Linear Spaces of Nilpotent Matrices

Ben Mathes
Department of Mathematics
Colby College
Waterville, Maine 04901

Matjaž Omladič
Department of Mathematics
E.K. University of Ljubljana
61000 Ljubljana, Jadranska 19, Yugoslavia

and

Heydar Radjavi
Department of Mathematics, Statistics and Computing Science
Dalhousie University
Halifax, Nova Scotia, Canada B3H 3J5

Submitted by Richard A. Brualdi

ABSTRACT

We consider several questions on spaces of nilpotent matrices. We present sufficient conditions for triangularizability and give examples of irreducible spaces. We give a necessary and sufficient condition, in terms of the trace, for all linear combinations of a given set of operators to be nilpotent. We also consider the question of the dimension of a space $\mathscr{L}$ of nilpotents on $\mathbb{F}^n$. In particular, we give a simple new proof of a theorem due to M. Gerstenhaber concerning the maximal dimension of such spaces.

---

1. INTRODUCTION

Collections of nilpotent matrices with various structures have been studied by many authors. For certain structures, e.g., a multiplicative semi-

655 Avenue of the Americas, New York, NY 10010 0024-3795/91/$3.50

group or a Lie algebra, it is well known that the collection is (simultaneously) triangularizable. (See Levitzki's theorem [2], Engel's theorem, and its extension by Jacobson [3].) If $\mathscr{L}$ is merely a linear space of nilpotent matrices, however, it can be very far from triangularizable; in fact $\mathscr{L}$ may be irreducible, i.e., it may fail to have a common, nontrivial, invariant subspace.

In this paper we consider several questions on spaces of nilpotent matrices. We present sufficient conditions for triangularizability and give examples of irreducible spaces. We give a necessary and sufficient condition, in terms of the trace, for all linear combinations of a given set of operators to be nilpotent. We also consider the question of the dimension of a space $\mathscr{L}$ of nilpotents on $\mathbb{F}^n$. In particular we give a simple new proof of the following result of Gerstenhaber [1]: the dimension of $\mathscr{L}$ cannot exceed $n(n-1)/2$, the dimension of the strictly upper triangular matrices; furthermore, if the dimension of $\mathscr{L}$ is equal to $n(n-1)/2$, then $\mathscr{L}$ is triangularizable (and thus coincides with the algebra of all triangular nilpotents relative to some basis).

2. A SIMPLE PROOF

The proof of Gerstenhaber's result given in [1] needs the assumption that the field $\mathbb{F}$ has at least n elements, and Gerstenhaber speculates that this assumption may be unnecessary. It is proved in [6] that this is indeed the case, and Gerstenhaber's result is obtained with no conditions on the underlying field. Our proof is substantially different from, and we believe much simpler than, the proofs previously obtained.

Let $\operatorname{tr}(A)$ denote the trace of a matrix A.

LEMMA 1. *If A, B, and $A+B$ are nilpotent matrices over a field $\mathbb{F}$, then* $\operatorname{tr}(AB)=0$.

Proof. Choose a basis relative to which B is in Jordan form; thus

$$B=\begin{bmatrix} 0 & \delta_1 & 0 & \cdots & 0 \\ 0 & 0 & \delta_2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & \delta_{n-1} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $\delta_i = 0$ or 1 $(i = 1,\dots,n-1)$. Let $S_2(M)$ be the sum of the 2×2 principal minors of a matrix M; thus if M is nilpotent then $S_2(M) = 0$. If $A = (a_{ij})$ relative to our chosen basis, then a brief calculation verifies that

$$S_2(A) - S_2(A+B) = \sum_{i=1}^{n-1} \delta_i a_{i+1\,i} = \operatorname{tr}(AB).$$

It follows that $\operatorname{tr}(AB) = 0$, since both A and $A+B$ are assumed to be nilpotent. ■

THEOREM 1. *If $\mathscr{L}$ is a linear space of $n\times n$ nilpotent matrices over a field $\mathbb{F}$, then the dimension of $\mathscr{L}$ is no greater than $n(n-1)/2$.*

Proof. Let $\mathscr{T}$ be the space of all strictly upper triangular matrices, let $\mathscr{L}_1 = \mathscr{L}\cap\mathscr{T}$, and fix a complementary subspace $\mathscr{L}_2$ of $\mathscr{L}_1$ in $\mathscr{L}$ (so $\mathscr{L} = \mathscr{L}_1 + \mathscr{L}_2$ and $\mathscr{L}_1\cap\mathscr{L}_2 = \{0\}$, which will be denoted by $\mathscr{L} = \mathscr{L}_1 \oplus \mathscr{L}_2$). For any set $\mathscr{S}$ of $n\times n$ matrices over $\mathbb{F}$, define

$$\mathscr{S}^{\perp} \equiv \{A \mid \operatorname{tr}(AB) = 0 \text{ for all } B\in\mathscr{S}\};$$

thus we have $\dim\mathscr{S} + \dim\mathscr{S}^{\perp} = n^2$. Note that $\mathscr{T}^{\perp}$ is the set of all upper triangular matrices and, since the elements of $\mathscr{L}_2$ are nilpotent, $\mathscr{T}^{\perp}\cap\mathscr{L}_2 = \{0\}$. For any $A\in\mathscr{L}_1$, $B\in\mathscr{L}_2$, and $C\in\mathscr{T}^{\perp}$ we observe that $\operatorname{tr}(AC) = 0$ and, by Lemma 1, $\operatorname{tr}(AB) = 0$; so we have $\mathscr{T}^{\perp}\oplus\mathscr{L}_2 \subset \mathscr{L}_1^{\perp}$. It follows that $\dim\mathscr{L}_2 + n(n+1)/2 \leqslant n^2 - \dim\mathscr{L}_1$. ■

Our proof for the second half of Gerstenhaber's result works as long as the underlying field $\mathbb{F}$ is not the two-element field. The proof is based on the following lemma and a theorem of Jacobson. A brief and elementary proof of Jacobson's theorem may be found in [5].

LEMMA 2. *If A and B are matrices over a field $\mathbb{F}$ with more than two elements, and if every linear combination of A and B is nilpotent, then* $\operatorname{tr}(AB^2) = 0$.

Proof. Write B in its Jordan form as in the proof of Lemma 1. Let $S_3(M)$ be the sum of the 3×3 principal minors of a matrix M; thus $S_3(A+zB) = 0$ for every $z\in\mathbb{F}$, since $A+zB$ is nilpotent for all $z\in\mathbb{F}$. Viewing $S_3(A+zB)$ as a quadratic polynomial in z, we must have that each coefficient vanishes, since $\mathbb{F}$ has at least three elements. A calculation reveals that the coefficient of z^2 is $\operatorname{tr}(AB^2)$. ∎

JACOBSON'S THEOREM [3]. *If $\mathscr{N}$ is a set of nilpotent matrices such that for every $A, B \in \mathscr{N}$ there exists $c \in \mathbb{F}$ with $AB - cBA \in \mathscr{N}$, then $\mathscr{N}$ is triangularizable.*

THEOREM 2. *If $\mathscr{L}$ is a linear space of nilpotent matrices over a field $\mathbb{F}$ with more than two elements, and if the dimension of $\mathscr{L}$ is $n(n-1)/2$, then $\mathscr{L}$ is triangularizable.*

Proof. Adopting the notation of Theorem 1, we must have that $\mathscr{T}^{\perp} \oplus \mathscr{L}_2 = \mathscr{L}_1^{\perp}$. It follows that $\mathscr{L}_1 = \mathscr{L}_2^{\perp} \cap \mathscr{T} = \mathscr{L}^{\perp} \cap \mathscr{T}$. Given $B \in \mathscr{L}$, choose a basis relative to which $B \in \mathscr{T}$. This yields $B^2 \in \mathscr{L}^{\perp}$ by Lemma 2, and since we also have $B^2 \in \mathscr{T}$, we conclude that $B^2 \in \mathscr{L}_1 \subset \mathscr{L}$. Therefore, $\mathscr{L}$ contains $BC + CB = (B+C)^2 - B^2 - C^2$ for all $B, C \in \mathscr{L}$, and the theorem follows from Jacobson's theorem. ∎

3. A TRACE CONDITION ON SETS OF NILPOTENTS

This section provides a result that characterizes those sets of matrices that generate linear spaces of nilpotents. Let S_k denote the group of permutations on the set $\{1, 2, \ldots, k\}$.

THEOREM 3. *Suppose $\mathscr{E}$ is a set of $n \times n$ matrices over a field with characteristic zero. The following are equivalent:*

(i) *The additive semigroup generated by $\mathscr{E}$ consists of nilpotents.*
(ii) *The linear space generated by $\mathscr{E}$ consists of nilpotents.*
(iii) *For every finite sequence $(E_i)_{i=1}^k$ in $\mathscr{E}$,*

$$\sum_{\sigma \in S_k} \operatorname{tr}\left(E_{\sigma(1)} E_{\sigma(2)} \cdots E_{\sigma(k)}\right) = 0.$$

Proof. The implication (ii) ⇒ (i) is trivial, and the converse is easy: if $(\lambda_i)_{i=1}^k$ is a sequence of scalars, then $A \equiv \sum_{i=1}^k \lambda_i E_i$ is nilpotent if and only if

$\mathrm{tr}(A^m)=0$ for all positive integers m (here we use the hypothesis that the underlying field has characteristic zero). Fixing m and viewing

$$\mathrm{tr}\left(\left(\sum_{i=1}^{k}\lambda_i E_i\right)^m\right)$$

as a polynomial p in k indeterminates, we note that if

$$p(m_1,m_2,\dots,m_k)=0$$

for all sequences $(m_i)_{i=1}^k$ of positive integers, then $p=0$.

We now prove the equivalence of (i) and (iii). Assume (i) holds, and let $(E_i)_{i=1}^k$ be a finite sequence in $\mathscr{E}$. For every sequence of positive integers $(m_i)_{i=1}^k$, one has that

$$\mathrm{tr}\left(\left(\sum_{i=1}^{k}m_i E_i\right)^k\right)=0.$$

Viewing this as a polynomial in k indeterminates, we see that the coefficient of $m_1m_2\cdots m_k$ is $\sum_{\sigma\in S_k}\mathrm{tr}(E_{\sigma(1)}E_{\sigma(2)}\cdots E_{\sigma(k)})$, and (iii) follows. Assume now that (iii) holds, and let $(E_i)_{i=1}^k$ be a finite sequence in $\mathscr{E}$. We must prove that $\sum_{i=1}^k E_i$ is nilpotent, i.e. that $\mathrm{tr}\left(\left(\sum_{i=1}^k E_i\right)^m\right)=0$ for every positive integer m. If $0\leqslant r_i\leqslant m$ $(i=1,\dots,k)$ and $\sum_{i=1}^k r_i=m$, then define $B(r_1,r_2,\dots,r_k)$ to be the sum of all distinct products of m matrices in which r_i of the factors are E_i $(i=1,\dots,k)$. (We mean that two products are distinct provided they are distinct as words, rather than distinct matrices.) Thus we have

$$r_1!r_2!\cdots r_k!B(r_1,r_2,\dots,r_k)=\sum_{\sigma\in S_m}A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(m)},$$

where $(A_j)_{j=1}^m$ is any sequence that maps exactly r_i integers to E_i $(i=1,\dots,k)$. By our hypothesis we must have that $\mathrm{tr}(B(r_1,r_2,\dots,r_k))=0$. It now follows that

$$\mathrm{tr}\left(\left(\sum_{i=1}^{k}E_i\right)^m\right)=\mathrm{tr}\left(\sum_{\substack{0\leqslant r_i\leqslant m\\ r_1+\cdots+r_k=m}}B(r_1,r_2,\dots,r_k)\right)=0. \qquad \blacksquare$$

Condition (iii) of the theorem above should be contrasted with the trace condition that $\mathscr{E}$ is simultaneously triangularizable (see Theorem 1 of [4]). If $\mathscr{E}$ consists of nilpotents, then $\mathscr{E}$ is simultaneously triangularizable if and only if each summand of

$$\sum_{\sigma \in S_k} \operatorname{tr}\left(E_{\sigma(1)} E_{\sigma(2)} \cdots E_{\sigma(k)}\right)$$

vanishes.

We comment that Lemma 2 is false when $\mathbb{F}$ is the two-element field. A counterexample is given by the following matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This example and the following corollary show that the equivalence of parts (ii) and (iii) of Theorem 3 is false in general if we impose no restriction on the underlying field. If $\mathbb{G} \equiv \mathbb{F}[\alpha]$, where α is a root of $x^2 + x + 1$, and we view A and B as matrices over $\mathbb{G}$, then we see that the additive semigroup generated by $\{A, B\}$ consists of nilpotents while the linear space generated by $\{A, B\}$ does not. Thus the equivalence of (i) and (ii) is also false in general if we impose no restriction on the underlying field.

Corollary 1. *Suppose $\mathscr{L}$ is a linear space of nilpotent matrices over a field with characteristic zero. If $A, B \in \mathscr{L}$ and $k \geqslant 1$, then* $\operatorname{tr}(A^k B) = 0$.

Proof. Let $(E_i)_{i=1}^{k+1}$ be the sequence defined by $E_i = A$ $(i = 1, \dots, k)$ and $E_{k+1} = B$. Then $0 = \sum_{\sigma \in S_{k+1}} \operatorname{tr}(E_{\sigma(1)} E_{\sigma(2)} \cdots E_{\sigma(k+1)}) = (k+1)! \operatorname{tr}(A^k B)$. ∎

4. TRIANGULARIZABLE SPACES

It is sometimes the case that if a linear space of nilpotent matrices enjoys some additional property, then one may conclude that the space is triangularizable. For example, in the proof of Gerstenhaber's result presented in [1], the following statement is established: if $\mathscr{L}$ is a linear space of nilpotent matrices (over an arbitrary field) that contains all the powers of a matrix S with maximal index of nilpotence, then $\mathscr{L}$ is triangularizable. Another example is given in [6], which says if $\mathscr{L}$ is a linear space of nilpotent matrices (over an arbitrary field) that is generated by its rank-one matrices, then $\mathcal{S}$ is triangularizable; the following is our generalization.

THEOREM 4. *If $\mathcal{S}$ is an additive semigroup of nilpotent matrices (over an arbitrary field) and $\mathcal{S}$ is generated by its rank-one matrices, then $\mathcal{S}$ is triangularizable.*

Proof. Let $\mathcal{E}$ be the set of rank-one matrices in $\mathcal{S}$. To show that $\mathcal{S}$ is triangularizable it suffices to prove that the multiplicative semigroup generated by $\mathcal{E}$ consists of nilpotents (by Levitzki's theorem [2]). We proceed by induction on k, the length of a word $E_1 E_2 \cdots E_{k-1} E_k$ with $E_i \in \mathcal{E}$. If $k = 1$ there is nothing to prove, so assume the induction hypothesis for $i < k$. Viewing the matrices as linear transformations of the vector space $\mathcal{V}$, since each E_i has rank one there exist $e_i \in \mathcal{V}$ and $\varphi_i \in \mathcal{V}^*$ (the dual space of $\mathcal{V}$) such that $E_i = e_i \otimes \varphi_i$ $(i = 1, \dots, k)$, where

$$e_i \otimes \varphi_i(v) \equiv \varphi_i(v) e_i .$$

Assume, by way of contradiction, that $E_1 E_2 \cdots E_{k-1} E_k$ is not nilpotent. It follows that $\varphi_j(e_i) \neq 0$ whenever $i = j + 1 \bmod k$. We assert that $\varphi_j(e_i) = 0$ if $i \neq j + 1 \bmod k$. To see this, select i and j such that $i \neq j + 1 \bmod k$, and note that either $E_i E_{i+1} \cdots E_k E_1 \cdots E_j$ (if $j < i$) or $E_i E_{i+1} \cdots E_j$ if $(j \geqslant i)$ is a word of length less than k. Thus,

$$\varphi_i(e_{i+1}) \varphi_{i+1}(e_{i+2}) \cdots \varphi_{j-1}(e_j) e_i \otimes \varphi_j$$

is nilpotent by the induction hypothesis. Since $\varphi_j(e_i) \neq 0$ if $i = j + 1 \bmod k$, we must have $\varphi_j(e_i) = 0$, which establishes the assertion. Now observe that $E_1 + \cdots + E_k(e_1) = \varphi_k(e_1) e_k$, and continue to find that

$$(E_1 + \cdots + E_k)^k (e_1) = \varphi_k(e_1) \varphi_{k-1}(e_k) \cdots \varphi_1(e_2) e_1,$$

so $E_1 + \cdots + E_k$ is not nilpotent, our contradiction. ■

The previous theorem is addressing the pathology that arises when the underlying field does not have characteristic zero: it says that Theorem 3 is true in full generality for sets $\mathcal{E}$ consisting of matrices of rank one.

LEMMA 3. *Assume L and E are nilpotent $n \times n$ matrices over a field with characteristic zero, and E has rank one. Then $\{E, L\}$ generates a linear space of nilpotents if and only if $\{E, L\}$ is triangularizable.*

Proof. It is obvious that if $\{E, L\}$ is triangularizable, then $\{E, L\}$ generates a linear space of nilpotents. Thus assume that $\{E, L\}$ generates a linear space of nilpotents. We will prove that $\operatorname{tr}(E_1 E_2 \cdots E_k) = 0$ for every finite sequence $(E_i)_{i=1}^k$ in $\{E, L\}$ (and the lemma will be obtained by the trace condition for triangularizability given in [4]). Since $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and $E^2 = 0$, we may assume without loss of generality that

$$E_1 E_2 \cdots E_k = L^{t_1} E L^{t_2} E \cdots L^{t_p} E.$$

But for some scalar $\lambda \in \mathbb{F}$, $\operatorname{tr}(L^{t_1} E L^{t_2} E \cdots L^{t_p} E) = \lambda \operatorname{tr}(L^{t_1} E)$, and thus by Corollary 1, $\operatorname{tr}(E_1 E_2 \cdots E_k) = \lambda \operatorname{tr}(L^{t_1} E) = 0$. ■

THEOREM 5. *Suppose $\mathscr{L}$ is a linear space of nilpotent $n \times n$ matrices over a field with characteristic zero. If $\mathscr{L}$ is spanned by the set $\{F_1, \ldots F_p, S\}$, with F_i rank-one matrices $(i = 1, \ldots, p)$ and S a nilpotent of index n, then $\mathscr{L}$ is triangularizable.*

Proof. By Lemma 3, F_i and S are simultaneously triangularizable $(i = 1, \ldots, p)$. Since S has a unique maximal chain of invariant subspaces, it follows that $\{F_1, \ldots F_p, S\}$, and hence $\mathscr{L}$, is triangularizable. ■

5. IRREDUCIBLE SPACES

We now consider some examples of spaces of nilpotents that are as far from triangularizable as possible, those with no common, nontrivial, invariant subspace. We begin with the following question.

QUESTION. What is the maximal dimension of an irreducible linear space of $n \times n$ nilpotents?

EXAMPLE 1. If $n = 3m$, then there exists an irreducible $(m^2 + 1)$-dimensional linear space of $n \times n$ nilpotents.

Let $\mathscr{L}(\mathscr{V})$ denote the algebra of operators on the n-dimensional vector space $\mathscr{V}$, and consider the linear space

$$\mathscr{L} \equiv \left\{ \begin{bmatrix} 0 & A & 0 \\ \lambda 1 & 0 & A \\ 0 & -\lambda 1 & 0 \end{bmatrix} \middle| \lambda \in \mathbb{F}, A \in \mathscr{L}(\mathbb{F}^m) \right\} \subset \mathscr{L}(\mathscr{V}),$$

where 1 is the identity map on $\mathbb{F}^m$. It is easy to see that $T^3 = 0$ for every $T \in \mathscr{L}$. The invariant subspaces of

$$\left\{ \begin{bmatrix} 0 & A & 0 \\ 0 & 0 & A \\ 0 & 0 & 0 \end{bmatrix} \middle| A \in \mathscr{L}(\mathbb{F}^m) \right\}$$

are all of the form $\mathscr{M} \oplus (0) \oplus (0)$, $\mathbb{F}^m \oplus \mathscr{M} \oplus (0)$, and $\mathbb{F}^m \oplus \mathbb{F}^m \oplus \mathscr{M}$, where $\mathscr{M}$ is any subspace of $\mathbb{F}^m$. Since only the trivial ones of these are invariant under

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix},$$

it follows that $\mathscr{L}$ is irreducible.

One interpretation of Theorem 1 is that every $n(n-1)/2$-dimensional linear space of $n \times n$ nilpotents is a maximal element in the family of all linear spaces of $n \times n$ nilpotents (ordered by inclusion).

Example 2. If $n \geqslant 3$, there is an irreducible $(n-1)$-dimensional linear space of $n \times n$ nilpotents which is a maximal element in the family of all linear spaces of $n \times n$ nilpotents.

Suppose $\{e_1, e_2, \dots, e_n\}$ is some fixed basis of a vector space $\mathscr{V}$, and $\{e_i^* \mid i = 1, \dots, n\}$ is the corresponding dual basis. Define S by

$$S(x) = \sum_{i=2}^{n} e_i^*(x) e_{i-1},$$

and define R_k for $k = 1, \dots, n-2$ by

$$R_k(x) = e_1^*(x) e_{k+1} - e_2^*(x) e_{k+2}.$$

If $\mathscr{M}_k = \operatorname{span}\{e_1, \dots, e_k\}$ for $k = 1, \dots, n$ and $\mathscr{M}_0 = \{0\}$, then the only invariant subspaces of S are $\{\mathscr{M}_k : k = 0, \dots, n\}$. It is clear that none of the nontrivial ones of these are invariant under R_{n-2}, so $\{S, R_1, \dots, R_{n-2}\}$ spans an irreducible linear space. We assert that this is a linear space of nilpotents.

If $\alpha, \beta_1, \dots, \beta_{n-2} \in \mathbb{F}$, then the matrix of $T \equiv \alpha S + \sum_{i=1}^{n-2} \beta_i R_i$ is

$$T = \begin{bmatrix} 0 & \alpha & 0 & \cdot & \cdot & \cdot & 0 \\ \beta_1 & 0 & \alpha & 0 & & & \cdot \\ \beta_2 & -\beta_1 & 0 & \alpha & \ddots & & \cdot \\ \beta_3 & -\beta_2 & 0 & 0 & \ddots & 0 & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \alpha & 0 \\ \beta_{n-2} & -\beta_{n-3} & 0 & 0 & \cdots & 0 & \alpha \\ 0 & -\beta_{n-2} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

which illustrates that the orbit of e_n under T is

$$e_n \mapsto \alpha e_{n-1} \mapsto \alpha^2 e_{n-2} \mapsto \cdots \mapsto \alpha^{n-3} e_3 \mapsto \alpha^{n-2} e_2.$$

Since $e_1 \mapsto \sum_{k=2}^{n-1} \beta_{k-1} e_k$, it follows that

$$e_2 \mapsto \alpha e_1 - \sum_{k=3}^{n} \beta_{k-2} e_k \mapsto \sum_{k=2}^{n-1} \alpha\beta_{k-1} e_k - \sum_{k=3}^{n} \alpha\beta_{k-2} e_k - 1 = 0.$$

Thus $T^n(e_n) = 0$, and since e_k appears in the above chain for $2 \leqslant k \leqslant n-1$, it is clear that $T^{n-1}(e_k) = 0$ for $k = 2, \dots, n-1$. Finally, $T(e_1)$ is a linear combination of these vectors, which implies $T^n = 0$.

Our proof that this space is maximal requires that the underlying field have characteristic zero. Let A be a matrix such that the set $\{A, S, R_1, \dots, R_{n-2}\}$ generates a linear space of nilpotents; we will prove that $\{A, S, R_1, \dots, R_{n-2}\}$ is dependent. Without loss of generality we may assume that the first column and the second entry of the first row of A are zero (by subtracting these entries away with appropriate multiples of S and the R_k's). It is easy to see that $(R_1 + R_k)^2 - (R_1)^2$ $(k = 1, \dots, n-2)$ has exactly one nonzero entry on the $k+2$nd place of the first column. It follows from Corollary 1 that the whole first row of the matrix of A is zero.

Fix now an index k, $1 \leqslant k \leqslant n-2$, and compute the matrix of

$$T_j \equiv (S + R_k)^j - S^j$$

for all positive integers j. An induction argument shows that for $1 \leqslant j \leqslant k$, T_j has a one entry at the $k - j + 2$nd place of the first column, a minus one entry at the $j + 1$st place of the $k + 2$nd row, and zeros everywhere else. It follows that T_{k+1} has a one at the first place of the first row, two minus ones at the first and the $k + 2$nd place of the $k + 2$nd row, and zeros everywhere else. If $k + 2 < n$, the induction argument proceeds over j, $k + 2 \leqslant j < n$, to obtain that T_j has only one nonzero entry left (which equals minus one), at the $j + 1$st place of the $k + 2$nd row. It follows from Corollary 1 that A has all the columns from the third to the nth equal to zero. Use $\operatorname{tr}(S^k A) = 0$ to see that the second column has all entries under the main diagonal equal to zero. Since A is nilpotent, the diagonal entry is also zero, which implies $A = 0$.

One might ask what the minimal dimension of an irreducible linear space of nilpotents is. The next example answers this question.

EXAMPLE 3. If $n \geqslant 3$, then there is an irreducible, two-dimensional linear space of $n \times n$ nilpotents.

This example is obtained by considering the span of $\{S, R_{n-2}\}$, where S and R_{n-2} are defined in the previous example.

REFERENCES

1 M. Gerstenhaber, On nilalgebras and linear varieties of nilpotent matrices, I, *Amer. J. Math.* 80:614–622 (1958).
2 J. Levitzki, Über nilpotente unterringe, *Math. Ann.* 105:620–622 (1931).
3 N. Jacobson, *Lie Algebras*, Interscience, New York, 1962.
4 H. Radjavi, A trace condition equivalent to simultaneous triangularizability, *Canad. J. Math.* 38:376–386 (1986).
5 H. Radjavi, The Engel-Jacobson theorem revisited, *J. Algebra* 111:427–430 (1987).
6 V. N. Serežkin, Linear transformations preserving nilpotency, *Vestī Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk*, 1985, No. 6, pp. 46–50, 125.

Received 2 April 1990; final manuscript accepted 23 August 1990