Diffusion of chemically reactive species in a porous medium over a stretching sheet

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Abstract

Solutions for a class of nonlinear second-order differential equations, arising in diffusion of chemically reactive species of a non-Newtonian fluid immersed in a porous medium over a stretching sheet, are obtained. Furthermore, using the Brouwer fixed point theorem, existence results are established. Moreover, the exact analytical solutions (for some special cases) are obtained. The results obtained for the diffusion characteristics reveal many interesting behaviors that warrant further study of the effects of reaction rate on the transfer of chemically reactive species.

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1. Introduction

The transport of heat, mass and momentum in laminar boundary layers on moving (stretching or inextensible) surfaces is a very important process in many engineering applications, such as in polymer processing (see [2–5] and the references therein) and electrochemistry (see [6,7]). Furthermore, materials manufactured by extrusion processes

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and heat-treated materials traveling between a feed roll and a wind-up roll or on conveyor belts possess the characteristics of a moving continuous surface. In view of these applications, this transport problem has received the attention of many researchers—starting with the pioneering work of Sakiadis [8], and its extension to a stretching sheet by Crane [9]. To learn more about the origins and applications of this transport problem the interested reader could consult the works of Rajagopal et al. [10], Abel et al. [11], Mcleod and Rajagopal [12], Troy et al. [13], Andersson et al. [14], Chambre and Young [15], Siddappa and Abel [16], Crane [9] and the references therein.

In this work we are interested in the transport and diffusion of chemically reactive species over a stretching sheet in a non-Newtonian fluid (fluid of differential type, second grade).

The Cauchy stress $\mathbf{T}$ in an incompressible homogeneous fluid of second grade has the form (see Ref. [17])

$$\mathbf{T} = -p\mathbf{I} + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2,$$

(1.1)

where

$$\mathbf{A}_1 = (\text{grad} \mathbf{v}) + (\text{grad} \mathbf{v})^T$$

and

$$\mathbf{A}_2 = d\mathbf{A}_1/dt + \mathbf{A}_1 (\text{grad} \mathbf{v}) + (\text{grad} \mathbf{v})^T \mathbf{A}_1.$$

(1.2)

In the above equations, the spherical stress $-p\mathbf{I}$ is due to the constraint of incompressibility, $\mu$ is the viscosity, $\alpha_1$ and $\alpha_2$ are material moduli and usually referred to as the normal stress moduli, $d/dt$ denotes the material time derivative, $\mathbf{v}$ denotes the velocity field, and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first two Rivlin–Ericksen tensors.

The above model has been studied in great detail. The sign of the coefficient $\alpha_1$ has been a subject of much controversy, and a thorough discussion of the issues involved can be found in the recent critical review of Dunn and Rajagopal [18]. We shall not get into a discussion of these issues here. In this study we shall assume that Eq. (1.1) models the fluid exactly. If the fluid modeled by Eq. (1.1) is to be compatible with thermodynamics, in the sense that all motions of the fluid meet the Clausius–Duhem inequality and the assumption that the specific Helmholtz free energy of the fluid is a minimum when the fluid is locally at rest (see for details Dunn and Fosdick [19], and Fosdick and Rajagopal [20]), then

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 = 0.$$

(1.3)

In 1984, Rajagopal et al. [21] studied the Falkner–Shan flows of a non-Newtonian fluid. Later, Vajravelu and Rollins [23] studied the flow and heat transfer in an incompressible second-order fluid due to stretching of a plane elastic surface. Vajravelu and Rollins examined the effects of viscous dissipational and internal heat generation or absorption in a viscoelastic boundary layer flow. Recently, Prasad et al. [24] obtained numerical solutions and analyzed the diffusion of chemically reactive species of a non-Newtonian fluid immersed in a porous medium over a stretching sheet, and brought out several interesting aspects of the problem. In Refs. [20,21], the sign for the material constant $\alpha_1$ (in Eq. (1.3)) was taken as negative; however, this is not compatible with the stability criteria (see Ref. [18]).
Hence, in this paper, we study the flow and mass transfer of a chemically reactive species of a viscoelastic fluid over a stretching sheet, using the proper sign for the material constant ($\alpha_1 \geq 0$). Furthermore, we analyze the salient features of the flow and mass transfer characteristics by obtaining exact, analytical, and numerical solutions with existence results for the resulting coupled nonlinear differential equations. The concentration $c$ of the reactive species is the solution to a nonlinear boundary value problem over the infinite domain $(0, +\infty)$. We introduce a change of variable which allows us to work in a bounded domain. The nonlinearity is not affected by this change of variable, though we end up with a degenerate nonlinear boundary value problem. A refined analysis allows us to overcome the singular nature of the resulting nonlinear boundary value problem. Using sophisticated tools of functional analysis we are able to show the existence of a solution. Our change of variable has a significant added benefit in computing numerical solutions. Indeed, as is well known, solving problems numerically in unbounded domains requires cutting the domains and the introduction of artificial boundary conditions. But since we transformed the problem into a boundary value problem on a bounded domain we are able to use the general method whose performance is well documented. The analytical estimates we establish in the proof of existence yield convergence results for the finite differences used here and guarantee the validity of our numerical solutions.

2. Flow analysis

Consider the flow of a second-order fluid obeying Eqs. (1.1)–(1.3) past a flat sheet coinciding with the plane $y = 0$, the flow being confined to $y > 0$. Two equal and opposite forces are applied along the $x$-axis so that the wall is stretched, keeping the origin fixed. The steady two-dimensional boundary layer equations for this fluid in usual notation are (for details see Vajravelu and Roper [25] and Prasad et al. [24])

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$  
$$\frac{u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + \lambda \left[ \frac{\partial}{\partial x} \left( u \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial u}{\partial y} \frac{\partial^2 v}{\partial y^2} + v \frac{\partial^3 u}{\partial y^3} \right] - \nu k_0 u,$$  

where $\nu = \mu / \rho$, $\lambda = \alpha_1 / \rho$ and $k_0$ is the permeability of the porous medium. In deriving these equations, it was assumed that the contribution due to the normal stress is of the same order of magnitude as that due to the shear stress (in addition to the usual boundary layer approximations). Thus both $\nu$ and $\lambda$ are $O(\delta^2)$, were $\delta$ is the boundary layer thickness.

The appropriate boundary conditions for the problem are

$$u = Bx, \quad v = 0 \quad \text{at} \quad y = 0, \quad B > 0,$$  
$$u \to 0, \quad \frac{\partial u}{\partial y} \to 0 \quad \text{at} \quad y \to \infty,$$  

where $\partial u / \partial y \to 0$ as $y \to \infty$ is the augmented condition. Equations (2.1) and (2.2) admit a self-similar solution of the form
where prime denotes differentiation with respect to \( \eta \). Clearly, \( u \) and \( v \) defined above satisfy the continuity equation (2.1). Substituting Eqs. (2.4) and (2.5) in Eq. (2.2) gives

\[
(f')^2 - ff'' = f''' + \lambda_1\left[2f'f''' - (f'')^2 - ff'''ight] - k_1 f',
\]

(2.6)

where \( \lambda_1 = \lambda B/v \) is the viscoelastic parameter and \( k_1 = v/k_0 B \) is the porosity parameter. The boundary conditions (2.3) become

\[
\begin{align*}
&f' = 1, \quad f = 0 \quad \text{at } \eta = 0, \\
&f' \to 0, \quad f'' \to 0 \quad \text{as } \eta \to \infty.
\end{align*}
\]

(2.7)

The exact solution for the differential equation (2.6) satisfying conditions (2.7) is

\[
f(\eta) = \left(1 - e^{-m\eta}\right)/m, \quad m = \sqrt{(1 + k_1)/(1 + \lambda_1)}.
\]

(2.8)

This gives the velocity components

\[
\begin{align*}
u &= Bxe^{-m\eta}, \\
v &= -(B/v)^{1/2}(1 - e^{-m\eta})/m.
\end{align*}
\]

(2.9)

For \( \lambda_1 = 0 \) and \( k_1 = 0 \), \( f(\eta) = 1 - e^{-\eta} \) is the unique solution of the problem. By a standard mathematical argument, it can be shown that the differential equation (2.6) subject to boundary conditions (2.7) fails to have a unique solution. That is, for \( k_1 = 0 \), when \( \lambda_1 \in (0, \infty) \) we get an exponential solution and when \( \lambda_1 \in (-1, 0) \) we obtain a solution containing an exponential, and sine and cosine terms (for details see Ref. [22, Eq. (25)]. However, for \( \lambda_1 > 0 \),

\[
f(\eta) = \left(1 - e^{-m\eta}\right)/m, \quad m = \sqrt{(1 + k_1)/(1 + \lambda_1)}
\]

(2.10)

is the only solution which is physically meaningful. Hence, in heat and mass transfer analyses we use this solution for the function \( f \). From Eq. (2.10), we can say that \( f' \) is an increasing function of the viscoelastic parameter \( \lambda_1 \) and a decreasing function of the porosity parameter \( k_1 \).

3. Heat and mass transfer analyses

By knowing the mathematical equivalence of the mass concentration boundary layer problem with the thermal boundary layer analogue, results obtained for mass transfer characteristics can be carried directly to the heat transfer characteristics by replacing Schmidt number with the Prandtl number. Hence, for brevity we do not present the heat transfer results here. For the flow problem discussed in Section 2, the concentration field \( c(x, y) \) is governed by the boundary layer diffusion equation (see Ref. [12]) reduces to

\[
\begin{align*}
u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} &= D \frac{\partial^2 c}{\partial y^2} - k_n c^n,
\end{align*}
\]

(3.1)
where \( c \) being the concentration of the species of the fluid, \( D \) is the diffusion coefficient of the diffusing species in the fluid and \( k_n \) denotes the reaction rate constant of the \( n \)th-order homogeneous and irreversible reaction. Since the concentration of the reactant is maintained at a prescribed value \( c_w \) at the sheet and is assumed to vanish far away from the sheet, the relevant boundary conditions for the concentration equation (3.1) become

\[
\begin{align*}
  c(x, y) &= c_w \quad \text{at } y = 0, \\
  c(x, y) &\to 0 \quad \text{as } y \to \infty.
\end{align*}
\]

(3.2)

Knowing the mathematical nature of the physical problem the similarity solution can be obtained for the mass concentration. For that end, we introduce the transformation

\[
c = c_w \phi(\eta),
\]

(3.3)

where \( \phi(\eta) \) is the dimensionless concentration field. The nonlinear partial differential equation (3.1) can now be transformed to

\[
\phi'' + \alpha \phi' = \alpha \beta \phi^n,
\]

(3.4)

where a prime denotes differentiation with respect to \( \eta \) and \( \alpha = S_c = \nu / D \) is the Schmidt number and \( \beta = k_n c_w^{n-1} / B \) is the reaction rate parameter. The boundary conditions (3.2) become

\[
\begin{align*}
  \phi &= 1 \quad \text{at } \eta = 0, \\
  \phi &\to 0 \quad \text{as } \eta \to \infty.
\end{align*}
\]

(3.5)

Evidently, the concentration field \( \phi \) is coupled to the velocity field through the dimensionless stream function \( f \) in the nonlinear mass transfer equation (3.4). Furthermore, in the special case of reactive species of first order \( (n = 1) \), the nonlinear term on the right-hand side of Eq. (3.4) becomes \( \alpha \beta \phi(\eta) \) and the present concentration boundary value problem becomes formally equivalent with the analogous thermal boundary layer problem in which the Prandtl number replaces the Schmidt number. The analytical solution of Eq. (3.4) for the first-order reaction \( (n = 1) \) with respect to the boundary conditions (3.5) can be written in the form of Kummer’s function:

\[
\phi(\eta) = \frac{e^{-m(A+B)\eta} M(A+B, 1+2B, -(Sc/m^2)e^{-m\eta})}{M(A+B, 1+2B, (-Sc/m^2))},
\]

where \( A = S_c / 2m^2, B = (4\beta m^2 S_c + S_c^2)^{1/2} / 2m^2 \) and \( M \) is the Kummer’s function (see Ref. [24]). The exact analytical solution of the complete concentration equation (3.4) for the case when \( \beta = 0 \) is expressed in terms of incomplete gamma function as noted in Andersson et al. [14]. In addition, if \( Sc = m^2 \), the exact solution is given by \( \phi(\eta) = e(1 - e^{-e^{-m\eta}})/(e - 1) \). It is worth mentioning that the exact analytical solution of the concentration equation for the general case \( n > 1 \) and \( \beta = 1/(n - 1), Sc = m^2/(n - 1) \), is given by \( \phi(\eta) = e^{-m\eta/(n-1)} \). For practical purposes the parameters \( m, n, \alpha \) are positive and \( \beta \) can be a real number. Due to the nonlinearity in Eq. (3.4) we cannot obtain exact solution for all values of the parameters \( m, n, \alpha \) and \( \beta \). Hence in Section 4 we shall prove the existence results; and in Section 5 we shall present the numerical solution (through graphs) and the discussion of the results.
4. Existence results

Using (2.10) Eq. (3.4) can be written as

\[ u'' + \alpha \left( \frac{1 - e^{-m\eta}}{m} \right) u' = \alpha \beta u^n, \] (4.1)

where \( u \) is a function of \( \eta \). Introducing the new variable \( y = m\eta \) we then have that

\[ \frac{du}{d\eta} = \frac{du}{dy} \cdot \frac{dy}{d\eta} = \frac{du}{dy} \cdot m \]

and

\[ \frac{d^2u}{d\eta^2} = \frac{d^2u}{dy^2} m^2 \]

in terms of the variable \( y \) Eq. (4.1) then becomes

\[ m^2 \frac{d^2u(y)}{dy^2} + \alpha \left( 1 - e^{-y} \right) \frac{du}{dy} = \alpha \beta u^n \]

which can be rewritten as

\[ \frac{d^2u(y)}{dy^2} + \left( \frac{\alpha}{m^2} \right) \left( 1 - e^{-y} \right) \frac{du}{dy} = \frac{\alpha \beta u^n}{m^2}. \]

Now set \( t = e^{-y} \); where \( \theta = \frac{\alpha}{m^2} \) then:

\[ \frac{du}{dy} = \frac{du}{dt} \cdot \frac{dt}{dy} = -\theta t \frac{du}{dt} \]

and

\[ \frac{d^2u}{dy^2} = \theta^2 \frac{du}{dt} + \theta^2 t^2 \frac{d^2u}{dt^2}. \] (4.2)

Equation (4.1) then becomes

\[ \theta^2 t^2 \frac{d^2u}{dt^2} + \theta^2 t \frac{du}{dt} - \frac{\theta \alpha}{m^2} t \frac{du}{dt} + \frac{\theta \alpha}{m^2} t^{1/\theta+1} \frac{du}{dt} = \frac{\alpha \beta u^n}{m^2}, \] (4.3)

where we used that \( e^{-y} = t^{1/\theta} \). Setting \( \theta = \alpha/m^2 \) Eq. (4.3) can be simplified to read

\[ \frac{d^2u}{dt^2} + \frac{1}{\theta t^{\theta-1}} \frac{du}{dt} = \frac{1}{\theta t^{2\theta-1}} \beta u^n \quad \forall t \in (0, 1). \] (4.4)

The boundary values problem (3.4), (3.5) is then equivalent to Eq. (4.4) with the following boundary conditions:

\[ u(0) = 0, \quad u(1) = 1. \] (4.5)

We will now concentrate on studying the boundary value problem (4.4), (4.5).

The differential equation (4.4) is degenerate. Indeed for \( t \) near zero (\( x \) near infinity) the term \( 1/t^2 \) becomes infinite. This represents a major difficulty. Because of this difficulty the existence of a solution to the problem (4.4), (4.5) cannot be deduced just from standard
existence results for boundary value problems (see [26]). We will overcome this difficulty by using the fact that \( u \) vanishes at zero. By careful estimates of \( u \) we are able to show that the singularity near zero is integrable. This will make it possible to show existence of a solution to this problem. Finally, we remark that if we considered Eq. (3.4) on a bounded interval \((0, R)\) this difficulty would have been completely avoided since in that case \( t \) would take only values in the interval \((e^{-R}, 1)\) and would therefore be bounded away from zero.

We will show that the problem (4.4), (4.5) has a solution. The proof is in two steps. In the first step we study a related linear boundary value problem. In the second step we will use the results of the first step and fixed point theorem to show that (4.4) has a solution \( u \).

Using integration factor \( \exp\left( \int t^{1/\theta-1} dt \right) = \exp(\theta t^{1/\theta}) \) we find that (4.4) can be written as

\[
(e^{\theta t^{1/\theta}} u_t)_t = \frac{1}{\theta} \beta e^{\theta t^{1/\theta}} u^n \cdot \frac{1}{t^2}.
\]  

(4.6)

When \( t \) is between 0 and 1 the function on \( g(t) \) is positive and is bounded from above it also is bounded from below by a strictly positive constant. Precisely

\[
1 = g(0) \leq g(t) \leq g(1) = e^\theta.
\]  

(4.7)

We start by considering the following related linear equation.

For \( f \) a given function of \( L^2 \)

\[
(g(t) u_t)_t = f \in L^2,
\]  

(4.8)

\[
u(0) = 0, \quad u(1) = 1.
\]  

(4.9)

We will be using the usual Sobolev spaces \( W^{k,p} \) with the standard notations such as can be found in [1,4].

**Lemma 1.** For any \( f \in L^2 \) the problem (4.8)–(4.9) has a unique solution \( u \in W^{1,2} \).

**Proof.** We will turn this problem into a problem with homogeneous boundary conditions. Set

\[
\Psi = (u - t).
\]

Note that \( \Psi(0) = \Psi(1) = 0 \). Then

\[
u_t = (\Psi_t + 1)
\]

and Eq. (4.8) becomes

\[
\left[ g(t)(\Psi_t + 1) \right]_t = (g(t)\Psi_t)_t + g_t = f
\]  

(4.10)

and

\[
(g(t)\Psi_t)_t = f - g_t,
\]  

(4.11)

\[
\Psi(0) = \Psi(1) = 0.
\]  

(4.12)

Since \( g \) is a continuous function it is in \( L^2 \) therefore its derivative will belong to \( W^{-1,2} \). Given that \( f \) is in \( L^2 \subset W^{-1,2} \) the right-hand side of (4.11) \( f - g_t \) belongs to \( W^{-1,2} \).
By Lax–Milgram lemma it easily follows that the problem (4.11)–(4.12) has a unique solution \( \Psi \in W^{1,2}_0(0, 1) \). We then deduce that problem (4.8)–(4.9) also has a unique solution \( u \). \( \square \)

**Lemma 2.** There are constants \( C_1 > 0, C_2 > 0 \) such that the solution of \( u \) of problem (4.8)–(4.9) satisfy

\[
\int_0^1 u_t^2 \, dt \leq C_1 \int_0^1 f^2 \, dt + C_2. \tag{4.13}
\]

**Proof.** We will make use of the following well-known Friedrichs’ inequality (see [1], for example), for \( u(0) = 0 \) we have that

\[
\int_0^1 u^2 \, dt \leq \frac{1}{2} \int_0^1 u_t^2 \, dt. \tag{4.14}
\]

Multiply (4.8) by \(-u(t) + t\) and integrate the left-hand side by parts over the interval \((0, 1)\), we get

\[
\int_0^1 g(t)u_t(u_t - 1) \, dt = -\int_0^1 f(u - t) \, dt. \tag{4.15}
\]

Hence,

\[
\int_0^1 g(t)u_t^2 \, dt = -\int_0^1 f(u - t) \, dt + \int_0^1 g(t)u_t \, dt. \tag{4.16}
\]

Using Cauchy–Schwarz inequality on the left-hand side we get

\[
\left| -\int_0^1 f(u - t) \, dt \right| \leq \int_0^1 f u \, dt + \int_0^1 (f \cdot t) \, dt
\]

\[
\leq \left( \int_0^1 f^2 \, dt \right)^{1/2} \left( \int_0^1 u^2 \, dt \right)^{1/2} + \left( \int_0^1 f^2 \, dt \right)^{1/2} \left( \int_0^1 t^2 \, dt \right)^{1/2}
\]

\[
\leq \frac{1}{2} \epsilon \int_0^1 f^2 \, dt + \frac{\epsilon}{2} \int_0^1 u^2 \, dt + \frac{1}{2} \int_0^1 f^2 \, dt + \frac{1}{2} \int_0^1 t^2 \, dt, \tag{4.17}
\]

where \( \epsilon \) is a positive number.
Also by Cauchy–Schwarz and an elementary inequality we have
\[
\left| \int_0^1 g(t)u_t \, dt \right| \leq \left( \int_0^1 g(t) \, dt \right)^{1/2} \left( \int_0^1 g(t)u_t^2 \, dt \right)^{1/2} \leq \frac{1}{2} \int_0^1 g(t) \, dt + \frac{1}{2} \int_0^1 g(t)u_t^2 \, dt. \tag{4.18}
\]

Combining all of these inequalities with (4.16) we find that
\[
\int_0^1 g(t)u_t^2 \, dt \leq \frac{1}{2\epsilon} \int_0^1 f^2 \, dt + \frac{\epsilon}{2} \int_0^1 u^2 \, dt + \frac{1}{2} \int_0^1 f^2 \, dt + \frac{1}{2} \int_0^1 t^2 \, dt + \frac{1}{2} \int_0^1 g(t) \, dt + \frac{1}{6}. \tag{4.19}
\]

Hence,
\[
\frac{1}{2} \int_0^1 g(t)u_t^2 \, dt \leq \left( \frac{1}{2\epsilon} + \frac{1}{2} \right) \int_0^1 f^2 \, dt + \frac{\epsilon}{2} \int_0^1 u^2 \, dt + \frac{1}{2} \int_0^1 g(t) \, dt + \frac{1}{6}. \tag{4.20}
\]

Using that \(1 \leq g(t)\) we then have that
\[
\frac{1}{2} \int_0^1 u_t^2 \, dt \leq \left( \frac{1}{2\epsilon} + \frac{1}{2} \right) \int_0^1 f^2 \, dt + \frac{\epsilon}{2} \int_0^1 u^2 \, dt + \frac{1}{2} \int_0^1 g(t) \, dt + \frac{1}{6}. \tag{4.21}
\]

Setting \(\epsilon = 1/2\) and using (4.14) it follows that
\[
\frac{1}{4} \int_0^1 u_t^2 \, dt \leq \frac{3}{2} \int_0^1 f^2 \, dt + \frac{1}{2} \int_0^1 g(t) \, dt + \frac{1}{6}. \tag{4.22}
\]

Next we will prove some regularity results for \(u\). We will be using the usual Hölder spaces with the usual notation such as can be found in [27, p. 52] or [4, p. 19], for example. Let \(\sigma = \min\{\theta, 1/2\}\).

**Lemma 3.** There are constants \(C_4 > 0, C_5 > 0\) such that for any \(f \in L^2\) the solution \(u\) of problem (4.8)–(4.9) satisfy \(u \in C^{1,\sigma}(0, 1)\) and its norm satisfies the estimate
\[
\|u\|_{C^{1,\sigma}} \leq C_4 \|f\|_{L^2} + C_5. \tag{4.23}
\]

Set
\[
v(t) = \int_0^t g(s)u'(s) \, ds. \tag{4.24}
\]
clearly \( v'(t) = g(t)u'(t) \in L^2(0, 1) \) (recall that \( g \) is continuous on \([0, 1]\) and that \( u' \in L^2 \)). In view of the fact that \( v(0) = 0 \) and Friedrichs’ inequality (see (4.14)) it follows that \( v \in W^{1,2} \). In view of (4.8) \( v'' = f \in L^2 \), therefore \( v \in W^{2,2} \). This yields that \( v' \in W^{1,2} \) and it follows from Sobolev embedding theorems that \( v' \) is in the Hölder space \( C^{0,1/2}([0, 1]) \). Furthermore, in view of (4.8) and (4.13) and \( v'' = f \in L^2 \) we have that the norm of \( v \) in \( W^{2,2} \) is bounded by \( C_3(\|f\|_{L^2} + 1) \), where \( C_3 \) is a positive constant. From the Sobolev embedding theorem it follows that the norm of \( v' \in C_0^{1/2}([0, 1]) \). We then deduce that

\[
u'(t) = \frac{1}{g(t)} v'(t) \in C^{0,\sigma}(0, 1), \tag{4.25}\]

where we used that \( \frac{1}{g(t)} \in C^{0,\theta} \) this is an elementary result which can be checked directly.

The estimate (4.23) easily follows from the arguments above and (4.14).

Now we will start the second step in our proof of the existence of a solution to the problem (4.4), (4.5).

We are interested in positive solutions for which (4.6) is equivalent to

\[
d^2u + t^{1/\theta - 1} \frac{du}{dt} = \frac{1}{t^2} \beta |u|^n \text{sign}(u). \tag{4.26}\]

We intend to use a fixed point technique to prove the existence of a solution. For this purpose we introduce the following notations:

\[E_1 = \{ u \in C^0([0, 1]), \ u' \in C^0([0, 1]), \ u(0) = 0 \}.\]

We will use the \( C^1 \) norm in \( E_1 \), i.e.,

\[
\|u\|_E_1 = \sup_{t \in [0,1]} |u(t)| + \sup_{t \in [0,1]} |u'(t)|. \tag{4.27}
\]

We set \( \gamma = \beta/\theta \) and consider the following linear BVP:

\[
d^2u + t^{1/\theta - 1} \frac{du}{dt} = \frac{\gamma |v|^n}{t^2} \text{sign}(v), \tag{4.28}\]

\[
u(0) = 0, \quad u(1) = 1, \tag{4.29}\]

where \( v \in E_1 \). We will prove that for a given \( v \in E_1 \) the BVP (4.28), (4.29) has a unique solution \( u \in E_1 \).

This will then establish the existence of a mapping \( T \) from \( E_1 \) to \( E_1 \).

We will show also that the mapping \( v \to T(v) = u \) is compact from \( E_1 \) to \( E_1 \). We start with the following lemma.

**Lemma 4.** For \( n > 3/2 \) and \( v \in E_1 \), \( |v|^n/t^2 \) is in \( L^2(0, 1) \) and

\[
\left\| \frac{|v|^n}{t^2} \right\|_{L^2(0,1)} \leq c \|v\|_{E_1}^n,
\]

where \( c \) is independent of \( v \).
Proof. For $\forall v \in E_1$

$$|v(t) - v(0)| \leq c_1 t,$$

where $c_1$ is the $C^0$ norm of $v'$ which is bounded by the norm of $v$ in $E_1$. Since $v(0) = 0$ we then have that

$$|v(t)| \leq c_1 t,$$  \hspace{1cm} (4.30)

Therefore, there exists $c > 0$ independent of $v$ such that

$$|v(t)|^n \leq c_1^n \|v\|^n_{E_1},$$

It then follows that

$$\frac{|v(t)|^n}{t^2} \leq c_1^n \|v\|^n_{E_1} t^{n-2},$$

now $(t^{n-2}) \in L^2$ as long as $n > 3/2$. □

Lemma 5. For any $v$ in $E_1$ the BVP (4.28), (4.29) has a unique solution $u \in C^{1,\sigma} \cap E_1$.

Proof. It easily follows from Lemma 4 that if $v \in E_1$ then $\frac{\gamma |v|^n}{t^2} \text{sign}(v) \in L^2$. The existence, uniqueness and regularity then follow from Lemmas 1 and 3. □

Lemma 6. $T$ is a compact operator from $E_1$ to $E_1$.

Proof. Let $v_1, v_2$ be two functions in $E_1$ and $w = T(v_1) - T(v_2)$. Then $w$ is a solution to the boundary value problem

$$
\begin{align*}
(g(t)w)_{t} &= \frac{\gamma |v_1|^n}{t^2} \text{sign}(v_1) - \frac{\gamma |v_2|^n}{t^2} \text{sign}(v_2) \in L^2, \\
w(0) &= w(1) = 0.
\end{align*}
$$

(4.31)
(4.32)

From classical regularity results for elliptic equations (see [27], for example), we have that $w \in C^{1,\alpha}$ for some positive $\alpha$ and that

$$\|w\|_{C^{1,\alpha}} \leq C \left\| \frac{\gamma |v_1|^n}{t^2} \text{sign}(v_1) - \frac{\gamma |v_2|^n}{t^2} \text{sign}(v_2) \right\|_{L^2}. \hspace{1cm} (4.33)$$

Using the elementary inequality

$$|a^n \text{sign}(a) - b^n \text{sign}(b)| \leq C |a - b| (|a|^{n-1} + |b|^{n-1}), \hspace{1cm} (4.34)$$

where $n > 1$ and $C$ is a positive constant independent of $a$ and $b$ we get from (4.33)

$$\|w\|_{C^{1,\alpha}} \leq C_1 \|v_1 - v_2\|_{C^1} \left\| \frac{n-1}{2} \|v_1\|_{C^1}^{n-1} + \|v_2\|_{C^1}^{n-1} \right\| t^{-2+n} \|L^2. \hspace{1cm} (4.35)$$

This proves that $T$ is a continuous operator from $E_1$ to $C^{1,\alpha}$. The compactness follows from the fact that for $\alpha > 0$, the embedding from $C^{1,\alpha}$ into $C^1$ is compact. □
To prove the existence of a solution to the problem (3.4), (3.5) we will use a fixed point theorem which generalizes Brouwer fixed point theorem. For the convenience of the reader we will state the fixed point theorem below.

**Theorem 1.** Let $T$ be a compact mapping of a Banach space $B$ into itself, and suppose there exists a constant $M$ such that

$$
\|X\|_B \leq M
$$

(4.36)

for all $X \in B$ and $\sigma \in [0, 1]$ satisfying $X = \sigma TX$. Then $T$ has a fixed point in $B$.

**Proof.** See [27, p. 280].

We will prove some facts about solutions of the equation $u = \sigma Tu$.

**Lemma 7.** Let $u \in E_1$ be such that $u = \sigma Tu$ where $\sigma \in (0, 1)$. Then $0 \leq u \leq 1$.

**Proof.** From the definition of $T$ we have that if $u = \sigma Tu$ then $u$ satisfy the differential equation

$$
\frac{d^2u}{dt^2} + t^{1/\theta - 1} \frac{du}{dt} = \sigma \frac{1}{\theta t^2} \beta |u|^n \text{sign}(u).
$$

(4.37)

Assume that $u$ takes a negative minimum in the interval $[0, 1]$. Since $u(0) = 0$ and $u(1) = 1$ this minimum will be achieved in the interior at $x_0 \in (0, 1)$ consequently

$$
u'(x_0) = 0, \quad u''(x_0) \geq 0,$$

and $|u(x_0)|^n \text{sign} u(x_0) < 0$. This contradicts the differential equation (4.37). Therefore, $u \geq 0$ in $(0, 1)$.

Assume that $u$ achieves a positive maximum at a point $x_1 \in (0, 1)$. Then $u'(x_1) = 0$, $u''(x_1) \leq 0$ and $|u|^n \text{sign} u > 0$ which contradicts the differential equation (4.37).

Using that $u(0) = 0$ and $u(1) = 1$ we find that $0 \leq u \leq 1$.

Since $u \geq 0$ we can rewrite (4.37) as

$$
\frac{d^2u}{dt^2} + t^{1/\theta - 1} \frac{du}{dt} = \sigma \frac{1}{\theta t^2} \beta u^n.
$$

(4.38)

**Lemma 8.** Let $u \in E_1$ be such that $u = \sigma Tu$ where $\sigma \in (0, 1)$. Then $0 \leq u'(t) \leq e^{1-t}u'(1)$.

**Proof.** From $u = \sigma Tu$ it follows that $u$ solves the differential equation (4.38). We deduce from (4.6) that

$$
(e^{\theta t^{1/\theta}} u_t)_t = \sigma \frac{1}{\theta} \beta e^{\theta t^{1/\theta}} u_t^n \cdot \frac{1}{t^2} \geq 0.
$$

(4.39)

Therefore, $(e^{\theta t^{1/\theta}} u_t)$ is an increasing function for $t \in (0, 1)$. Hence $u'(0) \leq e^{\theta t^{1/\theta}} u_t(t) \leq e^\theta u_t(1)$. Now, since $u$ takes it minimum value at $t = 0$ it follows that $u'(0) \geq 0$.

□
Lemma 9. Let \( v \) be a solution to the problem
\[
v''(x) + x^{1/\theta - 1}v'(x) = f(x), \quad x \in (1/2, 1),
\]
\[
v(1/2) = a, \quad v(1) = 1.
\]
Assuming that \( a \in (0, 1) \) and that \( f \) is a continuous function and that \( |f(x)| \leq M \) for all \( x \in (1/2, 1) \). Then there exists a constant \( C > 0 \), independent of \( a \) and \( f \) such that the norm of \( v \) in \( W_{2,2} \) is bounded by \( C \), i.e., \( \|v\|_{2,2} \leq C \).

Proof. Let \( h(x) = 2(1-a)(x-1) + 1 \) and \( w = v - h \). Then we find that \( w \) solves the problem
\[
w'' + x^{1/\theta - 1}w' = f + 2(a-1)x^{1/\theta - 1}, \quad x \in (1/2, 1),
\]
\[
w(1/2) = w(1) = 0.
\]
The boundary value problem (4.42), (4.43) is an elliptic boundary value problem with coefficients that are \( C^1 \); it then follows from the usual regularity results (see [27] or [4, p. 21]) that its solution \( w \) satisfies
\[
\|w\|_{2,2} \leq c_1 \left\| f + 2(a-1)x^{1/\theta - 1} \right\|_{L^2},
\]
where \( c_1 \) is a constant independent of \( f \). The lemma then easily follows from the fact that \( |f(x)| \leq M \) for all \( x \in (1/2, 1) \).

Lemma 10. There exists a constant \( A > 0 \) such that any \( u \in E_1 \), and \( u = \sigma Tu \), where \( \sigma \in (0, 1) \), satisfy \( |u'(t)| \leq A \) for all \( t \in [0, 1] \).

Proof. Since \( u \) satisfies the boundary value problem
\[
\frac{d^2u}{dt^2} + t^{1/\theta - 1} \frac{du}{dt} = \sigma \frac{\gamma}{t^2} u'',
\]
\[
u(0) = 0, \quad u(1) = 1
\]
by Lemma 7 and \( 0 \leq u \leq 1 \) we then have that when \( u \) is restricted to the interval \( (1/2, 1) \) it satisfies all the assumptions of Lemma 9 with \( M = 4\gamma \sigma \). It then follows from Lemma 9 that
\[
\|u\|_{2,2} \leq C,
\]
where \( C \) is independent of \( u \). We then deduce from the Sobolev embedding theorem that there exists a positive constant \( C_7 \) independent of \( u \) such that
\[
\sup_{t \in (1/2,1)} |u'(t)| \leq C_7.
\]
Setting \( A = 3C_7 \) the estimate (4.48) is extended to all of \([0, 1]\) using Lemma 8.

Lemma 11. Let \( u \in E_1 \) be such that \( u = \sigma Tu \) where \( \sigma \in (0, 1) \). Then there exists a positive constant \( B \) independent of \( u \) such that the \( L^2(0, 1) \) norm of \( \frac{\sigma \gamma}{t^2} u'' \) is bounded by \( B \).
Proof. Since we have proved that \(|u'(t)| \leq A\) we have that \(0 \leq u(t) \leq A \cdot t\). Therefore,
\[
0 \leq \frac{\sigma \gamma}{r^2} u^n \leq \gamma A^n t^{n-2}.
\]
Hence
\[
\int_0^1 \left( \frac{\sigma \gamma}{r^2} u^n \right)^2 \, dt \leq \gamma^2 A^{2n} \int_0^1 t^{2(n-2)} \, dt \leq B,
\]
where \(B\) is independent of \(u\). □

Theorem 2. For any \(n > 3/2\) the boundary value problem (4.5), (4.4) has a solution \(u\) which satisfy the following:

(i) \(0 \leq u \leq 1\) and
(ii) \(u\) is an increasing function on the interval \((0, 1)\).

Proof. Let \(u \in E_1\) be such that \(u = \sigma Tu\) where \(\sigma \in (0, 1)\). From the definition of \(T\), \(u\) is a solution to the differential equation (4.45) and satisfy the boundary conditions (4.46). Therefore, by Lemma 3 the norm of \(u\) in \(E_1\) is bounded by the \(L^2\) norm of the left-hand side of (4.45) plus a constant. On the other hand, by the Lemma 11 the norm of the left-hand side of (4.45) is independent of \(u\). Therefore, the norm of \(u\) in \(E_1\) is uniformly bounded. This satisfies all the conditions of the fixed point theorem we stated above. Therefore, \(T\) has a fixed point \(u\) which is then a solution to the boundary value problem (4.4), (4.5). □

5. Numerical solution and discussion of the results

We now consider the boundary value problem (4.4) for the particular values \(m = \alpha = \theta = 1\) of the parameters. The boundary value problem
\[
\begin{align*}
\frac{d^2 u}{dt^2} + \frac{du}{dt} &= \frac{\gamma}{r^2} u^n, \\
u(0) &= 0, \quad u(1) = 1
\end{align*}
\]
is solved numerically for several sets of values of the parameters \(n\) and \(\gamma\). Some of the qualitatively interesting results are presented in Figs. 1–4.

In Fig. 1, for \(\gamma = 2\), we plotted the nondimensional concentration profiles \(u(t)\) for several values of \(n\). We also included in Fig. 1 the graph of the function \(u(t) = t\). In order to visualize the solutions better we also plotted in Fig. 2 the graphs of \(u(t) - t\) to see how \(u(t)\) differs from the function \(u(t) = t\). From Fig. 1 it is evident that the fluid concentration increases with an increase in the reaction-order parameter \(n\) when the reaction rate parameter \(\gamma\) is positive (\(\gamma = 2\)). This phenomenon is quite opposite when \(\gamma\) is negative—this is in conformity with the physical fact that the destructive chemical reaction takes place when \(\gamma > 0\) and generative chemical reaction takes place when \(\gamma < 0\).

In Fig. 3, for \(n = 8\), we plotted \(u(t)\) for several values of \(\gamma\), along with the plot for \(u(t) = t\). Also, we plotted in Fig. 4 (for better visualization) the graphs of \(u(t) - t\). From Fig. 3 it is evident that the fluid concentration decreases with an increase in the reaction rate parameter \(\gamma\). That is, the effect of \(\gamma\) is to decrease \(u\) considerably. Physically it means
Fig. 1. $u$ versus $t$ for $\gamma = 2$ and for various values of $n$.

Fig. 2. $u(t) - t$ versus $t$ for $\gamma = 2$ and for various values of $n$. 
Fig. 3. $u$ versus $t$ for $n = 8$ and for various values of $\gamma$.

Fig. 4. $u(t) - t$ versus $t$ for $n = 8$ and for various values of $\gamma$. 
that the thickness of the concentration boundary layer decreases with the reaction rate parameter $\gamma$.

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