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Signed mahonians on some trees and parabolic quotients

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ABSTRACT

We study the distribution of the major index with sign on some parabolic quotients of the symmetric group, extending and generalizing simultaneously results of Gessel–Simion and Adin–Gessel–Roichman, and on the labellings of some special trees that we call rakes. We further consider and compute the distribution of the flag-major index on some parabolic quotients of wreath products and other related groups. All these distributions turn out to have very simple factorization formulas.

1. Introduction

Let W be a finite reflection group of rank n and S be a set of simple reflections for W as a Coxeter group. If ℓ denotes the length function on W with respect to S, the distribution of ℓ on W is called the Poincaré polynomial and it is a classical result of Chevalley [7] and Solomon [12] in different level of generalizations that

$$\sum_{u \in W} q^{\ell(u)} = [d_1]_q [d_2]_q \cdots [d_n]_q, \tag{1}$$

where $d_1, \ldots, d_n$ are the fundamental degrees of W. The main problems faced in this work are variations of this identity (with statistics other than the length function) to parabolic quotients of W and labellings of trees in the sense of [5].

If $J \subset S$ we denote by W_J and ${}^J W$ the corresponding parabolic subgroup and (left) parabolic quotient. It is well known that given $u \in W$ there exists a unique decomposition $u = u_J \cdot {}^J u$, where $u_J \in W_J$ and ${}^J u \in {}^J W$; it is also well known that in this decomposition we have

$$\ell(u) = \ell(u_J) + \ell\left({}^J u\right). \tag{2}$$

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It follows from Eq. (2) that the distribution of the length function on the parabolic quotient ${}^J W$ is given by

$$\sum_{u \in {}^J W} q^{\ell(u)} = \frac{\sum_{u \in W} q^{\ell(u)}}{\sum_{u \in W_J} q^{\ell(u)}}. \tag{3}$$

In particular, if $W = S_n$ is the symmetric group on n letters, S is identified with the set of positive integers $\{1, 2, \ldots, n-1\}$ and $J = \{n-k+1, \ldots, n-1\}$, then W_J is clearly isomorphic as a Coxeter group to S_k and therefore

$$\sum_{u \in \mathscr{S}_{n,k}} q^{\ell(u)} = [k+1]_q [k+2]_q \cdots [n]_q,$$

where $\mathscr{S}_{n,k} \stackrel{\text{def}}{=} {}^J W$, since the fundamental degrees of S_n are $2, 3, \ldots, n$.

It is a classical result of MacMahon [10] that the major index is equidistributed with the length function on S_n, i.e. $\sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} = \sum_{\sigma \in S_n} q^{\ell(\sigma)}$. As one can easily verify, Eq. (2) is no longer satisfied with maj in the place of ℓ. Nevertheless, as an immediate consequence of classical results of Stanley [13] and Foata and Schützenberger [8] the statistics maj and inv remain equidistributed on all parabolic quotients ${}^J W$, and in particular

$$\sum_{u \in \mathscr{S}_{n,k}} q^{\mathrm{maj}(u)} = [k+1]_q [k+2]_q \cdots [n]_q. \tag{4}$$

Major index and inversion number can also be defined for labellings $w \in \mathscr{W}(F)$ of a forest F (see [5]). In particular, one can consider a particular tree $R_{n,k}$, that we call a rake, and the set of equivalence classes of labellings $\mathscr{R}_{n,k}$, by the action of the automorphism group of $R_{n,k}$ (see Section 4 for more details), such that the distribution of the major index is still

$$\sum_{w \in \mathscr{R}_{n,k}} q^{\mathrm{maj}(w)} = [k+1]_q [k+2]_q \cdots [n]_q. \tag{5}$$

We observe that Eqs. (4) and (5) are trivially equivalent for $k = 1$ only, while there is no bijective explanation for this equidistribution for $k > 1$. The main target of this work is to study the signed versions of the distribution of maj on $\mathscr{S}_{n,k}$ and on $\mathscr{R}_{n,k}$, i.e. we study the polynomials

$$s_{n,k}(q) \stackrel{\text{def}}{=} \sum_{\sigma \in \mathscr{S}_{n,k}} (-1)^{\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)} \quad \text{and} \quad r_{n,k}(q) \stackrel{\text{def}}{=} \sum_{w \in \mathscr{R}_{n,k}} (-1)^{\mathrm{inv}(w)} q^{\mathrm{maj}(w)}.$$

If $k = 1$ these polynomials are computed by a well-known formula of Gessel and Simion (see [15] for an elegant bijective proof based on its unsigned version (1)):

$$s_{n,1}(q) = r_{n,1}(q) = \sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)} = [2]_{-q} [3]_q \cdots [n]_{(-1)^{n-1} q},$$

which is a sort of an alternating version of Eq. (1), and the first main result here is the following alternating version of Eq. (4) that include the Gessel–Simion formula as a special case

$$s_{n,k}(q) = [k+1]_{(-1)^{nk+n+k} q} [k+2]_{(-1)^{k+1} q} [k+3]_{(-1)^{k+2} q} \cdots [n]_{(-1)^{n-1} q}. \tag{6}$$

As for the polynomial $r_{n,k}(q)$, we show that $r_{n,k}(q) = s_{n,k}(q)$ unless n is odd and k is even (in which case $s_{n,k}(q)$ does not seem to factorize nicely at all) strengthening the fact that these two families of combinatorial objects are strictly related. The nice combinatorial and bijective methods used in [8,13,15] for the corresponding unrestricted results cannot be easily generalized to the present context in the computation of $s_{n,k}(q)$ whilst an idea appearing in [1] will be of some help. We also mention here that the present proof of Eq. (6) is rather involved and full of technicalities (many of which will be omitted), so that one could say that it proves the result without really explaining it; and it would be really desirable to find an algebraic explanation for it, or at least a simple combinatorial proof.

Another possible generalization of Eq. (4) considers special classes of "parabolic" subgroups for complex reflection groups. In fact, if W is a complex reflection group, although one can define a "length function" with respect to some generating set of pseudo-reflections, this concept lacks algebraic significance and in particular Eq. (1) is no longer valid. Nevertheless, for wreath products $G(r,n)$ of the cyclic group C_r with S_n, and in particular for Weyl groups of type B, there is a natural counterpart of the major index called the flag-major index. This index has been introduced in [2] and has the following distribution

$$\sum_{g\in G(r,n)} q^{\mathrm{fmaj}(g)} = [d_1]_q[d_2]_q\cdots[d_n]_q,$$

where $d_i = ri$, $i = 1, 2, \ldots, n$ are the fundamental degrees of $G(r,n)$; it is therefore natural to ask whether one can extend Eq. (4) to wreath products $G(r,n)$. This problem is solved in this paper as a particular case of a more general result involving a wider class of groups and using some machinery developed by the author in [6] in the study of some aspects of the invariant theory of complex reflection groups.

2. Notation and preliminaries

In this section we collect the notations that are used in this paper. If $r, n \in \mathbb{N}$ we let $[n] \stackrel{\mathrm{def}}{=} \{1, 2, \ldots, n\}$ and $\mathbb{Z}_r \stackrel{\mathrm{def}}{=} \mathbb{Z}/r\mathbb{Z}$. We let $\mathcal{P}_n \stackrel{\mathrm{def}}{=} \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n\colon\ \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n\}$ be the set of *partitions* of *length* at most n. If q is an indeterminate and $n \in \mathbb{N}$ we let $[n]_q \stackrel{\mathrm{def}}{=} \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$ be the *q-analogue* of n and $[n]_q! \stackrel{\mathrm{def}}{=} [1]_q[2]_q\cdots[n]_q$. A permutation $\sigma \in S_n$ will be denoted by $\sigma = [\sigma(1), \ldots, \sigma(n)]$. We denote by $\mathrm{inv}(\sigma) \stackrel{\mathrm{def}}{=} |\{(i,j)\colon\ i<j \text{ and } \sigma(i) > \sigma(j)\}|$ the number of *inversions* of σ and by $\mathrm{maj}(\sigma) \stackrel{\mathrm{def}}{=} \sum_{i\colon\ \sigma(i)>\sigma(i+1)} i$ the *major index* of σ.

According to [5] we say that a poset P is a *forest* if every element of P is covered by at most one element. If P is a finite forest with n elements we let $\mathscr{W}(P) \stackrel{\mathrm{def}}{=} \{w : P \to \{1, 2, \ldots, n\}$ such that w is a bijection$\}$ be the set of *labellings* of P. If w is a labelling of a forest P we say that a pair (x, y) of elements of P is an inversion of w if $x < y$ and $w(x) > w(y)$ and we let $\mathrm{inv}(w)$ be the number of inversions of w; we say that $x \in P$ is a *descent* of w if x is covered by an element y and $w(x) > w(y)$, and we denote by $\mathrm{Des}(w)$ the set of descents of w; if $x \in P$ we let $h_x = |\{a \in P\colon\ a \leqslant x\}|$ be the *hook length* of x and

$$\mathrm{maj}(w) = \sum_{x\in \mathrm{Des}(w)} h_x$$

be the *major index* of w. Finally, if w is a labelling of a forest P we let

$$\mathscr{L}(w) = \bigl\{\sigma \in S_n\colon\ \text{if } x < y \text{ then } \sigma^{-1}\bigl(w(x)\bigr) < \sigma^{-1}\bigl(w(y)\bigr)\bigr\}$$

be the set of *linear extensions* of w. The main results in [5] we are interested in are summarized in the following result.

Theorem 2.1. *If P is a forest then*

$$\sum_{w\in\mathscr{W}(P)} q^{\mathrm{maj}(w)} = \frac{n!}{\prod_{x\in P} h_x} \prod_{x\in P} [h_x]_q \tag{7}$$

and if $w \in \mathscr{W}(P)$ then

$$\sum_{\sigma\in\mathscr{L}(w)} q^{\mathrm{maj}(\sigma)} = q^{\mathrm{maj}(w)} \frac{[n]_q!}{\prod_{x\in P} [h_x]_q}. \tag{8}$$

If $r > 0$, an *r-colored integer* is a pair (i, z), denoted also i^z, where $i \in \mathbb{N}$ and $z \in \mathbb{Z}_r$, and we let $|i^z| \stackrel{\text{def}}{=} i$. The *wreath product* $G(r,n) \stackrel{\text{def}}{=} \mathbb{Z}_r \wr S_n$ is the group of permutations g of the set of r-colored integers i^z, where $i \in [n]$ and $z \in \mathbb{Z}_r$ such that, if $g(i^0) = j^z$ then $g(i^{z'}) = j^{z+z'}$. An element $g \in G(r,n)$ is therefore uniquely determined by the r-colored integers $g(1^0), \dots, g(n^0)$ and we usually write $g = [\sigma_1^{z_1}, \dots, \sigma_n^{z_n}]$, where $g(i^0) = \sigma_i^{z_i}$. Note that in this case we have that $|g| \stackrel{\text{def}}{=} [\sigma_1, \dots, \sigma_n] \in S_n$. When it is not clear from the context, we will denote z_i by $z_i(g)$.

For $p|r$ the *complex reflection group* $G(r,p,n)$ is the normal subgroup of $G(r,n)$ defined by

$$G(r,p,n) := \left\{\left[\sigma_1^{z_1}, \dots, \sigma_n^{z_n}\right] \in G(r,n) \;\middle|\; z_1 + \dots + z_n \equiv 0 \bmod p\right\}, \tag{9}$$

and its *dual group*

$$G(r,p,n)^* = G(r,n)/C_p, \tag{10}$$

where C_p is the cyclic subgroup of $G(r,n)$ of order p generated by $[1^{r/p}, 2^{r/p}, \dots, n^{r/p}]$.

The study of permutation statistics has found a new interest in the more general setting of complex reflection groups after the work of Adin and Roichman [2]. Some of these results have been generalized in [6, §5] in the following way.

For $g = [\sigma_1^{z_1}, \dots, \sigma_n^{z_n}] \in G(r,p,n)^*$ we let

$$\begin{aligned} &\mathrm{HDes}(g) \stackrel{\text{def}}{=} \left\{i \in [n-1]\colon\ z_i = z_{i+1},\ \text{and}\ \sigma(i) > \sigma(i+1)\right\}, \\ &h_i(g) \stackrel{\text{def}}{=} \#\left\{j \geqslant i\colon\ j \in \mathrm{HDes}(g)\right\}. \end{aligned}$$

We also let $(k_1(g), \dots, k_n(g))$ be the smallest element in $\mathcal{P}_n$ (with respect to the entrywise order) such that $g = [\sigma_1^{k_1(g)}, \dots, \sigma_n^{k_n(g)}]$, where we make slight abuse of notation identifying an integer with its residue class in $\mathbb{Z}_r$. In other words, $(k_1(g), \dots, k_n(g))$ is a partition characterized by the following property: if $(\beta_1, \dots, \beta_n) \in \mathcal{P}_n$ is such that $g = [\sigma_1^{\beta_1}, \dots, \sigma_n^{\beta_n}]$, then $\beta_i \geqslant k_i(g)$, for all $i \in [n]$. For example, let $g = [2^2, 7^3, 6^3, 4^5, 8^1, 1^1, 5^3, 3^2] \in G(6,3,8)^*$. Then $\mathrm{HDes}(g) = \{2,5\}$, $(h_1, \dots, h_8) = (2,2,1,1,1,0,0,0)$ and $(k_1, \dots, k_8) = (18,13,13,9,5,5,1,0)$.

We note that if we let $\lambda_i(g) \stackrel{\text{def}}{=} r \cdot h_i(g) + k_i(g)$, then the sequence $\lambda(g) \stackrel{\text{def}}{=} (\lambda_1(g), \dots, \lambda_n(g))$ is a partition such that $g = [\sigma_1^{\lambda_1(g)}, \dots, \sigma_n^{\lambda_n(g)}]$. The *flag-major index* of an element $g \in G(r,p,n)^*$ is defined by $\mathrm{fmaj}(g) \stackrel{\text{def}}{=} |\lambda(g)|$. All these definitions are valid for wreath products $G(r,n)$ just by letting $p = 1$ and one can easily verify that for $r = p = 1$ $\mathrm{fmaj}(\sigma) = \mathrm{maj}(\sigma)$ for all $\sigma \in S_n$.

3. Signed mahonians in parabolic quotients of symmetric groups

For $n > 0$ and $k = 0, \dots, n$ we let $J = \{n-k+1, \dots, n-1\}$ and, following [4, §2.4], we let

$$\mathscr{S}_{n,k} \stackrel{\text{def}}{=} {}^J S_n = \left\{\sigma \in S_n\colon\ \sigma^{-1}(n-k+1) < \sigma^{-1}(n-k+2) < \dots < \sigma^{-1}(n)\right\},$$

be the corresponding left parabolic quotient. We observe that if $k = 0, 1$ then $J = \emptyset$ and so $\mathscr{S}_{n,k} = S_n$; moreover, if $\sigma \in \mathscr{S}_{n,k}$ then $\sigma(n) \in \{1, 2, \dots, n-k, n\}$ and we set

$$s(\sigma) \stackrel{\text{def}}{=} \begin{cases} \sigma(n) - 1 & \text{if}\ \sigma(n) \in [n-k]; \\ n-k & \text{otherwise}. \end{cases}$$

We consider the following generating function

$$s_{n,k}(q,z) \stackrel{\text{def}}{=} \sum_{\sigma \in \mathscr{S}_{n,k}} (-1)^{\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)} z^{s(\sigma)}.$$

We will be mainly interested in the special evaluation $s_{n,k}(q) \stackrel{\text{def}}{=} s_{n,k}(q,1)$ but we will eventually find an explicit formula for the whole bivariate generating function $s_{n,k}(q,z)$. Observe that, by definition,

$s_{n,0}(q,z) = s_{n,1}(q,z)$ and we recall that this generating function has already been computed by Adin, Gessel and Roichman [1]. We also extend an idea appearing in [1] to prove the following result, where we let $\epsilon \stackrel{\text{def}}{=} -1$.

Theorem 3.1. *Let $n > 1$ and $k \in [n-1]$. Then*

$$\begin{aligned} s_{n,k}(q,z) = {} & \frac{1}{1+z}\big(\big(\epsilon^k z^{n-k} + (-q)^{n-1}\big)s_{n-1,k}(q,1) + \epsilon^n z\big(1-q^{n-1}\big)s_{n-1,k}(q,-z)\big) \\ & + z^{n-k} s_{n-1,k-1}(q,1). \end{aligned}$$

Proof. We need to construct the set $\mathscr{S}_{n,k}$ from analogous sets in S_{n-1}. For this it will be helpful the following notation: if $\sigma \in S_n$ we let $\sigma_0 \in S_{n-1}$ given by

$$\sigma_0(i) \stackrel{\text{def}}{=} \begin{cases} \sigma(i) & \text{if } \sigma(i) < \sigma(n); \\ \sigma(i) - 1 & \text{if } \sigma(i) > \sigma(n), \end{cases}$$

for all $i \in [n-1]$. In other words σ_0 is obtained from σ by deleting the last entry and rescaling the others. Now it is clear that the map $\sigma \mapsto \sigma_0$ establishes a bijection between $\{\sigma \in \mathscr{S}_{n,k}\colon \sigma(n) = n\}$ and $\mathscr{S}_{n-1,k-1}$, and also between the sets $\{\sigma \in \mathscr{S}_{n,k}\colon \sigma(n) = i < n\}$ and $\mathscr{S}_{n-1,k}$, for all $i \in [n-k]$. So, the map $\sigma \mapsto (\sigma_0, \sigma(n))$ establishes an explicit bijection

$$\mathscr{S}_{n,k} \leftrightarrow \big(\mathscr{S}_{n-1,k} \times [n-k]\big) \sqcup \big(\mathscr{S}_{n-1,k-1} \times \{n\}\big),$$

where $\sqcup$ denotes disjoint union. In this bijection, if $\sigma \leftrightarrow (\sigma_0, i)$, with $\sigma_0 \in \mathscr{S}_{n-1,k}$ and $i \in [n-k]$, then

$$\begin{cases} \operatorname{inv}(\sigma) = \operatorname{inv}(\sigma_0) + n - i, \\ \operatorname{maj}(\sigma) = \begin{cases} \operatorname{maj}(\sigma_0) + n - 1 & \text{if } i \leqslant \sigma_0(n-1), \\ \operatorname{maj}(\sigma_0) & \text{if } i > \sigma_0(n-1), \end{cases} \\ s(\sigma) = i - 1, \end{cases}$$

and if $\sigma \leftrightarrow (\sigma_0, n)$ with $\sigma_0 \in \mathscr{S}_{n-1,k-1}$ then

$$\begin{cases} \operatorname{inv}(\sigma) = \operatorname{inv}(\sigma_0), \\ \operatorname{maj}(\sigma) = \operatorname{maj}(\sigma_0), \\ s(\sigma) = n - k. \end{cases}$$

We use the bijection above and these equations to compute recursively the polynomial $s_{n,k}(q,z)$: we easily obtain

$$\begin{aligned} s_{n,k}(q,z) = {} & \sum_{\sigma_0 \in \mathscr{S}_{n-1,k}} \epsilon^{\operatorname{inv}(\sigma_0)+n-1} q^{\operatorname{maj}(\sigma_0)} \left(q^{n-1} \sum_{j=0}^{s(\sigma_0)} (\epsilon z)^j + \sum_{j=s(\sigma_0)+1}^{n-k-1} (\epsilon z)^j \right) \\ & + z^{n-k} s_{n-1,k-1}(q,1). \end{aligned}$$

Computing the geometric sums on the right-hand side we then conclude that

$$\begin{aligned} s_{n,k}(q,z) = {} & \sum_{\sigma \in \mathscr{S}_{n-1,k}} \epsilon^{\operatorname{inv}(\sigma)+n-1} q^{\operatorname{maj}(\sigma)} \left(q^{n-1} \frac{1-(\epsilon z)^{s(\sigma)+1}}{1-\epsilon z} + \frac{(\epsilon z)^{s(\sigma)+1} - (\epsilon z)^{n-k}}{1-\epsilon z} \right) \\ & + z^{n-k} s_{n-1,k-1}(q,1) \\ = {} & \frac{1}{1+z}\big(\big(\epsilon^k z^{n-k} + (-q)^{n-1}\big)s_{n-1,k}(q,1) + \epsilon^n z\big(1-q^{n-1}\big)s_{n-1,k}(q,\epsilon z)\big) \\ & + z^{n-k} s_{n-1,k-1}(q,1). \qquad \square \end{aligned}$$

Theorem 3.1 can be used to compute the polynomials $s_{n,k}(q,z)$ taking as initial condition $s_{n,n}(q,z)=1$ for all $n>0$ and recalling that $s_{n,0}(q,z)=s_{n,1}(q,z)$. As mentioned in the introduction, the special evaluation of the polynomials $s_{n,k}(q,z)$ at $z=1$ will have the following nice factorization

$$s_{n,k}(q)=[k+1]_{\epsilon^{k+n+nk}q}[k+2]_{\epsilon^{k+1}q}[k+3]_{\epsilon^{k+2}q}\cdots[n]_{\epsilon^{n-1}q}.$$

In particular, for $k=1$, we find the Gessel–Simion formula $s_{n,1}(q)=[2]_{-q}[3]_q\cdots[n]_{\epsilon^{n-1}q}$.

Unfortunately, the polynomials $s_{n,k}(q,z)$ do not factorize in general as nicely as their specializations at $z=1$ (at least if k is even). Nevertheless, Theorem 3.1 allows us to prove explicit formulas for these polynomials. We start with a simple example.

Lemma 3.2. *For all $n>0$ we have $s_{n,n-1}(q,z)=z[n-1]_{-q}+(-q)^{n-1}$ and in particular $s_{n,n-1}(q,1)=[n]_{-q}$.*

Proof. This can be proved by induction using Theorem 3.1 but it is simpler to provide a direct proof. In fact we clearly have $\mathscr{S}_{n,n-1}=\{[1\,2\cdots n],[2\,1\,3\cdots n],\ldots,[2\,3\cdots n-1\,1\,n],[2\,3\cdots n\,1]\}$ and the result follows immediately from the definition. $\square$

In the following result we have a completely explicit description of the polynomials $s_{n,k}(q,z)$.

Theorem 3.3. *If $k<n$ is odd we have*

$$s_{n,k}(q,z)=[k+1]_{-q}[k+2]_q\cdots[n-1]_{\epsilon^n q}\left(\sum_{i=0}^{n-k-1}\epsilon^{(n+1)(n-i-1)}z^iq^{n-i-1}+z^{n-k}[k]_{\epsilon^{n-1}q}\right).$$

If $k<n-1$ is even we have

$$s_{n,k}(q,z)=[k+2]_{-q}\cdots[n-1]_{\epsilon^n q}\cdot\Bigg([k+1]_{\epsilon^n q}[n]_{\epsilon^{n-1}q}+(z-1)$$
$$\cdot\left(\sum_{i=0}^{n-k-1}[k+1]_{\epsilon^n q}[n-i-1]_{\epsilon^{n+1}q}z^i+\sum_{\substack{i=0\\ i\text{ even}}}^{n-k-1}q^{n-i-1}z^i\big([k]_{-q}-[k]_q\big)\right)\Bigg),$$

where the last sum runs through all nonnegative even integers smaller than $n-k$.

Proof. The result is readily verified for $n=1,2$. We may also easily verify that the claimed expression for $k=1$ agrees with the one for $k=0$. These initial conditions together with the recurrence given by Theorem 3.1 uniquely determine the polynomials $s_{n,k}(q,z)$, and hence we only have to verify that the claimed expressions actually satisfy this recurrence.

We provide some sketches of the proof only if $k\neq n-2$ is even. If k is odd or $k=n-2$ the proof is similar (and simpler) and is left to the reader.

So let $2\leqslant k\leqslant n-3$, with k even. If we substitute the claimed expressions in the right-hand side of the recursion in Theorem 3.1, and we delete the factor let $[k+2]_{-q}\cdots[n-1]_{\epsilon^n q}$ for notational convenience, we obtain the polynomial

$$g_{n,k}(q,z)\stackrel{\text{def}}{=}\frac{1}{1+z}\big(z^{n-k}-\epsilon^nq^{n-1}+\epsilon^nz-\epsilon^nzq^{n-1}\big)[k+1]_{\epsilon^{n-1}q}-\epsilon^nz\big(1-\epsilon^nq\big)$$
$$\cdot\left(\sum_{i=0}^{n-k-2}[k+1]_{\epsilon^{n-1}q}[n-i-2]_{\epsilon^n q}(-z)^i+\sum_{\substack{i=0\\ i\text{ even}}}^{n-k-2}q^{n-i-2}z^i\big([k]_{-q}-[k]_q\big)\right)$$
$$+z^{n-k}[k]_{-q}[k+1]_q,$$

where we have also used that $1-q^{n-1}=(1-\epsilon^n q)[n-1]_{\epsilon^n q}$. So the proof will be achieved if we show that

$$g_{n,k}(q,z)=[k+1]_{\epsilon^n q}[n]_{\epsilon^{n-1}q}+(z-1) \cdot\left(\sum_{i=0}^{n-k-1}[k+1]_{\epsilon^n q}[n-i-1]_{\epsilon^{n+1}q}z^i+\sum_{\substack{i=0\\ i \text{ even}}}^{n-k-1} q^{n-i-1}z^i\big([k]_{-q}-[k]_q\big)\right). \tag{11}$$

By means of the identity $(1-\epsilon^n q)([k]_{-q}-[k]_q)=-2q[k]_{\epsilon^{n-1}q}$ we can deduce that

$$\begin{aligned} g_{n,k}(q,z)=&\big(\epsilon^n z+\epsilon^{n-1}z^2+\cdots+z^{n-k-1}-\epsilon^n q^{n-1}\big)[k+1]_{\epsilon^{n-1}q}\\ &-\epsilon^n z\left(\sum_{i=0}^{n-k-2}[k+1]_{\epsilon^{n-1}q}\big(1-(\epsilon^n q)^{n-i-2}\big)(-z)^i-2\sum_{\substack{i=0\\ i \text{ even}}}^{n-k-2} q^{n-i-1}z^i[k]_{\epsilon^{n-1}q}\right)\\ &+z^{n-k}[k]_{-q}[k+1]_q. \end{aligned}$$

Now we make the following observation. If we expand $g_{n,k}(q,z)=\sum_{i\geqslant 0}a_i(q)z^i$ then we also have $g_{n,k}(q,z)=b_{-1}(q)+(z-1)\sum_{i\geqslant 0}b_i(q)z^i$ where

$$b_i(q)=\sum_{j>i}a_j(q) \quad \text{for all } i\geqslant -1. \tag{12}$$

To complete the proof we only have to compute the polynomials $b_i(q)$ using Eq. (12) and verify that they agree with the expressions given in (11). Unfortunately, we need to split the proof again, according to the parity of i and n. We treat the case where i and n are both even, leaving the other similar cases to the reader.

We have

$$\begin{aligned} b_i(q)&=[k+1]_{-q}\left(1-\sum_{j=i}^{n-k-2}\big(1-q^{n-j-2}\big)(-1)^j\right)+2[k]_{-q}\sum_{\substack{j=i\\ j \text{ even}}}^{n-k-2}q^{n-j-1}+[k]_{-q}[k+1]_q\\ &=[k+1]_{-q}\big(q^k-q^{k+1}+\cdots+q^{n-i-2}\big)+2[k]_{-q}\big(q^{k+1}+q^{k+3}+\cdots+q^{n-i-1}\big)\\ &\quad+[k]_{-q}[k+1]_q\\ &=[k]_{-q}[n-i]_q+q^k[n-i-1]_{-q}+q^{n-i-1}[k]_{-q}. \end{aligned}$$

Now we make the simple observation that if r and s are both even then $[r]_q[s]_{-q}=[r]_{-q}[s]_q$ and so

$$\begin{aligned} b_i(q)&=[k]_q[n-i]_{-q}+q^k[n-i-1]_{-q}+q^{n-i-1}[k]_{-q}\\ &=[k]_q[n-i-1]_{-q}-q^{n-i-1}[k]_q+q^k[n-i-1]_{-q}+q^{n-i-1}[k]_{-q}\\ &=[k+1]_q[n-i-1]_{-q}+q^{n-i-1}\big([k]_{-q}-[k]_q\big), \end{aligned}$$

and the proof is complete. □

Corollary 3.4. *We have*

$$s_{n,k}(q)=\sum_{\sigma\in\mathscr{S}_{n,k}}\epsilon^{\mathrm{inv}(\sigma)}q^{\mathrm{maj}(\sigma)}=[k+1]_{\epsilon^{k+n+nk}q}[k+2]_{\epsilon^{k+1}q}[k+3]_{\epsilon^{k+2}q}\cdots[n]_{\epsilon^{n-1}q},$$

where $\epsilon=-1$.

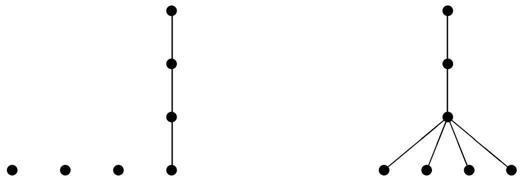

Fig. 1. $T_{7,4}$ (left) and $R_{7,4}$ (right).

Proof. This follows immediately from Theorem 3.3. $\square$

We close this section by observing that Corollary 3.4 can also be interpreted as an alternating version of (a special case of) Eq. (8). In fact, consider the poset $T_{n,k} = \{x_1, x_2, \ldots, x_n\}$ with the ordering given by $x_i < x_j$ if and only if $n - k < i < j$. The Hasse diagram of $T_{n.k}$ is a forest consisting of $n - k$ disjoint vertices and of a linear tree of length k (see Fig. 1 (left)). If we consider the natural labelling for $T_{n,k}$ given by $w(x_i) = i$, then we clearly have $\mathscr{L}(w) = \mathscr{S}_{n,k}$ and so Corollary 3.4 can also be reformulated as

$$\sum_{\sigma \in \mathscr{L}(w)} \epsilon^{\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)} = [k+1]_{\epsilon^{k+n+nk} q} [k+2]_{\epsilon^{k+1} q} [k+3]_{\epsilon^{k+2} q} \cdots [n]_{\epsilon^{n-1} q}.$$

4. Signed mahonian distributions on rakes' labellings

For $k < n$ we denote by $R_{n,k}$ the poset consisting of n elements $x_1, \ldots, x_n$ with the ordering given by $x_i < x_j$ if and only if $i, k < j$. The Hasse diagram of $R_{n,k}$ is shown in Fig. 1 (right) and because of its shape we call $R_{n,k}$ a *rake* with k *teeth*. It is clear that the action of $\mathrm{Aut}(R_{n,k}) \cong S_k$ on $\mathscr{W}(R_{n,k})$ by permutations of the labels of the teeth preserves inversion index and major index; therefore we can consider these indices also on the set $\mathscr{R}_{n,k}$ of equivalence classes of labellings.

It follows in particular from (8) that

$$\sum_{w \in \mathscr{R}_{n,k}} q^{\mathrm{maj}(w)} = [k+1]_q \cdots [n]_q, \tag{13}$$

and we denote by $r_{n,k}(q)$ its signed version

$$r_{n,k}(q) = \sum_{w \in \mathscr{R}_{n,k}} \epsilon^{\mathrm{inv}(w)} q^{\mathrm{maj}(w)}.$$

As for the polynomials $s_{n,k}$ we need to introduce a further catalytic parameter. If $w \in \mathscr{R}_{n,k}$ we let $r(w)$ be $n - w(x_n)$, where x_n is the top element of $R_{n,k}$. We let

$$r_{n,k}(q,t) \stackrel{\text{def}}{=} \sum_{w \in \mathscr{R}_{n,k}} \epsilon^{\mathrm{inv}(w)} q^{\mathrm{maj}(w)} t^{r(w)}.$$

These polynomials satisfy the following recursion.

Lemma 4.1. *For all $n \geqslant k+2$ we have*

$$r_{n,k}(q,t) = \frac{1}{1+t} \big(t\big(1 - q^{n-1}\big) r_{n-1,k}(q,-t) + \big(1 + t(-qt)^{n-1}\big) r_{n-1,k}(q,1) \big).$$

Proof. This proof is similar (and actually much simpler) to that of Theorem 3.1 and so we briefly sketch it. There is a bijection $w \to (w_0, w(x_n))$ between $\mathscr{R}_{n,k}$ and $\mathscr{R}_{n-1,k} \times [n]$ satisfying $\mathrm{inv}(w) = \mathrm{inv}(w_0) + n - w(x_n)$, $r(w) = n - w(x_n)$, $\mathrm{maj}(w) = \mathrm{maj}(w_0)$ if $w_0(x_{n-1}) < w(x_n)$, and $\mathrm{maj}(w) = \mathrm{maj}(w_0) + n - 1$ otherwise. Therefore

$$\begin{aligned} r_{n,k}(q,t) &= \sum_{w_0 \in \mathscr{R}_{n-1,k}} \epsilon^{\mathrm{inv}(w_0)} q^{\mathrm{maj}(w_0)} \left(q^{n-1} \sum_{i=1}^{w_0(n-1)} (-t)^{n-i} + \sum_{i=w_0(n-1)+1}^{n} (-t)^{n-i} \right) \\ &= \sum_{w_0 \in \mathscr{R}_{n-1,k}} \epsilon^{\mathrm{inv}(w_0)} q^{\mathrm{maj}(w_0)} \left(q^{n-1} \frac{(-t)^{n-w_0(n-1)} - (-t)^n}{1+t} + \frac{1-(-t)^{n-w_0(n-1)}}{1+t} \right) \\ &= \frac{1}{1+t} \big(t\big(1-q^{n-1}\big) r_{n-1,k}(q,-t) + \big(1+t(-qt)^{n-1}\big) r_{n-1,k}(q,1) \big). \qquad \square \end{aligned}$$

Despite the polynomials $s_{n,k}$, the polynomials $r_{n,k}(q,t)$ admit a nice factorization only if k is odd.

Proposition 4.2. *Let k be odd and $n > k$. Then*

$$r_{n,k}(q,t) = [k+1]_{-q}[k+2]_q \cdots [n-1]_{\epsilon^n q}[n]_{\epsilon^{n+1} qt}.$$

Proof. The case $n = k+1$ is an easy verification. Since this initial condition plus the recursion of Lemma 4.1 uniquely determine $r_{n,k}(q,t)$ we only need to show that the claimed expression satisfies the recurrence. Substituting $r_{n,k}(q,t)$ as given by the proposition and cancelling a common factor, it remains to check that

$$[n]_{\epsilon^{n+1} qt} = \frac{1}{t+1} \big(t\big(1+\epsilon^{n-1} q\big)[n-1]_{\epsilon^{n-1} qt} + 1 + t(-qt)^{n-1} \big),$$

where we have also used that $(1-q^{n-1}) = (1+\epsilon^{n-1}q)[n-1]_{\epsilon^n q}$. But the above right-hand side may be rewritten as

$$\begin{aligned} &\frac{1}{1+t} \big(t\big([n-1]_{\epsilon^{n-1} qt} + (-qt)^{n-1}\big) + \epsilon^{n-1} qt [n-1]_{\epsilon^{n-1} qt} + 1 \big) \\ &\quad = \frac{1}{1+t} \big(t[n]_{\epsilon^{n-1} qt} + [n]_{\epsilon^{n-1} qt} \big) \\ &\quad = [n]_{\epsilon^{n-1} qt}. \qquad \square \end{aligned}$$

If k is even the polynomial $r_{n,k}(q,t)$ does not admit a nice factorization in general. Nevertheless, if n is also even the specialization $r_{n,k}(q)$ can be easily computed thanks to the following result, whose proof is valid for all k.

Proposition 4.3. *Let n be even. Then*

$$r_{n,k}(q) = [k+1]_{\epsilon^k q}[k+2]_{\epsilon^{k+1} q} \cdots [n-1]_q [n]_{-q}.$$

Proof. We generalize here an idea appearing in [15]. Consider the following bijection ϕ of $\mathscr{R}_{n,k}$. If $w \in \mathscr{R}_{n,k}$ is such that the vertices labelled by $2i-1$ and $2i$ are either adjacent or are not comparable for all $i \in [\frac{n}{2}]$ we let $\phi(w) = w$. Otherwise let i be the minimum index such that $2i-1$ and $2i$ are comparable but not adjacent and we let $\phi(w)$ be the labelling obtained by exchanging the labels $2i$ and $2i-1$. It is clear that ϕ is an involution on $\mathscr{R}_{n,k}$ and that if $\phi(w) \neq w$ then $\epsilon^{\mathrm{inv}(w)} = -\epsilon^{\mathrm{inv}(\phi(w))}$ and $\mathrm{maj}(w) = \mathrm{maj}(\phi(w))$. Therefore, computing $r_{n,k}(q)$ we can restrict the sum on the fixed points of ϕ. If w is fixed by ϕ then among the $\frac{n}{2}$ pairs of labels of the form $\{2i-1, 2i\}$ there are $\lfloor \frac{n+1-k}{2} \rfloor$ pairs in adjacent positions, and $\lfloor \frac{k}{2} \rfloor$ pairs which appear in the teeth of the rake. Therefore we can construct an element $\bar{w} \in \mathscr{R}_{\frac{n}{2}, \lfloor \frac{k}{2} \rfloor}$ in the following way: if $2i-1$ and $2i$ appear in the teeth of w then i is a label of a tooth of $\bar{w}$. The other labels are inserted in such way that $w^{-1}(2i) < w^{-1}(2j)$ if and only if $\bar{w}^{-1}(i) < \bar{w}^{-1}(j)$. We also define a 0–1 vector $a(w) = (a_1(w), \ldots, a_d(w))$, with $d = \lfloor \frac{n+1-k}{2} \rfloor$, in the following way. Let $\{2i_1-1, 2i_1\}, \ldots, \{2i_d-1, 2i_d\}$ be the pairs of labels of adjacent vertices ordered in such way that $w^{-1}(2i_1) < w^{-1}(2i_2) < \cdots < w^{-1}(2i_d)$, and we let $a_j(w) = 1$ if and only if

$w^{-1}(2i_j) < w^{-1}(2i_j - 1)$. It is a straightforward verification that the map $w \to (\bar{w}, a(w))$ is a bijection between the fixed points of ϕ and $\mathscr{R}_{\frac{n}{2}, \lfloor \frac{k}{2} \rfloor} \times \{0,1\}^d$ such that

- $\mathrm{inv}(w) \equiv \sum a_i(w) \bmod 2$;
- $\mathrm{maj}(w) = 2\,\mathrm{maj}(\bar{w}) + a_1 + \sum_{i=2}^{d}(k + 2i - 2)a_i$ if k is odd;
- $\mathrm{maj}(w) = 2\,\mathrm{maj}(\bar{w}) + \sum_{i=1}^{d}(k + 2i - 1)a_i$ if k is even.

Therefore, if k is odd, applying (13), we have

$$\begin{aligned} r_{n,k}(q) &= \sum_{\bar{w} \in \mathscr{R}_{\frac{n}{2}, \lfloor \frac{k}{2} \rfloor}} q^{2\,\mathrm{maj}(\bar{w})}(1-q)\big(1-q^{k+2}\big)\big(1-q^{k+4}\big)\cdots\big(1-q^{n-1}\big) \\ &= \left[\frac{k+1}{2}\right]_{q^2}\left[\frac{k+3}{2}\right]_{q^2}\cdots\left[\frac{n}{2}\right]_{q^2}(1-q)\big(1-q^{k+2}\big)\cdots\big(1-q^{n-1}\big) \\ &= [k+1]_q[k+2]_{-q}\cdots[n]_{-q}. \end{aligned}$$

Similarly, if k is even, we have

$$\begin{aligned} r_{n,k}(q) &= \sum_{\bar{w} \in \mathscr{R}_{\frac{n}{2}, \lfloor \frac{k}{2} \rfloor}} q^{2\,\mathrm{maj}(\bar{w})}\big(1-q^{k+1}\big)\big(1-q^{k+3}\big)\cdots\big(1-q^{n-1}\big) \\ &= \left[\frac{k+2}{2}\right]_{q^2}\left[\frac{k+4}{2}\right]_{q^2}\cdots\left[\frac{n}{2}\right]_{q^2}\big(1-q^{k+1}\big)\big(1-q^{k+3}\big)\cdots\big(1-q^{n-1}\big) \\ &= [k+1]_{-q}[k+2]_q\cdots[n]_{-q}. \qquad \square \end{aligned}$$

5. Mahonian distribution on parabolic quotients in complex reflection groups

In this section we consider the infinite family of complex reflection groups $G(r,p,n)$. We let $G = G(r,p,n)$ and $G^* = G(r,n)/C_p$ and we recall from [6] that

$$\sum_{g \in G^*} q^{\mathrm{fmaj}(g)} = [d_1]_q[d_2]_q\cdots[d_n]_q,$$

where $d_i = ri$ if $i < n$ and $d_n = \frac{rn}{p}$ are the fundamental degrees of G. It is therefore natural to look at the distribution of the flag-major index on sets of cosets representatives for some special subgroups of G^*, in order to generalize Eq. (4) to these groups. With this in mind we need to extend the ideas appearing in [13], and in particular the use of P-partitions, in this context, using some of the tools developed in [6] and further exploited in [3]. In particular, we recall the following result (see [6, Theorem 8.3] and [3, Lemma 5.1]).

Lemma 5.1. *The map*

$$\begin{aligned} &G^* \times \mathcal{P}_n \times \{0,1,\ldots,p-1\} \to \mathbb{N}^n, \\ &(g,\lambda,h) \mapsto f = (f_1,\ldots,f_n), \end{aligned}$$

where $f_i = \lambda_{|g^{-1}(i)|}(g) + r\lambda_{|g^{-1}(i)|} + h\frac{r}{p}$ for all $i \in [n]$, is a bijection. And in this case we say that f is g-compatible.

For $g \in G^*$ we let S_g be the set of g-compatible vectors in $\mathbb{N}^n$.

Lemma 5.2. *We have*

$$F_g(x_1,\dots,x_n) \stackrel{\text{def}}{=} \sum_{f\in S_g} x_1^{f_1}\cdots x_n^{f_n}$$
$$= \frac{x_{|g_1|}^{\lambda_1(g)}\cdots x_{|g_n|}^{\lambda_n(g)}}{(1-x_{|g_1|}^r)(1-x_{|g_1|}^r x_{|g_2|}^r)\cdots(1-x_{|g_1|}^r\cdots x_{|g_{n-1}|}^r)\cdot(1-x_{|g_1|}^{r/p}\cdots x_{|g_n|}^{r/p})}.$$

Proof. By Lemma 5.1 we have

$$\sum_{f\in S_g} x_1^{f_1}\cdots x_n^{f_n} = \sum_{\lambda\in\mathcal{P}_n}\sum_{h=0}^{p-1} x_{|g_1|}^{\lambda_1(g)+r\lambda_1+h\frac{r}{p}}\cdots x_{|g_n|}^{\lambda_n(g)+r\lambda_n+h\frac{r}{p}}$$
$$= x_{|g_1|}^{\lambda_1(g)}\cdots x_{|g_n|}^{\lambda_n(g)}\sum_{\lambda\in\mathcal{P}_n} x_{|g_1|}^{r\lambda_1}\cdots x_{|g_n|}^{r\lambda_n}\sum_{h=0}^{p-1}(x_1\cdots x_n)^{h\frac{r}{p}}$$
$$= x_{|g_1|}^{\lambda_1(g)}\cdots x_{|g_n|}^{\lambda_n(g)}\frac{1}{(1-x_{|g_1|}^r)(1-x_{|g_1|}^r x_{|g_2|}^r)\cdots(1-x_{|g_1|}^r\cdots x_{|g_n|}^r)}\frac{1-x_1^r\cdots x_n^r}{1-x_1^{\frac{r}{p}}\cdots x_n^{\frac{r}{p}}},$$

and the result follows. □

Let

$$\mathbb{N}^n_{(r,p)} \stackrel{\text{def}}{=} \left\{(f_1,\dots,f_n)\in\mathbb{N}^n\colon\ f_1\equiv f_2\equiv\cdots\equiv f_n\equiv h\frac{r}{p} \bmod r \text{ for some } h=0,1,\dots,p-1\right\}$$

and $\mathcal{A}=\{(f_1,\dots,f_n)\in\mathbb{N}^n\colon\ f_1\geqslant f_2\geqslant\cdots\geqslant f_k \text{ and } (f_1,\dots,f_k)\in\mathbb{N}^k_{(r,p)}\}$. We will show that the set $\mathcal{A}$ consists of all g-compatible vectors in $\mathbb{N}^n$ as g varies in a suitable subset of G^*. Before proving this we need the following preliminary result.

Lemma 5.3. *Let $g\in G^*$. Then $(\lambda_1(g)+\lambda_{|g(1)|}(g^{-1}),\dots,\lambda_n(g)+\lambda_{|g(n)|}(g^{-1}))\in\mathbb{N}^n_{(r,p)}$.*

Proof. We recall that if $|g|=\sigma$ then $g=[\sigma_1^{\lambda_1(g)},\dots,\sigma_n^{\lambda_n(g)}]$ and hence also $g^{-1}=[\tau_1^{\lambda_1(g^{-1})},\dots,\tau_n^{\lambda_n(g^{-1})}]$, where $\tau=\sigma^{-1}$. Therefore

$$g^{-1}g=\left[1^{\lambda_1(g)+\lambda_{\sigma_1}(g^{-1})},\dots,n^{\lambda_n(g)+\lambda_{\sigma_n}(g^{-1})}\right].$$

Since this is the identity element in the group G^*, we deduce that there exists $h\in\{0,1,\dots,p-1\}$ such that $\lambda_i(g)+\lambda_{\sigma_i}(g^{-1})\equiv h\frac{r}{p}$ for all $i\in[n]$, and the proof is complete. □

For $k<n$ we let $C_k \stackrel{\text{def}}{=} \{[\sigma_1^0,\sigma_2^0,\dots,\sigma_k^0,g_{k+1},\dots,g_n]\in G^*\colon\ \sigma_1<\cdots<\sigma_k\}$. We observe that the subgroup of G^* given by $\{g\in G^*\colon\ g=[\sigma_1^{z_1},\dots,\sigma_k^{z_k},(k+1)^0,(k+2)^0,\dots,n^0]\}$ is isomorphic to $G(r,k)$ for all $k<n$. Moreover, we may observe that C_k contains exactly p representatives for each right coset of $G(r,k)$ in G^* (we prefer here to consider right instead of left cosets to be consistent with the notation and the results in [9] and [11] that we are going to generalize). In particular, if $p=1$, we have that C_k is a complete system of representatives of the cosets of $G(r,k)$ in $G(r,n)$.

Proposition 5.4. *Let $f\in\mathbb{N}^n$. Then $f\in\mathcal{A}$ if and only if f is g^{-1}-compatible for some $g\in C_k$.*

Proof. Recall that, if $|g|=\sigma$, then $g=[\sigma_1^{\lambda_1(g)},\sigma_2^{\lambda_2(g)},\dots,\sigma_n^{\lambda_n(g)}]$. In particular, $g\in C_k$ if and only if $\sigma_1<\sigma_2<\cdots<\sigma_k$ and

$$\big(\lambda_1(g),\lambda_2(g),\dots,\lambda_k(g)\big)\in\mathbb{N}^k_{(r,p)}. \tag{14}$$

Let $f \in \mathbb{N}^n$ be g^{-1}-compatible for some $g \in G^*$. Then, by Lemma 5.1, there exist $\lambda \in \mathcal{P}_n$ and $h \in \{0, \dots, p-1\}$ such that

$$f_i = \lambda_{\sigma_i}\big(g^{-1}\big) + r\lambda_{\sigma_i} + h\frac{r}{p}, \quad 1 \leqslant i \leqslant k. \tag{15}$$

The proof will follows immediately from the following two claims and Eq. (14).

Claim 1. *$f_1 \geqslant f_2 \geqslant \cdots \geqslant f_k$ if and only if $\sigma_1 < \sigma_2 < \cdots < \sigma_k$. Since $\lambda(g^{-1})$ and λ are both partitions, it is clear that Eqs. (15) imply that if $\sigma_1 < \sigma_2 < \cdots < \sigma_k$ then $f_1 \geqslant f_2 \geqslant \cdots \geqslant f_k$. For the same reason, if $f_i > f_{i+1}$ then $\sigma_i < \sigma_{i+1}$. Finally, assume that $f_i = f_{i+1}$. This immediately implies that $\lambda_{\sigma_i}(g^{-1}) = \lambda_{\sigma_{i+1}}(g^{-1})$ and a moment's thought based on the definition of the statistics $\lambda_i(g^{-1})$ will show that this also implies $\sigma_i < \sigma_{i+1}$.*

Claim 2. *$(f_1 \geqslant f_2 \geqslant \cdots \geqslant f_k) \in \mathbb{N}^k_{(r,p)}$ if and only if $(\lambda_1(g), \dots, \lambda_k(g)) \in \mathbb{N}^k_{(r,p)}$. By Eqs. (15) we clearly have that $(f_1, \dots, f_k) \in \mathbb{N}^k_{(r,p)}$ if and only if $(\lambda_{|g_1|}(g^{-1}), \lambda_{|g_2|}(g^{-1}), \dots, \lambda_{|g_k|}(g^{-1})) \in \mathbb{N}^k_{(r,p)}$. By Lemma 5.3 this is also equivalent to $(\lambda_1(g), \dots, \lambda_k(g)) \in \mathbb{N}^k_{(r,p)}$ and the proof is complete.* $\square$

We are now ready to state and prove the main result of this section.

Theorem 5.5. *Let $G = G(r, p, n)^*$. Then*

$$\sum_{g \in C_k} q^{\mathrm{fmaj}(g^{-1})} = [p]_{q^{kr/p}}\big[r(k+1)\big]_q \cdots \big[r(n-1)\big]_q [rn/p]_q.$$

Proof. Consider the formal power series

$$G(q) \stackrel{\mathrm{def}}{=} \sum_{f \in \mathcal{A}} q^{|f|},$$

where, if $f = (f_1, \dots, f_n)$, we let $|f| = f_1 + \cdots + f_n$. We compute the series $G(q)$ in two different ways. First, by Lemma 5.2 and Proposition 5.4 we have

$$\begin{aligned} G(q) &= \sum_{g \in C_k} F_{g^{-1}}(q, \dots, q) \\ &= \sum_{g \in C_k} \frac{q^{\lambda_1(g^{-1})} \cdots q^{\lambda_n(g^{-1})}}{(1-q^r)(1-q^{2r}) \cdots (1-q^{r(n-1)})(1-q^{rn/p})} \\ &= \frac{\sum_{g \in C_k} q^{\mathrm{fmaj}(g^{-1})}}{(1-q^r)(1-q^{2r}) \cdots (1-q^{r(n-1)})(1-q^{rn/p})}. \end{aligned}$$

Now we compute directly $G(q)$ using the definition of $\mathcal{A}$. We have

$$G(q) = \frac{1}{(1-q^r)(1-q^{2r}) \cdots (1-q^{kr})} \frac{(1-q^{kr})}{(1-q^{kr/p})} \frac{1}{(1-q)^{n-k}}$$

and therefore

$$\sum_{g \in C_k} q^{\mathrm{fmaj}(g^{-1})} = \frac{1-q^{kr}}{1-q^{kr/p}} \frac{(1-q^{(k+1)r}) \cdots (1-q^{(n-1)r})(1-q^{nr/p})}{(1-q)^{n-k}}. \qquad \square$$

Corollary 5.6. *If $G = G(r, n)$, then C_k is a system of coset representatives for the subgroup $G(r, k)$ and*

$$\sum_{g \in C_k} q^{\mathrm{fmaj}(g^{-1})} = \big[r(k+1)\big]_q \big[r(k+2)\big]_q \cdots [rn]_q.$$

Eq. (4) has been rediscovered by Panova in [11] to present a simple combinatorial proof of a related result that was discovered by Garsia in [9], using nice relations of these objects with symmetric functions. Corollary 5.6 can be easily used to find a natural $G(r,n)$ counterpart of these results. We first need some further notation. If $g \in G(r,n)$ we let

$$is_z(g) \stackrel{\text{def}}{=} \max\bigl\{j\colon \exists 1 \leqslant i_1 < \cdots < i_j \leqslant n \text{ with } z_{i_1}(g) = \cdots = z_{i_j}(g) = z \\ \text{and } \bigl|g(i_1)\bigr| < \cdots < \bigl|g(i_j)\bigr|\bigr\},$$

be the maximum length of a homogeneous increasing subsequence of g of color z. Then we let

$$\Pi_{r,n,k} \stackrel{\text{def}}{=} \bigl\{g = \bigl[\sigma_1^0, \ldots, \sigma_{n-k}^0, \sigma_{n-k+1}^{z_{n-k+1}}, \ldots, \sigma_n^{z_n}\bigr] \in G(r,n)\colon\ \sigma_1 < \cdots < \sigma_{n-k} \\ \text{and } is_0(g) = n-k\bigr\}.$$

The following result generalizes [11, Theorems 1 and 2] and the main results in [9].

Theorem 5.7. *If $n \geqslant 2k$ we have that*

$$\sum_{g \in \Pi_{r,n,k}} q^{\mathrm{fmaj}(g^{-1})} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \bigl[r(n-i+1)\bigr]_q \bigl[r(n-i+2)\bigr]_q \cdots [rn]_q.$$

The proof of this result for $r=1$ in [11] makes use of Eq. (4) and on a cute use of the principle of inclusion–exclusion on some sets of standard tableaux based on the Robinson–Schensted correspondence. This proof can be generalized to the general case of groups $G(r,n)$ using Corollary 5.6 and the generalized version of the Robinson–Schensted correspondence [14]. This is why we do not present this proof and we refer the reader to [6, §10] for further details on the generalized Robinson–Schensted correspondence.

We conclude with some open problems arising from this work.

Problem 5.8. Let $J' = [k]$ and $\mathscr{S}'_{n,k} \stackrel{\text{def}}{=} {}^{J'}S_n = \{\sigma \in S_n\colon\ \sigma^{-1}(1) < \cdots < \sigma^{-1}(k)\}$. Numerical evidence suggests that

$$\sum_{\sigma \in \mathscr{S}'_{n,k}} (-1)^{\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)} = \sum_{u \in \mathscr{S}_{n,k}} (-1)^{\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)}$$

if n is even or k is odd. These are the same cases for which $s_{n,k}(q) = r_{n,k}(q)$. Give a (possibly bijective) proof of these equidistributions.

Problem 5.9. Unify the main results of this work in a unique statement, i.e. compute the polynomials

$$\sum_{g \in C_k} \epsilon^{\mathrm{inv}(|g|)} q^{\mathrm{fmaj}(g^{-1})},$$

where C_k is a system of coset representatives of $G(r,k)$ in $G(r,n)$ including as particular cases Theorems 3.3 and Corollary 5.6.

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