A comment on unions of rings

by D. A. Overdijk, F. H. Simons and J. G. F. Thiemann

Eindhoven University of Technology, Dept. of Mathematics

Communicated by Prof. N. G. de Bruijn at the meeting of 25 November 1978

Recently, Broughton and Huff [1] showed that the union of a strictly increasing sequence of σ-fields cannot be a σ-field. It is most remarkable that this fact seems not to have been noted before. However, since the conditions for a class of subsets to be a field are weaker than those to be a σ-field, the statement that the union of a strictly increasing sequence of fields cannot be a σ-field is even more plausible. Unfortunately, the proof of Broughton and Huff makes (only at one place) essentially use of the fact that they consider a sequence of σ-fields.

In this note we shall give an even simpler proof of the theorem that the union of a strictly increasing sequence of rings cannot be a σ-ring. This obviously implies that the union of a strictly increasing sequence of (σ-)fields cannot be a σ-field.

Throughout, X will be a fixed set. A sequence $(\mathscr{A}_n)$ of rings of subsets of X is said to be increasing if $\mathscr{A}_n \subset \mathscr{A}_{n+1}$ for all n, and an increasing sequence of rings $(\mathscr{A}_n)$ is said to be stationary if eventually we have $\mathscr{A}_n = \mathscr{A}_{n+1}$.

For any class $\mathscr{A}$ of subsets of X and any $F \subset X$ we define

$$\mathscr{A}|F = \{A \in \mathscr{A} | A \subset F\}.$$

If $(\mathscr{A}_n)$ is an increasing sequence of rings, then for every $F \subset X$ the sequence $(\mathscr{A}_n|F)$ is again an increasing sequence of rings.

LEMMA 1. Let $(\mathscr{A}_n)$ be a non-stationary increasing sequence of rings, and put $\mathscr{A}=\bigcup_{n=1}^{\infty}\mathscr{A}_n$. Then for every N there exists a set $F\in\mathscr{A}\backslash\mathscr{A}_N$ such that the sequence $(\mathscr{A}_n|X\backslash F)$ is non-stationary.

PROOF. Choose N. Since the sequence $(\mathscr{A}_n)$ is non-stationary, there exist an integer $N_1>N$, a set $G_1\in\mathscr{A}_{N_1}\backslash\mathscr{A}_N$, an integer $N_2>N_1$ and a set $G_2\in\mathscr{A}_{N_2}\backslash\mathscr{A}_{N_1}$. It is easily verified that at least two of the three disjoint sets $G_1\backslash G_2$, $G_1\cap G_2$ and $G_2\backslash G_1$ do not belong to $\mathscr{A}_N$. Therefore there exist two disjoint sets F_1 and F_2 in $\mathscr{A}$ not belonging to $\mathscr{A}_N$.

For two rings $\mathscr{R}_1$ and $\mathscr{R}_2$ we define

$$\mathscr{R}_1\vee\mathscr{R}_2=\{A\cup B|A\in\mathscr{R}_1 \text{ and } B\in\mathscr{R}_2\}.$$

Then for all $n>N_2$ we have

$$\mathscr{A}_n=\mathscr{A}_n|X\backslash F_1\vee\mathscr{A}_n|X\backslash F_2.$$

Since the sequence $(\mathscr{A}_n)$ is non-stationary, at least one of the sequences $(\mathscr{A}_n|X\backslash F_1)$, $(\mathscr{A}_n|X\backslash F_2)$ is non-stationary. Now we define $F=F_1$ if the sequence $(\mathscr{A}_n|X\backslash F_1)$ is non-stationary, and $F=F_2$ otherwise. □

LEMMA 2. Let $(\mathscr{A}_n)$ be a non-stationary increasing sequence of rings, and put $\mathscr{A}=\bigcup_{n=1}^{\infty}\mathscr{A}_n$. Then there exists a sequence of disjoint sets (C_n) in $\mathscr{A}$ such that $C_n\notin\mathscr{A}_n$ for every n.

PROOF. By lemma 1 there exists a set $C_1\in\mathscr{A}$ such that $C_1\notin\mathscr{A}_1$ and the sequence $(\mathscr{A}_n|X\backslash C_1)$ is non-stationary. Now suppose that the disjoint sets $C_1, C_2, \dots, C_k$ in $\mathscr{A}$ have been found such that $C_i\notin\mathscr{A}_i$ for $1\leqslant i\leqslant k$, and the sequence $(\mathscr{A}_n|X\backslash(C_1\cup\dots\cup C_k))$ is non-stationary. Then again by lemma 1 there exists a set $C_{k+1}\in\mathscr{A}|X\backslash(C_1\cup\dots\cup C_k)$ with $C_{k+1}\notin$ $\notin\mathscr{A}_{k+1}|X\backslash(C_1\cup\dots\cup C_k)$ and the sequence $(\mathscr{A}_n|X\backslash(C_1\cup\dots\cup C_k)|X\backslash C_{k+1})$ is non-stationary. The first condition implies that the sets $C_1, \dots, C_{k+1}$ are disjoint and $C_{k+1}\notin\mathscr{A}_{k+1}$. Since we have

$$\mathscr{A}_n|X\backslash(C_1\cup\dots\cup C_k)|X\backslash C_{k+1}=\mathscr{A}_n|X\backslash(C_1\cup\dots\cup C_{k+1}),$$

the second condition implies that the sequence $(\mathscr{A}_n|X\backslash(C_1\cup\dots\cup C_{k+1}))$ is non-stationary as well. □

THEOREM. Let $(\mathscr{A}_n)$ be a non-stationary increasing sequence of rings, and put $\mathscr{A}=\bigcup_{n=1}^{\infty}\mathscr{A}_n$. Then $\mathscr{A}$ is not a σ-ring.

PROOF. Suppose that $\mathscr{A}$ is a σ-ring. Let the sequence (C_n) be as in lemma 2. Let $\{N_1, N_2, \dots\}$ be a partition of the set of natural numbers into infinite sets, and put

$$X_p=\bigcup_{n\in N_p} C_n.$$

By our assumption the sets X_p belong to $\mathscr{A}$ and therefore for every p there exists an integer n_p such that $X_p \in \mathscr{A}_{n_p}$. Since $(\mathscr{A}_n)$ is increasing, we may assume that the sequence (n_p) is strictly increasing.

For every p we choose an integer $m_p \in N_p$ such that $m_p > n_p$, and we put

$$D = \bigcup_{p=1}^{\infty} C_{m_p}.$$

Then by assumption we have $D \in \mathscr{A}$, hence eventually the set D belongs to every $\mathscr{A}_n$, and therefore there exists an integer q such that $D \in \mathscr{A}_{n_q}$. Because of the construction of the sets X_p we now have

$$X_q \cap D = C_{m_q} \in \mathscr{A}_{n_q}.$$

Since $n_q < m_q$ this implies $C_{m_q} \in \mathscr{A}_{m_q}$. This is a contradiction, and therefore the assumption that $\mathscr{A}$ were a σ-ring is false. □

REFERENCE

1. Broughton, A. and B. W. Huff – A comment on unions of sigma-fields. Amer. Math. Monthly **84**, 553–554 (1977).