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# Planar nonautonomous polynomial equations II. Coinciding sectors

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ABSTRACT

We give a few sufficient conditions for the existence of periodic solutions of the equation $\dot{z} = \sum_{j=0}^{n} a_j(t) z^j$ where $a_j$'s are complex-valued. We prove the existence of one up to two periodic solutions and heteroclinic ones.



## 1. Introduction

The presented paper is a continuation of [31,32]. We study a planar nonautonomous differential equations of the form

$$\dot{z} = v(t, z) = \sum_{j=0}^{n} a_j(t) z^j, \tag{1}$$

where $n \geqslant 3$ and $a_j \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ are $T$-periodic.

An extensive study of the set of periodic solutions of Eq. (1) was initiated in [25] and continued in many papers e.g. [1–3,11–14,17–19,21,23,24,26]. In those papers the coefficients $a_j$ are real. One of the most important problem is to examine the structure of the set of periodic solutions. The second one is the investigation of a centre which is motivated by the Poincaré centre–focus problem. The third one is connected with the XVIth Hilbert problem for degree two equations in the plane which can be reduced to the problem of finding the maximal number of closed solutions of Eq. (1) with $n = 3$ and special coefficients $a_j$. This leads to investigations of the maximal number of periodic solutions of (1). It is proved in [20] that in the general case there is no upper bound for this number provided that $n \geqslant 3$.

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The complex coefficients are considered in [6,7] and a few sufficient conditions for the existence of periodic solutions are presented. The upper bound of the number of periodic solutions and structure of the centre variety is considered in [10]. The problem of nonexistence of periodic solutions is investigated in [34] where it is proved that there exist coefficients $a_j$ such that Eq. (1) has no periodic solutions.

In the presented paper we develop the ideas from [31,32] and give a few sufficient conditions for the existence of one or two periodic solutions. We deal with condition of geometric type which corresponds to the ones from [32, Section 3.1]. Namely, we try to include sets $a_j(\mathbb{R})$ in some sectors of the complex plane. The point is, we try to find one sector appropriate for all $a_j$'s. By the existence of greater number of coefficients $a_j$ than in the case of the Riccati equation (i.e. Eq. (1) with $n = 2$), it is convenient to consider sectors which axis of symmetry is the real axis. Other sectors need a change of variables of the form $w = e^{i\alpha} z$. We prove that there exists in every of such sectors exactly one $T$-periodic solution.

After the change of variables $w = \frac{1}{z}$ Eq. (1) has the form

$$\dot{w} = -\sum_{j=0}^{n} a_j(t) w^{2-j}.$$

Thus unlike the Riccati equation it is not well defined in the point $z = \infty$ and the Poincaré map is more complicated than a Möbius transformation. In fact, complexity of the Poincaré map unables us to cover the whole plane with sectors, so we cannot provide an upper bound for the number of periodic solutions.

We consider behaviour of the vector field on the boundary of the sectors which allows us to construct some sets which are almost periodic isolating segments and detect periodic solutions inside them (see [28–30] for the notion of isolating segments). By the special properties of holomorphic functions we can use the Denjoy–Wolff fixed point theorem (cf. [9,27]) instead of the Brouwer one. It allows us to obtain asymptotic stability or asymptotic unstability of detected periodic solutions. Moreover, they are attracting or repelling in the whole sector, which leads to the heteroclinic solutions connecting periodic ones. By using Ważewski method, we are able to prove the existence of solutions which blow up and are contained in the considered sectors.

The paper is organised as follows. In Section 2 we give definitions and introduce notion. The next section is devoted to (1) with $a_0 \not\equiv 0$. In the next one we deal with the equation in the very simple form, namely $\dot{z} = z^n + a_0(t)$, and state theorems on the lower bound of the number of periodic solutions depending on whether or not $n$ is even. Section 5 provides considerations in the case $a_0 \equiv 0$ and $a_1 \not\equiv 0$. In the last section we assume $a_0 = a_1 \equiv 0$ and $a_2 \not\equiv 0$.

## 2. Definitions

### *2.1. Dynamical systems and Ważewski method*

Let $X$ be a topological space and $W$ be a subset of $X$. Denote by $\operatorname{cl} W$ the closure of $W$. The following definitions come from [29].

Let $D$ be an open subset of $\mathbb{R} \times X$. By a *local flow* on $X$ we mean a continuous map $\phi : D \to X$, such that three conditions are satisfied:

(i) $I_x = \{t \in \mathbb{R}\colon\ (t, x) \in D\}$ is an open interval $(\alpha_x, \omega_x)$ containing 0, for every $x \in X$,
(ii) $\phi(0, x) = x$, for every $x \in X$,
(iii) $\phi(s + t, x) = \phi(t, \phi(s, x))$, for every $x \in X$ and $s, t \in \mathbb{R}$ such that $s \in I_x$ and $t \in I_{\phi(s,x)}$.

In the sequel we write $\phi_t(x)$ instead of $\phi(t, x)$.

Let $\phi$ be a local flow on $X$, $x \in X$ and $W \subset X$. We call the set

$$\phi^+(x) = \phi\big([0, \omega_x) \times \{x\}\big)$$

the *positive semitrajectory* of $x \in X$.

We distinguish three subsets of $W$ given by

$$W^- = \{x \in W\colon\ \phi([0,t] \times \{x\}) \not\subset W,\ \text{for every } t > 0\},$$
$$W^+ = \{x \in W\colon\ \phi([-t,0] \times \{x\}) \not\subset W,\ \text{for every } t > 0\},$$
$$W^* = \{x \in W\colon\ \phi(t,x) \notin W,\ \text{for some } t > 0\}.$$

It is easy to see that $W^- \subset W^*$. We call $W^-$ the *exit set of* $W$, and $W^+$ the *entrance set of* $W$.

We call $W$ a *Ważewski set* provided

(1) if $x \in W$, $t > 0$, and $\phi([0,t] \times \{x\}) \subset \operatorname{cl} W$ then $\phi([0,t] \times \{x\}) \subset W$,
(2) $W^-$ is closed relative to $W^*$.

**Proposition 1.** *If both $W$ and $W^-$ are closed subsets of $X$ then $W$ is a Ważewski set.*

The function $\sigma : W^* \to [0,\infty)$

$$\sigma(x) = \sup\{t \in [0,\infty)\colon\ \phi([0,t] \times \{x\}) \subset W\}$$

is called the *escape-time function of* $W$.

The following lemma is called the Ważewski lemma.

**Lemma 2.** *(See [29, Lemma 2.1(iii)].) Let $W$ be a Ważewski set and $\sigma$ be its escape-time function. Then $\sigma$ is continuous.*

Finally, we state one version of the Ważewski theorem.

**Theorem 3.** *(See [29, Corollary 2.3].) Let $\phi$ be a local flow on $X$, $W \subset X$ be a Ważewski set and $Z \subset W$. If $W^-$ is not a strong deformation retract of $Z \cup W^-$ in $W$ then there exists an $x_0 \in Z$ such that $\phi^+(x_0) \subset W$.*

(For the definition of the strong deformation retract see e.g. [15].)

*2.2. Processes*

Let $X$ be a topological space and $\Omega \subset \mathbb{R} \times X \times \mathbb{R}$ be an open set.

By a *local process* on $X$ we mean a continuous map $\varphi : \Omega \to X$, such that three conditions are satisfied:

(i) $I_{(\sigma,x)} = \{t \in \mathbb{R}\colon\ (\sigma,x,t) \in \Omega\}$ is an open interval containing 0, for every $\sigma \in \mathbb{R}$ and $x \in X$,
(ii) $\varphi(\sigma,\cdot,0) = \mathrm{id}_X$, for every $\sigma \in \mathbb{R}$,
(iii) $\varphi(\sigma,x,s+t) = \varphi(\sigma+s,\varphi(\sigma,x,s),t)$, for every $x \in X$, $\sigma \in \mathbb{R}$ and $s,t \in \mathbb{R}$ such that $s \in I_{(\sigma,x)}$ and $t \in I_{(\sigma+s,\varphi(\sigma,x,s))}$.

For abbreviation, we write $\varphi_{(\sigma,t)}(x)$ instead of $\varphi(\sigma,x,t)$.

Local process $\varphi$ on $X$ generates a local flow $\phi$ on $\mathbb{R} \times X$ by the formula

$$\phi(t,(\sigma,x)) = (\sigma+t,\varphi(\sigma,x,t)).$$

Let $M$ be a smooth manifold and let $v : \mathbb{R} \times M \to TM$ be a time-dependent vector field. We assume that $v$ is so regular that for every $(t_0,x_0) \in \mathbb{R} \times M$ the Cauchy problem

$$\dot{x} = v(t, x), \tag{2}$$

$$x(t_0) = x_0 \tag{3}$$

has unique solution. Then Eq. (2) generates a local process $\varphi$ on $X$ by $\varphi_{(t_0,t)}(x_0) = x(t_0, x_0, t + t_0)$, where $x(t_0, x_0, \cdot)$ is the solution of the Cauchy problem (2), (3).

Let $T$ be a positive number. In the sequel $T$ denotes the period. We assume that $v$ is $T$-periodic in $t$. It follows that the local process $\varphi$ is $T$-periodic, i.e.,

$$\varphi_{(\sigma+T,t)} = \varphi_{(\sigma,t)} \quad \text{for all } \sigma, t \in \mathbb{R},$$

hence there is a one-to-one correspondence between $T$-periodic solutions of (2) and fixed points of the Poincaré map $\varphi_{(0,T)}$.

### 2.3. *Periodic isolating segments*

Let $X$ be a topological space. We assume that $\varphi$ is a $T$-periodic local process on $X$.

For any set $Z \subset \mathbb{R} \times X$ and $t \in \mathbb{R}$ we put

$$Z_t = \{x \in X:\ (t, x) \in Z\}.$$

Let $\pi_1 : \mathbb{R} \times X \to \mathbb{R}$ be the projection on the time variable.

We call a compact set $W \subset [a, b] \times X$ an *isolating segment over* $[a, b]$ *for* $\varphi$ if the exit and entrance sets $W^-$, $W^+$ of $W$ are also compact and there exist compact subsets $W^{--}, W^{++} \subset W$ (called, respectively, the *proper exit set* and the *proper entrance set*) such that

(1) $\partial W = W^- \cup W^+$,
(2) $W^- = W^{--} \cup (\{b\} \times W_b)$,
(3) $W^+ = W^{++} \cup (\{a\} \times W_a)$, $W_a^{++} = \mathrm{cl}(\partial(W_a) \setminus W_a^{--})$,
(4) there exists homeomorphism $h : [a, b] \times W_a \to W$ such that $\pi_1 \circ h = \pi_1$ and $h([a, b] \times W_a^{--}) = W^{--}$, $h([a, b] \times W_a^{++}) = W^{++}$.

An isolating segment $W$ over $[a, b]$ is said to be $(b - a)$*-periodic* (or simply *periodic*) if $W_a = W_b$, $W_a^{--} = W_b^{--}$ and $W_a^{++} = W_b^{++}$.

The definition of periodic isolated segment from [29] does not contain the second equality from the point (3). Presented definition is more general then the ones from [28,30]. All segments which appear in the paper satisfy also definitions introduced in [28–30].

The simplest isolating segments are of the form $W = [0, T] \times B$, where $W^{--} = [0, T] \times \partial B$, $W^{++} = \emptyset$ or $W^{--} = \emptyset$, $W^{++} = [0, T] \times \partial B$ and $B$ is some arbitrary compact subset of $X$. All segments in the sequel are of one of this form with $B$ such that $\mathrm{int}\, B$ is holomorphic equivalent to the unit disc.

Let a local process $\varphi$ be generated by Eq. (2). To prove that a set $W \subset \mathbb{R} \times M$ is an isolating segment for $\varphi$ it is enough to check the behaviour of the vector field $(1, v)$ on the boundary of $W$. Then, by an appropriate fixed point theorem, there exists a periodic solution inside the segment. The Lefschetz fixed point theorem is used in the general case but, by the simplicity of our segments, we need only the Brouwer one. In fact, we use the Denjoy–Wolff fixed point theorem because the Poincaré map $\varphi_{(0,T)}$ is holomorphic.

### 2.4. *Basic notions*

We make the general assumptions about Eq. (1) that its coefficients $a_j \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ are $T$-periodic.

Let $g : M \to M$ and $n \in \mathbb{N}$. We denote by $g^n$ the $n$th iterate of $f$, and by $g^{-n}$ the $n$th iterate of $g^{-1}$ (if exists).

We say that the point $z_0$ is *attracting* (*repelling*) *for* $g$ *in the set* $W \subset M$ if the equality $\lim_{n\to\infty} g^n(w) = z_0$ ($\lim_{n\to\infty} g^{-n}(w) = z_0$) holds for every $w \in W$.

We call a $T$-periodic solution of (2) *attracting* (*repelling*) *in the set* $W \subset M$ if the corresponding fixed point of the Poincaré map $\varphi_{(0,T)}$ is attracting (repelling) in the set $W$.

Let $-\infty \leqslant \alpha < \omega \leqslant \infty$ and $s : (\alpha, \omega) \to \mathbb{C}$ be a full solution of (1). We call $s$ *forward blowing up* (shortly *f.b.*) or *backward blowing up* (*b.b.*) if $\omega < \infty$ or $\alpha > -\infty$, respectively.

We call an isolated periodic solution *simple* if it corresponds to the simple fixed point of the Poincaré map $\varphi_{(0,T)}$.

We define the sector

$$\mathcal{S}(\alpha, \beta) = \{z \in \mathbb{C}\colon\ \alpha < \operatorname{Arg}(z) < \beta\},$$

where $-\pi \leqslant \alpha < \beta \leqslant \pi$. Moreover, for $0 < \alpha \leqslant \pi$ we define $\mathcal{S}(\alpha) = \mathcal{S}(-\alpha, \alpha)$ and $\widehat{\mathcal{S}}(\alpha)$ to be a set symmetric with respect to the origin to sector $\mathcal{S}(\alpha)$. Obviously, $0 \notin \mathcal{S}(\alpha, \beta)$.

Let $n \geqslant 2$ and $j \in \mathbb{Z}$, $l \in \mathbb{N}$. We define sectors

$$\mathcal{S}^n_j = e^{i\frac{\pi}{n-1}j}\mathcal{S}\left(0, \frac{\pi}{n-1}\right),$$

$$\mathcal{S}^n_{j,j+l} = \operatorname{int}\bigcup_{k=j}^{j+l} \operatorname{cl}\mathcal{S}^n_k.$$

It is easy to see that $\mathcal{S}^n_j = \mathcal{S}^n_k$ iff $k \equiv j \mod 2n-2$.

Let $I \subset \mathbb{R}$. We denote the *angular width* of function $f : I \to \mathbb{C}$ by

$$\sphericalangle(f) = \inf\{\beta - \alpha\colon\ \text{there exists } \theta \in \mathbb{R} \text{ such that } e^{i\theta} f(I) \subset \mathcal{S}(\alpha, \beta) \cup \{0\}\}.$$

It is easy to see that for $f, g, h : \mathbb{R} \to \mathbb{C}$ with $f(t) = e^{i\sin(t)}$ we have $\sphericalangle(f) = 2$, for $g(t) = 1 + e^{it}$ it is $\sphericalangle(g) = \pi$ and for $h(t) = e^{it}$ the angular width $\sphericalangle(h)$ is not defined.

Let us recall that the inner product of two vectors $a, b \in \mathbb{C}$ is given by the formula $\langle a, b\rangle = \mathfrak{Re}(a\bar{b}) = \mathfrak{Re}(\bar{a}b)$.

Denote by $K(w, r)$, $O(w, r)$ the *open disc, circle,* respectively, *of radius r and centre w*.

In the sequel we write $\operatorname{Arg}(0) = 0$. Let $a_j \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ be the coefficients of the vector field $v$ from Eq. (1). We define $\tau_j \geqslant 0$ and $\widehat{\tau}_j \geqslant 0$ to be the smallest numbers such that $|\operatorname{Arg}[a_j(t)]| \leqslant \tau_j$ and $|\operatorname{Arg}[-a_j(t)]| \leqslant \widehat{\tau}_j$ hold for every $t \in \mathbb{R}$.

## 3. Nontrivial free term

We investigate the case $a_0 \not\equiv 0$. This allows us to control the solution starting from the origin.

We also assume that $a_1 \in \mathcal{C}(\mathbb{R}, \mathbb{R})$. It is possible to consider $a_1 \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ (cf. [32, Section 3.1]), provided the inequality

$$\int_0^T \mathfrak{Im}[a_1(s)]\, ds = 0 \tag{4}$$

holds. In this case we make the change of variables

$$w = B(t)z, \tag{5}$$

where

$$B(t) = e^{-i\int_{t_0}^t \mathfrak{Im}[a_1(s)]\, ds}$$

for some fixed $t_0 \in \mathbb{R}$ and get the equation

$$\dot{w} = B(t)a_0(t) + \mathfrak{Re}\big[a_1(t)\big]w + \sum_{j=2}^{n} B^{1-j}(t)a_j(t)w^j.$$

The following theorem corresponds to [32, Theorem 1] but it does not give a full description of dynamics because it describes dynamics in at most two sectors which do not cover for $n \geqslant 3$ the whole plane $\mathbb{C}$.

**Theorem 4.** *Let $n \geqslant 3$, $a_1 \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $a_j \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ for $j \in \{0, 2, 3, \dots, n\}$ be $T$-periodic. If there exists number $M \geqslant n$ such that*

$$a_0 \not\equiv 0 \quad \text{and} \quad \widehat{\tau}_0 < \frac{\pi}{M-1}, \tag{6}$$

$$\begin{cases} \tau_j < \dfrac{j-1}{M-1}\pi, & \text{for } 2 \leqslant j \leqslant \dfrac{M+1}{2}, \\ \tau_j \leqslant \dfrac{M-j}{M-1}\pi, & \text{for } \dfrac{M+1}{2} < j \leqslant n, \end{cases} \tag{7}$$

$$\sum_{j=2}^{n} |a_j| \not\equiv 0 \tag{8}$$

*hold, then Eq. (1) has in the sector $\mathcal{S}(\frac{\pi}{M-1})$*

- *exactly one $T$-periodic solution $\xi$. It is asymptotically unstable and repelling in the whole sector,*
- *infinitely many forward blowing up solutions.*

*Moreover, the equation*

$$\dot{z} = v_1(t, z) = \sum_{j=0}^{n} (-1)^j a_j(t) z^j \tag{9}$$

*has in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$*

- *exactly one $T$-periodic solution $\chi$. It is asymptotically stable and attracting in the whole sector,*
- *infinitely many backward blowing up solutions.*

*If in addition we assume that the equalities*

$$a_j \equiv 0, \quad \text{for all odd numbers } j \geqslant 3 \tag{10}$$

*hold, then Eq. (1) has*

- *exactly one $T$-periodic solution $\chi$ in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$. It is asymptotically stable and attracting in the whole sector,*
- *infinitely many backward blowing up solutions in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$,*
- *infinitely many solutions which are heteroclinic from $\xi$ to $\chi$.*

**Proof.** Our goal is to define a compact set $E \subset \mathbb{C}$, such that $\operatorname{int} E$ is holomorphic equivalent to unit disc and there exists $t_0 \in \mathbb{R}$ satisfying $\varphi^{-1}_{(t_0,t_0+T)}(E) \subset \operatorname{int} E$. It allows us to apply the Denjoy–Wolff fixed point theorem and get the existence of asymptotically unstable periodic solution inside the set $E$.

We set

$$0 < \varepsilon < \min\left\{\frac{\pi}{M-1} - \widehat{\tau}_0, \frac{\frac{j-1}{M-1}\pi - \tau_j}{j-1}, \text{ for } 2 \leqslant j \leqslant n\right\}$$

and

$$A(\varepsilon) = \left\{z \in \mathbb{C}\colon \left|\operatorname{Arg}(z)\right| \leqslant \frac{\pi}{M-1} - \varepsilon\right\}.$$

Let us recall that $0 \in A(\varepsilon)$. We show that the vector field $(1, v)$ points outward or is tangent to the set $[0, T] \times A(\varepsilon)$ in every point from $[0, T] \times [\partial A(\varepsilon) \setminus \{0\}]$.

We make the following calculations mod $2\pi$.

Let $a_j(t) \neq 0$ and $z = re^{i(\frac{\pi}{M-1}-\varepsilon)}$, where $r > 0$. By (7), we get for every $2 \leqslant j \leqslant \frac{M+1}{2}$ and $t \in \mathbb{R}$ the following inclusion

$$\arg\bigl(a_j(t)\bigr) \in \left(-\frac{j-1}{M-1}\pi + (j-1)\varepsilon, \frac{j-1}{M-1}\pi - (j-1)\varepsilon\right),$$

so

$$\begin{aligned}\arg\bigl(a_j(t)z^j\bigr) &\in \left(\frac{\pi}{M-1} - \varepsilon, \frac{2j-1}{M-1}\pi - (2j-1)\varepsilon\right)\\ &\subset \left(\frac{\pi}{M-1} - \varepsilon, \frac{\pi}{M-1} - \varepsilon + \pi\right).\end{aligned} \tag{11}$$

The last inclusion holds due to $3 \leqslant 2j-1 \leqslant M$. Similarly, by (7), we get for every $2 \leqslant \frac{M+1}{2} < j \leqslant n$ and $t \in \mathbb{R}$ inclusion

$$\arg\bigl(a_j(t)\bigr) \in \left(-\frac{j-1}{M-1}\pi + (j-1)\varepsilon, \frac{M-j}{M-1}\pi\right],$$

so

$$\begin{aligned}\arg\bigl(a_j(t)z^j\bigr) &\in \left(\frac{\pi}{M-1} - \varepsilon, \frac{M}{M-1}\pi - j\varepsilon\right]\\ &\subset \left(\frac{\pi}{M-1} - \varepsilon, \frac{\pi}{M-1} - \varepsilon + \pi\right).\end{aligned} \tag{12}$$

We have just proved that every term of the form $(1, a_j(t)z^j)$ for $2 \leqslant j \leqslant n$ does not point at any point of the set $\mathbb{R} \times \{z \in \mathbb{C}\colon \operatorname{Arg}(z) = \frac{\pi}{M-1} - \varepsilon\}$ inward the set $\mathbb{R} \times A(\varepsilon)$ (it points outward or is tangent provided that $a_j(t) = 0$). The term $(1, a_1(t)z)$ is tangent since $a_1$ is real. Moreover, by (6) and the definition of $\varepsilon$, the term $(1, a_0(t))$ points outward or is tangent provided that $a_0(t) = 0$. If the vector field $(1, v(t, z))$ is tangent for $t$ in some time interval, then parts of some trajectories are subsets of the set $\mathbb{R} \times \{z \in \mathbb{C}\colon \operatorname{Arg}(z) = \frac{\pi}{M-1} - \varepsilon\}$. But, by (8), the time interval is shorter than $T$.

Finally, every trajectory starting in the set $\mathbb{R} \times \{z \in \mathbb{C}\colon \operatorname{Arg}(z) = \frac{\pi}{M-1} - \varepsilon\}$ leaves $\mathbb{R} \times A(\varepsilon)$ in time shorter than $T$. The same is true for $\mathbb{R} \times \{z \in \mathbb{C}\colon \operatorname{Arg}(z) = -\frac{\pi}{M-1} + \varepsilon\}$.

We fix $\varepsilon_0 > 0$ such that the set $A(\varepsilon_0)$ has properties described above. We set $A = A(\varepsilon_0)$.

The vector field $(1, v)$ points at the point $(t_0, 0)$ outward or is tangent to $\mathbb{R} \times A$ provided $a_0(t_0) \neq 0$ or $a_0(t_0) = 0$, respectively. But, by (6), every trajectory starting in $\mathbb{R} \times \{0\}$ leaves the set $\mathbb{R} \times A$ in time shorter than $T$.

Let $m = \frac{M-1}{2} \geqslant 1$. We make the change of variables in the set $\{z \in \mathbb{C}\colon\ |\operatorname{Arg}(z)| < \frac{\pi}{M-1}\} \setminus \{0\}$ given by $\xi = f(z) = z^m := e^{m \operatorname{Log} z}$. It gives

$$\dot{\xi} = m \sum_{j=0}^{n} a_j(t) \xi^{\frac{m+j-1}{m}}.$$

We set $B = f(A \setminus \{0\})$. Thus $B \subset \{\xi \in \mathbb{C}\colon\ |\operatorname{Arg}(\xi)| \leqslant \frac{\pi}{2} - m\varepsilon_0\}$. Another change of variables $w = g(\xi) = \frac{1}{\xi}$ gives

$$\dot{w} = u(t, w) = -m \sum_{j=0}^{n} a_j(t) w^{\frac{m-j+1}{m}}. \tag{13}$$

Let $\gamma > 0$. We set

$$C(\gamma) = g(B) \cap \big\{ w \in \mathbb{C}\colon\ \mathfrak{Re}(w) \geqslant \gamma \big\},$$
$$\widehat{C}(\gamma) = C(\gamma) \cap \big\{ w \in \mathbb{C}\colon\ \mathfrak{Re}(w) = \gamma \big\}.$$

It is easy to see that $C(\gamma) \subset \{w \in \mathbb{C}\colon\ |\operatorname{Arg}(w)| \leqslant \frac{\pi}{2} - m\varepsilon_0\}$.

We show that the vector field $(1, u)$ points in every point of the set $[0, T] \times \widehat{C}(\gamma)$ outward $[0, T] \times C(\gamma)$, provided $\gamma$ is sufficiently small and the inequality

$$\sum_{j=2}^{n} |a_j| > 0 \tag{14}$$

holds.

Let $w \in \widehat{C}(\gamma)$. Then $w = re^{i\psi}$ for some $r > 0$ and $\psi \in [-\frac{\pi}{2} + m\varepsilon_0, \frac{\pi}{2} - m\varepsilon_0]$. By (7), the inclusion

$$\operatorname{Arg}\big(a_j(t) w^{\frac{m-j+1}{m}}\big) \in \left( -\frac{j-1}{2m}\pi + (j-1)\varepsilon_0 + \frac{m-j+1}{m}\psi, \frac{j-1}{2m}\pi - (j-1)\varepsilon_0 + \frac{m-j+1}{m}\psi \right)$$
$$\subset \left( -\frac{\pi}{2} + m\varepsilon_0, \frac{\pi}{2} - m\varepsilon_0 \right)$$

holds for every $2 \leqslant j \leqslant \frac{M+1}{2}$ and $t \in \mathbb{R}$.

Similarly, (7) and $\frac{M+1}{2} < j$ imply

$$\operatorname{Arg}\big(a_j(t)\big) \in \left[ -\frac{M-j}{M-1}\pi, \frac{M-j}{M-1}\pi \right] = \left[ -\frac{2m+1-j}{2m}\pi, \frac{2m+1-j}{2m}\pi \right]$$

and

$$\operatorname{Arg}\big(a_j(t) w^{\frac{m-j+1}{m}}\big) \in \left[ -\frac{\pi}{2} - (m-j+1)\varepsilon_0, \frac{\pi}{2} + (m-j+1)\varepsilon_0 \right],$$

where $m - j + 1 < 0$.

An outward normal vector is in every point of the set $[0,T]\times\widehat{C}(\gamma)$ equal to $[0,-1]^T$. It follows by the above calculations that

$$\left\langle -1, -a_j(t)w^{\frac{m-j+1}{m}}\right\rangle \geqslant \kappa_j\left|a_j(t)\right||w|^{\frac{m-j+1}{m}} \tag{15}$$

holds where $j\in\{2,\dots,n\}$, $(t,w)\in[0,T]\times\widehat{C}(\gamma)$ while $\kappa_j=\sin(m\varepsilon_0)>0$ for $2\leqslant j\leqslant\frac{M+1}{2}$ and $\kappa_j=\sin(-(m-j+1)\varepsilon_0)>0$ for $\frac{M+1}{2}<j\leqslant n$. Let $\kappa=\min\{\kappa_j\colon\ 2\leqslant j\leqslant n\}$. It is easy to see that $|\langle -1,-a_0(t)w^{\frac{m+1}{m}}\rangle|\leqslant|a_0(t)||w|^{\frac{m+1}{m}}$ and $|\langle -1,-a_1(t)w\rangle|\leqslant|a_1(t)||w|$ hold.

It follows by (14) that there exists $\lambda>0$ such that for every $t\in\mathbb{R}$ there exists $j\in\{2,\dots,n\}$ satisfying $|a_j(t)|\geqslant\lambda$. Let $(t,w)\in[0,T]\times\widehat{C}(\gamma)$. Thus the inequality

$$\mathfrak{Re}\big[u(t,w)\big]\leqslant\left|a_0(t)\right||w|^{\frac{m+1}{m}}+\left|a_1(t)\right||w|-m\kappa\lambda|w|^{\frac{m-j+1}{m}}<0$$

holds, provided $\gamma$ is sufficiently small. We fix $\gamma_0>0$ such that the above estimation holds.

Finally, the vector field $(1,u)$ points on $[0,T]\times\widehat{C}(\gamma_0)$ outward the set $[0,T]\times C(\gamma_0)$.

We define $E=(f^{-1}\circ g^{-1})(C(\gamma_0))\cup\{0\}$. Then the Poincaré mapping $\varphi_{(0,T)}^{-1}: E\to\operatorname{int}E$ is well defined.

Now we prove the same without assuming (14). To obtain a contradiction, we suppose that for every $\gamma>0$ there exists a solution $\eta$ of Eq. (13) such that $\eta(t_0)\in\widehat{C}(\gamma)$ and $\eta(t_0+T)\in C(\gamma)$ hold.

Write

$$\iota>\frac{m}{\sin(m\varepsilon_0)}\big(\left|a_0(t)\right|+\left|a_1(t)\right|\big)\quad\text{for every } t\in\mathbb{R}$$

and

$$R=\mathbb{R}\times\left\{w\in\mathbb{C}\setminus\{0\}\colon\ \left|\operatorname{Arg}(w)\right|\leqslant\frac{\pi}{2}-m\varepsilon_0,\ \mathfrak{Re}(w)\leqslant\gamma e^{\iota T}\right\},$$

where $\gamma$ is fixed and so small that $R\subset\mathbb{R}\times K(0,1)$. By (8), we find $t_0$ such that $a_j(t_0)\neq0$ holds for some $j\geqslant2$. Let $\nu,\mu>0$ be such that $|a_j(t)|>\mu>0$ is satisfied for every $t\in[t_0-\nu,t_0]$ and the inequality

$$\mathfrak{Re}\big[u(t,w)\big]<-\frac{m\mu\kappa}{2}|w|^{1-\frac{j-1}{m}}\leqslant-\beta|w|^q\leqslant-\beta\big(\mathfrak{Re}[w]\big)^q<0 \tag{16}$$

holds for every $(t,w)\in R\cap([t_0-\nu,t_0]\times\mathbb{C})$ where $\beta=\frac{m\mu\kappa}{2}>0$, $1>q=\frac{m-1}{m}\geqslant0$, provided $\gamma$ is small enough. By (15),

$$\mathfrak{Re}\big[u(t,w)\big]\leqslant\iota\mathfrak{Re}(w) \tag{17}$$

is true for every $(t,w)\in R$.

By (17), the inequality

$$\mathfrak{Re}\big[\eta(t_0+T)\big]\leqslant\gamma e^{\iota T}$$

holds. But by (16), the solution is swept out the set $C(\gamma)$, provided $\gamma$ is small enough. Indeed, the solution of the scalar equation

$$\dot{p}=-\beta p^q \tag{18}$$

with the initial condition $p(t_2)=\varpi$ is given by

$$p(t_2+t)=\left[(q-1)\beta t+\varpi^{1-q}\right]^{\frac{1}{1-q}}.$$

Let $t_2=t_0+T-\nu$, $\varpi=\gamma e^{\iota T}$, then $p(t_2+t)=\gamma$, provided

$$t=\frac{\gamma^{1-q}[e^{\iota T(1-q)}-1]}{(1-q)\beta}.$$

Taking $\gamma$ small enough we get $t<\nu$, so $\mathfrak{Re}[\eta(t_0+T-\nu+t)]\leqslant\gamma$ and, if defined, $\mathfrak{Re}[\eta(t_0+T)]<\gamma$, which gives the desired contradiction.

We now fix $0<\gamma_0$ small enough and define $E$ as above. It follows that $\varphi^{-1}_{(t_0,T)}(E)\subset\operatorname{int}E$.

Finally, $\operatorname{int}E$ is holomorphic equivalent to the unit disc then, by the Denjoy–Wolff fixed point theorem, there exists exactly one $T$-periodic solution of Eq. (1) which image is contained in $E$. Moreover, it is asymptotically unstable and repelling in $E$.

For every $z\in\mathcal{S}(\frac{\pi}{M-1})$ there exist $\varepsilon$ and $\gamma$ such small that by using them in construction of $E$ we get $z\in E$. Thus there exists exactly one $T$-periodic solution in $\mathcal{S}(\frac{\pi}{M-1})$ and it is asymptotically unstable and repelling in $\mathcal{S}(\frac{\pi}{M-1})$.

We now prove the existence of infinitely many f.b. solutions inside $\mathcal{S}(\frac{\pi}{M-1})$. It is enough to prove the existence of solutions $\eta:(-\infty,\omega_\eta)\to\mathbb{C}\setminus\{0\}$ of Eq. (13) inside the sector $\mathcal{S}(\frac{\pi}{2})$ satisfying $\omega_\eta<\infty$ and

$$\lim_{t\to\omega_\eta^-}\eta(t)=0.$$

By (8), there exist $j\geqslant 2$, $t_0\in\mathbb{R}$ and $\nu,\mu>0$ such that $|a_j(t)|>\mu>0$ for every $t\in[t_0,t_0+2\nu]$. Let $\zeta$ be the solution of (18) satisfying $\zeta(t_0)=\gamma>0$. Then $\zeta(t_0+t)>0$ iff $t<\frac{1}{\beta(1-q)}\gamma^{1-q}$. Let $\gamma$ be such small that

$$\frac{1}{\beta(1-q)}\gamma^{1-q}<\nu \tag{19}$$

holds. Write

$$D(\gamma)=\left\{w\in\mathbb{C}\setminus\{0\}\colon\ \left|\operatorname{Arg}(w)\right|\leqslant\frac{\pi}{2}-m\varepsilon_0,\ \mathfrak{Re}(w)\leqslant\gamma\right\}$$

and $K=[t_0,t_0+2\nu]\times D(\gamma)$. Here

$$K^-=[t_0,t_0+2\nu]\times\left[D(\gamma)\cap\left\{w\in\mathbb{C}\colon\ \left|\operatorname{Arg}(w)\right|=\frac{\pi}{2}-m\varepsilon_0\right\}\right]\cup\{t_0+2\nu\}\times D(\gamma),$$
$$K^+=[t_0,t_0+2\nu]\times\left[D(\gamma)\cap\left\{w\in\mathbb{C}\colon\ \mathfrak{Re}(w)=\gamma\right\}\right]\cup\{t_0\}\times D(\gamma),$$

provided $\gamma$ is so small that (16) holds for every $(t,w)\in K$. Thus $K$, $K^-$ are closed relative to $\mathbb{R}\times\mathbb{C}\setminus\{0\}$, so, by Proposition 1, $K$ is a Ważewski set for the local flow $\phi$ on $\mathbb{R}\times[\mathbb{C}\setminus\{0\}]$ generated by (13). We fix $\vartheta\in(t_0,t_0+\nu]$ and set $Q(\vartheta)=K^+_\vartheta$. To obtain a contradiction, we suppose that $Q(\vartheta)\subset K^*$. By Lemma 2, the map $\Lambda:Q(\vartheta)\ni p\mapsto\phi(\sigma(\vartheta,p),(\vartheta,p))\in K^-$ is continuous (here $\sigma$ is the escape-time function of $K$). Let $\pi_1$ be the projection on the time variable. Then $\pi_1(\phi(\sigma(\vartheta,p),(\vartheta,p)))=\vartheta+\sigma(\vartheta,p)$ holds. The inequality (19) implies $\sigma|_{\{\vartheta\}\times Q(\vartheta)}<\nu$, so connected set $\Lambda(Q(\vartheta))$ is contained in disconnected $K^-\cap(t_0,t_0+2\nu)\times\mathbb{C}$. Moreover, $\Lambda(Q(\vartheta))$ has nonempty intersection with both its connected components which is the desired contradiction. Thus there exists $p\in Q(\vartheta)$ such that

$\phi^+(\vartheta, p) \subset K$. By (16), the solution $\eta : (-\infty, \omega_{(\vartheta,p)}) \to \mathbb{C} \setminus \{0\}$ of (13) with initial condition $\eta(\vartheta) = p$ satisfies

$$\lim_{t \to \omega^-_{(\vartheta,p)}} \eta(t) = 0.$$

Since $\omega_{(\vartheta,p)} < \infty$ holds, it follows that $\eta$ is f.b. solution. Let us observe that $\eta$ is defined on the whole $(-\infty, \vartheta)$, because

$$\eta\big((\varsigma, \vartheta)\big) \subset \bigcup_{\gamma > 0} C(\gamma)$$

holds for every $\varsigma < \vartheta$. By the arbitrariness of the choice of $\vartheta \in (t_0, t_0 + \nu]$, there are infinitely many such solutions. This finishes the proof of the main part of the theorem.

By the equality $v(t, -z) = v_1(t, z)$, there exists $t_0 \in \mathbb{R}$ such that the Poincaré mapping $\varphi_{(t_0,T)} : (-E) \to \operatorname{int}(-E)$ of (9) is well defined. Thus there exists exactly one $T$-periodic solution of Eq. (9) inside the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$ and it is attracting. Moreover, there are infinitely many b.b. solutions inside the sector.

Let now $a_1 \equiv 0$ hold. Then the assumption (10) implies the equalities $v(t, z) = v_1(t, z)$ and $v(t, -z) = v_1(t, z)$ hold for every $(t, z) \in \mathbb{R} \times \mathbb{C}$. Thus the properties of the sets $E$ and $-E$ defined as above are preserved. If we allow $a_1 \not\equiv 0$ then the behaviour of the vector field $v$ on the boundaries of the sets is qualitatively the same: the term $a_1(t)z$ is tangent or dominated by $\sum_{j=2}^{n} a_j(t)z^j$. Finally, Eq. (1) has in every sector $\mathcal{S}(\frac{\pi}{M-1})$ and $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$ exactly one $T$-periodic repelling and attracting, respectively, solution and infinitely many f.b. and b.b., respectively, ones.

Let us fix $t_0 \in \mathbb{R}$ such that $a_0(t_0) \neq 0$. We denote by $\eta$ the solution of (1) such that $\eta(t_0) = 0$. It follows that $\eta'(t_0) = a_0(t_0)$ which, by (6), implies $\eta(t_0 + \nu) \in \widehat{\mathcal{S}}(\frac{\pi}{M-1})$ and $\eta(t_0 - \nu) \in \mathcal{S}(\frac{\pi}{M-1})$ for every $\nu > 0$. Thus $\eta$ is heteroclinic solution from $\xi$ to $\chi$. By the continuity of $a_0$, there are infinitely many such $t_0$ inside the interval $[0, T]$ so there are infinitely many heteroclinic solutions. □

The following examples are straightforward applications of Theorem 4.

**Example 5.** The equation

$$\dot{z} = -1 + \cos(t) + e^{i\frac{\pi}{5}\sin(t)} z^2 + z^5$$

has inside the sector $\mathcal{S}(\frac{\pi}{4})$ exactly one $2\pi$-periodic solution (it is asymptotically unstable) and infinitely many f.b. solutions. Here $M = 5$, $\widehat{\tau}_0 = 0$, $\tau_2 = \frac{\pi}{5} < \frac{\pi}{4}$. It is worth to mention that the coefficient at $z^2$ allows only $M < 6$. Thus Theorem 4 does not imply the existence of periodic solution inside $\mathcal{S}(\frac{\pi}{5})$.

**Example 6.** The equation

$$\dot{z} = -1 - z^3 + \big(2 + \sqrt{3}e^{it}\big)z^4$$

does not fulfil the assumptions of the main part of Theorem 4 (because of the coefficient at $z^3$). But the auxiliary equation

$$\dot{z} = -1 + z^3 + \big(2 + \sqrt{3}e^{it}\big)z^4$$

fulfils them for $M = 7$. Here $\tau_3 = 0 < \frac{\pi}{3}$ and $\tau_4 = \frac{\pi}{3} < \frac{\pi}{2}$. Thus the main equation has exactly one $2\pi$-periodic (asymptotically stable) solution inside $\widehat{\mathcal{S}}(\frac{\pi}{6})$ and the solution is attracting in the whole sector.

**Example 7.** The equation

$$\dot{z} = -1 - \cos(t)z + \left(1 + \varepsilon + e^{it}\right)z^4$$

has for every $\varepsilon > 0$ exactly one $2\pi$-periodic solution in every sector $\mathcal{S}(\frac{\pi}{6})$ and $\widehat{\mathcal{S}}(\frac{\pi}{6})$. They are asymptotically unstable and asymptotically stable, respectively. Moreover, there are infinitely many solutions heteroclinic to them and infinitely many blowing up ones. Here $M = 7$, $\tau_4 < \frac{\pi}{2}$ and $a_4 \not\equiv 0$.

**Remark 8.** In Theorem 4, all monomials $a_j(t)z^j$ coincide with respect to the sector which symmetry axis is the positive part of the real axis. If there is coincidence with respect to other sector one can use the change of variables

$$w = e^{i\mu}z. \tag{20}$$

**Example 9.** The equation

$$\dot{z} = -3i + e^{it} + iz^3 + z^4$$

does not fulfil the assumptions of Theorem 4, because $\tau_3 = \frac{\pi}{2}$ and it should be $\tau_3 \leqslant \frac{\pi}{3}$ for $M = 4$ and $\tau_3 < \frac{2}{M-1}\pi \leqslant \frac{\pi}{2}$ for $M \geqslant 5$. But, by the change of variables $w = e^{-\frac{2\pi i}{3}}z$, we get the equation

$$\dot{w} = \left(-3i + e^{it}\right)e^{-\frac{2\pi i}{3}} + e^{-\frac{\pi i}{6}}w^3 + w^4.$$

Here $\tau_3 = \frac{\pi}{6} < \frac{\pi}{3}$ and $\widehat{\tau}_0 < \frac{\pi}{3}$ for $M = 4$. Then the main equation has in the sector $\mathcal{S}(\frac{\pi}{3}, \pi)$ exactly one $2\pi$-periodic asymptotically unstable solution and infinitely many f.b. ones.

**Example 10.** The equation

$$\dot{z} = -1 + ie^{i\frac{t}{T}}z + z^3$$

has at least one $T$-periodic asymptotically unstable solution provided that $0 < T < \frac{\pi}{4}$. By the change of variables (5) where $t_0 = 0$ and $B(t) = e^{-iT\sin(\frac{t}{T})}$, we get

$$\dot{w} = -B(t) + \mathfrak{Re}\left[ie^{i\frac{t}{T}}\right]w + B^{-2}(t)w^3.$$

Then $\tau_3 = 2T$, $\widehat{\tau}_0 = T$. The assumptions of Theorem 4 are fulfilled provided that $M = 5$ and $0 < T < \frac{\pi}{4}$.

## 4. Free term and degree of the vector field

In the present section we deal with the special case of (1) given by

$$\dot{z} = z^n + a_0(t), \tag{21}$$

where $n \geqslant 3$ and the free term $a_0 \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ is $T$-periodic.

As we will show below, in the case of $n$ being odd, a straightforward use of Theorem 4 helps to find only one periodic solution of (21). But, it is sometimes possible to prove the existence of two ones. Moreover, if the angular width of the free term $a_0$ is small enough, then there exists at least one periodic solution.

The case of even $n$ is quite different. If Theorem 4 helps to find a periodic solution, it helps to find at least two ones. But, if $a_0(\mathbb{R})$ lies in some regions of the plane, then we know nothing about periodic solutions irrespective of the angular width of $a_0$.

**Theorem 11.** *Let $n \geqslant 3$ be odd and $a_0 \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ be $T$-periodic.*
*If there exists $j \in \mathbb{Z}$ such that $a_0(\mathbb{R}) \subset \mathcal{S}^n_{j,j+1}$, then Eq.* (21) *has*

- *in sector $\mathcal{S}^n_{j,j+1}$ exactly one $T$-periodic asymptotically stable solution, provided $j$ is even,*
- *in sector $\mathcal{S}^n_{n+j-1,n+j}$ exactly one $T$-periodic asymptotically unstable solution, provided $j$ is odd.*

*If there exists $j \in \mathbb{Z}$ such that $a_0(\mathbb{R}) \subset \mathcal{S}^n_j$, then Eq.* (21) *has exactly one $T$-periodic asymptotically stable $\chi$, asymptotically unstable $\xi$ solution in sectors*

- *$\mathcal{S}^n_{j,j+1}$, $\mathcal{S}^n_{n+j-2,n+j-1}$, respectively, provided $j$ is even,*
- *$\mathcal{S}^n_{j-1,j}$, $\mathcal{S}^n_{n+j-1,n+j}$, respectively, provided $j$ is odd.*

*Moreover, there are infinitely many heteroclinic solutions from $\xi$ to $\chi$.*

**Proof.** Let $a_0(\mathbb{R}) \subset \mathcal{S}^n_{j,j+1}$. We consider the change of variables given by $w = e^{i\frac{\pi}{n-1}(n-2-j)}z$. Eq. (21) has the form

$$\dot{w} = e^{-i\pi(n-2-j)}w^n + e^{i\frac{\pi}{n-1}(n-2-j)}a_0(t), \tag{22}$$

where $e^{i\frac{\pi}{n-1}(n-2-j)}a_0(\mathbb{R}) \subset \widehat{\mathcal{S}}(\frac{\pi}{n-1})$.

If $j$ is odd, then $e^{-i\pi(n-2-j)} = 1$ holds and, by Theorem 4, Eq. (22) has $T$-periodic asymptotically unstable solution in $\mathcal{S}(\frac{\pi}{n-1}) = \mathcal{S}^n_{-1,0}$. Thus (21) has periodic solution in $\mathcal{S}^n_{n-1+j,n+j}$.

If $j$ is even, then $e^{-i\pi(n-2-j)} = -1 = (-1)^n$ holds, and, by Theorem 4 for the auxiliary equation (9), Eq. (22) has $T$-periodic asymptotically stable solution in $\widehat{\mathcal{S}}(\frac{\pi}{n-1}) = \mathcal{S}^n_{n-2,n-1}$. Thus (21) has periodic solution in $\mathcal{S}^n_{j,j+1}$.

If $a_0(\mathbb{R}) \subset \mathcal{S}^n_j$, then $a_0(\mathbb{R}) \subset \mathcal{S}^n_{j,j+1}$ and $a_0(\mathbb{R}) \subset \mathcal{S}^n_{j-1,j}$ hold, so the existence of periodic solutions follows by the previous arguments. Vector $v(t,0)$ points inward the sector $\mathcal{S}^n_j$ thus every solution passing close to the origin is attracted by the periodic asymptotically stable one (periodic solution is attracting in the whole sector). Analogously, they are repelled by the periodic asymptotically unstable one. Finally, they are the desired heteroclinic solutions. □

**Corollary 12.** *If $n \geqslant 3$ is odd, $a_0 \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ is $T$-periodic, $a_0 \not\equiv 0$ and $\sphericalangle(a_0) < \frac{\pi}{n-1}$ hold, then Eq.* (21) *has at least one $T$-periodic solution.*

**Proof.** There exists $j \in \mathbb{Z}$ such that after the change of variables $w = e^{i\frac{\pi}{n-1}j}z$ one gets $\dot{w} = w^n + A_0(t)$ with $\widehat{\tau}_0 < \frac{\pi}{n-1}$. It is now enough to apply Theorem 4. □

The following examples are straightforward applications of Theorem 11 and Corollary 12.

**Example 13.** The equation

$$\dot{z} = z^3 + 2i + e^{it}$$

has exactly one $2\pi$-periodic asymptotically stable solution inside the sector $\mathcal{S}^3_{0,1} = \{z \in \mathbb{C}\colon\ \mathfrak{Im}(z) > 0\}$ because $a_0(\mathbb{R}) \subset \mathcal{S}^3_{0,1}$ holds.

**Example 14.** The equation

$$\dot{z} = z^5 + 5 + 2i + e^{it}$$

has exactly one $2\pi$-periodic solution in every of the sectors $\mathcal{S}^5_{0,1}$ and $\mathcal{S}^5_{3,4}$. They are asymptotically stable and asymptotically unstable, respectively. There are also infinitely many solutions which are heteroclinic to them because $a_0(\mathbb{R}) \subset \mathcal{S}^5_0$ holds.

**Example 15.** The equation

$$\dot{z} = z^n + (1+i)(n-1) + e^{it}$$

has at least one $2\pi$-periodic solution, provided $n \geqslant 3$ is odd, because $\sphericalangle(a_0) = 2\arcsin(\frac{\sqrt{2}}{2(n-1)}) < \frac{\pi}{n-1}$ holds.

One can follow the proof of Theorem 11 and prove the following one.

**Theorem 16.** *Let $n \geqslant 3$ be even and $a_0 \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ be $T$-periodic. If there exists even $j \in \mathbb{Z}$ such that $a_0(\mathbb{R}) \subset \mathcal{S}^n_{j,j+1}$ holds, then Eq.* (21) *has exactly one $T$-periodic solution in every of the sectors $\mathcal{S}^n_{j,j+1}$ and $\mathcal{S}^n_{n+j-1,n+j}$. They are asymptotically stable and asymptotically unstable, respectively. There are also infinitely many solutions which are heteroclinic to them.*

**Example 17.** By Theorem 16, the equation

$$\dot{z} = z^8 + 7i + e^{it}$$

has exactly one $2\pi$-periodic solution in every of the sectors $\mathcal{S}^8_{2,3}$, $\mathcal{S}^8_{9,10}$ and infinitely many heteroclinic to them solutions, because $a_0(\mathbb{R}) \subset \mathcal{S}^8_{2,3}$ holds.

**Remark 18.** If $n \geqslant 3$ is even and

$$a_0(\mathbb{R}) \cap \mathcal{S}^n_j \neq \emptyset, \ a_0(\mathbb{R}) \cap \mathcal{S}^n_{j+1} \neq \emptyset \text{ hold for some odd } j, \tag{23}$$

then the presented method says nothing about the existence of periodic solutions. Moreover, in the case $n = 2$, i.e. the Riccati equation (the condition (23) can be interpreted as saying that the critical line condition for the Riccati equation is not satisfied) there can be no periodic solutions (cf. [8,16,21, 22,32–34]).

**Remark 19.** It is possible to state similar theorems for the equation

$$\dot{z} = z^n + a_1(t)z + a_0(t),$$

provided $a_1 \in \mathcal{C}(\mathbb{R}, \mathbb{R})$.

## 5. Trivial free term

In this section we investigate the equation

$$\dot{z} = v(t, z) = \sum_{j=1}^{n} a_j(t) z^j. \tag{24}$$

Linear term is the dominating one in the neighbourhood of the origin. If it satisfies the inequality

$$\int_0^T a_1(t)\,dt < 0, \tag{25}$$

then it is possible to adapt method from the proof of Theorem 4. In this case $a_1 z$ plays similar role to $a_0$.

As in Section 3, we consider only $a_1 \in \mathcal{C}(\mathbb{R},\mathbb{R})$. It is possible to allow $a_1$ to be complex, provided it satisfies (4).

We state the main theorem of the section.

**Theorem 20.** *Let $n \geqslant 3$, $a_1 \in \mathcal{C}(\mathbb{R},\mathbb{R})$ and $a_j \in \mathcal{C}(\mathbb{R},\mathbb{C})$ for $j \in \{2,3,\dots,n\}$ be $T$-periodic. If there exists $M \geqslant n$ such that* (25), (7) *and* (8) *are satisfied, then Eq.* (24) *has in the sector $\mathcal{S}(\frac{\pi}{M-1})$*

- *exactly one $T$-periodic solution $\xi$. It is asymptotically unstable and repelling in the whole sector,*
- *infinitely many f.b. solutions,*
- *infinitely many solutions which are heteroclinic from $\xi$ to the zero solution.*

*Moreover, the equation*

$$\dot z = v_1(t,z) = \sum_{j=1}^{n} (-1)^j a_j(t) z^j \tag{26}$$

*has in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$*

- *exactly one $T$-periodic solution $\chi$. It is asymptotically stable and attracting in the whole sector,*
- *infinitely many b.b. solutions,*
- *infinitely many solutions which are heteroclinic from the zero solution to $\xi$.*

**Proof.** We adopt notation and modify the proof of Theorem 4. Let $0 < \delta < 1$. We define $E(\delta) = E \cap \{z \in \mathbb{C}\colon |z| \geqslant \delta\}$ and $\widehat{E}(\delta) = E \cap \{z \in \mathbb{C}\colon |z| = \delta\}$.

We show that for every solution $\eta$ such that $\eta(0) \in \widehat{E}(\delta)$, the condition $|\eta(T)| < \delta$ holds, provided $\delta$ is small enough. We denote $d(t) = a_1(t) + \sum_{j=2}^{n} |a_j(t)|\delta_1$ where $\delta_1 > 0$. Then for $0 < r < \delta_1 < 1$ one gets

$$d(t)r \geqslant \sup\left\{\left\langle \frac{z}{|z|}, v(t,z)\right\rangle\colon\ |z| = r\right\}. \tag{27}$$

We fix $\delta_1$ so small that the inequality $\int_0^T d(t)\,dt < 0$ holds. Let $\zeta$ be the solution of

$$\dot r = d(t) r \tag{28}$$

satisfying $\zeta(0) = \delta$ for some $0 < \delta < \delta_1$. Solutions of (28) have the form

$$r(t) = r(0) e^{D(t)}, \quad \text{where } D(t) = \int_0^t d(s)\,ds,$$

so the condition $\zeta([0,T]) \subset (0,\delta_1)$ holds for $\delta$ small enough. Moreover, if $\eta$ is the solution of (24) with the property $|\eta(0)| = \delta$, then $|\eta(T)| \leqslant \zeta(T)$. Thus it is possible to fix $0 < \delta < \delta_1$ such small that for every solution of (24) satisfying $|\eta(0)| = \delta$ the inequality

$$\bigl|\eta(T)\bigr| < \bigl|\eta(0)\bigr| \tag{29}$$

holds.

Finally, there exist $\delta_0 > 0$ and $t_0 \in \mathbb{R}$ such that the Poincaré map $\varphi_{(t_0,-T)} : E(\delta_0) \to \operatorname{int} E(\delta_0)$ is well defined. Similar arguments to those used in the proof of Theorem 4 show that there exists in sector

$\mathcal{S}(\frac{\pi}{M-1})$ exactly one $T$-periodic asymptotically unstable solution $\xi$ which is repelling in the whole sector and infinitely many f.b. solutions.

By (29), there are solutions which are heteroclinic from $\xi$ to the trivial one. We prove that infinitely many of them are contained in $\mathcal{S}(\frac{\pi}{M-1})$.

Let us consider Eq. (24) on the set $\mathbb{C}\setminus\{0\}$. It generates the local flow $\phi$ on the set $X=\mathbb{R}\times[\mathbb{C}\setminus\{0\}]$. We fix $0<\delta<\delta_0$ and denote

$$L(\delta)=\left\{(t,z)\in\mathbb{R}\times\left[\mathbb{C}\setminus\{0\}\right]:\ \left|\operatorname{Arg}(z)\right|\leqslant\frac{\pi}{M-1}-\varepsilon_0\right\}$$
$$\cap\left[\left\{(t,z):\ |z|\leqslant\delta\right\}\cup\left\{(t,z):\ |z|>\delta \text{ and } z=\eta(t),\ \text{where } \eta \text{ is the solution of (24) such that } \left|\eta(t_1)\right|=\delta \text{ for some } t_1<t,\ |t-t_1|<T\right\}\right].$$

By the behaviour of the vector field on the set $\widehat{E}(\delta)$, it is easy to see that $L(\delta)$ is closed relative to $X$. Moreover, (11) and (12) imply

$$L(\delta)^-=L\cap\left[\mathbb{R}\times\left\{z\in\mathbb{C}:\ \left|\operatorname{Arg}(z)\right|=\frac{\pi}{M-1}-\varepsilon_0\right\}\right].$$

Since $L(\delta)^-$ is closed relative to $X$, it follows that $L(\delta)$ is the Ważewski set. We fix $\theta\in\mathbb{R}$. Let $R(\theta)=\{\theta\}\times\{z\in\mathbb{C}:\ |z|=\delta, |\operatorname{Arg}(z)|\leqslant\frac{\pi}{M-1}-\varepsilon_0\}$. By the Ważewski theorem (Theorem 3), there exists $z_\theta\in R(\theta)$ such that $\phi^+((\theta,z_\theta))\subset L(\delta)$ holds. Let $\eta:(-\infty,\omega_{(\theta,z_\theta)})\to\mathbb{C}$ be the full solution of (24) in $\mathbb{C}$ such that $\eta(\theta)=z_\theta$. Then

$$\lim_{t\to\omega^-_{(\theta,z_\theta)}}\eta(t)=0$$

holds. If $\omega_{(\theta,z_\theta)}<\infty$, then $\eta(\omega_{(\theta,z_\theta)})=0$, which is impossible, because the equation satisfies the uniqueness condition for solutions of the Cauchy problem. Thus $\omega_{(\theta,z_\theta)}=\infty$ and $\eta$ is the desired heteroclinic solution. By the arbitrariness of the choice of $\theta$, there are infinitely many heteroclinic solutions.

The equality $v(t,-z)=v_1(t,z)$ and analysis similar to the above finishes the proof in the case of (26). $\square$

The following examples are straightforward applications of Theorem 20.

**Example 21.** The equation

$$\dot{z}=\big(\sin(t)-1\big)z+z^3+e^{i\frac{\pi}{4}\cos(t)}z^8$$

has in sector $\mathcal{S}(\frac{\pi}{10})$ exactly one $2\pi$-periodic solution $\xi$ which is asymptotically unstable and infinitely many f.b. ones and heteroclinic solutions from $\xi$ to the trivial one. Here $M=11$ and $\tau_8=\frac{\pi}{4}\leqslant\frac{3}{10}\pi$.

**Example 22.** The equation

$$\dot{z}=\big(2+\sin(t)\big)z-z^3+e^{i\cos(t)}z^4 \tag{30}$$

does not fulfil the assumptions of Theorem 20 because signs of the real parts of terms at $z^3$ and $z^4$ are not the same. Let us consider the auxiliary equation

$$\dot{z}=-\big(2+\sin(t)\big)z+z^3+e^{i\cos(t)}z^4.$$

Now $a_1 < 0$ holds and for $M = 6$ one gets $\tau_3 = 0$ as well as $\tau_4 = 1 \leqslant \frac{2}{5}\pi$. Thus the auxiliary equation has in the sector $\mathcal{S}(\frac{\pi}{5})$ exactly one $2\pi$-periodic solution. Finally, Eq. (30) has in the sector $\widehat{\mathcal{S}}(\frac{\pi}{5})$ exactly one $2\pi$-periodic solution $\eta$ which is asymptotically stable and attracting in the whole sector, infinitely many solutions which are heteroclinic from the trivial one to $\eta$ and infinitely many b.b. solutions.

Now we investigate Eq. (24) where the condition

$$\int_0^T a_1(t)\,dt \geqslant 0 \tag{31}$$

is satisfied. In this case, the trivial solution is not necessarily a simple periodic solution (cf. [2,4,5,21]) and it is repelling or attracting in some sectors. The following statements can also be adapted to the case $a_1 \in \mathcal{C}(\mathbb{R}, \mathbb{C})$.

**Theorem 23.** *Let $n \geqslant 3$, $a_1 \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $a_j \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ for $j \in \{2, 3, \ldots, n\}$ be $T$-periodic. If there exists $M \geqslant n$ such that the conditions* (31), (7), (8) *are satisfied, then in $\mathcal{S}(\frac{\pi}{M-1})$ Eq.* (24)

- *has infinitely many f.b. solutions,*
- *the trivial solution is repelling in the whole sector.*

*Moreover, in $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$ Eq.* (26)

- *has infinitely many b.b. solutions,*
- *the trivial solution is attracting in the whole sector.*

**Proof.** We adopt notation from the proof of Theorem 20.

The existence of f.b. solutions follows by the same arguments as in the proof of Theorem 4.

Similarly to the proof of Theorem 20, one shows that $\varphi_{(t_0,-T)}(E \setminus \{0\}) \subset \operatorname{int} E$ holds for some $t_0 \in \mathbb{R}$ and $\varphi_{(t_0,-T)}(0) = 0$.

Let the inequality

$$\int_0^T a_1(t)\,dt > 0 \tag{32}$$

be satisfied. Write $Y = E \cup K(0, \delta_0)$. Then, by analysis similar to that from the proof of Theorem 20, the inclusion $\varphi_{(t_0,-T)}(\operatorname{cl} K(0, \delta_0)) \subset K(0, \delta_0)$ holds. Thus $\varphi_{(t_0,-T)}(Y) \subset Y$ is satisfied. By the Denjoy–Wolff fixed point theorem, there exists exactly one fixed point $z_0$ of $\varphi_{(t_0,-T)}$ inside $Y$, so it must be $z_0 = 0$. Finally, the trivial solution is repelling in the sector $\mathcal{S}(\frac{\pi}{M-1})$.

Let now the inequality

$$\int_0^T a_1(t)\,dt = 0 \tag{33}$$

be satisfied. To use the Denjoy–Wolff fixed point theorem for proving that the point $z = 0$ is attracting for $\varphi_{(t_0,-T)}$ in $\operatorname{int} E$ it is enough to show that $\varphi_{(t_0,-T)}$ has no fixed points in $\operatorname{int} E$.

Without loss of generality we can assume that $t_0 = 0$. To obtain a contradiction, we suppose that there exists a fixed point $z_0 \in \operatorname{int} E$ for $\varphi_{(0,-T)}$. It is also a fixed point of $\varphi_{(0,T)}$. Then there exists a fixed point $z_\mu$ of the Poincaré map $\varphi^{[\mu]}_{(0,T)}$ of the equation

$$\dot{z} = \big[a_1(t) + \mu\big]z + \sum_{j=2}^{n} a_j(t)z^j \tag{34}$$

such that $z_\mu \in \mathcal{S}(\frac{\pi}{M-1})$, provided $\mu > 0$ is small enough. Indeed, if $\varphi_{(0,T)} - \mathrm{id} \equiv 0$ in some neighbourhood of $z_0$ then $\varphi_{(0,-T)} = \mathrm{id}$ on the whole $E$ but it cannot be true since the vector field $v$ points outward $E$ on $\partial E(\delta) \setminus \{0\}$. Thus $z_0$ is an isolated zero of $\varphi_{(0,-T)} - \mathrm{id}$. Then the existence of $z_\mu$ follows by the continuous dependence of solutions on the vector field and the Rouche theorem, which contradicts the first part of the proof.

The statement for Eq. (26) follows by the equality $v(t,-z) = v_1(t,z)$ for $(t,z) \in \mathbb{R} \times \widehat{\mathcal{S}}(\frac{\pi}{M-1})$. □

The following examples are straightforward applications of Theorem 23.

**Example 24.** The trivial solution of the equation

$$\dot{z} = z + e^{i\frac{\pi}{10}\cos(t)}z^5 + z^6$$

is repelling in $\mathcal{S}(\frac{\pi}{5})$. Here $M = 6$ and $\tau_5 = \frac{\pi}{10} \leqslant \frac{\pi}{5}$.

**Example 25.** The trivial solution of the equation

$$\dot{z} = \cos(t)z - (1+i)z^3 + z^4$$

is attracting in the whole $\widehat{\mathcal{S}}(\frac{\pi}{3})$. To see this let us observe that the auxiliary equation

$$\dot{z} = -\cos(t)z + (1+i)z^3 + z^4$$

fulfils the assumptions of Theorem 23 with $M = 4$ and $\tau_3 = \frac{\pi}{4} \leqslant \frac{\pi}{3}$.

**Corollary 26.** *Let $n \geqslant 3$, $a_1 \in \mathcal{C}(\mathbb{R},\mathbb{R})$ and $a_j \in \mathcal{C}(\mathbb{R},\mathbb{C})$ for $j \in \{2,3,\dots,n\}$ be $T$-periodic. If*

$$a_j \equiv 0, \quad \textit{for all even numbers } j \geqslant 2$$

*and there exists $M \geqslant n$ such that the conditions* (7), (8) *are satisfied, then Eq.* (24)

- *has infinitely many f.b. solutions in every of the sectors $\mathcal{S}(\frac{\pi}{M-1})$, $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$,*
- *has exactly one $T$-periodic solution in every of the sectors (it is asymptotically unstable and repelling in the whole sector),*
- *has in every of the sectors infinitely many solutions which are heteroclinic from the periodic to the trivial one*

*provided* (25) *is satisfied.*

*Moreover, Eq.* (24)

- *has infinitely many f.b. solutions in every of the sectors $\mathcal{S}(\frac{\pi}{M-1})$, $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$,*
- *the trivial solution is repelling in every of the sectors*

*provided* (31) *holds.*

The following examples are straightforward applications of Corollary 26.

**Example 27.** The trivial solution of the equation

$$\dot{z} = z + e^{i\frac{\pi}{5}\cos(t)} z^3 + z^5$$

is repelling in $\mathcal{S}(\frac{\pi}{4})$ and $\widehat{\mathcal{S}}(\frac{\pi}{4})$. Here $M = 5$ and $\tau_3 = \frac{\pi}{5} < \frac{\pi}{2}$.

**Example 28.** The trivial solution of the equation

$$\dot{z} = e^{i\frac{\pi}{4}\cos(t)} z^3 + z^5 + z^7$$

is repelling in $\mathcal{S}(\frac{\pi}{6})$ and $\widehat{\mathcal{S}}(\frac{\pi}{6})$. Here $M = 7$ and $\tau_3 = \frac{\pi}{4} < \frac{\pi}{3}$.

**Example 29.** The equation

$$\dot{z} = -z + (1+i)z^3 + z^5$$

has in every of the sectors $\mathcal{S}(\frac{\pi}{4})$ and $\widehat{\mathcal{S}}(\frac{\pi}{4})$ exactly one $2\pi$-periodic solution and infinitely many solutions which are heteroclinic from the periodic to the trivial one. Here $M = 5$ and $\tau_3 = \frac{\pi}{4} < \frac{\pi}{2}$.

**Corollary 30.** *Let $n \geqslant 3$, $a_1 \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $a_j \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ for $j \in \{2, 3, \ldots, n\}$ be $T$-periodic. If* (10) *holds and there exists $M \geqslant n$ such that the conditions* (7), (8) *are satisfied, then Eq.* (24)

- *has exactly one $T$-periodic solution in the sector $\mathcal{S}(\frac{\pi}{M-1})$ (it is asymptotically unstable and repelling in the whole sector),*
- *has in the sector infinitely many solutions which are heteroclinic from the periodic to the trivial one,*
- *has in the sector infinitely many f.b. solutions,*
- *has in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$ infinitely many b.b. solutions,*
- *the trivial solution is attracting in the whole sector*

*provided* (25) *is satisfied.*

*Moreover, Eq.* (24)

- *has infinitely many f.b. solutions in the sector $\mathcal{S}(\frac{\pi}{M-1})$,*
- *the trivial solution is repelling in the whole sector,*
- *has infinitely many b.b. solutions in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$,*
- *the trivial solution is attracting in the whole sector*

*provided* (33) *holds.*

*Eq.* (24)

- *has in the sector $\mathcal{S}(\frac{\pi}{M-1})$ infinitely many f.b. solutions,*
- *the trivial solution is repelling in the whole sector,*
- *has exactly one $T$-periodic solution in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$ (it is asymptotically stable and attracting in the whole sector),*
- *has in the sector infinitely many solutions which are heteroclinic from the trivial to the periodic one,*
- *has in the sector infinitely many b.b. solutions*

*provided* (32) *is satisfied.*

The following examples are straightforward applications of Corollary 30.

**Example 31.** The equation

$$\dot{z} = -z + e^{i\cos(t)}z^4 + (1+i)z^6$$

has in the sector $\mathcal{S}(\frac{3}{20}\pi)$ exactly one $2\pi$-periodic solution and the trivial solution is attracting in $\widehat{\mathcal{S}}(\frac{3}{20}\pi)$. Here $M = \frac{23}{3}$, $\tau_4 = 1 < \frac{9}{20}\pi$ and $\tau_6 = \frac{\pi}{4} \leqslant \frac{\pi}{4}$.

**Example 32.** The trivial solution of the equation

$$\dot{z} = \cos(t)z + e^{i\frac{\pi}{8}\cos(t)}z^2 + z^4$$

is repelling in $\mathcal{S}(\frac{\pi}{3})$ and attracting in $\widehat{\mathcal{S}}(\frac{\pi}{3})$. Here $M = 4$ and $\tau_2 = \frac{\pi}{8} < \frac{\pi}{3}$.

**Example 33.** The equation

$$\dot{z} = z + e^{i\sin(t)}z^4 + z^8$$

has in the sector $\widehat{\mathcal{S}}(\frac{\pi}{7})$ exactly one $2\pi$-periodic solution and the trivial solution is repelling in $\mathcal{S}(\frac{\pi}{7})$. Here $M = 8$ and $\tau_4 = 1 < \frac{3}{7}\pi$.

## 6. Trivial free and linear terms

In Section 3 we proved the existence of periodic solutions provided the free term of the vector field was nontrivial. Then the same method was used in the case of the trivial free term and nontrivial linear one. It occurs that application of the method to Eq. (1) with the trivial free and linear terms meets with difficulties. Let $j \geqslant 2$ be the smallest index such that $a_j \not\equiv 0$. Then we need $\mathfrak{Re}(a_j) < 0$, so the term $a_j(t)z^j$ points on $\partial E \cap \{z \in \mathbb{C}\colon \operatorname{Arg}(z) = \alpha\}$ inward the set $E$. But $a_n(t)z^n$ points outwards. Thus inward tangency points occurs which preserves the application of the method of isolating segments. That is why we modify the construction of the set $E$ which gives results less general that the ones from Sections 3 and 5.

We construct an isolating segment $[0, T] \times E$ in such a way that $\partial E$ is not longer a subset of a line but a solution of the equation $\dot{z} = z^j$. To simplify calculations we assume that $j = 2$. It is possible to deal with $j > 2$ but this case is more difficult because solutions of $\dot{z} = z^m$ are given by complicated formulas, whereas for $m = 2$ they are just circles.

In this section we investigate a special case of (1) given by

$$\dot{z} = v(t, z) = a(t)z^2 + b(t)z^n, \tag{35}$$

where $n \geqslant 3$ and $a \in \mathcal{C}(\mathbb{R}, (-\infty, 0))$, $b \in \mathcal{C}(\mathbb{R}, \mathbb{C} \setminus \{0\})$ are $T$-periodic. We assume that $a$ is real-valued to simplify calculations. Let $\tau_b$ be the smallest number such that $|\operatorname{Arg}[b(t)]| \leqslant \tau_b$ holds for every $t \in \mathbb{R}$. Write

$$J = \min_{t\in\mathbb{R}} \left|\frac{a(t)}{b(t)}\right|^{\frac{1}{n-2}} \quad \text{and} \quad L = \max_{t\in\mathbb{R}} \left|\frac{a(t)}{b(t)}\right|^{\frac{1}{n-2}}.$$

The following lemma is crucial to the main theorem of the section.

**Lemma 34.** *Let $n \geqslant 3$ and mappings $a \in \mathcal{C}(\mathbb{R}, (-\infty, 0))$, $b \in \mathcal{C}(\mathbb{R}, \mathbb{C} \setminus \{0\})$ be $T$-periodic. If there exist constants $0 < p < 1$, $q > 1$ and $C > \frac{qL}{2}$ such that*

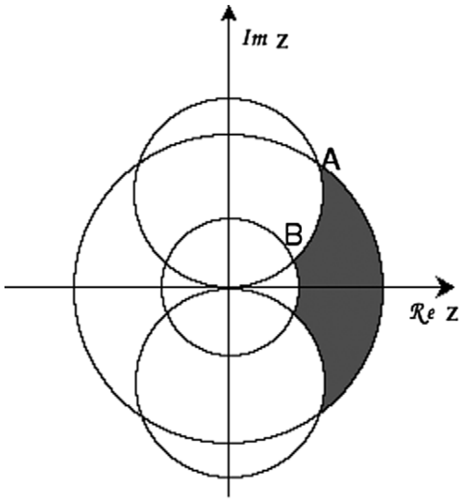


**Fig. 1.** The set $E$ is marked in grey.

$$\tau_b < (n-2)\arcsin\left(\frac{pJ}{2C}\right), \tag{36}$$

$$4C^2\left(1 - p^{2n-4}\right) > p^2 J^2, \tag{37}$$

$$\tau_b + (n-1)\arcsin\left(\frac{qL}{2C}\right) < \frac{\pi}{2}, \tag{38}$$

$$\cos\left[\tau_b + (n-1)\arcsin\left(\frac{qL}{2C}\right)\right] > q^{2-n} \tag{39}$$

*hold, then Eq.* (35) *has at least one $T$-periodic asymptotically unstable solution $\xi$ in the sector $\mathcal{S}(\arcsin(\frac{qL}{2C}))$.*

*If, in addition, we assume that*

$$J > 2Cp^{n-3} \tag{40}$$

*holds, then there exist in the sector $\mathcal{S}(\arcsin(\frac{qL}{2C}))$ infinitely many solutions which are heteroclinic from $\xi$ to the trivial one.*

**Proof.** Our goal is to construct isolating segment $W = [0, T] \times E$ such that the vector field $(1, v)$ points inward $W$ on $[0, T] \times \partial E$.

Let $E \subset \mathcal{S}(\frac{\pi}{2})$ be the closure of the set bounded by four circles $O(iC, C)$, $O(-iC, C)$, $O(0, pJ)$ and $O(0, qL)$ (cf. Fig. 1). Let $A, B \in E$ be the common points of $O(iC, C)$ and $O(0, qL)$, $O(iC, C)$ and $O(0, pJ)$, respectively. Let $\alpha = \operatorname{Arg}(A)$ and $\beta = \operatorname{Arg}(B)$. Obviously, $0 < \beta < \alpha$ holds.

We investigate the vector field $v$ on the common part of $\partial E$ and $O(iC, C)$. Let $z = |z|e^{i\theta}$. Then $\theta \in [\beta, \alpha]$ and $e^{2i\theta}$ is a vector tangent to $\partial E$ at $z$. The vector field $a(t)z^2$ is tangent to $\partial E$ and $b(t)z^n$ satisfies $\operatorname{Arg}[b(t)|z|^n e^{i\theta n}] \in [n\theta - \tau_b, n\theta + \tau_b]$. Thus $b(t)z^n$ points outward $E$ provided that

$$[n\theta - \tau_b, n\theta + \tau_b] \subset (2\theta, 2\theta + \pi).$$

But it is satisfied, provided the inequalities

$$\tau_b < (n-2)\beta, \tag{41}$$

$$\tau_b + (n-2)\alpha < \pi \tag{42}$$

hold. The situation is the same on the common part of $\partial E$ and $O(-iC, C)$. The inequality (38) gives $\tau_b + (n-1)\alpha < \frac{\pi}{2}$ which implies (42).

We parameterise the set $\partial E \cap O(0, pJ)$ by $s(o) = pJe^{io}$ where $o \in [-\beta, \beta]$. An outward normal vector is equal to $n(o) = -e^{io}$. Thus

$$\begin{aligned}\langle n(o), v(t, s(o))\rangle &= \mathfrak{Re}\left[-a(t)p^2J^2e^{io} - b(t)p^nJ^ne^{io(n-1)}\right]\\ &\geqslant |a(t)|p^2J^2\cos(\beta) - |b(t)|p^nJ^n = (\star),\end{aligned} \tag{43}$$

so $(\star) > 0$ holds provided that

$$\cos(\beta) > \frac{|b(t)|}{|a(t)|}p^{n-2}J^{n-2}.$$

Since the inequality

$$\frac{|b(t)|}{|a(t)|}p^{n-2}J^{n-2} \leqslant \max_{t\in\mathbb{R}} \frac{|b(t)|}{|a(t)|}p^{n-2}J^{n-2} = p^{n-2}$$

is satisfied, it is enough to

$$\cos(\beta) > p^{n-2} \tag{44}$$

hold. Since $\mathfrak{Re}[b(t)q^nL^ne^{i(n-1)o}] \geqslant \cos(\tau_b + (n-1)\alpha)|b(t)|q^nL^n$ holds, similar argument to that above shows that the vector field $v$ points outward $E$ on $\partial E \cap O(0, qL)$ provided that the inequality

$$\cos(\tau_b + (n-1)\alpha) > q^{2-n} \tag{45}$$

is satisfied.

Let us observe that

$$\sin(\alpha) = \frac{qL}{2C}, \tag{46}$$

$$\sin(\beta) = \frac{pJ}{2C} \tag{47}$$

hold. The inequality (36) is equivalent to (41) and (39) to (45). Similarly, since $\cos[\arcsin(x)] = \sqrt{1-x^2}$ holds, the inequalities (37) and (44) are also equivalent.

Finally, the vector field $(1, v)$ points on $[0, T] \times \partial E$ outward $W$. Thus the Poincaré map $\varphi_{(0,-T)}: E \to \operatorname{int} E$ is well defined. By the Denjoy–Wolff fixed point theorem, there exists $T$-periodic repelling solution $\xi$ inside $E$. Observation that $\operatorname{int} E \subset \mathcal{S}(\arcsin(\frac{qL}{2C}))$ finishes the main part of the proof.

Let $F = \operatorname{cl}[K(0, pJ) \cap \mathcal{S}(\beta)]$. We parameterise the set $\partial F \cap \{z \in \mathbb{C}\colon \operatorname{Arg}(z) = \beta\}$ by $s_1(o) = oe^{i\beta}$ where $o \in (0, pJ]$. An outward normal vector is equal to $n_1(o) = ie^{i\beta}$. Thus

$$\begin{aligned}\langle n_1(o), v(t, s_1(o))\rangle &= \mathfrak{Re}\left[-a(t)o^2ie^{i\beta} - b(t)o^nie^{i(n-1)\beta}\right]\\ &\leqslant -|a(t)|o^2\sin(\beta) + |b(t)|o^n < 0,\end{aligned}$$

where the latter inequality holds provided that

$$\sin(\beta) > p^{n-2} \tag{48}$$

which is equivalent to (40). Similarly, the vector field $v$ points inward $F$ on $\partial F \cap \{z \in \mathbb{C}\colon \operatorname{Arg}(z) = -\beta\}$.

Calculations similar to (43) and (44) show that $v$ points inward $F \cap K(0,r)$ on every arc $F \cap O(0,r)$ provided that $r \in (0, pJ]$. Thus $F$ is positive invariant and every solution starting in $F$ is attracted by the trivial one, i.e. is heteroclinic from $\xi$ to the zero solution. □

**Proposition 35.** *Let the assumption of Lemma* 34 *be satisfied. If, in addition, n is even, then Eq.* (35) *has at least one T-periodic asymptotically stable solution $\chi$ in the sector $\widehat{\mathcal{S}}(\arcsin(\frac{qL}{2C}))$. In this case,* (40) *implies the existence of infinitely many solutions which are heteroclinic from the trivial one to $\chi$ and contained in $\widehat{\mathcal{S}}(\arcsin(\frac{qL}{2C}))$.*

**Proof.** It is a straightforward consequence of the symmetry of the vector field $v(t,z) = v(t,-z)$ and argument similar to that from the proof of Theorem 4. □

**Proposition 36.** *Let the assumption of Lemma* 34 *be satisfied. If, in addition, n is odd, then the trivial solution of Eq.* (35) *is repelling in the sector $\widehat{\mathcal{S}}(\frac{\pi-\tau_b}{n-1})$. Moreover, the sector contains infinitely many f.b. solutions.*

**Proof.** By the change of variables $w = -z$ one gets

$$\dot{w} = -aw^2 + bw^n. \tag{49}$$

Thus $\tau_2 = 0$, $\tau_n = \tau_b$ and, by (38), $\tau_b < \frac{\pi}{2}$. Let us write $M = \frac{n\pi - \tau_b}{\pi - \tau_b}$, so $M \geqslant n$. Since (49) satisfies (31), then, by Theorem 23, the trivial solution is repelling in $\mathcal{S}(\frac{\pi-\tau_b}{n-1})$. □

Setting appropriate values to $p$, $q$ and $C$ in Lemma 34, we obtain the following theorem.

**Theorem 37.** *Let $n \geqslant 3$ and mappings $a \in \mathcal{C}(\mathbb{R}, (-\infty, 0))$, $b \in \mathcal{C}(\mathbb{R}, \mathbb{C} \setminus \{0\})$ be T-periodic. If*

$$\tau_b < \frac{n-2}{3(n-1)} \cos\left(\frac{\pi}{18}\right) \frac{J}{L} \tag{50}$$

*holds, then Eq.* (35) *has in the sector $\mathcal{S}(\arcsin(\frac{2}{3(n-1)}))$ at least one T-periodic asymptotically unstable solution.*

*Similarly, if*

$$\tau_b < \frac{19(n-2)J}{40(n-1)L} \quad \text{and} \quad n \geqslant 5 \tag{51}$$

*holds, then Eq.* (35) *has in the sector $\mathcal{S}(\arcsin(\frac{1}{n-1}))$ at least one T-periodic asymptotically unstable solution $\xi$.*

*If besides* (51) *the following condition*

$$J > 2(n-1)\left(\frac{19}{20}\right)^{n-3} L \tag{52}$$

*is satisfied, then there are infinitely many solutions which are heteroclinic from $\xi$ to the trivial one and contained in $\mathcal{S}(\arcsin(\frac{1}{n-1}))$.*

**Proof.** It is enough to show that the inequalities (38), (41), (44), (45) and in some cases (48) hold. Write $C = wL$ where $2w > q$.

Since (46), (47) hold, the conditions $\sin(\beta) = \frac{\cos(\frac{\pi}{18})J}{3(n-1)L} < \frac{1}{6}$ and $\sin(\alpha) = \frac{2}{3(n-1)}$ are satisfied provided that $p = \cos(\frac{\pi}{18})$, $q = 2$ and $w = \frac{3(n-1)}{2}$. Then (50) is equivalent to $\tau_b < (n-2)\sin(\beta)$ which implies (41). Moreover, $\beta < \frac{\pi}{3}\sin(\beta) < \frac{\pi}{18}$ holds and implies (44). By (50), it follows that $\tau_b < \frac{1}{3}$ holds. Since $\alpha < \frac{\pi}{3}\sin\alpha = \frac{2\pi}{9(n-1)}$, we have $\tau_b + (n-1)\alpha < \frac{3+2\pi}{9} < \frac{\pi}{3} < \frac{\pi}{2}$ which gives (38). Moreover, $\cos[\tau_b + (n-1)\alpha] > \cos\frac{\pi}{3} = 2^{-1} \geqslant 2^{2-n}$ which gives (45).

We now investigate the case where (51) is satisfied. Let us fix $p = \frac{19}{20}$, $q = 2$, $w = n-1$ which imply $\sin(\beta) = \frac{19J}{40(n-1)L} \leqslant \frac{19}{160}$ and $\sin(\alpha) = \frac{1}{n-1}$. Then the first inequality from (51) is equivalent to $\tau_b < (n-2)\sin(\beta)$ which implies (41). Moreover, $\beta < \frac{\pi}{3}\sin(\beta) \leqslant \frac{19\pi}{480}$ holds, and $\cos(\frac{19\pi}{480}) > (\frac{19}{20})^2$ is satisfied which gives (44).

By (51), it follows that $\tau_b < \frac{19}{40}$ holds. Then $\tau_b + (n-1)\alpha < \frac{19}{40} + \frac{\pi}{3} < \frac{\pi}{2}$ implies (38). Moreover, $\cos[\tau_b + (n-1)\alpha] > \cos(\frac{19}{40} + \frac{\pi}{3}) > 0.04 > 2^{2-n}$ is satisfied for $n \geqslant 7$. For $n = 6$ one gets $\tau_b \leqslant \frac{19}{50}$ and $\cos[\tau_b + 5\alpha] > \cos(\frac{19}{50} + \frac{\pi}{3}) > 0.14 > 2^{-4}$. Similarly, for $n = 5$ one gets $\tau_b \leqslant \frac{57}{160}$ and $\cos[\tau_b + 4\alpha] > \cos(\frac{57}{160} + \frac{\pi}{3}) > 0.16 > 2^{-3}$. Finally, (45) holds.

It is easy to see that (52) is equivalent to (48). □

The following examples are straightforward applications of Theorem 37 and Propositions 35, 36.

**Example 38.** The equation

$$\dot{z} = \big(\sin(t) - 2\big)z^2 + e^{i\cdot 0.05\cdot\cos(t)}z^3$$

has at least one $2\pi$-periodic asymptotically unstable solution in the sector $\mathcal{S}(\arcsin(\frac{1}{3}))$ and the trivial solution is repelling in the sector $\widehat{\mathcal{S}}(\frac{\pi-0.05}{2})$. Here $J = 1$, $L = 3$, $\tau_b = 0.05$ and the condition (50) is satisfied.

**Example 39.** The equation

$$\dot{z} = -z^2 + \big(\sin(t) + 3\big)e^{i\cdot 0.3\cdot\cos(t)}z^6$$

has at least one $2\pi$-periodic asymptotically unstable solution in $\mathcal{S}(\arcsin(\frac{1}{5}))$ and at least one $2\pi$-periodic asymptotically stable one in $\widehat{\mathcal{S}}(\frac{\pi}{2})$. Here $J = \frac{1}{\sqrt{2}}$, $L = \frac{1}{\sqrt[4]{2}}$, $\tau_b = 0.3$ and the condition (51) is satisfied.

**Example 40.** The equation

$$\dot{z} = -z^2 + 3e^{i\frac{\pi}{7}\cos(t)}z^{110}$$

has in $\mathcal{S}(\arcsin(\frac{1}{109}))$ at least one $2\pi$-periodic asymptotically unstable solution $\xi$ and infinitely many solutions which are heteroclinic from $\xi$ to the trivial one. Moreover, it has in $\widehat{\mathcal{S}}(\arcsin(\frac{1}{109}))$ at least one $2\pi$-periodic asymptotically stable solution $\chi$ and infinitely many solutions which are heteroclinic from the trivial one to $\chi$. Here $J = L = 3^{-\frac{1}{108}}$, $\tau_b = \frac{\pi}{7}$ and the conditions (51), (52) are satisfied.

**Remark 41.** The values of the constants $p$, $q$, $C$ are in the proof of Theorem 37 chosen to maximise $\tau_b$ for small $n$. For $n = 3$ the condition (50) implies $\tau_b < \frac{1}{6}$ but for $n \geqslant 5$ the inequality (51) gives $\tau_b < \frac{57}{160}$. In some cases, the straightforward application of Lemma 34 allows $\tau_b$ to be bigger, as exemplified below.

**Example 42.** The equation

$$\dot{z} = -z^2 + e^{i\cdot 0.7\cdot\cos(t)} z^{199}$$

has in $\mathcal{S}(\arcsin(\frac{309}{79600})) \subset \mathcal{S}(0.004)$ at least one $2\pi$-periodic asymptotically unstable solution $\xi$ and infinitely many solutions which are heteroclinic from $\xi$ to the trivial one. Moreover, the trivial solution is repelling in $\widehat{\mathcal{S}}(\frac{\pi-0.7}{2})$. Here $J = L = 1$, $\tau_b = 0.7$, $p = 0.97$, $q = 1.03$, $C = \frac{398}{3}$ and the conditions (36)–(40) hold.

In the following example we combine results from the previous and present sections.

**Example 43.** Let us consider the equation

$$\dot{z} = \varepsilon z^2 + e^{i\cdot 0.05\cdot\sin(t)} z^4.$$

If $\varepsilon \geqslant 0$, then, by Corollary 30, the trivial solution is repelling in the sector $\mathcal{S}(\frac{\pi}{4})$ and attracting in $\widehat{\mathcal{S}}(\frac{\pi}{4})$. Here $M = 5$ and $\tau_4 = 0.05 \leqslant \frac{\pi}{4}$.

If $\varepsilon < 0$, then by Lemma 34, there are in every of the sectors $\mathcal{S}(\arcsin(\frac{1}{3}))$, $\widehat{\mathcal{S}}(\arcsin(\frac{1}{3}))$ at least one $2\pi$-periodic solution and infinitely many solutions which are heteroclinic between the periodic and trivial ones. Here $J = L$, $\tau_b = 0.05$, $p = 0.2$, $q = 1.5$, $C = \frac{9}{4}L$ and the conditions (36)–(40) hold.

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