conjecture for the case where the walker started to the right.

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For more references see, for instance, references contained in Orsingher (1990) and Foong (1991). These recent developments prompted the present detailed study of the walk with the hope that it will be useful for subsequent applications. A related walk has been considered by Weiss (1973).

The details of a rapid communication by one of the authors (Foong, 1992), and a number of other new results are presented in this paper. For the distribution of the maximum displacement, Orsingher (1990) calculated the cases for which the walker has undergone \( N \leq 5 \) reversals in a given time interval, and conjectured a formula for the case of general value of \( N \). The distributions of the first passage time will be given, and then used in one of the detailed proofs of Orsingher’s conjecture.

From the densities of the first passage time and the displacement, the densities of the displacement in the presence of a trap may be obtained for both starting directions. An average of these results is consistent with the result given in the interesting work by Masoliver et al. (1992). Their approach, namely via partial differential equations, is different from our more direct probabilistic approach in this paper. In our approach, it is easy to explain a certain discontinuity feature of the density distribution as pointed out in Foong and Kanno (1992). Some of the results in this paper can also be obtained by yet another method (Kanno, 1993).

In Section 2, the telegrapher random process is discussed. We derive, in Section 3 the distributions of the first passage time, in Section 4 the formulae for the distributions of the maximum displacement, and in Section 5 the densities of the displacement in the presence of a trap. In Section 6, the relationships between the densities, first passage time and maximum displacement are given.

2. The telegrapher process

The telegrapher process on the lattice may be defined by the following simple random evolution of the motion of a particle on a one-dimensional lattice, where the sites are separated by \( \Delta x \). Let the particle move with a constant velocity \( v \), and let \( \Delta t = \Delta x/v \). At each site, it may reverse its direction of travel instantaneously. The probability of a reversal at a site is \( a\Delta t \), while the probability of continuing its direction is \( 1 - a\Delta t \). It is to be understood that the particle is a sample out of an ensemble.

When the continuum limit of this lattice random walk is taken, with \( v \) held constant, the probability for the number of reversals by time \( t \), \( N(t) \), is described by the Poisson distribution (Kac, 1956; Kaplan, 1964; Foong, 1991), namely

\[
P(N_t = k) = \frac{\left( \int_0^t d\theta a(\theta) \right)^k}{k!} \exp(- \int_0^t d\theta a(\theta)), \quad k = 0, 1, \ldots. \tag{1}
\]
which for $a = \text{constant}$, reduces to

$$P(N_t = k) = e^{-at}(at)^k/k!.$$  \hfill (2)

A quantity of interest to the telegrapher equation is $S(t)$ defined by $S(0) = r_0$, and $S(t) = r_0 + v_0\int_0^t (-1)^{i+1} \, \mathrm{d}t$. It is the displacement of the particle starting at $r_0$ with velocity $v_0$. If $v_0 = 1$ and $r_0 = 0$, it is also the "randomized time" of Kac (1956). We shall discuss the cases with different starting directions, namely to the right with $v_0 = 1$ (denoted by $v_+$) and to the left with $v_0 = -1$ (denoted $v_-$) separately.

We denote the density distribution of $S(t)$, given $S(0) = r_0$ and $v_0 = \pm 1$, by $g(t, r| r_0, v_\pm)$, its Laplace transform by $\mathcal{L}_a$, with $r' = r - r_0$, $u'_\pm = t \pm r'$, and $u' = \sqrt{t^2 - r'^2}$; then (see, for instance, Foong, 1990 or Zastawniak, 1993)

$$g(t, r| r_0, v_\mp) = e^{-at} \delta(t + r') + \frac{ae^{-at}}{2\theta(u'^2)} \left[ I_0(au') + \frac{u'_\mp}{u'} I_1(au') \right],$$ \hfill (3)

and from DeWitt-Morette and Foong (1989) we can write

$$\mathcal{L}_a[g(t, r| r_0, v_\pm)] = \begin{cases} \frac{1}{2}(1 + 2a/s)^{1/2} \pm 1) \exp(-r'(s^2 + 2as)^{1/2}), & r' > 0, \\ \frac{1}{2}(1 + 2a/s)^{1/2} \mp 1) \exp(r'(s^2 + 2as)^{1/2}), & r' < 0, \\ \frac{1}{2}(1 + 2a/s)^{1/2}, & r' = 0, \end{cases}$$ \hfill (4)

where $I_\nu$ denotes the modified Bessel functions, $\theta(t) = 1$ for $t \geq 0$, and $\theta(t) = 0$ for $t < 0$. If we set $y = at$, $x = ar$, and $w' \equiv y \pm x'$, and $w' \equiv \sqrt{y^2 - x'^2}$ then Eq. (3) is transformed to

$$\tilde{g}(y, x| x_0, v_\mp) = e^{-y} \delta(y \mp x') + \frac{e^{-y}}{2\theta(w'^2)} \left[ I_0(w') + \frac{w'_\mp}{w'} I_1(w') \right],$$ \hfill (5)

such that

$$\int_{-y}^{y} \tilde{g}(y, x| x_0, v_\pm) \, \mathrm{d}x = 1.$$ \hfill (6)

For the later calculations, it is also convenient to state here the formula for $g(t, r, N_t)$ (Zastawniak, 1993), for $N_t$ even,

$$g(t, r, N_t = 2k| r_0, v_\pm) = \theta(u'^2) C(t, k) \frac{u'_\pm u'^{2(k-1)}}{(k-1)!k!}, \quad k \geq 1.$$ \hfill (7)

and for $N_t$ odd,

$$g(t, r, N_t = 2k + 1| r_0, v_\pm) = \theta(u'^2) D(t, k) \frac{u'^{2k}}{[(k)!]^2}, \quad k \geq 0.$$ \hfill (8)

where $C(t, k) = a^{2k}e^{-at}/2^{2k}$ and $D(t, k) = a^{2k+1}e^{-at}/2^{2k+1}$. Note that for $N_t$ odd, $g$ is symmetric in $r'$, but not so for $N_t$ even.
3. Calculations of first passage times

Let the probability density that the particle, starting with \( v_+ \) from \( r_0 \), passing the point \( r_1 > r_0 \) for the first time at time \( t \) be \( f(r_1, t | r_0, v_+) \). Then, Siegert's (1951) formula relating \( f \) and \( g \) is

\[
g(t, r | r_0, v_+) = \int_0^t f(r_1, \tau | r_0, v_+) g(t - \tau, r | r_1, v_+) \, d\tau, \quad r > r_1, \tag{9}
\]

which is a convolution of the two densities. Note that the requirement \( r > r_1 \) ensures \( f \) is the first passage time because in order to arrive at such \( r \) starting at \( r_0 \), the value of \( S(t) \) must first cross \( r_1 \). If we had allowed \( r < r_1 \), this need not be the only possibility because \( S(t) \) could then arrive at \( r \) without passing \( r_1 \). Note also that regardless of the starting directions, the corresponding argument of \( g \) in the integrand is \( v_+ \).

Applying the convolution theorem of the Laplace transform to Eq. (9), the Laplace transform of \( f(r_1, t | r_0, v_+) \) is given by

\[
\mathcal{L}_f(s, r_1 | r_0, v_+) = \mathcal{L}_f(s, r | r_1, v_+) \frac{\mathcal{L}_g(s, r | r_0, v_+)}{\mathcal{L}_g(s, r | r_1, v_+)}. \tag{10}
\]

It follows from Eqs. (10) and (4) that the Laplace transform of the distribution of the first passage time is

\[
\mathcal{L}_f(s, r_1 | r_0, v_+) = \frac{[(1 + (2a/s)^{1/2} + 1]}{[(1 + (2a/s)^{1/2} + 1]} e^{-r_1(s^2 + 2as)^{1/2}}
\]

\[
- \left\{ \mathcal{L}_f(s, r_1 | r_0, v_+) - \frac{1}{a} [(s^2 + 2as)^{1/2} - s] e^{-r_1(s^2 + 2as)^{1/2}}, \quad v_+ ;
\]

\[
\mathcal{L}_f(s - a, r_1 | r_0, v_+) = \exp\left[ - r_1^2 [(s + a)^2 - a^2]^{1/2} \right] ,
\]

and replacing \( s \) by \( s - a \), we have

\[
\mathcal{L}_f(s - a, r_1 | r_0, v_+) = \exp\left[ - r_1^2 (s^2 - a^2)^{1/2} \right]
\]

\[
eq \{ \exp(- r_1(s^2 - a^2)^{1/2}) - \exp(- r_1 s) \} + \exp(- r_1 s). \tag{12}
\]

3.1. Case 1: \( v_0 = 1 \) \( (v_+) \)

For this case, the inverse Laplace transform may be calculated as follows (Foong, 1990b): rewriting Eq. (11) as

\[
\mathcal{L}_f(s, r_1 | r_0, v_+) = \exp\left[ - r_1^2 [(s + a)^2 - a^2]^{1/2} \right] ,
\]

and replacing \( s \) by \( s - a \), we have

\[
\mathcal{L}_f(s - a, r_1 | r_0, v_+) = \exp\left[ - r_1^2 (s^2 - a^2)^{1/2} \right]
\]

\[
eq \{ \exp(- r_1(s^2 - a^2)^{1/2}) - \exp(- r_1 s) \} + \exp(- r_1 s). \tag{12}
\]
The last step is convenient because the inverse Laplace transform of the term \( \{ \cdots \} \) in Eq. (12) is (Abramowitz and Stegun, p. 1027), with \( u'_{1 \pm} \equiv t \pm r'_1 \), and 
\[
a_t \frac{r'_1}{u'_1} I_1(u'_1) \theta(u'_1),
\]
and that of \( \exp(-r'_1s) \) is \( \delta(u'_1) \). By the shift theorem of the transform, it follows that the distribution for the first passage time \( t \) through \( r_1 \) for the particle starting at \( r_0 \) is given by
\[
f(r_1, t | r_0, v_+) = \exp(-at) \delta(u'_1) + \exp(-ar'_1) I_1(u'_1) \theta(u'_1).
\]

In the rest of the paper, we shall set \( r_0 = 0 \), and the case for \( r_0 \neq 0 \) is to be recovered by replacing \( r_1 \) by \( r_1 - r_0 \). Also, unless otherwise stated, \( \theta \) denotes \( \theta(u'_1) \). Eq. (13) is rewritten below for easy reference:
\[
f(r_1, t | v_+) = \exp(-at) \delta(u'_1) + \exp(-ar'_1) I_1(u'_1) \theta(u'_1).
\]

We can perform the following check on Eq. (14). If we wait long enough the particle will eventually pass \( r_1 \), namely \( \int_0^\infty dt f(r_1, t | v_+) = 1 \). This is indeed the case as shown in Appendix A.

The probability \( F \) that the first passage through \( r_1 \) occurs at some time in the time interval \([r_1, t]\) is given by
\[
F(r_1, t | v_+) = \int_0^t f(r_1, \tau | v_+) d\tau,
\]
and the complement
\[
\text{P}(\text{Max}\{S(\tau), 0 \leq \tau \leq t\} < r_1 | v_+) = 1 - F(r_1, t | v_+) = \int_{t+\epsilon}^{\infty} f(r_1, \tau | v_+) d\tau,
\]
where the last equality follows from \( \int_0^\infty f(r_1, \tau | v_+) d\tau = 1 \) is the probability that the particle is confined to the region \((-t, r_1)\) by time \( t \). The lower limit \( t + \epsilon \) of the integral in Eq. (16) serves as a reminder to exclude the contribution from the case of no reversal, since it is included in Eq. (15) already.

In fact, we can deduce some detail information from Eq. (14), that is the distribution \( f(r_1, t, N_t = 2k | v_+) \) of the first passage time for a path whose number of reversals by the first passage time is \( 2k \). To do this, first observe that
\[
f(r_1, t | v_+) = \sum_{k=0}^\infty f(r_1, t, N_t = 2k | v_+),
\]
since \( f(r_1, t, N_1 = \text{odd}|v_+) = 0 \). By expanding \( I_1 \) in \( f(r_1, t|v_+) \), using the series representation

\[
I_1(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+v}}{k!k(k+v+1)},
\]

we have

\[
\sum_{k=0}^{\infty} f(r_1, t, N_1 = 2k|v_+ = 0 = e^{-at}\delta(u_1 -), 0 \leq u \leq 1, 0 \leq t \leq r_1. \quad (18)
\]

Because each reversal associates with a factor \( u \), as we see in Eq. (2), and the factor \( e^{-at} \) is independent of the number of reversals, Eq. (18) suggests

\[
f(r_1, t, N_1 = 2k|v_+) = \begin{cases} e^{-at}\delta(u_1 -), & k = 0, \\ 02C(t, k)r_1 [(k - 1)!k!]^{-1}u_1^{2(k-1)}, & k \geq 1. \end{cases} \quad (19)
\]

Indeed, we can conclude Eq. (19) from Eq. (18) by the following argument. By the same reasoning that leads to Siegert’s formula (9), we can write, where now \( r' = r - r_1 \),

\[
g(t, r, N_i = n|v_+) = \sum_{i=0}^{[n/2]} \int_{r_1}^{r'-|r'|} f(r_1, t, N_1 = 2i 2t + 1|v_+) \times g(t - i, r, N_i - t) \frac{n - 2i}{n - 2i - 1} r_1u_1^{2(k-1)}dt, \quad r > r_1, \quad (20)
\]

where \([x]\) denotes the greatest integer not larger than \( x \). (By summing over \( n \) in Eq. (20), we obtain Siegert’s formula: thus, it may be said to be the elementary formula relating the distributions \( g \) and \( f \) in the case of the Poissonian walk.) Now observe that \( g(N_i = n) \) has the factor \( a^n \). On the right-hand side of Eq. (20), the exponent of the intensity \( a \) in each of the term must be \( n \), because \( a \) is an arbitrary positive constant, and the equation must hold for all values of \( a \). Hence, we conclude that \( f(N_i = 2k) \) has the factor \( a^{2k} \), and conclude Eq. (19) from Eq. (18). The case of \( k = 0 \) can be understood by the fact that the probability of no reversal is \( e^{-at} \), and the first passage time to pass \( r_1 \) is just \( r_1 \).

We end this subsection by writing the conditional probability given \( N_i \):

\[
f(r_1, t|N_i = 2k|v_+) = \begin{cases} \delta(u_1 -), & k = 0, \\ 02B(\frac{1}{2}, k)^{-1}r_1u_1^{2k-1}t^{2k}, & k \geq 1. \end{cases} \quad (21)
\]

where \( B \) is the beta function:

\[
B(\frac{1}{2}, k)^{-1} = \frac{\Gamma(k + \frac{1}{2})}{\Gamma(k)\Gamma(\frac{1}{2})} = \frac{(2k)!}{2^{2k}(k - 1)!k!}.
\]
3.2. Case 2: \( v_0 = -1 \) (\( v_- \))

We now continue with the evaluation of the first passage time for the left starting direction, \( v_- \). In order to invert the Laplace transform (11) (bottom line), we use the following result from the appendix of DeWitt-Morette and Foong (1990):

\[
\mathcal{L}^{-1}[(s^2 + 2as)^{1/2} \exp(-r_1(s^2 + 2as)^{1/2})] = e^{-at} \left( \frac{\partial^2}{\partial t^2} - a^2 \right) I_0(a u_1) \theta. \tag{22}
\]

By the standard Laplace transform formula \( \mathcal{L}(h'(t)) = s \mathcal{L}(h) - h(0) \), letting \( \mathcal{L}_h(r_1, s) = \exp(-r_1(s^2 + 2as)^{1/2}) \), and using Eqs. (11) (top line) and (14), we have

\[
h(r_1, t | v_+) = e^{-at} \delta(u_1 -) + e^{-at} \frac{ar_1}{u_1} I_1(a u_1) \theta. \tag{23}
\]

Therefore, we get \( h(r_1, 0) = 0 \) for \( r_1 > 0 \), and

\[
\mathcal{L}^{-1}[s \exp(-r_1(s^2 + 2as)^{1/2})] = \frac{\partial}{\partial t} \left[ e^{-at} \delta(t - r_1) + \theta e^{-at} \frac{ar_1}{u_1} I_1(a u_1) \right]. \tag{24}
\]

We shall be concerned with \( t > r_1 \) (since for \( t \leq r_1 \), \( f(r_1, t | v) = 0 \)), for which we may set \( \delta(u_1 -) = 0 \) and \( \theta(u_1 -) = 1 \) in Eqs. (22) and (24). Then straightforward calculation gives

\[
f(r_1, t | v_-) = \theta \frac{e^{-at}}{u_1 +} \left[ a r_1 I_0(a u_1) + \left( \frac{u_1 -}{u_1 +} \right)^{1/2} I_1(a u_1) \right]. \tag{25}
\]

Again, we can obtain the first passage time for walks with certain number of reversals by expanding \( I_0 \) and \( I_1 \):

\[
f(r_1, t | v_-) = \frac{e^{-at}}{u_1 +} \sum_{i=0}^{\infty} \frac{a^{2i+1} u^{2i}}{2^{2i+1} i! (i+1)!} \left[ (2i+1) r_1 + t \right] \theta,
\]
giving

\[
f(r_1, t, N_i = 2k + 1 | v_-) = \theta \frac{D(t, k)}{k!(k+1)!} \left[ t + (2k + 1) r_1 \right] u_1 - u_1^{2(k-1)}, \tag{26}
\]

and consequently,

\[
f(r_1, t | N_i = 2k + 1, v_-) = \theta \frac{\left[ t + (2k + 1) r_1 \right] u_1 - u_1^{2(k-1)}}{4(k+1) B(\frac{3}{2}, k+1) r_1^{2k+1}}. \tag{27}
\]

Fig. 1 gives the plots of the densities \( f(r_1, t | v_\pm) \), with the \( \delta \) function dropped, and the probabilities \( F(r_1, t | v_\pm) \), where the solid line corresponds to \( v_+ \), and the dashed line \( v_- \). This concludes our calculation of first passage time, and we now turn to the calculation of the distribution of the maximum displacement of the walk.
Fig. 1. Contrasts of the first passage time densities for both the starting directions, $f(r_1 = 1, t|v_\pm)$ with $a = 1$, minus the delta-function, and the corresponding probabilities $F(r_1 = 1, t|v_\pm)$, where the solid line corresponds to $v_+$, and the dashed line $v_-$. The curves which decrease (increase) with $t$ are those of $f(F)$.

4. Maximum displacement

Let the maximum displacement, to the right of the origin, of the walk during the time interval $[0, t]$ be denoted by $S_m(t)$, namely $S_m(t) = \max\{S(\tau), 0 \leq \tau \leq t\}$. We seek the distribution of $S_m(t)$ subject to $N_1$ reversals by time $t$. The probability that the maximum of a path is greater or equal to $r_1$ is given by the sum of the probabilities that it undergoes $2i$ ($i \geq 0$) reversals by the first passage time $\tau$ through $r_1$, and the remaining $N_1 - 2i$ reversals between $\tau$ and $t$. That is,

$$P(S_m(t) \geq r_1, N_1 = n|v_\pm) = \sum_{i=0}^{[n/2],[n-1]/2} \int_0^t f(r_1, \tau, N_1 = \left\{ \begin{array}{ll} 2i \\ 2i + 1 \end{array} \mid v_\pm \right\}) P(N_1 - t = \left\{ \begin{array}{ll} n - 2i \\ n - 2i - 1 \end{array} \right\}) \, \text{d}\tau,$$

(28)

where the first upper limit $[n/2]$ belongs to the upper alternative, i.e. $v_+$, $[n/2]$, $2i$, $v_+$, and $n - 2i$, of the equation. We shall study the $v_\pm$ cases separately.

4.1. Case I: $v_0 = 1$ ($v_+$)

In Eq. (28), separating the $i = 0$ term from the summand, and substituting for $f$ from Eq. (19) gives

$$P(S_m(t) \geq r_1, N_1 = n|v_+) = e^{-at} \frac{[(au_1 - 1)^n}{n!}.$$
and dividing by the probability of \( N_t \) reversals, we have the conditional probability

\[
P(S_m(t) \geq r_1 | N_t = n, v_+) = \left( \frac{u_1}{t} \right)^n \theta + \theta \frac{n! r_1}{2^m} \sum_{i=1}^{[n/2]} \int_{r_1}^{r} \frac{d\tau (\tau^2 - r_1^2)^{i-1} (t - \tau)^{-2i}}{2^{2(i-1)}(i-1)!i!(n-2i)!}.
\]

The complement of this gives the conditional probability that the maximum is less than \( r_1 \),

\[
P(S_m(t) < r_1 | N_t = n) = 1 - P(S_m(t) \geq r_1 | N_t)
\]

\[= \left[ 1 - \left( \frac{1 - r_1}{t} \right)^n \right] \theta - \theta \frac{n! r_1}{2^m} \sum_{i=1}^{[n/2]} \int_{r_1}^{r} \frac{d\tau (\tau^2 - r_1^2)^{i-1} (t - \tau)^{-2i}}{2^{2(i-1)}(i-1)!i!(n-2i)!},
\]

and the density is given by

\[
\rho(S_m(t) = r | N_t = n) = \frac{\partial}{\partial r} P(S_m(t) < r | N_t = n) = - \frac{\partial}{\partial r} P(S_m(t) \geq r | N_t = n).
\]

Substituting Eq. (30), with the \( i = 1 \) term separated from the summand for \( P \), we obtain

\[
\rho(S_m(t) = r | N_t = n, v_+) = \begin{cases} 
0, & n = 0, \\
1/t, & n = 1, \\
\theta \frac{1}{2^m} [nu_r^{n-1} + n(n-1)r u_r^{n-2} + n I(n)], & n \geq 2,
\end{cases}
\]

where

\[
I(n) = \sum_{i=2}^{[n/2]} \int_{r_1}^{r} \frac{d\tau [(2i-1)r^2 - \tau^2] (\tau^2 - r_1^2)^{i-2} (t - \tau)^{-2i}}{2^{2(i-1)}(i-1)!i!(n-2i)!}.
\]

By working out Eq. (33) for several values of \( n \), one is led to the two conjectures by Orsingher (1990):

\[
\rho(S_m(t) = r | N_t = 2k - 1, v_+) = \theta \frac{2n! r_1^{2k-1}}{B(\frac{1}{2}, k) t^{2k-1}} = \rho(S_m(t) = r | N_t = 2k, v_+), \quad k \geq 1,
\]

\[
\rho(S_m(t) = r | N_t = 2k, v_+) = \theta \frac{2n! r_1^{2k-1}}{B(\frac{1}{2}, k) t^{2k-1}} = \rho(S_m(t) = r | N_t = 2k, v_+), \quad k \geq 1,
\]
which we now proceed to prove. For \( N = \text{odd} \), Eq. (33) reads

\[
\rho(S_m(t) = r | N = 2k + 1, v_+) = \begin{cases} 1/r, & k = 0, \\ \frac{2r^{k-1}}{2k^r} [(2k + 1)u^{2k} + 2k(2k + 1)r u^{2k-1} + (2k + 1)!(2k + 1)], & k \geq 1. \end{cases}
\]

(36)

In other words, recalling Eq. (35), we need to prove for \( k > 2 \),

\[
I(2k + 1) = \frac{1}{(2k + 1)!} \left\{ \frac{4u^{2k}}{B(1/2, k + 1)} - (2k + 1)u^{2k-1}[t + (2k - 1)r] \right\} \equiv J(2k + 1),
\]

(37)

which we will do by induction. While checking for \( k = 2 \), namely

\[
I(5) - J(5) = \frac{3}{25} u^2 (t - 3r)(t + 5r),
\]

is easy, showing directly \( I(2k + 3) = J(2k + 3) \) for all \( k \) is not so obvious. We shall instead first show that their second derivative with respect to \( t \) are equal. We have

\[
I(2k + 3) = \sum_{i=2}^{k+1} \int_0^t d\tau \frac{[(2i - 1)r^2 - \tau^2](\tau^2 - r^2)^{k+1-i} (t - \tau)^{2k + 3 - 2i}}{2^{2i}(i-1)!i!(2k + 3 - 2i)!}.
\]

Differentiating this twice gives

\[
\frac{\partial^2 I(2k + 3)}{\partial t^2} = \frac{[(2k + 1)r^2 - t^2]u^{2k-1}}{2^{2k}k!(k + 1)!} + I(2k + 1),
\]

(38)

which can be reexpressed explicitly by the use of the conjecture (Eq. (37)) as

\[
\frac{\partial^2 I(2k + 3)}{\partial t^2} = \frac{(2k + 1)u^{2k}}{2^{2k}k!(k + 1)!} + \frac{r^2 u^{2k-1}}{2^{2k-1}(k - 1)!(k + 1)!} - \frac{u^{2k}}{(2k)!} - \frac{ru^{2k-1}}{(2k - 1)!}
\]

\[
= \frac{\partial^2 J(2k + 3)}{\partial t^2},
\]

(39)

where the last equality can be established by straightforward calculation. Now by inspection, we see that \( I(n) \) is a \( n - 1 \) degree polynomial, which together with Eq. (39) implies

\[
I(2k + 3, t, r) = J(2k + 3, t, r) + C(k)r^{2k+1}t + D(k)r^{2k+2}.
\]

(40)

Again by inspection, we see that \( I(2k + 3, t, t) = 0 = J(2k + 3, t, t) \); therefore, Eq. (40) becomes

\[
I(2k + 3, t, r) = J(2k + 3, t, r) + C(k)r^{2k+1}u_-
\]

\[
= \frac{2r^{k-1}}{2k^r} [(2k + 1)u^{2k} + 2k(2k + 1)r u^{2k-1} + (2k + 1)!(2k + 1)],
\]

(41)
In order to show $C(k) = 0$, we substitute for $I$ in Eq. (36) giving

$$p(N_t = 2k + 3 | v_+) = \frac{1}{2t^{2k+3}} \left[ \frac{4u^{2(k+1)}}{B(1/2, k + 1)} + (2k + 3)C(k)r^{2k+1}u_+ \right].$$

Next observe that the density $p$ given by Eq. (32) is normalized, namely $\int_0^\infty p \, \mathrm{d}r = 1$, for all $k$ because the probability $P(S_m(\tau) < r_1 | N_t = 2k + 1 | v_+)$ (Eq. (31)), unlike Eq. (43) in the next subsection, has no term independent of $r_1$ by inspection. By the integral representation of beta functions, the normalization of $p$ implies $C(k) = 0$. We have thus proved $I(2k + 3) = J(2k + 3)$, and hence the first equality of Eq. (35).

The proof for the second equality of Eq. (35) ($N_t = 2k$ case) is similar but simpler due to a result for $N_t = 2k + 1$. Instead of Eq. (37), we now need to prove

$$I(2k, t, r) = \frac{2}{(2k)!} \left[ 2B(1/2, k)^{-1}tu^{2(k-1)} - ku^{2(k-1)}(t + 2(k - 1)r) \right] = J(2k, t, r).$$

Again it is easily shown that

$$I(4) = -\frac{u}{24}(t^2 - 8r^2 + tr) = J(4),$$

and to show $I(2k + 2) = J(2k + 2)$, we separate the $(k + 1)$th term in the summand of $I(2k + 2)$ giving

$$I(2k + 2) = \int_0^r \frac{d\tau[2k + 1)(r^2 - \tau^2)](r^2 - r^2)^{k-1}}{2^{2k}k!(k + 1)!}$$

$$+ \sum_{i=2}^k \int_0^r \frac{d\tau[(2i - 1)(r^2 - \tau^2)(r^2 - r^2)^{i-2}(t - \tau)^{2k+2-2i}}{2^{2(i-1)}(i - 1)!i!(2k + 2 - 2i)!}.$$

Differentiating with respect to $t$ gives

$$\frac{\partial I(2k + 2)}{\partial t} = \frac{[(2k + 1)r^2 - t^2]u^{2(k-1)}}{2^{2k}k!(k + 1)!}$$

$$+ \sum_{i=2}^k \int_0^r \frac{dt[(2i - 1)(r^2 - \tau^2)(r^2 - r^2)^{i-2}(t - \tau)^{2k+1-2i}}{2^{2(i-1)}(i - 1)!i!(2k + 1 - 2i)!},$$

where the last term is just $I(2k + 1)$ (see Eq. (34)). Therefore, using Eq. (37) for $I(2k + 1)$, and simplifying gives

$$\frac{\partial I(2k + 2)}{\partial t} \approx \frac{(2k + 1)u^{2k} + 2kr^2u^{2(k-1)}}{2^{2k}k!(k + 1)!} - \frac{u^{2k-1}[t + (2k - 1)r]}{(2k)!},$$
which is easily shown to be $\partial J(2k + 2)/\partial t$, and since $I(n)$ is a polynomial of degree $n - 1$, we conclude that

$$I(2k + 2) = J(2k + 2) + C(k)r^{2k+1}.$$ 

Substituting this into the density, we have

$$\rho(N_t = 2k + 2 | v_+) = \frac{2u^{2k}}{B(1/2, k + 1)t^{2k+1}} + (2k + 2)!C(k)r^{2k+1}/2t^{2k+2},$$

which again by the normalization condition of $\rho$ implies $C(k) = 0$, and hence we have shown that the conjecture holds true under $k \to k + 1$. We have now completed the proof of both the conjectures, Eq. (35).

By integrating the density $\rho$, the probability of confinement to less than $r_1$ is obtained easily to be

$$P(S_m(t) < r_1 | N_t = \{2k, 2k - 1, v_+\}) = \frac{B_z(1/2, k)}{B(1/2, k)} \theta \equiv I_z(1/2, k)\theta,$$

where $z = r_1^2/t^2$ and $I_z$, the regularized beta function.

The density of $S_m(t)$ regardless of $N_t$ for $t \geq r_1$ is given by

$$\rho(S_m(t) = r | v_+) = \sum_{k=0}^{\infty} \rho(S_m(t) = r, N_t = k)$$

$$= e^{-at}\delta(t - r) + \theta ae^{-at}\left[I_o(au) + \frac{t}{u}I_1(au)\right]. \quad (42)$$

4.2. Case 2: $v_0 = -1$ ($v_-$)

We now turn to the case of $v_0 = -1$ the left starting direction. Substituting for $f$ (Eq. (26)) and $P$ (Eq. (2)) in Eq. (28) and dividing by $P(N_t = n)$, we obtain

$$P(S_m(t) \geq r_1 | N_t = n, v_-) = \frac{n!}{r^n} \sum_{i=0}^{[(n-1)/2]} \frac{\int_{r_1}^{r} d\tau (\tau^2 - r_1^2)^{i}(t - \tau)^{n-(2i+1)}}{2^{i+1}i!(i + 1)!\left[n - (2i + 1)\right]!} + r_1 \sum_{i=1}^{[(n-1)/2]} \frac{i!}{2^{2i}(i - 1)!(i + 1)\left[n - (2i + 1)\right]!}.$$

(43)

By inspection, we see that there is a constant term independent of $t$ and $r_1$, which can be obtained by setting $r_1 = 0$ in the above equation. Substituting $x = \tau/t$, the
constant part of the probability is given by

\[
P_{\text{const}}(S_m(t) \geq r_1 | N_i = n, v_-) = P(S_m(t) \geq 0 | N_i = n, v_-)
\]

\[
= n! \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2^{2i+1}i!(i+1)!} \frac{[(n-2)(2i+1)]!}{(2i)!} = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \frac{(2i)!}{2^{2i+1}i!(i+1)!},
\]

noting that the integral displayed in Eq. (44) is \(B(2i+1, n-2i)\). To evaluate this sum, we use the results of a similar sum in the \(v_+\) case. From Eq. (34), we have

\[
-I(2k, r_1 = 0) = k \int_0^{\infty} d\tau \frac{\tau^{2(i-1)(t-\tau)^2}}{2^{2i}i!(i+1)}
\]

\[
= \frac{t^{2k-1}}{(2k-1)!} \sum_{i=2}^{k} \frac{[2(i-1)]!}{2^{2i-1}(i-1)!}.
\]

and from Eq. (41),

\[
-I(2k, r_1 = 0) = \frac{2t^{2k-1}}{(2k)!} \left[ k - 2B\left(\frac{1}{2}, k\right) \right],
\]

hence giving

\[
\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \frac{(2i)!}{2^{2i+1}i!(i+1)!} = 1 - [(k + 1)B(1/2, k + 1)]^{-1}, \quad k \geq 0, \quad n = 2k + 1, \, 2k + 2.
\]

That is,

\[
P_{\text{const}}(S_m(t) < r_1 | N_i = n, v_-) = [(k + 1)B(1/2, k + 1)]^{-1}, \quad k \geq 0, \quad n = 2k + 1, \, 2k + 2,
\]

and the normalization for the density denoted as \(\tilde{\rho}\) is, instead of 1,

\[
\int_0^{\infty} \tilde{\rho}(S_m(t) = r | N_i = n, v_-) dr
\]

\[
= 1 - [(k + 1)B(1/2, k + 1)]^{-1}, \quad k \geq 0, \quad n = 2k + 1, \, 2k + 2.
\]

Just as the \(v_+\) case, it is easier to conjecture the formula for the density, rather than the probability, given by

\[
\tilde{\rho}(S_m(t) = r | N_i = n, v_-) = \begin{cases} 1/(2t), & n = 1, \\ u_-/t^2, & n = 2, \\ \theta_{\frac{1}{4}}^{n-2} [nu_n^{-2} [3t + (2n - 5)r] - \theta_{n58} nI(n)], & n \geq 3, \end{cases}
\]
where now

\[
I(n) = \sum_{i=2}^{(n-1)/2} \frac{\int \tau \left( \tau^2 - r^2 \right)^{i-2} \left( \tau - r \right)^{\frac{1}{2}(2i+1)} \left( \tau^2 - 2r \tau - (2i+1)r^2 \right)(t - \tau)^{n-(2i+1)}}{2^{2i}(i-1)!(i+1)! \left[ n - (2i+1) \right]!},
\]

(48)

and

\[
\theta_{nm} = \begin{cases} 
1, & n \geq m, \\
0, & n < m.
\end{cases}
\]

Evaluating explicitly the density for several values of \( n \) by algebraic computing, we are led to conjecture that for \( N_i \) odd,

\[
\tilde{\rho}(S_m(t) = r | N_i = 2k + 1, v_-) = A(k) \left( \frac{(2k + 1)t + r}{r^{2k+1}} \right)^{u-uz^{(k-1)}},
\]

(49)

where \( A(k) = \left[ (k + 1)B(1/2, k + 1) \right]^{-1} \), and for \( N_i \) even,

\[
\tilde{\rho}(S_m(t) = r | N_i = 2k, v_-) = B(k) \left( \frac{u-uz^{(k-1)}}{r^{2k}} \right).
\]

(50)

where \( B(k) = 2B(1/2, k)^{-1} \). The coefficient \( A(k) \) may be inferred, by inspection, from the fact that the coefficient \( A(k) \) equals the constant in \( P(S_m(t) < r_1 | N_i = 2k, v_-) \) for the cases computed, whereas \( B(k) \) is worked out from the requirement of Eq. (46):

\[
\int_0^{r_1} \tilde{\rho}(S_m(t) = r | N_i = 2k, v_-) dr = \frac{1}{2} B(k) \left[ B_z(1/2, k) - B_z(1, k) \right], \quad z = \frac{r_1^2}{t^2},
\]

where \( B_z \) denotes the incomplete beta function, giving

\[
B(k) = 2 \frac{1 - \frac{[kB(1/2, k)]^{-1}}{B(1/2, k) - B(1, k)}}{B(1/2, k) - B(1, k)} = 2B(1/2, k)^{-1}.
\]

The proofs of these conjectures are similar to the case for \( v_+ \), and is given in Appendix B. By integrating these densities with respect to \( \rho \), the probability of confinement to less than \( r_1 \) is given by, for \( N_i = 0 \), \( P(S_m(t) < r_1 | N_i = 0, v_-) = 1 \), of course, and for \( N_i > 1 \),

\[
P(S_m(t) < r_1 | N_i = n, v_-) = \left[ \frac{(n + 1)/2}{B(1/2, [(n + 1)/2])] \right]^{-1} \theta
+ \theta \int_0^{r_1} \tilde{\rho}(S_m(t) = r | N_i = n, v_-) dr.
\]

(51)

The integral may be evaluated in terms of the \( B_z \), giving

\[
P(S_m(t) < r_1 | N_i = 2k, v_-) = B(1/2, k)^{-1} \left[ \frac{1}{k} + B_z(1/2, k) - B_z(1, k) \right]
= B(1/2, k)^{-1} \left[ B_z(1/2, k) + \frac{1}{k}(1 - z)^k \right]
\]

(52)
Fig. 2. $P(S_m(t) < r_1|N_t = n, v_+)$, the probability of confinement to less than $r_1$ for $r_1 = 1$, as a function of $\sqrt{z} = r_t/t$ for several values of $N_t$. (For the purpose of identifying the lines, along $z^{1/2} = 0.4$, the values of $n$ in the order of increasing $P$, are $n = 1$ or 2, 3 or 4, 5 or 6, and 29 or 30 for the solid line, and $n = 1, 3, 5$ and 29 for the dashed line.) The solid line and the dashed line take the meaning as in Fig. 1.

and

$$P(S_m(t) < r_1|N_t = 2k + 1, v_-)$$

$$= \left[ (k + 1)B(1/2, k + 1) \right]^{-1} \left[ 1 + kB_z(1/2, k) - kB_z(1, k) + \frac{1}{2} B_z(1/2, k + 1) \right]$$

$$= \left[ (k + 1)B(1/2, k + 1) \right]^{-1} \left[ kB_z(1/2, k) + \frac{1}{2} B_z(1/2, k + 1) + (1 - z)^k \right],$$

(53)

where the relation $B_z(1, k) = \left[ 1 - (1 - z)^k \right]/k$ is used. These probabilities as a function of $z$ for several values of $N_t$ are shown in Fig. 2, where the solid line and the dashed line take the meaning as in Fig. 1.

By summing over $N_t$ we obtain the density of the maximum regardless of $N_t$:

$$\tilde{\rho}(S_m(t) = r|v_-) = \sum_{k=0}^{\infty} \rho(N_t = 2k + 1, v_-) + \sum_{k=1}^{\infty} \rho(N_t = 2k, v_-)$$

$$= ae^{-at} \left[ \frac{1}{au} I_1(au) + \left( \frac{u_-}{u_+} \right)^{1/2} I_1(au) + \frac{t}{u_+} I_2(au) \right],$$

(54)

which, by $I_2(z) = I_0 - (2/z)I_1(z)$, may also be written as

$$\tilde{\rho}(S_m(t) = r|v_-) = ae^{-at} \left\{ \frac{t}{u_+} I_0(au) + \left( 1 - \frac{1}{au_+} \right) \left( \frac{u_-}{u_+} \right)^{1/2} I_1(au) \right\}.$$
5. Distribution in the presence of a trap

The density of the randomized time in the presence of a trap can be obtained from the density of the randomized time and the first passage time. The strategy is to think of the trap as a virtual source. The density of $S(t)$ in the presence of a trap at $r_1$, denoted by $g(t, r, N_t; r_1 | v_{\pm})$, is given by $g(t, r, N_t | v_{\pm})$ less the contribution from the virtual source, denoted by $g_{r_1}(t, r, N_t | v_{\pm})$, which can be computed from the following formulae, with $r < r_1$ and $r' \equiv r - r_1$,

$$g_{r_1}(t, r, N_t = n | v_{\pm}) = \sum_{i=0}^{(n-1)/2} \left\lfloor \frac{n}{2} \right\rfloor - 1 \int_{r_1}^{t - r'} \mathcal{f}(r_1, \tau, N_t = \begin{cases} 2i & |v_+ \\ 2i + 1 & |v_- \end{cases})$$

$$\times g\left(t - \tau, r, N_{t-\tau} = \begin{cases} n - 2i & |v_+ \\ n - 2i - 1 & |v_- \end{cases} \bigg| r_1, v_{\pm}\right) d\tau. \quad (55)$$

Notice that we have excluded the term corresponding to $g(N_{t-\tau} = 0)$ in Eq. (55) because it would not contribute to the density for $r < r_1$. The evaluation of Eq. (55) is greatly simplified with the help of Eq. (20).
Consider the cases $N_t = 2k + 1$ with $v_+$ and $N_t = 2k$ with $v_-:

g_{r_1}(t, r, N_t) = \begin{cases} 
2k + 1 & v_+ \\
2k & v_-
\end{cases} = \frac{k(k - i) - 1}{2} \int_{r_1}^{r - |r'|} d\tau f(r_1, \tau, N_t) = \begin{cases} 
2i & v_+ \\
2i + 1 & v_-
\end{cases} \times g(t - \tau, r, N_{t-}) = \begin{cases} 
2(k - i) + 1 & v_+ \\
2(k - i) - 1 & v_-
\end{cases}, \quad r < r_1. \quad (56)

The RHS of Eq. (56) has the same form as Eq. (20), except for the region of validity. Recalling that for $N_t$ odd $g$ is symmetric in $r - r_1$ (Eq. (8)), and since $f$ is independent of $r$, we obtain

\[ g_{r_1}(t, r, N_t) = \begin{cases} 
2k + 1 & v_+ \\
2k & v_-
\end{cases} = g(t, 2r_1 - r, N_t) = \begin{cases} 
2k + 1 & v_+ \\
2k & v_-
\end{cases} = \frac{1}{2} \left[\frac{1}{(k - i)!} - \frac{1}{(k - 1)!(k - i)!}\right] C(t, k) \bar{u}_+ u_+^{2(k - i - 1)}, \quad (57)

where $\bar{u}_+ = t + (2r_1 - r), \bar{u}_- = t - (2r_1 - r)$ and $\bar{u} \equiv \sqrt{\bar{u}_+ \bar{u}_-}$. We write $\theta(\bar{u}_-)$ instead of $\theta(\bar{u})$ in (57) since $\bar{u}_+$ is positive for $r < r_1$.

For the remaining two cases the densities $g$ in the integrands are not symmetric, and their evaluations are not as quick. Substitute for $f$ from Eq. (19) and $g$ from Eq. (8), and writing $t - |r'| = x$ and $t + |r'| = y$, we have

\[ g_{r_1}(t, r, N_t) = \begin{cases} 
2k + 1 & v_+ \\
2k & v_-
\end{cases} = \sum_{i=0}^{k-1} \int_{r_1}^{x} d\tau f(r_1, \tau, N_t) = 2i |v_+| g(t - \tau, r, N_{t-}) = 2(k - i) |r_1, v_+|

= \theta(\bar{u}_-) C(t, k) \sum_{i=0}^{x} \int_{r_1}^{r_1} d\tau X (x - \tau)^{k-i-1} (y - \tau)^{k-i-1} \left(k - i - 1\right)! (k - i)! \quad (58)

where

\[ X = \begin{cases} 
\delta(\tau - r_1), & i = 0, \\
\theta(\tau - r_1) \theta(x - \tau) 2r_1 (t_2 - r_1)^{i-1} & i > 1.
\end{cases}\]

To evaluate the rather complicated sum of integrals in Eq. (58), we again make use of Eq. (20) from which we have

\[ g(t, r, N_t) = \begin{cases} 
2k + 1 & v_+ \\
2k & v_-
\end{cases} = \sum_{i=0}^{k-1} \int_{r_1}^{x} d\tau f(r_1, \tau, N_t) = 2i |v_+| g(t - \tau, r, N_{t-}) = 2(k - i) |r_1, v_+|

= \theta(\bar{u}_-) C(t, k) \sum_{i=0}^{x} \int_{r_1}^{r_1} d\tau X (x - \tau)^{k-i-1} (y - \tau)^{k-i} \left(k - i - 1\right)! (k - i)! \quad (59)\]
Replacing $r$ by $|r^*| + r_1$, we obtain from Eqs. (59) and (7),

$$\theta(\tilde{u}_-) \sum_{i=0}^{k} \int_{r_1}^{\infty} d\tau \frac{(x - \tau)^{k-i}(y - \tau)^{k-1}}{(k-i)!(k-i)!} = \frac{\theta(\tilde{u}_-)(y + r_1)^{k}(x - r_1)^{k-1}}{(k-1)!k!}.$$  

Differentiating this twice with respect to $y$, and letting $k \to k + 1$, we obtain the sum of the integrals in Eq. (58) to be

$$\theta(\tilde{u}_-) \sum_{i=0}^{k-1} \int_{r_1}^{\infty} d\tau \frac{(x - \tau)^{k-i}(y - \tau)^{k-1}}{(k-i)!(k-i)!} = \frac{\theta(\tilde{u}_-)(y + r_1)^{k-1}(x - r_1)^{k}}{(k-1)!k!},$$

and consequently,

$$g_{r_1}(t, r, N_t = 2k | v_+) = \theta(\tilde{u}_-) C(t, k) \frac{\tilde{u}_- \tilde{u}^{2(k-1)}}{(k-1)!k!} = g_{r_1}(t, r, N_t = 2k | v_-).$$  

By the same method, the last case is given by

$$g_{r_1}(t, r, N_t = 2k + 1 | v_-) = \theta(\tilde{u}_-) D(t, k) \frac{\tilde{u}_- \tilde{u}^{2(k-1)}}{(k-1)!(k+1)!}.$$  

With $g_{r_1}$ evaluated, the densities in the presence of a trap at $r_1$ may be written down directly from Eqs. (7), (8), (57), (60) and (61):

$$g(t, r, N_t = 2k; r_1 | v_+) = \frac{C(t, k)}{(k-1)!k!} \left[ u_+ u^{2(k-1)} \theta(u_2) - \tilde{u}_- \tilde{u}^{2(k-1)} \theta(\tilde{u}_-) \right],$$  

$$g(t, r, N_t = 2k + 1; r_1 | v_+) = \frac{D(t, k)}{(k-1)!k!} \left[ u^{2k} \theta(u_2) - \tilde{u}^{2k} \theta(\tilde{u}_-) \right]$$

and

$$g(t, r, N_t = 2k + 1; r_1 | v_-) = \frac{D(t, k)}{(k-1)!k!} \left[ u^{2k} \theta(u_2) - \frac{k}{k+1} \tilde{u}^{2(k-1)} \tilde{u}^{2k-1} \theta(\tilde{u}_-) \right].$$

where we restate here, for convenience, $\tilde{u}_- = t - (2r_1 - r)$, $C(t, k) = a^{2k}e^{-ar}/2^{2k}$ and $D(t, k) = a^{2k+1}e^{-ar}/2^{2k+1}$. The conditional densities [Eqs. (62)–(64) divided by $P(N_t = n)$] are plotted (continuous lines) in Fig. 4(a)–(d), where the dashed line corresponds to the case without the trap for $r \leq r_1$. Summing over $N_t$, the density regardless of $N_t$ in the presence of a trap is given by, for $v_+$,

$$g(t, r; r_1 | v_+) = \sum_{k=0}^{\infty} g(t, r, N_t = k; r_1 | v_+)$$

$$= \frac{e^{-ar} \delta(u_-)}{2} \left\{ \frac{I_0(u_-) + (u_+/u_-)^{1/2} I_1(u_-)}{I_0(u_-) + (u_+/u_-)^{1/2} I_1(u_-)} \right\} \theta(u^2)$$

$$- \left[ I_0(a\tilde{u}) + \left( \frac{\tilde{u}_-}{a\tilde{u}} \right)^{1/2} I_1(a\tilde{u}) \right] \theta(\tilde{u}_-).$$
Fig. 4a

Fig. 4b

Fig. 4. The conditional densities in the presence of a trap (continuous lines) at $r_1 = 1$ with $t = 3$: $g(t, r, r_1 | N_t = 2k, v_+)$ (a) and $g(t, r, r_1 | N_t = 2k, v_-)$ (b) for $k = 1, 2, 3, 4$ and 5, as well as $g(t, r, r_1 | N_t = 2k + 1, v_+)$ (c) and $g(t, r, r_1 | N_t = 2k + 1, v_-)$ (d) for $k = 0, 1, 2, 3, 4$ and 5. Dashed line corresponds to the case without the trap for $r \leq r_1$. 
and for \( v_- \),

\[
g(t, r, r_1 | v_-) = e^{-\alpha t} \delta(u_+)
+ \frac{\alpha}{2} e^{-\alpha t} \left\{ I_0(au) + \left( \frac{u_-}{u_+} \right)^{1/\gamma} I_1(au) \right\} \theta(u^2)
- \left( \frac{\bar{u}_-}{\bar{u}_+} \right)^{1/2} \left[ I_1(a\bar{u}) + \left( \frac{u_-}{\bar{u}_+} \right)^{1/2} I_2(a\bar{u}) \right] \theta(\bar{u}_-) \right\}.
\]  

(66)
Fig. 5. Densities in the presence of a trap at $r_1$, $g(t, r, r_1 | v_+)$, $g(t, r, r_1 | v_-)$ and the average $g(t, r; r_1)$, minus the delta-functions, are plotted in (a), (b) and (c), respectively, for $r_1 = 1$, and $t = 1, 2, 3, 4$ and 5.
These densities minus the delta-functions are plotted in Fig. 5(a) and (b) for \( r_1 = 1 \), and \( t = 1, 2, 3, 4, \) and 5. If the initial velocities are symmetric, that is \( P(v_+) = P(v_-) = \frac{1}{2} \), then by taking the average of Eqs. (65) and (66), we have

\[
g(t, r; r_1) = \frac{1}{2} \left[ g(t, r; r_1 | v_+) + g(t, r; r_1 | v_-) \right] \\
= \frac{1}{2} e^{-\alpha t} \delta(u^2) + \frac{a}{4} e^{-au} \left\{ 2I_0(au) + \frac{2t}{u} I_1(au) \right\} \theta(u) \\
- \left[ I_0(\overline{u}_- a\overline{u}_+) + 2 \left( \frac{\overline{u}_-}{\overline{u}_+} \right)^{1/2} I_1(\overline{u}_- a\overline{u}_+) + \frac{\overline{u}_-}{\overline{u}_+} I_2(\overline{u}_- a\overline{u}_+) \right] \theta(\overline{u}_-) \right\},
\]

which was obtained by Masoliver et al. (1992) using a different approach, namely via the partial differential equations satisfied by the densities. This average density is given in Fig. 5(c).

6. Relationship among the first passage time, maximum displacement and density distribution in the presence of a trap

We shall end this paper by showing two interesting relations: first between the first passage time and the density distribution in the presence of a trap, and second between the maximum displacement and the density. A particle situated just before the trap in the interval \( r_1 - \Delta t \) and \( r_1 \) will arrive at \( r_1 \) for the first time with probability of the order \( 1 - a\Delta t \). Therefore in the limit \( \Delta t \to 0 \) we have

\[
g(t, r_1, N_t; r_1 | v_+) = f(r_1, t, N_t | v_-).
\]
which is indeed satisfied by Eqs. (19), (26), (62) and (64). The other relation is the probability that the particle has not passed the point \( r_1 \) is equal to the survival probability of the particle in the presence of the trap:

\[
P(S_m(t) \leq r_1, N_t|v_\pm) = \int_{-\infty}^{r_1} dr g(t, r, N_t; r_1|v_\pm).
\]

Then differentiating with respect to \( r_1 \) gives

\[
\rho(S_m(t) = r_1, N_t|v_\pm) = g(t, r_1, N_t; r_1|v_\pm) + \int_{-\infty}^{r_1} dr \frac{\partial}{\partial r_1} g(t, r, N_t; r_1|v_\pm),
\]

which can be checked by Eqs. (35), (49), (50), (62), (63) and (64). The last integration is easily carried out because the \( r_1 \)-dependent term of \( g \) is a function of \( r - r_1 \).

### Appendix A

Here the proof of

\[
s \int_{r_1}^{\infty} e^{-ar_1} I_1(a t)I_1(a u_1) \, dt = 1 - e^{-ar_1}
\]

is given.

We first rewrite \( I_1 \), with \( x = a(t - r_1) \), as

\[
\frac{ar_1}{u_1} I_1(a u_1) = \frac{a^2 r_1}{2} \sum_{k=0}^{\infty} \frac{(a u_1)^{2k}}{2^{2k} k!(k+1)!}
\]

\[
= \frac{a^2 r_1}{2} \sum_{k=0}^{\infty} \frac{[x(x + 2ar_1)]^k}{2^{2k} k!(k+1)!}
\]

\[
= a \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{(ar_1)^{n+1} x^{2k-n}}{n! 2^{2k-n+1} (k-n)!(k+1)!}.
\]

Interchanging the order of the summation with respect to \( k \) and \( n \), and rewriting \( k - n \) as \( k \), we obtain

\[
\frac{ar_1}{u_1} I_1(a u_1) = \sum_{n=0}^{\infty} \frac{(ar_1)^{n+1} a}{n!} \sum_{k=0}^{\infty} \frac{(x/2)^{2k+n+1}}{k!(n+k+1)!} = \sum_{n=0}^{\infty} \frac{(ar_1)^{n+1} a}{n!} \frac{1}{x} I_{n+1}(x).
\]

Using this expression, we have the following final result:

\[
ar_1 \int_{r_1}^{\infty} \frac{e^{-ar_1}}{u_1} I_1(a u_1) \, dt = e^{-ar_1} \sum_{n=0}^{\infty} \frac{(ar_1)^{n+1} a}{n!} \int_{0}^{\infty} \frac{e^{-x}}{x} I_{n+1}(x) \, dx
\]

\[
e^{-ar_1} \sum_{n=0}^{\infty} \frac{(ar_1)^{n+1}}{n!} \frac{1}{n+1}
\]

\[
= 1 - e^{-ar_1},
\]

where the integral table by Moriguchi et al. (1965) was used.
Appendix B.

We present the proof for the conjectures, Eqs. (49) and (50), for the case of \( N \), odd and \( N \), even, respectively. For \( k = 0 \) and 1, verification of Eq. (49) is easily done by direct evaluations. For \( k \geq 2 \), from Eqs. (47)-(49), we see that proving the conjecture is equivalent to proving that

\[
I(2k + 1) = \frac{1}{(2k + 1)!} \{ k(2k + 1)u_{2k-1}^2 (3u_- + 4kr) \\
- A(k)u_- u^{2k-1} [(2k + 1)t + r] \} \equiv J(2k + 1).
\]

(69)

First, we check that the formula works for \( k = 2 \), for which we find

\[
I(5) = \frac{1}{3!2^6} u_-^2 (t^2 - 11r^2 - 2rt) = J(5).
\]

We now proceed to show that for \( k \to k + 1 \), \( I(2k + 3) = J(2k + 3) \), by first showing that their second time derivatives are equal, as in the case of \( v_+ \). Writing the \( i = k + 1 \) term separately, we have

\[
I(2k + 3) = \sum_{i=2}^{k} \int_r^t d\tau (\tau^2 - r^2)^{i-2} (\tau - r) [\tau^2 - 2r\tau - (2i + 1)r^2] (t - \tau)^{2(k-i) + 2} \\
\times 2^{2i}(i - 1)!{(i + 1)!}[2(k - i) + 2]!
\]

\[
+ \int_r^t d\tau (\tau^2 - r^2)^{k-1} (\tau - r) [\tau^2 - 2r\tau - (2k + 3)r^2] \\
\times 2^{2k+2} k!(k + 2)!
\]

The second time derivative of the first term on the right-hand side is \( I(2k + 1) \) which is \( J(2k + 1) \) by the conjecture, and consequently, we have

\[
\frac{\partial^2 I(2k + 3)}{\partial t^2} = [8(2k)!]^{-1} [3u_- + 4kr]u_{2k-1}^2 - [(2k + 1)!]^{-1} A(k) [(2k + 1)t + r]u_- u^{2(k-1)}
\]

\[
+ [2^{2k+2} k!(k + 2)!]^{-1} [(2k + 1)(t^3 - r^3) - (2k + 1)^2 r^2 t - (4k - 1)rt^2]
\times u_- u^{2(k-2)}
\]

By straightforward computation, it can be shown that in the above equation the first term (second and third terms) on the RHS is the second time derivative of the first (second) term of \( J(2k + 3) \) defined by Eq. (69); hence, \( \frac{\partial^2 I(2k + 3)}{\partial t^2} = \frac{\partial^2 J(2k + 3)}{\partial t^2} \). Then by the same arguments following Eq. (39) in the case of \( v_+ \), it may be concluded that \( I(2k + 3) = J(2k + 3) \), and hence the proof for the case of \( n \) odd is completed.
We now prove the case of $n$ even, namely

$$
\rho(S_m(t) = r \mid N_t = 2k, v_\perp) = \frac{2u_- u^{2(k-1)}}{B(1, k) t^{2k}}.
$$

For $k = 1$ and $2$, the formula checks. For $k \geq 3$, proving the conjecture is equivalent to proving

$$
I(2k) = \sum_{i=2}^{k-1} \int_r^t \frac{d \tau}{\tau^2 - (r^2)^{-2}} \left( \tau - r \right) \left[ \tau^2 - 2r \tau - (2i + 1)r^2 \right] (t - \tau)^{2(k-i)-1} \left( \tau - r \right) \left[ 2(k - i) - 1 \right]!
$$

$$
= \frac{1}{4(2k)!} \left\{ k \left[3t + (4k - 5)r\right] u^{2(k-1)} - \frac{8u_- u^{2(k-1)}}{B(1/2, k)} \right\} = J(2k).
$$

The formula works for $k = 3$, for which we have

$$
I(6) = u^3 \frac{(t^2 - 19r^2 - 2tr)(2715)}{2} = J(6).
$$

To show $I(2(k + 1)) = J(2(k + 1))$, we first note that

$$
\frac{\partial I(2k + 2)}{\partial t} = I(2k + 1) = J(2k + 1),
$$

and it is easy to show that $\partial I(2k + 2)/\partial t = \partial J(2k + 2)/\partial t$; in other words, $I(2k + 2) = J(2k + 2) + C(k)r^{2k+1}$. Again by the normalization of the density $\rho$, it may be concluded that $C(k) = 0$, and the proof for the case of $n$ even is completed.

Appendix C

This appendix checks the normalization condition $P(S_m(t) < t \mid v_-) = 1$. Summing over $N_t$ the constant part of the probability is

$$
P_{\text{const}}(S_m(t) < t \mid v_-) = 1.
$$

Summing over $N_t$, the constant part of the probability is

$$
P_{\text{const}}(S_m(t) < t \mid v_-) = \sum_{n=0}^{\infty} P_{\text{const}}(S_m(t) < r_1, N_t = n \mid v_-)
$$

$$
= P_{\text{const}}(N_t = 0 \mid v_-) + \sum_{k=0}^{\infty} P_{\text{const}}(N_t = 2k + 1 \mid v_-) + \sum_{k=1}^{\infty} P_{\text{const}}(N_t = 2k \mid v_-)
$$

$$
= e^{-at} + e^{-at} \sum_{k=0}^{\infty} [(k + 1)B(1/2, k + 1)]^{-1} \frac{(at)^{2k+1}}{(2k + 1)!}
$$

$$
+ e^{-at} \sum_{k=1}^{\infty} [kB(1/2, k)]^{-1} \frac{(at)^{2k}}{(2k)!}
$$

$$
= e^{-at}[I_0(at) + I_1(at)].
$$
Now we integrate Eq. (54). With the integral table in Foong (1993), adapted from Prunikov et al. (1986), we have

\[ \int_0^t dr \frac{I_1(au)}{u} = \frac{\cosh(at) - 1}{at}, \]  
(71)

\[ \int_0^t dr \left( \frac{u}{u_+} \right)^{1/2} I_1(au) = \frac{1}{a} \left[ \cosh(at) - I_0(at) \right]. \]  
(72)

and using Prunikov (item 2.15.2 no. 10) and Gradshteyn and Ryzhik (item 6.561 no. 11), we have

\[ \int_0^t dr \frac{I_2(au)}{u_+} = \int_0^t \frac{I_2(au)}{u \sqrt{t^2 - u^2}} du - \int_0^t \frac{I_2(au)}{u} du \]

\[ = \frac{\pi}{\sqrt{2at}} L_{3/2}(at) - \frac{I_1(at)}{at} + \frac{1}{2} \]

\[ = \frac{1}{(at)^2} [at \sinh at - \cosh at + 1 - at I_1(at)]. \]  
(73)

where we have used the relation

\[ L_{3/2}(z) = \sqrt{\frac{2}{\pi}} z^{-3/2} \left( z \sinh z - \cosh z + 1 - \frac{z^2}{2} \right). \]

With these results, Eqs. (71)–(73), the integrated Eq. (54) is

\[ \int_0^t \rho(S_m(t) = r | v_-) dr = 1 - e^{-at} \left[ I_0(at) + I_1(at) \right]. \]  
(74)

Hence, adding Eqs. (70) and (74), we verify that \( P(S_m(t) < t | v_-) = 1. \)

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