Topological structures of the sets of composition operators on the Bloch spaces

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Abstract

We study properties of the topological sets of composition operators on Bloch and little Bloch spaces in the operator topology.

Keywords: Composition operator; Bloch space; Little Bloch space

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane with the unit circle $\partial \mathbb{D}$ as its boundary, and let $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. Denote by $S(\mathbb{D})$ the set of analytic self-maps of $\mathbb{D}$. Every self-map $\varphi \in S(\mathbb{D})$ induces the composition operator $C_\varphi$ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$. 

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Recall that the Bloch space $\mathcal{B}$ consists of all $f \in H(\mathbb{D})$ such that 
\[ \| f \| = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty. \]
Then $\| \cdot \|$ defines a complete semi-norm on $\mathcal{B}$. Let the little Bloch space $\mathcal{B}_o$ denote the subspace of $\mathcal{B}$ consisting of those functions $f$ such that 
\[ \lim_{|z| \to 1} (1 - |z|^2) f'(z) = 0. \]
Hence $\mathcal{B}$ is a Banach space under the norm 
\[ \| f \|_B = |f(0)| + \| f \| \]
and $\mathcal{B}_o$ is a closed subspace of $\mathcal{B}$. In particular, $\mathcal{B}_o$ is the closure in $\mathcal{B}$ of the polynomials.

It is well known that for any $w \in \mathbb{D}$, 
\[ \sup \{(1 - |w|^2) |f'(w)| : f \in \mathcal{B}, \| f \| \leq 1\} = 1. \tag{1} \]
See [2,8,12] for more information on composition operators and Bloch spaces.

Let $H^\infty = H^\infty(\mathbb{D})$ be the set of all bounded analytic functions on $\mathbb{D}$. Then $H^\infty$ is the Banach algebra with the supremum norm 
\[ \| f \|_\infty = \sup_{z \in \mathbb{D}} |f(z)|. \]

Note that $H^\infty \subset \mathcal{B}$ and that $\| f \| \leq \| f \|_\infty$ if $f \in H^\infty$.

For a Banach space $X$ of analytic functions on $\mathbb{D}$, let $\mathcal{C}(X)$ be the set of composition operators on $X$ with the operator norm topology. We write $C_\varphi \sim_X C_\psi$ if $C_\varphi$ and $C_\psi$ are in the same path component of $\mathcal{C}(X)$. In this paper, we investigate the topological structure of $\mathcal{C}(\mathcal{B})$ and $\mathcal{C}(\mathcal{B}_o)$. Here, by the Schwarz–Pick inequality, $C_\varphi$ is always bounded on $\mathcal{B}$ and $C_\varphi$ is bounded on $\mathcal{B}_o$ if and only if $\varphi \in \mathcal{B}_o$ [6]. We can easily obtain that if $C_\varphi \sim_{H^\infty} C_\psi$, then $C_\varphi \sim_{\mathcal{B}} C_\psi$ [4, Corollary 4.3]. Originally Sundberg and Shapiro [9] posed the topological structure of the set $\mathcal{C}(H^2)$ of composition operators on the Hilbert–Hardy space $H^2$ and asked conditions to characterize components and isolated elements of the set $\mathcal{C}(H^2)$. These problems are so hard. Instead, MacCluer, Zhao and the second author [5] considered the above problems on $H^\infty$. In the setting of $\mathcal{C}(H^\infty)$, path components and isolated points have been completely characterized in [3,5]. Then this work gave a relationship between a component problem and the behavior of the difference of two composition operators acting from $\mathcal{B}$ to $H^\infty$. The authors [4] studied properties of the differences of two composition operators on $\mathcal{B}$ and $\mathcal{B}_o$. Continuously, we here consider properties of the path components of both $\mathcal{C}(\mathcal{B})$ and $\mathcal{C}(\mathcal{B}_o)$ in the operator topology.

In Section 2, we have the inequalities estimating the differences of two Bloch-type derivatives which would be useful tools to obtain our main results. In Section 3, we will consider the problem whether the set of compact composition operators forms a path component in $\mathcal{C}(\mathcal{B})$ and $\mathcal{C}(\mathcal{B}_o)$. In the case of Bloch space, Toews proved in his thesis [11] that the compact composition operators form a path connected set in $\mathcal{C}(\mathcal{B})$. We here will have a similar result in $\mathcal{C}(\mathcal{B}_o)$ using a different method. After such a characterization, we will present analytic self-maps $\varphi, \psi$ of $\mathbb{D}$ such that corresponding composition operators $C_\varphi$ and $C_\psi$ are isolated in $\mathcal{C}(H^\infty)$ but lie in the same component in $\mathcal{C}(\mathcal{B}_o)$. In Section 4, we will show that the compactness of $C_\varphi - C_\psi$ implies $C_\varphi \sim_{\mathcal{B}_o} C_\psi$ and in the last section we give a remark to the isolation problem of $\mathcal{C}(\mathcal{B})$. 

2. Prerequisites

For \( w \in \mathbb{D} \), let \( \alpha_w \) be the Möbius transformation of \( \mathbb{D} \) defined by
\[
\alpha_w(z) = \frac{w - z}{1 - \overline{w}z}.
\]
For \( w, z \) in \( \mathbb{D} \), the pseudo-hyperbolic distance \( \rho(w, z) \) between \( z \) and \( w \) is given by
\[
\rho(w, z) = |\alpha_w(z)|,
\]
and the hyperbolic metric \( \beta(w, z) \) is given by
\[
\beta(w, z) = \frac{1}{2} \log \frac{1 + \rho(w, z)}{1 - \rho(w, z)}.
\]
To study properties of composition operator \( C_\varphi \) on \( B \) and \( B_\alpha \), we introduce the following derivative \( \varphi^\# \) induced by the Schwarz–Pick lemma:
\[
\varphi^\#(z) = \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \varphi'(z).
\]
This form will be an important tool in the sequel. Explicitly we will consider the behavior of \( \varphi^\# \) on the neighborhood of \( \partial \mathbb{D} \). To discuss the behavior, we need the following sets.

**Definition 2.1.** For \( \varphi \in S(\mathbb{D}) \), let \( \Gamma_r(\varphi) = \{ z \in \mathbb{D} : |\varphi(z)| > r \} \) for \( r \in (0, 1) \). Let \( \Gamma(\varphi) \) be the set of sequences \( \{ z_k \} \) in \( \mathbb{D} \) such that \( |\varphi(z_k)| \to 1 \), and let \( \Gamma^\#(\varphi) \) be the set of sequences \( \{ z_k \} \) in \( \mathbb{D} \) such that \( |\varphi(z_k)| \to 1 \) and \( \varphi^\#(z_k) \nrightarrow 0 \).

Then it is clear that \( \Gamma^\#(\varphi) \subset \Gamma(\varphi) \). It is well known that \( C_\varphi \) is compact on \( H^\infty \) if and only if \( \Gamma(\varphi) = \emptyset \). In [6], it is showed that \( C_\varphi \) is compact on \( B \) if and only if \( \Gamma^\#(\varphi) = \emptyset \). Montes-Rodríguez [7] determined the essential norm of composition operators on \( B \) and \( B_\alpha \). Recall that the essential norm \( \| T \|_e \) of a bounded linear operator \( T \) on a Banach space \( X \) is defined as
\[
\| T \|_e = \inf \{ \| T - K \| : K \text{ is compact on } X \}.
\]
That is, he obtained
\[
\| C_\varphi \|_e = \lim_{s \to 1^-} \sup_{|\varphi(z)| > s} |\varphi^\#(z)|.
\]
We can also estimate the “semi-operator norm” \( \| C_\varphi \| \) of \( C_\varphi \) on \( B \), which is defined by
\[
\| C_\varphi \| = \sup \{ \| C_\varphi f \| : \| f \| \leq 1 \}.
\]

**Proposition 2.2.** Let \( \varphi \in S(\mathbb{D}) \). Then \( \| C_\varphi \| = \| \varphi^\# \|_\infty \).

**Proof.** Let \( \varphi \in S(\mathbb{D}) \) and \( f \in B \) with \( \| f \| \leq 1 \). Then (1) implies that
\[
\| C_\varphi f \| = \sup_{z \in \mathbb{D}} (1 - |z|^2) |\varphi'(z)| |f'(\varphi(z))| \\
= \sup_{z \in \mathbb{D}} |\varphi^\#(z)| (1 - |\varphi(z)|^2) |f'(\varphi(z))| \leq \| \varphi^\# \|_\infty.
\]
Hence we have that \( \| C_\varphi \| \leq \| \varphi^\# \|_\infty \).
Conversely we can easily check that \( \| \alpha_w \| = 1 \) for \( w \in \mathbb{D} \). So

\[
\| C_\varphi \| \geq \| C_\varphi \alpha_\varphi(w) \| = \sup_{z \in \mathbb{D}} |\varphi^#(z)|(1 - |\varphi(z)|^2) \frac{1 - |\varphi(w)|^2}{|1 - \varphi(w)\varphi(z)|^2} \geq |\varphi^#(w)|. 
\]

Taking the supremum of \( w \) in \( \mathbb{D} \), we have that \( \| C_\varphi \| \geq \| \varphi^# \|_\infty \). \( \square \)

For any \( s \in (0, 1) \), we can always make composition operator \( C_\varphi \) satisfying \( \| C_\varphi \| \epsilon = s \) or \( \| \varphi^# \|_\infty = s \).

**Example 2.3.** Let \( \sigma(z) = (1 + z)/(1 - z) \) and

\[
\varphi_s(z) = \frac{\sigma(z)^s - 1}{\sigma(z)^s + 1} \quad \text{for } s \in (0, 1).
\]

This \( \varphi_s \) is called the lens map. Then \( \| C_{\varphi_s} \| \epsilon = \| \varphi_s^# \|_\infty = s \).

**Proof.** Put \( w = \sigma(z) = re^{i\theta} \). We have that

\[
|\varphi_s^#(z)| = \frac{1 - |z|^2}{1 - |\sigma(z)^s - 1|} \cdot \frac{2s|\sigma'(z)||\sigma(z)^{s-1}|}{|\sigma(z)^s + 1|^2} = \frac{1 - |z|^2}{1 - |1 - z|^2} \cdot \frac{2s|\sigma(z)^{s-1}|}{|\sigma(z)^s + \sigma(z)^s|}
\]

\[
= \frac{1 - |\frac{w}{w+1}|^2}{1 - |\frac{w}{w+1}|^2} \cdot s \frac{r^{s-1}}{r^s \cos s\theta} = \frac{s \cos \theta}{\cos s\theta}.
\]

Since \( \cos s\theta > \cos \theta > 0 \) for \( |\theta| < \pi/2 \), then \( |\varphi_s^#(z)| < s \). Moreover, we can see that \( \varphi_s^#(x) = s \) for all \( x \in (-1, 1) \). This implies that \( \| C_{\varphi_s} \| \epsilon = \| \varphi_s^# \|_\infty = s \). \( \square \)

Our results involve the difference of two Bloch-type derivatives, defined by

\[
\| (C_\varphi - C_\psi) f \| \leq \sup_{z \in \mathbb{D}} \left| \left( 1 - |z|^2 \right) f'(z) - \left( 1 - |w|^2 \right) f'(w) \right|.
\]

Hence we have the following estimations:

\[
\| (C_\varphi - C_\psi) f \| \leq \sup_{z \in \mathbb{D}} \left| \left[ |\varphi^#(z)| - |\psi^#(z)| \right] (1 - |\varphi(z)|^2) f'(\varphi(z)) - \left( 1 - |\psi(z)|^2 \right) f'(\psi(z)) \right|
\]

and

\[
\| C_\varphi - C_\psi \| \leq \sup_{z \in \mathbb{D}} \left( |\varphi^#(z) - \psi^#(z)| + |\varphi^#(z)| b(\varphi(z), \psi(z)) \right).
\]

In [4], the distance \( b(z, w) \) is estimated as

\[
\rho(z, w)^2 \leq b(z, w) \leq C \beta(z, w) + 2\rho(z, w),
\]

where \( C \) is a positive constant independent of \( z, w \). We remark that \( b(z, w) \leq 2 \) by the definition and that for \( \{z_n\}, \{w_n\} \subset \mathbb{D}, \rho(z_n, w_n) \to 0 \) if and only if \( b(z_n, w_n) \to 0 \).

The compactness of \( C_\varphi - C_\psi \) on \( \mathcal{B} \) and \( \mathcal{B}_\alpha \) was characterized in [4]. We here present such results for convenience.
Theorem 2.4 [4]. Let $\varphi, \psi \in S(\mathbb{D})$. Then the following are equivalent:

(i) $C_{\varphi} - C_{\psi}$ is compact on $\mathcal{B}$.

(ii) Both (a) and (b) hold:

(a) $\Gamma^*(\varphi)$ and $\Gamma^*(\psi)$ are included in $\Gamma(\varphi) \cap \Gamma(\psi)$.

(b) For $\{z_n\} \subset \Gamma(\varphi) \cap \Gamma(\psi)$,

$$\lim_{n \to \infty} (\varphi^*(z_n) - \psi^*(z_n)) = 0 \quad \text{and} \quad \lim_{n \to \infty} \varphi^*(z_n) \rho(\varphi(z_n), \psi(z_n)) = \lim_{n \to \infty} \psi^*(z_n) \rho(\varphi(z_n), \psi(z_n)) = 0.$$

Theorem 2.5 [4]. Let $\varphi, \psi \in S(\mathbb{D})$. Then the following are equivalent:

(i) $C_{\varphi} - C_{\psi}$ is compact on $\mathcal{B}_0$.

(ii) Both (a) and (b) hold:

(a) $\lim_{|z| \to 1} (\varphi^*(z) - \psi^*(z)) = 0$.

(b) $\lim_{|z| \to 1} \varphi^*(z) \rho(\varphi(z), \psi(z)) = \lim_{|z| \to 1} \psi^*(z) \rho(\varphi(z), \psi(z)) = 0$.

Relating to these results, we can add the following theorem, which follows from well-known facts in functional analysis using the Schur property.

Theorem 2.6. Let $\varphi, \psi \in S(\mathbb{D})$. Then the following are equivalent:

(i) $C_{\varphi} - C_{\psi} : \mathcal{B} \to \mathcal{B}_0$ is bounded.

(ii) $C_{\varphi} - C_{\psi} : \mathcal{B} \to \mathcal{B}_0$ is compact.

(iii) $C_{\varphi} - C_{\psi} : \mathcal{B}_0 \to \mathcal{B}_0$ is compact.

3. The sets of compact composition operators on the Bloch spaces

Let $\varphi_s(z) = \varphi(sz)$ for each $s \in [0, 1]$. Then

$$\varphi_s^*(z) = \frac{s(1 - |z|^2)}{1 - s^2 |z|^2} \varphi^*(sz).$$

Since $s(1 - |z|^2) \leq 1 - s^2 |z|^2$,

$$|\varphi_s^*(z)| \leq |\varphi^*(sz)|. \quad (3)$$

In this section, we will consider the curve $\{C_{\varphi_s} : s \in [0, 1]\}$ for a compact composition operator $C_{\varphi}$ on $\mathcal{B}_0$.

Proposition 3.1. For $\varphi \in S(\mathbb{D})$, suppose that $C_{\varphi}$ is compact on $\mathcal{B}_0$. Then the map $s \mapsto \varphi_s^*$ is continuous for $s \in [0, 1]$ under the supremum norm on $\mathbb{D}$.

Proof. It is easy to prove that the correspondence $s \mapsto \varphi_s^*$ is continuous for $s \in [0, 1)$. So it is sufficient to consider only the case $s = 1$. 
Suppose that \( s = 1 \). Since \( C_\psi \) is compact on \( B_o \), there exists a constant \( r \in (0, 1) \) such that for any \( 1 > |z| > r \), \(|\phi^#(z)| < \varepsilon/2\). Fix \( r' \in (r, 1) \). If \( t > r/r' \) and \( 1 > |z| > r' \), then \(|tz| > r\). By (3), we have that, for any \( t > r/r' \) and any \( 1 > |z| > r' \), \(|\phi^#(tz)| < \varepsilon/2\). Hence put \( \delta_1 = 1 - r/r' \), then we have that for \( t > 1 - \delta_1 \),

\[
\sup_{z \in \mathbb{D} \cap r' \mathbb{D}} |\phi^#(z) - \phi^#(z)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Similarly, we can estimate the case of \( r' \mathbb{D} \) and obtain that there exists some \( \delta_2 \) such that if \( 1 - t < \delta_2 \),

\[
\sup_{z \in r' \mathbb{D}} |\phi^#(z) - \phi^#(z)| < \varepsilon.
\]

Put \( \delta = \min\{\delta_1, \delta_2\} \). We have that \( \|\phi^# - \phi^#\|_{\infty} < \varepsilon \) for any \( t > 1 - \delta \). So the continuity at \( s = 1 \) is proved. \( \Box \)

**Remark 3.2.** Theorem 5.2.2 of [12] says that if \( f \in B_o \) if and only if \( \|f_r - f\| \to 0 \) as \( r \to 1- \).

Analogously to this result, we can say that \( C_\psi \) is compact on \( B_o \) if and only if \( \|\phi^# - \phi^#\|_{\infty} \to 0 \) as \( t \to 1- \).

In the case of the Bloch space, Toews proved in his thesis [11] that \( C_\psi \sim_{\mathcal{B}} C_\psi \) if both \( C_\psi \) and \( C_\psi \) are compact on \( \mathcal{B} \). Explicitly, he showed that the curve of composition operators \( \{C_\phi^s\} \) is continuous in \( \mathcal{C}(\mathcal{B}) \), so obtained that \( C_\psi \sim_{\mathcal{B}} C_0 \sim_{\mathcal{B}} C_\psi \). In the proof of the following theorem we will show that the dilation transform \( \{\psi_t\} \) induces a continuous curve in \( \mathcal{C}(B_o) \).

**Theorem 3.3.** The set of compact composition operators on \( B_o \) is path connected.

**Proof.** We will show that \( C_\psi \sim_{\mathcal{B}} C_\psi(0) \sim_{\mathcal{B}} C_\psi(0) \sim_{\mathcal{B}} C_\psi \).

Denote the point evaluations at \( p \in \mathbb{D} \) by \( C_p \), that is, \( C_p f(z) = f(p) \) for any \( f \in B_o \) and \( z \in \mathbb{D} \). Then \( C_{\psi_0} = C_{\psi(0)} \) and \( C_{\psi_0} = C_{\psi(0)} \).

At first we note \( C_{\psi_0} \sim_{B_o} C_{\psi_0} \). Indeed, put \( p_s = (1-s)\psi(0) + s\psi(0) \) for \( s \in [0, 1] \). Then \( \{C_{ps} : s \in [0, 1]\} \) is a continuous curve from \( C_{\psi_0} \) to \( C_{\psi_0} \).

Next we will prove \( C_\psi \sim_{B_o} C_{\psi_0} \) and \( C_\psi \sim_{B_o} C_{\psi_0} \), we will show that \( \|C_{\psi_s} - C_{\psi_r}\|_{B_o} \to 0 \) as \( s - t \to 0 \).

Suppose that \( f \in B_o \) and \( \|f\| \leq 1 \). By (2),

\[
\|\(C_{\psi_s} - C_{\psi_r}\) f\|_{B_o} = \|f \circ \psi_s(0) - f \circ \psi_r(0)\| + \|\(C_{\psi_s} - C_{\psi_r}\) f\| \\
\leq \|\phi_s^# - \phi_r^#\|_{\infty} + \sup_{z \in \mathbb{D}} |\phi^#(z)| |\psi_s(z) - \psi_r(z)|.
\]

(4)

From Proposition 3.1, the first term on the right side of (4) converges to 0 as \( t \to s \). So it is sufficient to prove our assertion that if \( t \to s \), the second term of (4) converges to 0.

Fix \( s < 1 \). Without loss of generality, we can suppose that \( t < s \). By [4, Proposition 2.2] and Schwarz’s lemma, we have that
\[
\sup_{z \in \mathbb{D}} |\varphi^#(z)| \beta(\varphi_s(z), \varphi_t(z)) \leq \sup_{z \in \mathbb{D}} (C\beta(\varphi_s(z), \varphi_t(z)) + 2\rho(\varphi_s(z), \varphi_t(z))) \\
\leq C' \sup_{z \in \mathbb{D}} \frac{\rho(\varphi_s(z), \varphi_t(z))}{1 - \rho(\varphi_s(z), \varphi_t(z))} \\
\leq C' \sup_{z \in \mathbb{D}} \frac{\rho(sz, tz)}{1 - \rho(sz, tz)} \\
\leq C' \frac{s - t}{1 - s},
\]

where \( C \) and \( C' \) are constant numbers independent of \( s, t \) and \( z \in \mathbb{D} \). This implies that \( \| C_{\varphi_s} - C_{\varphi_t} \|_{\mathcal{B}_o} \to 0 \) as \( t \to s \).

Next we suppose that \( s = 1 \). Fix any \( \varepsilon > 0 \). Since \( C_{\varphi} \) is compact on \( \mathcal{B}_o \), there exists \( r \in (0, 1) \) such that \(|\varphi^#(z)| < \varepsilon / 2\) for any \( z \in \mathbb{D} \setminus r\mathbb{D} \). By \( \beta(\varphi(z), \varphi_t(z)) \leq 2 \), we have that

\[
\sup_{z \in \mathbb{D} \setminus r\mathbb{D}} |\varphi^#(z)| \beta(\varphi(z), \varphi_t(z)) < \varepsilon.
\]

By the similar method as in the case that \( s < 1 \), there exists a constant \( C \) such that

\[
\sup_{z \in r\mathbb{D}} |\varphi^#(z)| \beta(\varphi(z), \varphi_t(z)) \leq C(1 - t).
\]

Thus, for \( t > 1 - \varepsilon / C \),

\[
\sup_{z \in \mathbb{D}} |\varphi^#(z)| \beta(\varphi(z), \varphi_t(z)) < \varepsilon.
\]

Hence we have that \( C_{\varphi} \sim_{\mathcal{B}_o} C_{\varphi_0} \).

By the similar way, we have that \( C_{\psi} \sim_{\mathcal{B}_o} C_{\psi_0} \). Our proof is completed. \( \square \)

Using this theorem, we can present the difference between the isolation problems on \( H^\infty \) and on \( \mathcal{B}_o \).

**Example 3.4.** There exist analytic self-maps \( \varphi, \psi \in S(\mathbb{D}) \) such that corresponding composition operators \( C_{\varphi} \) and \( C_{\psi} \) are isolated in \( \mathcal{C}(H^\infty) \) but lie in the same component in \( \mathcal{C}(\mathcal{B}_o) \).

**Proof.** Indeed, we can find out inner functions \( \varphi, \psi \) satisfying

\[
\varphi^#(z), \psi^#(z) \to 0
\]
as \(|z| \to 1\) in [1] or [10]. Then \( C_{\varphi} \) and \( C_{\psi} \) are isolated in \( \mathcal{C}(H^\infty) \) because \( \varphi \) and \( \psi \) are inner functions.

On the other hand, \( C_{\varphi} \) and \( C_{\psi} \) are compact on \( \mathcal{B}_o \) [6] and so lie in the same component in \( \mathcal{C}(\mathcal{B}_o) \) by Theorem 3.3. \( \square \)

4. Compact differences and components

In this section, we consider the relationship between the compact differences of two composition operators and the components of \( \mathcal{C}(\mathcal{B}) \) and \( \mathcal{C}(\mathcal{B}_o) \).
For \( s \in [0, 1] \), put \( z_s = (1 - s)z + sw \) and \( \varphi_s(z) = (1 - s)\varphi(z) + s\psi(z) \). Then it is easy to see that \( \Gamma(\varphi_s) \subset \Gamma(\varphi) \cap \Gamma(\psi) \) and that for \( z \in \mathbb{D} \),

\[
1 - |\varphi_s(z)| \geq (1 - s)(1 - |\varphi(z)|).
\]

We start from the following lemma.

**Lemma 4.1.** Let \( z, w \in \mathbb{D} \) and \( \rho(z, w) = \lambda < 1 \). Then the map \( s \mapsto \rho(z_s, w) \) is continuous and decreasing on \([0, 1]\).

**Proof.** We have that

\[
\rho(z_s, w) = \left| \frac{1 - s}{1 - zw} + s \frac{z}{w} \right| \leq \frac{1 - s}{\lambda - 1 - s} = \frac{(1 - s)\lambda}{1 - s\lambda}.
\]

This implies the continuity at \( s = 1 \), that is, \( \rho(z_s, w) \to 0 \) as \( s \to 1 \).

Next, suppose that \( 0 \leq s < t < 1 \). Put \( \tau = z_t \) and \( \tau_u = (1 - u)z + u\tau \). Then \( z_s = \tau_{s/t} \).

Since \( \rho(z_s, z_t) = \rho(\tau_{s/t}, \tau) \), we can see that \( \rho(z_s, z_t) \to 0 \) as \( s - t \to 0 \) and \( s \mapsto \rho(z_s, w) \) is decreasing by the inequality above. \( \square \)

Next, we give the following lemma.

**Lemma 4.2.** Suppose that \( \varphi, \psi \in S(\mathbb{D}) \) and \( s \in [0, 1] \). Then, for each \( z \in \mathbb{D} \),

\[
|\varphi^#(z) - \varphi_s^#(z)| \leq |\varphi^#(z) - \psi^#(z)| + |\varphi^#(z)|\rho(\varphi(z), \psi(z))^2.
\]

**Proof.** We have that

\[
\varphi_s^#(z) = (1 - s)\frac{1 - |\varphi(z)|^2}{1 - |\varphi_s(z)|^2}\varphi^#(z) + s\frac{1 - |\psi(z)|^2}{1 - |\varphi_s(z)|^2}\psi^#(z).
\]

Then

\[
|\varphi^#(z) - \varphi_s^#(z)| \leq |\varphi^#(z) - \psi^#(z)|\frac{s(1 - |\psi(z)|^2)}{1 - |\varphi_s(z)|^2} + |\varphi^#(z)|\left| 1 - (1 - s)\frac{1 - |\varphi(z)|^2}{1 - |\varphi_s(z)|^2} - s\frac{1 - |\psi(z)|^2}{1 - |\varphi_s(z)|^2} \right|.
\]

(5)

Here we estimate the first term on the right-hand side of (5),

\[
\frac{s(1 - |\psi(z)|^2)}{1 - |\varphi_s(z)|^2} \leq \left| \frac{s(1 - |\psi(z)|^2)}{1 - (1 - s)^2 - 2s(1 - s)|\psi(z)| - 2s(1 - s)|\psi(z)|^2} \right| \leq 1 - |\psi(z)|^2
\]

\[
= \frac{1 + |\psi(z)|}{2 - s(1 - |\psi(z)|)} \leq 1.
\]

(6)

Next, we estimate the second term of (5):
\[
1 - (1 - s) \frac{1 - |\varphi(z)|^2}{1 - |\varphi_s(z)|^2} - s \frac{1 - |\psi(z)|^2}{1 - |\varphi_s(z)|^2} \leq \left| 1 - \frac{(1 - s)|\varphi(z)|^2 + s|\psi(z)|^2 - |\varphi_s(z)|^2}{1 - (1 - s)^2 - s^2 - 2s(1 - s) \text{Re}(\varphi(z)\psi(z))} \right| \\
\leq \frac{|\varphi(z)|^2 + |\psi(z)|^2 - 2 \text{Re}(\varphi(z)\psi(z))}{2(1 - \text{Re}(\varphi(z)\psi(z)))} \\
= \frac{|\varphi(z) - \psi(z)|^2}{2(1 - \text{Re}(\varphi(z)\psi(z)))} \\
= \rho(\varphi(z), \psi(z))^2 \frac{1 - |\varphi(z)\psi(z)|^2}{2(1 - \text{Re}(\varphi(z)\psi(z)))} \\
= \rho(\varphi(z), \psi(z))^2 \frac{1 + |\varphi(z)|^2|\psi(z)|^2 - 2 \text{Re}(\varphi(z)\psi(z))}{2(1 - \text{Re}(\varphi(z)\psi(z)))} \\
\leq \rho(\varphi(z), \psi(z))^2. \\
\tag{7}
\]

By (5)–(7), we have that
\[
\left| \varphi_s'(z) - \varphi'(z) \right| \leq \left| \varphi'(z) - \psi'(z) \right| + \left| \varphi'(z) \right| \rho(\varphi(z), \psi(z))^2.
\]

\textbf{Proposition 4.3.} Let \( \varphi \) and \( \psi \) be in \( S(D) \) such that \( C_\varphi - C_\psi \) is compact on \( B \). Then for any \( s \in [0, 1] \), the following hold:

(i) \( \Gamma^#(\varphi_s) \subset \Gamma(\varphi) \cap \Gamma(\psi) \).

(ii) For any \( \{z_n\} \subset \Gamma(\varphi) \cap \Gamma(\psi) \),
\[
\lim_{n \to \infty} (\varphi'(z_n) - \varphi_s'(z_n)) = \lim_{n \to \infty} \varphi'(z_n) \rho(\varphi(z_n), \varphi_s(z_n)) = 0.
\]

Moreover, \( C_\varphi - C_\psi_s \) is compact on \( B \) for any \( s \in [0, 1] \).

\textbf{Proof.} (i) By the definition, \( \Gamma^#(\varphi_s) \subset \Gamma(\varphi_s) \). Since \( \Gamma(\varphi_s) \subset \Gamma(\varphi) \cap \Gamma(\psi) \), we have that \( \Gamma^#(\varphi_s) \subset \Gamma(\varphi) \cap \Gamma(\psi) \).

(ii) Since \( C_\varphi - C_\psi \) is compact on \( B \), Theorem 2.4 asserts that
\[
\lim_{n \to \infty} (\varphi'(z_n) - \psi'(z_n)) = \lim_{n \to \infty} \varphi'(z_n) \rho(\varphi(z_n), \psi(z_n)) = 0
\]
for any \( \{z_n\} \subset \Gamma(\varphi) \cap \Gamma(\psi) \). By Lemma 4.2, for any \( \{z_n\} \subset \Gamma(\varphi) \cap \Gamma(\psi) \),
\[
\left| \varphi'(z_n) - \psi'(z_n) \right| \leq \left| \varphi'(z_n) - \psi'(z_n) \right| + \left| \varphi'(z_n) \right| \rho(\varphi(z_n), \psi(z_n))^2 \to 0.
\]

Next, by Lemma 4.1, we have that
\[
\left| \varphi'(z_n) \right| \rho(\varphi(z_n), \varphi_s(z_n)) \leq \left| \varphi'(z_n) \right| \rho(\varphi(z_n), \psi(z_n)) \to 0.
\]

Hence we get (ii).

Then the last part of the assertion follows from (i) immediately. 

Similar consequence holds on \( B_0 \).
Proposition 4.4. Let \( \varphi \) and \( \psi \) be in \( S(D) \) such that \( C\varphi \) and \( C\psi \) are bounded on \( B_\alpha \). If \( C\varphi - C\psi \) is compact on \( B_\alpha \), then \( C\varphi - C\psi_{s} \) is compact on \( B_\alpha \) for any \( s \in [0, 1] \), that is,

\[
\lim_{|z| \to 1} (\varphi^#(z) - \varphi_s^#(z)) = \lim_{|z| \to 1} \varphi^#(z) \rho(\varphi(z), \varphi_s(z)) = 0.
\]

The following theorem is a main result in this section.

Theorem 4.5. Let \( \varphi, \psi \in S(D) \). Suppose that \( C\varphi - C\psi \) is compact on \( B \). Then the following are equivalent:

(i) \( \varphi^#(z_n) \to 0 \) on \( \Gamma(\psi) \setminus \Gamma(\varphi) \) and \( \psi^#(z_n) \to 0 \) on \( \Gamma(\varphi) \setminus \Gamma(\psi) \).

(ii) The map \( s \to C\varphi_s \) is continuous from \([0, 1]\) to \( C(B) \).

Proof. (i) \( \Rightarrow \) (ii). To prove this implication, it is sufficient to consider only the continuity at \( s = 0 \). By (2),

\[
\|C\varphi - C\varphi_{s}\|_{B} \leq \beta(\varphi(0), \varphi_s(0)) + \|\varphi^# - \varphi_s^#\|_{\infty} + \sup_{z \in \mathbb{D}} |\varphi^#(z)| \rho(\varphi(z), \varphi_s(z)).
\]

It is trivial that \( \beta(\varphi(0), \varphi_s(0)) \to 0 \) as \( s \to 0 \).

Fix \( \varepsilon > 0 \). By the assumption and Theorem 2.4, we have that for any \( \{z_n\} \subset \Gamma(\varphi) \),

\[
\lim_{n \to \infty} (\varphi^#(z_n) - \psi^#(z_n)) = \lim_{n \to \infty} \varphi^#(z_n) \rho(\varphi(z_n), \psi(z_n)) = 0.
\]

Then there exists some \( r_1 \in (0, 1) \) such that for any \( z \in \Gamma_1(\varphi) \), \( |\varphi^#(z) - \psi^#(z)| < \varepsilon / 2 \) and \( |\varphi^#(z_n)| \rho(\varphi(z_n), \psi(z_n)) < \varepsilon / 2 \). By Lemma 4.2, we have that for any \( z \in \Gamma_1(\varphi) \),

\[
|\varphi^#(z) - \varphi_s^#(z)| \leq |\varphi^#(z) - \psi^#(z)| + |\varphi^#(z)| \rho(\varphi(z), \psi(z))^2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

(8)

On \( \mathbb{D} \setminus \Gamma_1(\varphi) \), \( \varphi^#(z) - \varphi_s^#(z) \) converges uniformly to 0 as \( s \to 0 \). Thus there exists some \( s_1 \) so close to 0 that for any \( s < s_1 \),

\[
\sup_{z \in \mathbb{D} \setminus \Gamma_1(\varphi)} |\varphi^#(z) - \varphi_s^#(z)| < \varepsilon.
\]

(9)

Combining (8) and (9), we have that for any \( s < s_1 \),

\[
\sup_{z \in \mathbb{D}} |\varphi^#(z) - \varphi_s^#(z)| < \varepsilon.
\]

Hence we get

\[
\sup_{z \in \mathbb{D}} |\varphi^#(z) - \varphi_s^#(z)| \to 0 \quad \text{as} \quad s \to 0.
\]

Next we prove

\[
\sup_{z \in \mathbb{D}} \varphi^#(z) \rho(\varphi(z), \varphi_s(z)) \to 0 \quad \text{as} \quad s \to 0.
\]

(10)

as \( s \to 0 \). Fix \( \varepsilon > 0 \) again. By Proposition 4.3 and Theorem 2.4, for any \( \{z_n\} \subset \Gamma(\varphi) \),

\[
\lim_{n \to \infty} \varphi^#(z_n) \rho(\varphi(z_n), \varphi_s(z_n)) = 0.
\]
This implies that there exists some $r_2 \in (0, 1)$ such that for any $z \in \Gamma_{r_2}(\varphi)$,
\[ |\varphi^#(z)| \beta(\varphi(z), \varphi_s(z)) < \varepsilon. \]
Since $\beta(\varphi(z), \varphi_s(z))$ converges uniformly to 0 on $\mathbb{D} \setminus \Gamma_{r_2}(\varphi)$ as $s \to 0$, there exists some $s_2$ so close to 0 that for any $s < s_2$,
\[ \sup_{z \in \mathbb{D} \setminus \Gamma_{r_2}(\varphi)} |\varphi^#(z)| \beta(\varphi(z), \varphi_s(z)) < \varepsilon. \]

Now we get (10).

Consequently, by the inequality (2), we have that $\|C\varphi - C\varphi_s\|$ converges to 0 as $s \to 0$. Similarly, we can prove the continuity at $s = 1$ using the condition that $\psi^#(z_n) \to 0$ on $\Gamma(\varphi) \setminus \Gamma(\psi)$.

(ii) $\Rightarrow$ (i). Suppose that there exists a sequence $\{z_n\} \in \Gamma(\psi) \setminus \Gamma(\varphi)$ such that $\varphi^#(z_n) \to \omega \neq 0$.

Since $\|\alpha_w\| = 1$ for each $w \in \mathbb{D}$, we have that
\[ \|C\varphi - C\varphi_s\| \geq \| (C\varphi - C\varphi_s) \alpha(\varphi(z_n)) \| \]
\[ = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \varphi'(z) \frac{1 - |\varphi(z_n)|^2}{(1 - \varphi(z_n)\overline{\varphi}(z))^2} - \frac{1 - |\varphi(z_n)|^2}{(1 - \varphi(z_n)\varphi_s(z))^2} \right| \]
\[ \geq (1 - |z_n|^2) \left| \frac{1}{1 - |\varphi(z_n)|^2} \varphi'(z_n) - \varphi_s'(z_n) \right| \frac{1 - |\varphi(z_n)|^2}{(1 - \varphi(z_n)\varphi_s(z))^2} \]
\[ \geq |\varphi^#(z_n)| - |\varphi_s^#(z_n)| \frac{(1 - |\psi(z_n)|^2)(1 - |\varphi(z_n)|^2)}{|1 - \varphi(z_n)\varphi_s(z_n)|^2}. \]
\[ \geq |\varphi^#(z_n)| - |\varphi_s^#(z_n)| \left(1 - \rho(\varphi(z_n), \varphi_s(z_n))^2 \right). \] (11)

So, taking the limit, we obtain that
\[ \|C\varphi - C\varphi_s\| \geq |w| > 0 \quad \text{for} \ s \in (0, 1]. \]

This implies that the map $s \to C\varphi_s$ is not continuous at $s = 0$. This contradicts the condition (ii).

**Corollary 4.6.** Let $\varphi$ and $\psi$ be in $S(\mathbb{D})$ such that neither $C\varphi$ nor $C\psi$ are compact but $C\varphi - C\psi$ is compact on $B$. Suppose $\Gamma(\psi) = \Gamma(\varphi)$. Then $C\varphi \sim_B C\psi$.

In the case of $H^\infty$, if $C\varphi - C\psi$ is compact on $H^\infty$, then $\Gamma(\varphi) = \Gamma(\psi)$ holds. But in the case of $B$, this implication is not always true. For example, let $I$ be an inner function satisfying $\lim_{|z| \to 1} |I^#(z)| = 0$. For $\xi \in \partial\mathbb{D}$, we put $\varphi_\xi(z) = (1 + \overline{\xi} I(z))/2$. Then we can easily check that $\lim_{|z| \to 1} |\varphi_\xi(z)| = 0$ and that $C\varphi_\xi$ is compact on $B$. So when we take two different points $\xi_1, \xi_2 \in \partial\mathbb{D}$, both $C\varphi_{\xi_1}$ and $C\varphi_{\xi_2}$ are compact on $B$ and so that $C\varphi_{\xi_1} \sim_B C\varphi_{\xi_2}$. On the other hand, $\Gamma(\varphi_{\xi_1}) \neq \Gamma(\varphi_{\xi_2})$.

About the converse of Corollary 4.6, Toews [11] presented an example which shows that there exist composition operators with non-compact differences that lie in the same component of $\mathcal{C}(B)$.

Similarly, we can prove the following theorem.
Theorem 4.7. Let \( \varphi \) and \( \psi \) be in \( S(\mathbb{D}) \) such that \( C_\varphi \) and \( C_\psi \) are bounded on \( \mathcal{B}_\alpha \). If \( C_\varphi - C_\psi \) is compact on \( \mathcal{B}_\alpha \), then \( C_\varphi \sim C_\psi \) in \( \mathcal{C}(\mathcal{B}_\alpha) \).

5. Isolation

Finally we give a remark to the isolation of \( \mathcal{C}(\mathcal{B}) \). To compare the topology, we can see that if \( C_\varphi \) is isolated in \( \mathcal{C}(\mathcal{B}) \), then \( C_\varphi \) is isolated in \( \mathcal{C}(\mathcal{H}_\infty) \). But Example 3.4 shows that the converse does not always hold.

We will give a sufficient condition for isolated points in \( \mathcal{C}(\mathcal{B}) \). We denote by \( \Gamma^\#(\varphi) \) the limit point of \( \Gamma^\#(\varphi) \). Then \( \Gamma^\#(\varphi) \) is a subset of \( \partial \mathbb{D} \).

Theorem 5.1. If \( \Gamma^\#(\varphi) \) has positive measure, then \( C_\varphi \) is isolated in \( \mathcal{C}(\mathcal{B}) \).

Proof. Suppose that there exists a positive constant \( \varepsilon \) such that \( E = \{ \omega \in \Gamma^\#(\varphi) : |\varphi^\#(\omega)| > \varepsilon \} \) has positive measure. Hence, if \( \psi \) is in \( S(\mathbb{D}) \) and \( \psi \neq \varphi \), then there exists a sequence \( \{z_n\} \subset \mathbb{D} \) such that \( z_n \to \omega \in E \) and \( \varphi(z_n) - \psi(z_n) \not\to 0 \).

So we have that

\[
\|C_\varphi - C_\psi\| \geq \|\alpha_{\varphi(z_n)}\| \geq |\varphi^\#(z_n)| - |\psi^\#(z_n)| \left(1 - \rho(\varphi(z_n), \psi(z_n))^2\right).
\]

As \( z_n \to w \),

\[
\|C_\varphi - C_\psi\| \geq |\varphi^\#(w)| > \varepsilon.
\]

This means that \( C_\psi \) is far from any other \( C_\psi \) at least \( \varepsilon \). \( \Box \)

If \( \varphi \) has finite angular derivatives on a set of positive measure, then \( \varphi \) satisfies the condition of Theorem 5.1. So \( \varphi(z) = z \) and Möbius transformations of \( \mathbb{D} \) induce isolated composition operators in \( \mathcal{C}(\mathcal{B}) \) respectively.

References