DISCRETE MATHEMATICS

ELSEVIER Discrete Mathematics 180 (1998) 3–22

# $q$-Hypergeometric solutions of $q$-difference equations

Sergei A. Abramov[a,*,1], Peter Paule[b,2], Marko Petkovšek[c,3]

[a] Computer Center of the Russian Academy of Science, Vavilova 40, Moscow 117967, Russia
[b] Institut für Mathematik, RISC, Johannes Kepler University, A-4040 Linz, Austria
[c] Department of Mathematics and Mechanics, University of Ljubljana, Jadranska 19, 1111 Ljubljana, Slovenia

Received 18 July 1995; received in revised form 1 October 1996; accepted 9 December 1996

**Abstract**

We present algorithm `qHyper` for finding all solutions $y(x)$ of a linear homogeneous $q$-difference equation, such that $y(qx)=r(x)y(x)$ where $r(x)$ is a rational function of $x$. Applications include construction of basic hypergeometric series solutions, and definite $q$-hypergeometric summation in closed form.

## 1. Introduction

As a motivating example, consider the following second-order $q$-difference equation

$$y_{n+2}-(1+q)x\,y_{n+1}+x^2y_n=0 \tag{1}$$

where $x=q^n$. This is a homogeneous linear equation with coefficients which are polynomials in $x$. It is easy to check that

$$y_n^{(1)}=q^{\binom{n}{2}} \quad \text{and} \quad y_n^{(2)}=q^{\binom{n}{2}-n}$$

both solve (1). Note that their consecutive-term ratios, $y_{n+1}^{(1)}/y_n^{(1)}=q^n=x$ and $y_{n+1}^{(2)}/y_n^{(2)}=q^{n-1}=x/q$, are rational functions of $x$. We call such solutions *$q$-hypergeometric*. This paper describes an algorithm for finding all $q$-hypergeometric solutions of linear $q$-difference equations with polynomial coefficients, and presents some of its applications.

The algebraic framework that we use is the following. Let $\mathbb{F}$ be a computable field of characteristic zero, $q\in\mathbb{F}$ a nonzero element which is not a root of unity, and $x$

* Corresponding author. E-mail: abramov@ccas.ru, sabramov@cs.msu.su.

[1] The research described in this publication was made possible in part by Grant J12100 from the International Science Foundation and Russian Government.
[2] Supported in part by grant P7720 of the Austrian FWF.
[3] Supported in part by grant J2-6193-0101-94 of the Slovenian Ministry of Science and Technology.


PII S0012-365X(97)00106-4

transcendental over $\mathbb{F}$. Denote by $\varepsilon$ the unique automorphism of $\mathbb{F}(x)$ which fixes $\mathbb{F}$ and satisfies $\varepsilon x = qx$. Then $\mathbb{F}(x)$ together with the *q-shift operator* $\varepsilon$ is a difference field [7].

Let $\rho$ be a nonnegative integer and $p_i \in \mathbb{F}(x)$, for $i = 0, 1, \ldots, \rho$, rational functions such that $p_\rho, p_0 \neq 0$. Then

$$L = \sum_{i=0}^{\rho} p_i \varepsilon^i \tag{2}$$

is a *linear q-difference operator* of order $\rho$ with rational coefficients. Two such operators may be multiplied by using the commutation relation

$$\varepsilon x = q\, x\, \varepsilon$$

and extending it by distributivity. Division of operators can be performed using the rule

$$f(x)\, \varepsilon^k = \left( \frac{f(x)}{g(q^{k-m}x)}\, \varepsilon^{k-m} \right) g(x)\, \varepsilon^m$$

for right-dividing a monomial $f(x)\,\varepsilon^k$ by another monomial $g(x)\,\varepsilon^m$ $(m \leqslant k)$. As with ordinary polynomials, for any two operators $L_1$, $L_2$ where $L_2 \neq 0$, there are operators $Q$ and $R$ such that $L_1 = QL_2 + R$ and $\operatorname{ord} R < \operatorname{ord} L_2$. Thus one can compute greatest common right divisors (and also least common left multiples, see [6]) of $q$-difference operators by the right-Euclidean algorithm.

We are interested in nonzero solutions $y$ of the homogeneous equation

$$Ly = 0 \tag{3}$$

where $L$ is as in (2). In general such solutions cannot be found within the coefficient field $\mathbb{F}(x)$. Rather, we look for them in some *difference extension ring* $M$ of $\mathbb{F}(x)$. Thus the problem of finding $q$-hypergeometric solutions of a $q$-difference equation contains two 'parameters': the ground field $\mathbb{F}$, and the difference extension ring $M$.

We call an element $a \in M$ *polynomial*[1] if $a \in \mathbb{F}[x]$, *rational*[1] if $a \in \mathbb{F}(x)$, and a *q-hypergeometric term* if $a \neq 0$ and $\varepsilon a = ra$ for some $r \in \mathbb{F}(x)$. Note that $q$-hypergeometric terms form a multiplicative group.

Let $y \in M$ with $\varepsilon y = ry$. Then it is easy to see that $y$ is a $q$-hypergeometric solution of (3) if and only if $L_1 = \varepsilon - rI$ (where $I$ is the identity operator) is a right divisor of $L$. This splits the search for $q$-hypergeometric solutions of (3) into two steps: 1. find first-order right divisors $L_1$ of $L$ (the nontrivial part), 2. solve the corresponding first-order equations $L_1 y = 0$ in $M$. Note that step 1 does not depend on the choice of $M$.

The overview of the paper is as follows. An algorithm `qHyper` for finding first-order right divisors of linear $q$-difference operators with rational coefficients is presented in

[1] All these concepts are relative to the field $\mathbb{F}$.

Section 4. It is a $q$-analogue of algorithm Hyper for finding hypergeometric solutions of difference equations described in [13]. Note that by clearing denominators in (3) we can restrict attention to operators $L$ with polynomial coefficients $p_i \in \mathbb{F}[x]$. In preparation, we show how to find polynomial solutions of (3) in Section 2, and give a suitable normal form for rational functions in Section 3. In Section 5, we discuss solutions of the first-order equation $\varepsilon y = ry$ in two specific instances of the extension ring $M$: the ring of germs of sequences over $\mathbb{F}$, and the ring of formal power series over $\mathbb{F}$. In Section 6 we show how to solve nonhomogeneous equations $Ly = f$ with $q$-hypergeometric right-hand side $f$. In Section 7, we describe applications of qHyper to the problem of closed-form evaluation of definite $q$-hypergeometric sums.

In our examples, we use the following $q$-calculus notation. Let $(z;q)_n = (1-z)(1-zq)\cdots(1-zq^{n-1})$ for $n \geqslant 1$ and $(z;q)_0 = 1$ be the $q$-shifted factorials, and $(z;q)_\infty = \prod_{k=0}^{\infty}(1-zq^k)$ the corresponding infinite product. For integer $m$ let $[m] = (1-q^m)/(1-q)$. Note that $-q^m[-m] = [m] = 1 + q + \cdots + q^{m-1}$ for $m \geqslant 1$. Thus $[m]$ turns into $m$ as $q \to 1$. The same applies to $[m]_d$ defined as $[m]$ with $q$ replaced by $q^d$, $d$ a positive integer. Let $[m]! = [1][2]\cdot \ldots \cdot [m]$ for $m \geqslant 1$ and $[0]! = 1$. For integer $n$ and nonnegative integer $k$, the Gaussian polynomial or the $q$-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = [n][n-1]\cdot \ldots \cdot [n-k+1]/[k]!.$$

These definitions are introduced to make the analogy with the case $q \to 1$ more transparent. For instance, when $q \to 1$ the $q$-factorial $[m]!$ turns into the ordinary factorial $m!$, and the $q$-binomial coefficient $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ turns into the ordinary binomial coefficient $\binom{n}{k}$. Again $[m]_d!$ and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_d$ denote the corresponding versions with $q$ replaced by $q^d$. The $q$-shifted factorials relate to the Gaussian polynomials via

$$(z;q)_n = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} z^k, \tag{4}$$

a $q$-analogue of the binomial theorem.

We use $\mathbb{N}$ to denote the set of nonnegative integers.

Sometimes we need to find the largest $n \in \mathbb{N}$ (if any) such that $q^n$ is a root of a given polynomial with coefficients in $\mathbb{F}$. Therefore we assume that $\mathbb{F}$ is a *$q$-suitable field*, meaning that there exists an algorithm which given $p \in \mathbb{F}[x]$ finds all $n \in \mathbb{N}$ such that $p(q^n) = 0$. Since by assumption $q$ is neither a root of unity nor zero, the set of all such $n$ is finite.

**Example 1.** Let $\mathbb{K}$ be any computable field. Then $\mathbb{F} = \mathbb{K}(q)$ where $q$ is transcendental over $\mathbb{K}$ is $q$-suitable, as shown by the following algorithm: Let $p(x) = \sum_{i=0}^{d} c_i x^i$ where $c_i \in \mathbb{K}[q]$. Compute $s = \min\{i;\ c_i \neq 0\}$ and $t = \max\{j;\ q^j \mid c_s\}$. Then $p(q^n) = 0$ only if $n \leqslant t$, and the set of all such $n$ can be found by consecutively testing the values $n = t, t-1, \ldots, 0$.

It is easy to see that if $\mathbb{F}$ is a $q$-suitable field and $\alpha$ is either transcendental or algebraic over $\mathbb{F}$, then the extension $\mathbb{F}(\alpha)$ is also $q$-suitable.

## 2. Polynomial solutions

First we show how to find solutions $y \in \mathbb{F}[x]$ of $Ly = 0$ where $L$ is as in (2) but with $p_i \in \mathbb{F}[x]$. Let $p_i = \sum_{k=0}^{d} c_{ik}x^k$ where not all $c_{id}$ are zero. Assume that $y = \sum_{j=0}^{n} a_j x^j$ where $a_n \neq 0$. Substituting these expressions into $Ly = 0$ and replacing $k$ by $l = j + k$ yields

$$\sum_{i,l,j} c_{i,l-j} a_j q^{ij} x^l = 0$$

which implies that

$$\sum_{j=\max\{l-d,0\}}^{\min\{l,n\}} \sum_{i=0}^{\rho} c_{i,l-j} a_j q^{ij} = 0 \quad \text{for } 0 \leqslant l \leqslant n+d. \tag{5}$$

In particular, for $l = n + d$,

$$\sum_{i=0}^{\rho} c_{id} q^{in} = 0, \tag{6}$$

and for $l = 0$,

$$a_0 \sum_{i=0}^{\rho} c_{i0} = 0. \tag{7}$$

From (6) it follows that $q^n$ is a root of the polynomial $p(x) = \sum_{i=0}^{\rho} c_{id} x^i$. Let $n_0$ be the largest $n \in \mathbb{N}$ such that $p(q^n) = 0$. Since $\mathbb{F}$ is $q$-suitable there is an algorithm to compute $n_0$. All polynomial solutions $y$ of $Ly = 0$ can now be found by the method of undetermined coefficients. Ultimately, the problem is reduced to a system of linear algebraic equations over $\mathbb{F}$ with $n_0 + 1$ unknowns. A more efficient method leading to a system with at most $\rho$ unknowns is described in [2].

## 3. A normal form for rational functions

**Theorem 1.** *Let $r \in \mathbb{F}(x) \backslash \{0\}$. Then there are $z \in \mathbb{F}$ and monic polynomials $a, b, c \in \mathbb{F}[x]$ such that*

$$r(x) = z \frac{a(x)}{b(x)} \frac{c(qx)}{c(x)}, \tag{8}$$

$$\gcd(a(x), b(q^n x)) = 1 \quad \textit{for all } n \in \mathbb{N}, \tag{9}$$

$$\gcd(a(x), c(x)) = 1, \tag{10}$$

$$\gcd(b(x), c(qx)) = 1, \tag{11}$$

$$c(0) \neq 0. \tag{12}$$

**Proof.** Write $r(x) = f(x)/g(x)$ where $f, g$ are relatively prime polynomials. We start by finding the set $\mathscr{S}$ of all $n \in \mathbb{N}$ such that $f(x)$ and $g(q^n x)$ have a nonconstant common factor. To this end consider the polynomial $R(w) = \text{Resultant}_x(f(x), g(wx))$. By the well-known properties of polynomial resultants, $\mathscr{S} = \{n \in \mathbb{N};\ R(q^n) = 0\}$.

Assume that $\mathscr{S} = \{n_1, n_2, \ldots, n_t\}$ where $t \geqslant 0$ and $n_1 < n_2 < \cdots < n_t$. In addition, let $n_{t+1} = +\infty$. Define polynomials $f_i$ and $g_i$ inductively by setting

$$f_0(x) = f(x), \qquad g_0(x) = g(x),$$

and for $i = 1, 2, \ldots, t$,

$$s_i(x) = \gcd(f_{i-1}(x), g_{i-1}(q^{n_i}x)),$$
$$f_i(x) = f_{i-1}(x)/s_i(x),$$
$$g_i(x) = g_{i-1}(x)/s_i(q^{-n_i}x).$$

Now take

$$z = \alpha/\beta, \qquad a(x) = f_t(x)/\alpha, \qquad b(x) = g_t(x)/\beta, \qquad c(x) = \prod_{i=1}^{t} \prod_{j=1}^{n_i} s_i(q^{-j}x),$$

where $\alpha$ and $\beta$ denote the leading coefficients of $f_t(x)$ and $g_t(x)$, respectively. Before proving (8)–(12) we state a lemma.

**Lemma 1.** *Let $n \in \mathbb{N}$. If $0 \leqslant l \leqslant i, j \leqslant t$ and $n < n_{l+1}$, then* $\gcd(f_i(x), g_j(q^n x)) = 1$.

**Proof.** Assume first that $n \notin \mathscr{S}$. Then $R(q^n) \neq 0$, hence $\gcd(f(x), g(q^n x)) = 1$. Since $f_i(x) \mid f(x)$ and $g_j(x) \mid g(x)$ it follows that $\gcd(f_i(x), g_j(q^n x)) = 1$, too.

To prove the lemma for $n \in \mathscr{S}$ we use induction on $l$.

$l = 0$: In this case there is nothing to prove since there is no $n \in \mathscr{S}$ such that $n < n_1$.

$l > 0$: Assume that the lemma holds for all $n < n_l$. It remains to show that it also holds for $n = n_l$. Since $f_i(x) \mid f_l(x)$ and $g_j(x) \mid g_l(x)$, it follows that $\gcd(f_i(x), g_j(q^{n_l}x))$ divides $\gcd(f_l(x), g_l(q^{n_l}x)) = \gcd(f_{l-1}(x)/s_l(x), g_{l-1}(q^{n_l}x)/s_l(x))$. By the definition of $s_l(x)$ the latter gcd is 1, completing the proof. $\square$

Now we proceed to verify properties (8)–(12).

(8)

$$z\frac{a(x)}{b(x)}\frac{c(qx)}{c(x)} = \frac{f_t(x)}{g_t(x)} \prod_{i=1}^{t} \prod_{j=1}^{n_i} \frac{s_i(q^{1-j}x)}{s_i(q^{-j}x)}$$

$$= \frac{f_0(x)}{\prod_{i=1}^{t} s_i(x)} \frac{\prod_{i=1}^{t} s_i(q^{-n_i}x)}{g_0(x)} \prod_{i=1}^{t} \frac{s_i(x)}{s_i(q^{-n_i}x)} = \frac{f(x)}{g(x)} = r(x).$$

(9) Let $i=j=l=t$ in Lemma 1. Then $\gcd(f_t(x), g_t(q^n x))=1$ for all $n<n_{t+1}=+\infty$. In other words, $\gcd(a(x), b(q^n x))=1$ for all $n\in\mathbb{N}$.

(10) If $a(x)$ and $c(x)$ have a nonconstant common factor then so do $f_t(x)$ and $s_i(q^{-j}x)$, for some $i$ and $j$ such that $1\leqslant i\leqslant t$ and $1\leqslant j\leqslant n_i$. Since $g_{i-1}(q^{n_i-j}x)=g_i(q^{n_i-j}x)s_i(q^{-j}x)$, it follows that $g_{i-1}(q^{n_i-j}x)$ contains this factor as well. As $n_i-j<n_i$, this contradicts Lemma 1. Hence $a(x)$ and $c(x)$ are relatively prime.

(11) If $b(x)$ and $c(qx)$ have a nonconstant common factor then so do $g_t(x)$ and $s_i(q^{-j}x)$, for some $i$ and $j$ such that $1\leqslant i\leqslant t$ and $1\leqslant j+1\leqslant n_i$. Since $f_{i-1}(q^{-j}x)=f_i(q^{-j}x)s_i(q^{-j}x)$, it follows that $f_{i-1}(x)$ and $g_t(q^j x)$ contain this factor as well. As $j<n_i$, this contradicts Lemma 1. Hence $b(x)$ and $c(qx)$ are relatively prime.

(12) It is easy to see that $s_i(x)$ divides both $f(x)$ and $g(q^{n_i}x)$. Hence $s_i(0)=0$ would imply that $f(0)=g(0)=0$, contrary to the assumption that $f$ and $g$ are relatively prime. It follows that $s_i(0)\neq 0$ for all $i$, and consequently $c(0)\neq 0$. □

**Example 2.** Let

$$r(x)=\frac{(x-1)(q^3x-1)}{(qx-1)(q^4x-1)}.$$

Rewriting this as

$$r(x)=\frac{1}{q^4}\,\frac{x-1}{x-q^{-4}}\,\frac{(qx-q^{-2})(qx-q^{-1})}{(x-q^{-2})(x-q^{-1})}$$

we can read off

$$z=q^{-4},\qquad a(x)=x-1,\qquad b(x)=x-q^{-4},\qquad c(x)=(x-q^{-2})(x-q^{-1}).$$

The representation described in Theorem 1 is unique and thus a normal form. In addition, it has $c(x)$ of least degree among all factorizations of $r(x)$ satisfying (8) and (9). A proof of this can be found in [3].

**Remark.** The factorization of Theorem 1 satisfying (8) and (9) is used by Koornwinder [9] in his Maple implementation of a $q$-analogue of Zeilberger's algorithm. In a similar context this representation is discussed by Paule and Strehl in [12] where a different normalization has been chosen.

## 4. $q$-Hypergeometric solutions

Now we derive the algorithm for finding first-order right divisors of linear $q$-difference operators with polynomial coefficients. Any such divisor has a non-trivial kernel in some suitable difference extension ring $M$ (see, e.g., Section 5.1), therefore it is permissible to think rather in terms of finding $q$-hypergeometric solutions

$y$ of $Ly=0$. Let $\varepsilon y = ry$ where $r \in \mathbb{F}(x)$, then $\varepsilon^i y = \prod_{j=0}^{i-1} r(q^j x)y$. We look for $r(x)$ in the normal form described in Theorem 1. After inserting (8) into $Ly=0$, clearing denominators and cancelling $y$ we obtain

$$\sum_{i=0}^{\rho} z^i f_i(x) c(q^i x) = 0 \tag{13}$$

where

$$f_i(x) = p_i(x) \prod_{j=0}^{i-1} a(q^j x) \prod_{j=i}^{\rho-1} b(q^j x).$$

Since all terms in (13) except for $i=0$ are divisible by $a(x)$ it follows that $a(x)$ divides $p_0(x)c(x)\prod_{j=0}^{\rho-1} b(q^j x)$. Because of (9) and (10), $a(x)$ divides $p_0(x)$. Similarly, all terms in (13) except for $i=\rho$ are divisible by $b(q^{\rho-1}x)$, therefore $b(q^{\rho-1}x)$ divides $z^\rho p_\rho(x) c(q^\rho x) \prod_{j=0}^{\rho-1} a(q^j x)$. Because of (9) and (11), $b(q^{\rho-1}x)$ divides $p_\rho(x)$. Thus we have a finite number of choices for $a(x)$ and $b(x)$.

For each choice of $a(x)$ and $b(x)$, Eq. (13) is a $q$-difference equation for the unknown polynomial $c(x)$. However, $z \in \mathbb{F}$ is also not known yet. Let $u_{ik}$ denote the coefficient of $x^k$ in $f_i$. Since $c(0) \neq 0$, we have $a_0 \neq 0$ in (7), hence applying (7) to (13) we obtain

$$\sum_{i=0}^{\rho} u_{i0} z^i = 0. \tag{14}$$

We may assume that not all $u_{i0}$ are zero, or else we start by first cancelling a power of $x$ from the coefficients of (13). Thus $z$ is a nonzero root of $f(z) = \sum_{i=0}^{\rho} u_{i0} z^i$, and is algebraic over $\mathbb{F}$.

If $n = \deg c(x)$ then by (6),

$$\sum_{i=0}^{\rho} u_{id} z^i q^{in} = 0, \tag{15}$$

hence $w = zq^n$ is a nonzero root of $g(w) = \sum_{i=0}^{\rho} u_{id} w^i$. It follows that $q^n$ is a root of $p(x) = \mathrm{Resultant}_w(f(w), g(wx))$, thus to obtain an upper bound on $n$ computation in algebraic extensions of $\mathbb{F}$ is not necessary.

In summary, we find the factors of $r(x)$ as follows:

1. $a(x)$ is a monic factor of $p_0(x)$,
2. $b(x)$ is a monic factor of $p_\rho(q^{1-\rho}x)$,
3. $z$ is a root of Eq. (14),
4. $c(x)$ is a nonzero polynomial solution of (13).

Then $r = z(a/b)(\varepsilon c/c)$, and the nonzero $y \in M$ satisfying $\varepsilon y = ry$ are $q$-hypergeometric solutions of $Ly=0$. Conversely, for any $q$-hypergeometric solution $y$ of $Ly=0$, its consecutive-term ratio $r = \varepsilon y/y$ can be obtained in this way. Our algorithm called `qHyper` works by finding, for each admissible triple $(a(x), b(x), z)$, a basis of polynomial solutions of the corresponding equation (13).

Alternatively, after finding one $q$-hypergeometric solution $u$ with $\varepsilon u/u=r$, we can divide $L$ by $L_1=\varepsilon-rI$ to obtain $L=L_2L_1$, and use the algorithm recursively on the reduced equation $L_2z=0$. If $z$ solves the new equation then any solution $y$ of the nonhomogeneous first-order equation

$$\varepsilon y-ry=z \tag{16}$$

solves the original equation. To solve (16), we can use the algorithm of Section 6. Instead, we can also make the substitution $y=uv$, and use either the $q$-analogue of Gosper's algorithm, or again the algorithm of Section 6, on the resulting equation $\varepsilon v-v=z/(ru)$. This process is equivalent to the standard method of reduction of order.

Our *Mathematica* implementation[2] of `qHyper` finds, in its basic form, at least one $q$-hypergeometric solution over $\mathbb{Q}(q)$ (if any such exists). With the option `Solutions -> All`, it finds a generating set for the space spanned by $q$-hypergeometric solutions over $\mathbb{Q}(q)$, and with the option `Quadratics -> True`, it works over quadratic extension fields of $\mathbb{Q}(q)$. It returns a list of rational functions $r_1,r_2,\dots,r_k$ which represent solutions $y_1,y_2,\dots,y_k$ such that $r_i=\varepsilon y_i/y_i$.

**Example 3.** Let us find a first-order right divisor of

$$L=x\varepsilon^3-q^3x^2\varepsilon^2-(x^2+q)\varepsilon+qx(x^2+q)I.$$

The candidates for $a(x)$ are

$$1,x,x^2+q,x(x^2+q),$$

and the candidates for $b(x)$ are

$$1,x.$$

Here we explore only the choice $a(x)=x$ and $b(x)=1$. The corresponding equation (13) is, after cancelling one $x$,

$$z^3q^3x^3c(q^3x)-z^2q^4x^3c(q^2x)-z(x^2+q)c(qx)+q(x^2+q)c(x)=0, \tag{17}$$

whence $f(z)=-qz+q^2$ with unique root $z=q$, and $g(w)=q^3w^3-q^4w^2$ with unique nonzero root $w=q=zq^n=q^{n+1}$. It follows that $n=0$ is the only possible degree for $c$. Eq. (17) is satisfied by $c=1$. Thus we have found $r=z(a/b)(\varepsilon c/c)=qx$, and the corresponding right divisor $L_1=\varepsilon-qx$ of $L$.

To find other first-order right divisors, the remaining combinations for $a(x)$ and $b(x)$ could be tried. Using our *Mathematica* implementation to carry this out, it turns out

[2] Available at `ftp://www.ijp.si/pub/math/qHyper.m`.

that there are in fact no other such divisors:

```
In[1]:= qHyper[x y[q^3 x] - q^3 x^2 y[q^2 x] - (x^2 + q) y[q x] +
              q x (x^2 + q) y[x] == 0, y[x],
              Solutions -> All, Quadratics -> True]

Out[1]= {q x}
```

**Example 4.** Consider the operator $L = \varepsilon^2 - (1+q)\varepsilon + q(1-qx^2)I$. As shown by qHyper,

```
In[2]:= qHyper[y[q^2 x] - (1 + q) y[q x] + q (1 - q x^2) y[x] == 0,
              y[x], Solutions -> All]

 Warning: irreducible factors of degree > 1 in trailing
 coefficient; some solutions may not be found

Out[2]= {}
```

it has no first-order right divisors over $\mathbb{F} = \mathbb{Q}(q)$. However, allowing quadratic factors to be split,

```
In[3]:= qHyper[y[q^2 x] - (1 + q) y[q x] + q (1 - q x^2) y[x] == 0,
              y[x], Solutions -> All, Quadratics -> True]

Out[3]= {1 - Sqrt[q] x, 1 + Sqrt[q] x}
```

we obtain two such divisors, namely $\varepsilon - (1 \pm \sqrt{q}\,x)$, over the splitting field $\mathbb{F} = \mathbb{Q}(\sqrt{q})$ of $1 - qx^2$.

**Example 5.** The operator $L = \varepsilon^2 - (1+2x)\varepsilon + xI$ has no first-order right divisors over $\mathbb{F} = \mathbb{Q}(q)$:

```
In[4]:= qHyper[y[q^2 x] - (1 + 2 x) y[q x] + x y[x], y[x]]

Out[4]= {}
```

Here $q$ was considered transcendental over the rational number field $\mathbb{Q}$. But when $q = 2$

```
In[5]:= q = 2;

In[6]:= qHyper[y[q^2 x] - (1 + 2 x) y[q x] + x y[x], y[x]]

Out[6]= {x}
```

we do get one such divisor over $\mathbb{F} = \mathbb{Q}$, namely $\varepsilon - x$.

```
In[7]:= Clear[q]
```

## 5. Examples of specific extension rings

### 5.1. Germs of sequences over $\mathbb{F}$

Let $\mathbb{F}^{\mathbb{N}}$ be the ring of sequences over $\mathbb{F}$, where addition and multiplication ('Hadamard product') are defined componentwise. Let $x=(q^n)_{n=0}^{\infty}$ denote the sequence of powers of $q$. If $E$ denotes the shift operator on $\mathbb{F}^{\mathbb{N}}$, i.e., $Ea_n=a_{n+1}$, then $Ex=qx$. Since $\lambda \in \mathbb{F}$ can be identified with the constant sequence $(\lambda, \lambda, \ldots)$, we can regard $\mathbb{F}$, $\mathbb{F}[x]$ and $\mathbb{F}(x)$ as subrings of $\mathbb{F}^{\mathbb{N}}$.

Unfortunately, $E$ is not an automorphism of $\mathbb{F}^{\mathbb{N}}$ because it annihilates nonzero sequences of the form $(\lambda, 0, 0, \ldots)$. To remedy this situation, we identify such sequences with zero; moreover, we identify any two sequences which agree from some point on. Formally we define $M$ as the quotient ring $S(\mathbb{F})=\mathbb{F}^{\mathbb{N}}/J$ where $J$ is the ideal of sequences with only finitely many nonzero terms. In particular, this means that equalities of the form $a_n=b_n$ are interpreted as being valid for all but finitely many $n$ (in short: for almost all $n$). Define $\varepsilon$ on $S(\mathbb{F})$ by requiring that $\varepsilon(a+J)=Ea+J$ for all $a \in \mathbb{F}^{\mathbb{N}}$. Then $\varepsilon$ is an automorphism of $S(\mathbb{F})$, and $S(\mathbb{F})$ is a difference extension ring of $\mathbb{F}(x)$. The elements of $S(\mathbb{F})$ are called the *germs of sequences over* $\mathbb{F}$ [15]. To simplify notation, we will identify the germ $a+J \in S(\mathbb{F})$ with its representative sequence $a \in \mathbb{F}^{\mathbb{N}}$.

In this domain the $q$-difference equation $Ly=0$ where $y=(y_n)_{n=0}^{\infty} \in S(\mathbb{F})$, translates into

$$\sum_{i=0}^{\rho} p_i(q^n) y_{n+i}=0$$

for almost all $n$. In particular, the first-order equation $\varepsilon y=ry$ where $r \in \mathbb{F}(x)$ can be rewritten as

$$y_{n+1}=r(q^n) y_n. \tag{18}$$

Let $n_0$ be the largest $n \in \mathbb{N}$ such that $q^n$ is a pole of $r$, or $-1$ if no such $n$ exists. Then, clearly, the sequence

$$y_n=C \prod_{k=n_0+1}^{n-1} r(q^k) \quad \text{for } n>n_0, \tag{19}$$

where $C \in \mathbb{F}$ is an arbitrary constant, satisfies (18) for almost all $n$. Thus every homogeneous first-order equation has a one-dimensional space of $q$-hypergeometric solutions in $S(\mathbb{F})$.

If $r(x)$ factors into linear factors over $\mathbb{F}$:

$$r(x)=z \frac{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_r)}{(x-\beta_1)(x-\beta_2)\cdots(x-\beta_s)} x^u$$

where $z, \alpha_i$, $\beta_j \in \mathbb{F}$, $u \in \mathbb{Z}$, and $\alpha_i, \beta_j \neq 0, q^k$, for all $k \in \mathbb{N}$, then we can also express the solution with $q$-shifted factorials as

$$y_n = C\frac{(1/\alpha_1;q)_n(1/\alpha_2;q)_n\cdots(1/\alpha_r;q)_n}{(1/\beta_1;q)_n(1/\beta_2;q)_n\cdots(1/\beta_s;q)_n}q^{u\binom{n}{2}}w^n$$

where

$$w = (-1)^{r+s}z\frac{\alpha_1\alpha_2\cdots\alpha_r}{\beta_1\beta_2\cdots\beta_s},$$

and $C \in \mathbb{F}$ is an arbitrary constant. This is the reason why $q$-hypergeometric solutions are considered to be expressible in closed form.

**Example 6.** Let $L$ be the linear $q$-difference operator of Example 3. Then the $q$-hypergeometric solutions $y \in S(\mathbb{F})$ of $Ly = 0$ satisfy $y_{n+1} = q^{n+1}y_n$. Hence

$$y_n = q^{\binom{n+1}{2}}$$

is a $q$-hypergeometric solution of $Ly = 0$ in $S(\mathbb{F})$, that is,

$$q^n y_{n+3} - q^{2n+3}y_{n+2} - q(1+q^{2n-1})y_{n+1} + q^{n+2}(1+q^{2n-1})y_n = 0$$

for $n \geqslant 0$.

Let $L$ be the linear $q$-difference operator of Example 4. Then the $q$-hypergeometric solutions $y \in S(\mathbb{F})$ of $Ly = 0$ satisfy $y_{n+1} = (1 \pm q^{n+1/2})y_n$. Hence $y_n^{(1)} = (\sqrt{q};q)_n$ and $y_n^{(2)} = (-\sqrt{q};q)_n$ are two linearly independent $q$-hypergeometric solutions of $Ly = 0$ in $S(\mathbb{F})$.

Let $L$ be the linear $q$-difference operator of Example 5, and let $q = 2$. Then the 2-hypergeometric solutions $y \in S(\mathbb{F})$ of $Ly = 0$ satisfy $y_{n+1} = 2^n y$. Hence

$$y_n = 2^{\binom{n}{2}}$$

is a 2-hypergeometric solution of $Ly = 0$ in $S(\mathbb{F})$.

### 5.2. *Formal power series over* $\mathbb{F}$

Let $M = \mathbb{F}[\![x]\!]$, the ring of formal power series over $\mathbb{F}$. Note that $\mathbb{F}$, $\mathbb{F}[x]$, and $\mathbb{F}(x)$ are embedded in $M$ in a natural way. For $y(x) = \sum_{k=0}^{\infty} y_k x^k \in \mathbb{F}[\![x]\!]$, define

$$\varepsilon\sum_{k=0}^{\infty} y_k x^k = \sum_{k=0}^{\infty} y_k q^k x^k.$$

Then $\varepsilon$ is an automorphism of $\mathbb{F}[\![x]\!]$, and $\mathbb{F}[\![x]\!]$ is a difference extension ring of $\mathbb{F}(x)$. Unlike in $S(\mathbb{F})$, a homogeneous first-order $q$-difference equation with rational coefficients does not always have a nonzero solution in $\mathbb{F}[\![x]\!]$.

**Theorem 2.** *The equation $y(qx) = r(x)y(x)$ where $r(x) = \sum_{k=0}^{\infty} r_k x^k \in \mathbb{F}[\![x]\!]$ has a nonzero solution $y(x) \in \mathbb{F}[\![x]\!]$ if and only if $r_0 = q^n$ for some $n \in \mathbb{N}$.*

**Proof.** Let $y(x)=\sum_{k=0}^{\infty} y_k x^k$ be a solution of $y(qx)=r(x)y(x)$. Then, comparing coefficients of like powers of $x$, we have

$$q^k y_k = \sum_{i=0}^{k} y_i r_{k-i} \quad \text{for } k=0,1,\ldots,$$

or, equivalently,

$$y_k(q^k - r_0) = \sum_{i=0}^{k-1} y_i r_{k-i} \quad \text{for } k=0,1,\ldots. \tag{20}$$

Assume first that $r_0 \neq q^k$ for all $k \in \mathbb{N}$. Then (20) implies that

$$y_0 = 0,$$
$$y_k = \sum_{i=0}^{k-1} y_i r_{k-i}/(q^k - r_0) \quad \text{for } k=1,2,\ldots.$$

From this it follows by induction on $k$ that $y_k=0$ for all $k \in \mathbb{N}$, hence that $y(x)=0$.

Now let $r_0=q^n$ for some $n \in \mathbb{N}$. Then we see as above that $y_k=0$ for $k<n$, and

$$y_k = \sum_{i=n}^{k-1} y_i r_{k-i}/(q^k - q^n) \quad \text{for } k=n+1,n+2,\ldots, \tag{21}$$

which is a recurrence allowing us to express all $y_k$ with $k>n$ in terms of $y_n$. □

**Example 7.** In Examples 3 and 5 we have $r(x)=qx$ and $r(x)=x$, respectively. In both cases $r_0=0$, so the condition of Theorem 2 is not fulfilled. Hence the corresponding equations have no $q$-hypergeometric solutions in $\mathbb{F}[\![x]\!]$.

In Example 4, $r(x)=1 \pm \sqrt{q}x$, and the condition of Theorem 2 is fulfilled with $n=0$. Using recurrence (21) we find two linearly independent $q$-hypergeometric solutions in $\mathbb{F}[\![x]\!]$:

$$y^{(1)}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k/2}}{(q;q)_k} x^k,$$

$$y^{(2)}(x) = \sum_{k=0}^{\infty} \frac{q^{k/2}}{(q;q)_k} x^k.$$

Both solutions are instances of a $q$-analogue of the exponential function $e^z$, namely $e_q(z)=\sum_{n=0}^{\infty} z^n/(q;q)_n = 1/(z;q)_\infty$ [8, Eq. (1.3.15)]. This classical product expansion allows an easy verification of the statement above.

We remark that Laurent series solutions can be handled in an analogous way. In that case, Theorem 2 still holds, provided that $n$ is allowed to be any integer.

### 5.3. Basic hypergeometric series

The algorithm qHyper allows us also to find power series solutions in $M = \mathbb{F}[[x]]$ whose coefficients form a $q$-hypergeometric sequence in $S(\mathbb{F})$. If $y(x) = \sum_{j=0}^{\infty} y_j x^j$ is such a series then $y_{j+1} = r(q^j) y_j$ for some $r(x) \in \mathbb{F}(x)$ and for almost all $j \in \mathbb{N}$. These series are usually called *basic hypergeometric series* [8].

Let $Ly(x) = b(x)$ where $b(x) = \sum_{j=0}^{\infty} b_j x^j$. As in (5), we obtain

$$\sum_{j=\max\{l-d,0\}}^{l} \sum_{i=0}^{\rho} c_{i,l-j} y_j q^{ij} = b_l \quad \text{for } l \geqslant 0. \tag{22}$$

We separate the cases $0 \leqslant l < d$ and $l \geqslant d$. In the former case, (22) yields initial conditions

$$\sum_{j=0}^{l} y_j \sum_{i=0}^{\rho} c_{i,l-j} q^{ij} = b_l \quad \text{for } 0 \leqslant l < d, \tag{23}$$

while in the latter, substitutions $m = l - d$, $s = j - m$, and $X = q^m$ transform (22) into the *associated q-difference equation*

$$\sum_{s=0}^{d} y_{m+s} \sum_{i=0}^{\rho} c_{i,d-s} q^{is} X^i = b_{m+d} \quad \text{for } m \geqslant 0, \tag{24}$$

for the unknown sequence $(y_m)_{m=0}^{\infty}$.

If $b(x) = 0$ we use qHyper on (24). Among the obtained solutions, we select those which are defined for all $m \in \mathbb{N}$, and satisfy initial conditions (23). If $b(x) \neq 0$, we use the algorithm of Section 6 instead.

**Example 8.** Let us find basic hypergeometric solutions $y(x)$ of

$$q^2 x^2 \varepsilon^3 y + (1+q) x \varepsilon^2 y + (1-x) \varepsilon y - y = 0. \tag{25}$$

The associated equation (24) in this case is

$$(q^2 X - 1) y_{m+2} + (q^2(q+1) X^2 - qX) y_{m+1} + q^2 X^3 y_m = 0 \tag{26}$$

and qHyper finds two solutions:

```
In[8]:= qHyper[(q^2 X - 1) y[q^2 X]+(q^2 (1 + q) X^2 - q X) y[q X]+
                q^2 X^3 y[X] = = 0, y[X], Solutions -> All]
```

Out[8] = $\{-X, \frac{qX^2}{1-qX}\}$

Thus the general solution of (26) in $S(\mathbb{F})$ is

$$y_m = C(-1)^m q^{\binom{m}{2}} + D q^{m^2} / (q;q)_m$$

where $C$ and $D$ are arbitrary constants. Eqs. (23) imply that $C=0$. Hence $y(x)=\sum_{m=0}^{\infty} q^{m^2}x^m/(q;q)_m$ is a basic hypergeometric solution of (25).

Note that running qHyper on Eq. (25) itself we obtain $r(x)=-1/x$ which does not belong to $\mathbb{F}[\![x]\!]$. Hence (25) has no $q$-hypergeometric solution in $\mathbb{F}[\![x]\!]$.

## 6. Nonhomogeneous equations

Consider the problem of finding $q$-hypergeometric solutions $y\in M$ of the nonhomogeneous equation $Ly=b$ where $b\in M\backslash\{0\}$. Let $\varepsilon y=ry$ where $r\in\mathbb{F}(x)$. Then $Ly=fy$ where $f=\sum_{i=0}^{\rho} p_i\prod_{j=0}^{i-1}\varepsilon^j r$ is a rational function of $x$. This simple fact has two important consequences:

1. $b=fy$ is $q$-hypergeometric,
2. $y=b/f$ is a rational multiple of $b$.

Let $\varepsilon b=sb$ where $s\in\mathbb{F}(x)$ is given. We look for $y$ in the form $y=fb$ where $f\in\mathbb{F}(x)$ is an unknown rational function. Substituting this into $Ly=b$ gives

$$\sum_{i=0}^{\rho} p_i\left(\prod_{j=0}^{i-1}\varepsilon^j s\right)\varepsilon^i f=1.$$

Now rational solutions of this equation can be found using the algorithm given in [1].

**Example 9.** Let $y(x)\in\mathbb{F}[\![x]\!]$ satisfy

$$\varepsilon^2 y(x)-(1-qx)\varepsilon y(x)+qy(x)=b(x) \tag{27}$$

where

$$b(x)=\sum_{i=0}^{\infty}\frac{x^i}{(q;q)_i}.$$

Here $b(qx)=(1-x)b(x)$, as can easily be verified. Thus $s(x)=1-x$ and $y(x)=f(x)b(x)$ where $f(x)$ satisfies

$$(1-qx)(1-x)\varepsilon^2 f(x)-(1-qx)(1-x)\varepsilon f(x)+qf(x)=1$$

with rational solution $f(x)=1/q$. Hence $y(x)=b(x)/q$ is a $q$-hypergeometric solution of (27) in $\mathbb{F}[\![x]\!]$. As in Example 7 the result is easily verified by using the product representation of the $q$-exponential function $e_q(x)=b(x)$.

We can also look for basic hypergeometric solutions of (27). The associated nonhomogeneous equation for $y_m$ is by (26)

$$(qX^2-X+1)y_{m+1}+Xy_m=\frac{1}{q(q;q)_{m+1}},$$

and we find its $q$-hypergeometric solutions in $S(\mathbb{F})$. Here $s(X)=(q(q;q)_{m+1})/(q(q;q)_{m+2})=1/(1-q^{m+2})=1/(1-q^2X)$, and the equation for $f(X)$

$$\frac{1-X+qX^2}{1-q^2X}\varepsilon f(X)+Xf(X)=1$$

is satisfied by the rational function $f(X)=1-qX$. Thus $y_m=(1-qX)/(q(q;q)_{m+1})=1/(q(q;q)_m)$, and we find the same solution $y(x)=b(x)/q$ as before.

The algorithm for finding $q$-hypergeometric solutions of nonhomogeneous equations can also be used to solve the problem of *indefinite q-hypergeometric summation*: Given a $q$-hypergeometric sequence $b=(b_n)_{n=0}^{\infty}$ over $\mathbb{F}$, decide if the telescoping recurrence $y_{n+1}-y_n=b_n$ has a $q$-hypergeometric sequence solution $(y_n)_{n=0}^{\infty}$. If so, the indefinite sum of $b$ can be expressed in closed form, namely, $\sum_{j=0}^{n-1} b_j=y_n-y_0$. Since we are interested in $q$-hypergeometric solutions, we rewrite the telescoping recurrence as $\varepsilon y-y=b$ and use the algorithm of this section to find solutions $y\in S(\mathbb{F})$.

**Example 10.** To evaluate the sum $\sum_{j=0}^{n-1} b_j$ where $b_n=q^n(q;q)_n$, we look for $q$-hypergeometric solutions $y$ of the nonhomogeneous equation

$$\varepsilon y-y=b. \tag{28}$$

Here $s=\varepsilon b/b=q(1-qx)$, and $f$ satisfies the equation

$$q(1-qx)\varepsilon f-f=1$$

which has a unique rational solution $f=-1/(qx)$. Hence $y_n=-b_n/q^{n+1}$ satisfies (28), and $\sum_{j=0}^{n-1} b_j=y_n-y_0=(1-(q;q)_n)/q$.

## 7. Applications to $q$-hypergeometric summation

It is well known that Zeilberger's 'fast' algorithm [18], or the more general WZ-machinery described in [17], does not always deliver a representing difference equation of minimal order for the given sum. For instance, as pointed out in [11] one can prove that the Zeilberger recurrence for the sum expression on the left-hand side of

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{dk}{n}=(-d)^n \tag{29}$$

for a fixed positive integer $d$ is of order $d-1$ instead of order 1 according to its hypergeometric evaluation. Here one applies algorithm `Hyper` of [13] to the recurrence in order to find its hypergeometric solutions. In this section a brief discussion of applications of `qHyper` in connection with definite $q$-hypergeometric summation is given.

### 7.1. A new q-summation identity

Let $d$ and $n$ be positive integers, then

$$\sum_{k=0}^{n}(-1)^k q^{d\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_d \begin{bmatrix} dk \\ n \end{bmatrix} = (-1)^n q^{(d-1)\binom{n}{2}} \frac{[n]_d!}{[n]!}[d]^n, \tag{30}$$

which for $q \to 1$ specializes to identity (29). The proof of (30), an identity we could not find in the literature, is elementary by using

$$\sum_{k=0}^{n}(-1)^k q^{d\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_d (q^{dk})^l = \begin{cases} 0, & \text{if } 0 \leqslant l \leqslant n-1, \\ q^{d\binom{n}{2}}(q^d;q^d)_n, & \text{if } l = n \end{cases}$$

which follows immediately from (4), and by observing that $[{dk \atop n}]$ is $1/(q;q)_n$ times a polynomial in $q^{dk}$ of degree $n$.

Denote by SUM($n$) the sum expression on the left-hand side of (30). Applying the $q$-analogue qZeil of Zeilberger's algorithm implemented in Mathematica by Riese [14] one obtains, for instance, for $d=3$ a recursion of order 2. This means, as in the case $q \to 1$, that the minimal order is missed by 1:

```
In[9]:= qZeil[(-1)^kq^(3 Binomial[n-k,2])qBinomial[n,k,q^3]*
         qBinomial[3k,n,q], {k,-Infinity,Infinity}, n, 2]

Out[9]=SUM[n]==(q^(-5 + 4n)(-1 + q^(-1 + n))(1 + q^(-1 + n)+ q^(-2 + 2n))
        >(1 + q^n+ q^2n)SUM[-2 + n]) / ((1 + q^n)(1 - q^(-1 + 2n))) +
        >(q^(-2 + 2n)(1 + q^n+ q^2n)(-1 - q + q^(-1 + 2n)+ q^(-1 + 3n))
        >SUM[-1 + n]) / ((1 + q^n) (1 - q^(-1 + 2n)))
```

The algorithm qHyper now finds the $q$-hypergeometric solution of this recurrence (after replacing $q^n$ by $x$, and `SUM[n+k]` by `Y[q^k x]`):

```
In[10]:=% /.{SUM[n + k_.] -> Y[q^k x],
            q^(a_. n + b_.) -> x^a q^b};

In[11]:= qHyper[%, Y[x]]

Warning: irreducible factors of degree >1 in leading
coefficient; some solutions may not be found
```

```
Warning: irreducible factors of degree >1 in trailing
coefficient; some solutions may not be found

Out[11]={-(x^2 (1 + qx + q^2 x^2))}
```

This means that for one solution $y_n = Y(q^n)$, we have

$$\frac{y_{n+1}}{y_n} = -q^{2n}(1+q^{n+1}+q^{2n+2}) = -q^{2n}\frac{1-q^{3n+3}}{1-q^{n+1}}.$$

From this information together with the initial values one computes as the $q$-hypergeometric evaluation of the sum the expression on the right-hand side of (30) for $d=3$.

### 7.2. Increased orders

In most instances the orders of the Zeilberger recurrences for $q$-analogues and their $q \to 1$ specializations are the same. This also applies when Zeilberger's algorithm fails to deliver the minimal order, for instance, as in the previous example.

But this is not true in general. Running the $q$-version of Zeilberger's algorithm on certain $q$-analogues of classical hypergeometric summation and transformation formulas one observes that the orders increase in comparison to the recurrence orders obtained in the case $q \to 1$. A relatively simple but important example is the following identity due to Rogers,

$$\sum_{k=-n}^{n} (-1)^k q^{k(3k-1)/2} \begin{bmatrix} 2n \\ n+k \end{bmatrix} = \frac{[2n]!}{[n]!}(1-q)^n. \tag{31}$$

Besides playing a key role in proving identities of the Rogers–Ramanujan type [5], in the limit as $n \to \infty$ it yields the celebrated Eulerian pentagonal number theorem. Despite its fundamental importance with respect to $q$-hypergeometric identities, for $q \to 1$ it specializes to a trivial instance of the binomial theorem,

$$\sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k} = (-1)^n \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} = \delta_{n,0},$$

which could be treated also with Gosper's algorithm.

Applying qZeil to the left-hand side of (31) surprisingly results in a $q$-difference equation of order 3 (!),

```
In[12]:= qZeil[(-1)^k q^(k(3k-1)/2) qBinomial[2n,n+k,q],
         {k,-Infinity,Infinity}, n, 3]
```

```
Out[12]= SUM[n] == ((-q + q^n) (q + q^n) (-q^2+ q^n) (q^2+ q^n)
        > (-q^3+ q^2n) (-q^5+ q^2n) SUM[-3 + n]) / q^11+ (q - q^n)
        > (q + q^n) (-q^3+ q^2n) (q^2+ q^3+ q^4+ q^2n) SUM[-2 + n] / q^6 +
        > (q^2 +q^3 +q^4 - q^3n - q^2+2n - q^1 + 3n) SUM[-1 + n] / q^2
```

Applying qHyper we find one $q$-hypergeometric solution of this recurrence (after replacing $q^n$ by $x$, and SUM[n+k] by Y[q^k x]):

```
In[13]:=% /. {SUM[n + a_.] -> Y[q^a x],
              q^(a_. n + b_.) -> x^a q^b};

In[14]:= qHyper[%, Y[x]]

Warning: irreducible factors of degree >1 in trailing
coefficient; some solutions may not be found

 Out[14] = {(1 + q x) (1 - q x^2)}
```

This means that for one solution $y_n = Y(q^n)$, we have

$$\frac{y_{n+1}}{y_n} = (1+q^{n+1})(1-q^{2n+1}).$$

From this together with the initial values the right-hand side evaluation of (31) is easily computed.

Another way to treat the increase of recurrence orders in the $q$-case was found by Paule [10]. His method of 'summing the even part', or variations of it, consists in rewriting the given sum by exploiting symmetries of the summand. After this preprocessing the $q$-version of Zeilberger's algorithm delivers the recursion of minimal order. We give a brief illustrating example. Let $f(n,k)$ denote the summand of the Rogers identity (31). Since $\sum_k f(n,k) = \sum_k f(n,-k)$, it follows that

$$\sum_k f(n,k) = \frac{1}{2}\sum_k (f(n,k) + f(n,-k)) = \frac{1}{2}\sum_k (1+q^k) f(n,k).$$

The extra factor $1 + q^k$ increases the chance that the $q$-Zeilberger algorithm finds a recurrence of lower order. Indeed, now one gets by applying qZeil the minimal recurrence of order 1.

```
In[15]:=qZeil[(-1)^k (1+q^k)/2 q^(k(3k-1)/2) qBinomial[2n,n+k,q],
        {k,-Infinity,Infinity}, n, 1]

Out[15]=SUM[n] == (1 + q^n) (1 - q^(-1 + 2 n)) SUM[-1 + n]
```

Paule's method is of special importance with respect to the theory of $q$-WZ pairs [16]. There are various applications, [10] or [14], where 'summing the even part' enables one to manufacture the dual or companion identities. We only mention two examples for which the $q$-Zeilberger algorithm delivers a recurrence of increased order 3, namely the Rogers identity (31) and the $q$-analogue of Dixon's formula,

$$\sum_k (-1)^k q^{k(3k-1)/2} \begin{bmatrix} n+b \\ n+k \end{bmatrix} \begin{bmatrix} n+c \\ c+k \end{bmatrix} \begin{bmatrix} b+c \\ b+k \end{bmatrix} = \frac{[n+b+c]!}{[n]![b]![c]!}.$$

Despite the fact that Paule's method applies to most 'increased order' cases of definite $q$-hypergeometric sums one finds in standard literature (and also to nontrivial $q \to 1$ examples as recently found by Zeilberger and Petkovšek), the symmetry-preprocessing up to now algorithmically has not been fully understood. For instance, it is not at all obvious how one could apply the method in the case of identity (30). Therefore the algorithm `qHyper` is currently the only tool that constructively decides about existence of $q$-hypergeometric evaluation of a definite $q$-hypergeometric sum for which the $q$-Zeilberger algorithm delivers a recurrence of order greater than 1.

## References


[1] S.A. Abramov, Rational solutions of linear difference and $q$-difference equations with polynomial coefficients, Programming and Comput. Software 6 (1995) 273–278; transl. from Programmirovanie 6 (1995) 3–11.

[2] S.A. Abramov, M. Bronstein, M. Petkovšek, On polynomial solutions of linear operator equations, in: T. Levelt (Ed.), Proc. ISSAC '95, ACM Press, New York, 1995, pp. 290–296.

[3] S.A. Abramov, M. Petkovšek, Finding all $q$-hypergeometric solutions of $q$-difference equations, in: B. Leclerc, J.-Y. Thibon (Eds.), Proc. FPSAC '95, Univ. de Marne-la-Vallée, Noisy-le-Grand, 1995, pp. 1–10.

[4] G.E. Andrews, The Theory of Partitions, Addison-Wesley, Reading, MA, 1976.

[5] G.E. Andrews, $q$-series: their development and application in analysis, number theory, combinatorics, physics, and computer algebra, CBMS Regional Conference Series, No. 66, AMS, Providence, RI, 1986.

[6] M. Bronstein, M. Petkovšek, An introduction to pseudo-linear algebra, Theoret. Comput. Sci. 157 (1996) 3–33.

[7] R.M. Cohn, Difference Algebra, Interscience Publishers, New York, 1965.

[8] G. Gasper, M. Rahman, Basic Hypergeometric Series, Cambridge Univ. Press, Cambridge, 1990.

[9] T.H. Koornwinder, On Zeilberger's algorithm and its $q$-analogue: a rigorous description, J. Comput. Appl. Math. 48 (1993) 91–111.

[10] P. Paule, Short and easy computer proofs of the Rogers–Ramanujan identities and of identities of similar type, Electronic J. Combin. 1 (1994) # R10.

[11] P. Paule, M. Schorn, A Mathematica version of Zeilberger's algorithm for proving binomial coefficient identities, J. Symbolic Comput. 20 (1995) 673–698.

[12] P. Paule, V. Strehl, Symbolic summation — some recent developments, in: Computer Algebra in Science and Engineering — Algorithms, Systems, and Applications, J. Fleischer, J. Grabmeier, F. Hehl, W. Küchlin (Eds.), World Scientific, Singapore, 1995, pp. 138–162.

[13] M. Petkovšek, Hypergeometric solutions of linear recurrences with polynomial coefficients, J. Symbolic Comput. 14 (1992) 243–264.

[14] A. Riese, A Mathematica $q$-analogue of Zeilberger's algorithm for proving $q$-hypergeometric identities, Diploma Thesis, University of Linz, Linz, 1995.

[15] R.P. Stanley, Differentiably finite power series, European J. Combin. 1 (1980) 175–188.

[16] H.S. Wilf, D. Zeilberger, Rational functions certify combinatorial identities, J. Amer. Math. Soc. 3 (1990) 147–158.

[17] H.S. Wilf, D. Zeilberger, An algorithmic proof theory for hypergeometric (ordinary and '$q$') multisum/integral identities, Invent. Math. 108 (1992) 575–633.

[18] D. Zeilberger, A fast algorithm for proving terminating hypergeometric identities, Discrete Math. 80 (1990) 207–211.