Exchange property and the natural preorder between simple modules over semi-Artinian rings ♠

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Abstract

We prove that if \( R \) is a right semi-Artinian ring, then \( R \) is an exchange ring and every irredundant set \( \text{Simp}_R \) of representatives of simple right \( R \)-modules carries a canonical structure of an Artinian poset, which is a Morita invariant. We investigate several basic features of this order structure and, for a wide class of right semi-Artinian rings, which we call nice, we establish a link between those (two-sided) ideals which are pure as left ideals and some upper subsets of \( \text{Simp}_R \). If \( R \) is nice and \( \text{Simp}_R \) does not contain infinite antichains, then that link realizes an anti-isomorphism from the lattice of upper subsets of \( \text{Simp}_R \) to the set of all ideals which are pure as left ideals. Further we show that every Artinian poset (possibly after adding a suitable maximal element if it is infinite) is order isomorphic to \( \text{Simp}_R \) for some nice right semi-Artinian ring \( R \). © 2002 Elsevier Science (USA). All rights reserved.

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0. Introduction

During the last quarter of the past century the exchange property has drawn the interest of many authors working in the theory of rings and modules. We recall that a (unital) right module $A$ over a ring $R$ (with identity) satisfies the exchange property if, whenever we have a right $R$-module $M$ and decompositions

$$M = A' \oplus B = \bigoplus_{i \in I} M_i$$

with $A' \cong A$, there is for each $i \in I$ a submodule $M'_i$ of $M_i$ such that

$$M = A' \oplus \left( \bigoplus_{i \in I} M'_i \right).$$

Note that, by the modular law, each $M'_i$ must be a direct summand of $M_i$. The module $A$ is said to satisfy the finite exchange property if it satisfies the above property for every finite index set $I$; of course, if $A$ is finitely generated, then it satisfies the exchange property if and only if it satisfies the finite exchange property. According to Warfield [25] the ring $R$ is an exchange ring if the module $RR$ satisfies the exchange property; this condition is right/left symmetric by [25, Corollary 2]. A nice ring-theoretical characterization of exchange rings was found independently by Goodearl and Nicholson: the ring $R$ is an exchange ring if and only if, for every $a \in R$, there is an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$; in addition Nicholson proved in [21] that $R$ is an exchange ring if and only if idempotents lift modulo every left (respectively right) ideal. Another interesting feature of exchange rings is that every finitely generated projective module over them is isomorphic to a finite direct sum of principal right ideals generated by idempotents (see [25, Theorem 1]); in other words, if $R$ is an exchange ring, then the abelian monoid $V(R)$ of isoclasses of finitely generated projective right $R$-modules is generated by the set $\{ [eR] \mid e = e^2 \in R \}$.

Semi-Artinian modules and rings too have been extensively studied over the years. The starting point is the the Loewy chain of a right $R$-module $M$, namely the nondecreasing chain of submodules $(\text{Soc}_{\alpha}(M))_{\alpha \geq 0}$ defined as follows: set $\text{Soc}_0(M) = 0$ and, recursively, $\text{Soc}_{\alpha+1}(M)/\text{Soc}_{\alpha}(M) = \text{Soc}(M/\text{Soc}_{\alpha}(M))$ (we denote by $\text{Soc}(M)$ the socle of $M$) for each ordinal $\alpha$ and $\text{Soc}_{\alpha}(M) = \bigcup_{\beta < \alpha} \text{Soc}_{\beta}(M)$ if $\alpha$ is a limit ordinal. The module $M/\text{Soc}_{\alpha}(M)$ is called the $\alpha$th Loewy factor of $M$, the first ordinal $\xi$ such that $\text{Soc}_{\xi}(M) = \text{Soc}_{\xi+1}(M)$ is called the Loewy length of $M$ (denoted by $L(M)$) and one says that $M$ is semi-Artinian or a Loewy module if $\text{Soc}_{\xi}(M) = M$. The ring $R$ is right semi-Artinian if the module $RR$ is semi-Artinian or, equivalently, if every nonzero right $R$-module contains a simple submodule. Of course, right Artinian rings are right semi-Artinian; more generally, since the Jacobson radical of a right semi-Artinian ring is left T-nilpotent (see [1, Remark 28.5]), then left perfect rings are precisely the semiperfect and right semi-Artinian rings. Camillo and Fuller proved in [12]
that if $R$ is a right semi-Artinian ring and $L(R_R) = n$ is a natural number, then $R$ is also a left semi-Artinian ring with $L(R_R) \leq 2^n - 1$. Construction of either one-sided or two-sided semi-Artinian rings with arbitrary Loewy length were given (in order of time) by Fuchs [16], Ososky [22], Camillo and Fuller [12], and Dung and Smith [15]. More recently we gave additional constructions of noncommutative semi-Artinian rings in [8] (with emphasis on the Von Neumann regular case) and in [9].

The first objective of the present work is to show that the class of exchange rings includes the class of right semi-Artinian rings. The decisive tool will be essentially a recent result of Ara, who extended the concept of an exchange ring to rings without unit; among others he proved that a (possibly nonunital) ring $R$ is an exchange ring if and only if, for every ideal $I$, both $R/I$ and $I$ are exchange rings and idempotents lift modulo $I$ (see [2, Theorem 2.2]). In addition we show that if $R$ is a right semi-Artinian ring, then the monoid $V(R)$ satisfies a restricted form of cancelability, namely, given $x, y \in V(R)$, if $x + x = x + y$ then $x = y$; abelian monoids satisfying such condition are known as strongly separative monoids (see [3,10]).

In the second section, for a given right semi-Artinian ring $R$ we define the natural preorder in the class $S_R$ of all simple right $R$-modules, and hence the natural partial order in $\text{Simp}_R$. The idea is to associate to every $U \in \text{Simp}_R$ a particular $U$-peak ideal $I(U)$ (in the sense that the right socle of $R/I(U)$ is essential, projective and $U$-homogeneous) in the following manner. First, set $L_\alpha = \text{Soc}_\alpha(R_R)$ for all $\alpha$ and, for every module $M_R$, define $h(M)$ as the smallest ordinal such that $M_{\alpha+1} = M$; note that $h(M)$ is nonlimit if $M$ is finitely generated. Given a simple module $U_R$, if $\alpha + 1 = h(U)$, then $U$ is $R/L_\alpha$-projective (see [9, Lemma 1.2]); define $I(U)$ as the left annihilator of the $U$-homogeneous component of the right socle of $R/L_\alpha$. Next declare that $U \preceq V$ in case $I(U) \subseteq I(V)$; then $\preceq$ is what we call the natural preorder in $S_R$. Since $U \simeq V$ if and only if $I(U) = I(V)$, then $\preceq$ induces the natural partial order in $\text{Simp}_R$ and we prove that it is a Morita invariant of the ring $R$. With respect to its natural partial order $\text{Simp}_R$ is an Artinian poset in which every maximal chain has a maximum.

It is worth to observe that, since the class of right semi-Artinian rings is closed by factor rings, for every $U \in \text{Simp}_R$ the primitive ring $R/rR(U)$ has nonzero socle and $U$ is the unique (up to an isomorphism) simple and faithful right $R/rR(U)$-module; thus the assignment $U \mapsto rR(U)$ defines a bijection from $\text{Simp}_R$ to the set $\text{Prim}_R$ of (right) primitive ideals of $R$. In view of this fact it would appear quite obvious to define a “natural” partial order in $\text{Simp}_R$ by declaring that $U \preceq V$ in case $rR(U) \subseteq rR(V)$; in addition we must record that Camillo and Fuller already observed in [13] that the set $\text{Prim}_R$, ordered by inclusion, is always Artinian when $R$ is right semi-Artinian. The point is that in many interesting cases $\text{Prim}_R$ is just the set of all maximal (two-sided) ideals and the above partial order becomes the trivial one, giving thus no information
on the structure of $R$. For instance this happens if $R$ is left perfect (in particular when $R$ is right Artinian); in this case the partial order we introduce is trivial if and only if $R$ is semisimple.

For every $U \in \text{Simp}_R$ it is true that $I(U) \subseteq r_R(U)$ and the equality holds if $R$ is either commutative or is regular (always in the sense of Von Neumann in the present paper). Consequently, if $R$ is commutative, then $\{I(U) \mid U \in \text{Simp}_R\}$ is simply the set of all maximal ideals of $R$ and the natural partial order of $\text{Simp}_R$ is the discrete one, therefore our investigations will not add any insight to commutative semi-Artinian ring theory. The regular case is, on the contrary, of a great interest and will be investigated in a forthcoming paper.

Our concern in the third section are idempotent ideals of semi-Artinian rings. It is well known that if $P$ is a projective right module over some ring $R$, then the trace ideal $\text{Tr}_R(P)$ of $P$ is idempotent, but the converse may fail, namely an idempotent ideal need not be the trace of some projective module. It is not difficult to prove that if $R$ is a semiprimary ring (that is, the Jacobson radical $J(R)$ of $R$ is nilpotent and $R/J(R)$ is semisimple), then the converse holds; specifically, if $\mathbb{E}$ is a basic set of primitive idempotents of $R$, then every idempotent ideal of $R$ has the form $Re_1R + \cdots + Re_nR$ for some $e_1, \ldots, e_n \in \mathbb{E}$. The role of idempotent ideals in the theory of quasi-hereditary semiprimary rings has been well established by Burgess and Fuller in [14] and by Dlab and Ringel in [11], while Auslander, Platzer and Todorov explored in [4] the homological properties of idempotent ideals of Artin algebras. However, earlier investigations on idempotent ideals of perfect rings were made, over thirty years ago, by Michler in [20] where, among others, he proved that the above outlined property of a semiprimary ring $R$ still holds when $R$ is left perfect and $R$ has $2^n$ idempotent ideals, where $n = |\text{Simp}_R|$. Thanks to the exchange property, we can extend to right semi-Artinian rings the result of Michler. We wish to remark that, however, the above connection between the number of isoclasses of simple modules and the number of idempotent ideals may fail for nonperfect right semi-Artinian rings; in fact, with the construction we gave in [8, Example 4.3], we see that for any given integer $n \geq 2$ there is a semi-Artinian Von Neumann regular ring $R$ such that $|\text{Simp}_R| = n$, but $R$ has exactly $n + 1$ ideals, which are necessarily idempotent. A particularly interesting situation is when $R$ is a right NLF-ring, that is all Loewy factor rings of $R$ are right nonsingular or, equivalently, when $\text{Soc}_\alpha(RR)$ is a pure left ideal for each $\alpha$ (see [9]); in fact, if it is the case, then for every $U \in \text{Simp}_R$ the ideal $I(U)$ is idempotent and is the smallest $U$-peak ideal.

Beside the natural partial order $\preceq$ in $\text{Simp}_R$ we consider the following relation: if $U, V \in \text{Simp}_R$, write $U \dashv V$ whenever $V \cdot I(U) = 0$. It turns out that always $U \preceq V$ implies $U \dashv V$ and we show by an example that the converse may fail. However, if $R$ is regular, then the converse holds since $I(U)$ is the annihilator of $U$.

The theme of the fourth section is the investigation of what we call nice right semi-Artinian rings; these are the right NLF-rings for which the relations $\preceq$ and
We prove that a right semi-Artinian ring \( R \) is nice if and only if \( I(U) \) is a pure left ideal for every \( U \in \text{Simp}_R \). Without any assumption on the right semi-Artinian ring \( R \), the assignment \( I \mapsto \text{Simp}_R/I \) defines an injective and strictly decreasing map from the set \( \mathbb{L}_2^p(R) \) of all ideals which are pure in \( _RR \) and the complete lattice \( \uparrow \text{Simp}_R \) of all upper subsets of \( \text{Simp}_R \). If \( R \) is nice, an element \( U \in \text{Simp}_R \) is maximal if and only if \( UR \) is injective and the vector space \( \text{End}(UR) \) is finite dimensional; if, in addition, \( \text{Simp}_R \) has no infinite antichains, we prove that the above map is an anti-isomorphism and, as a consequence, \( \mathbb{L}_2^p(R) \) is a complete Artinian lattice.

In the fifth section, after introducing an extension to rings without identity of the concept of a piecewise domain given by Gordon and Small in [18] and applying it to semi-Artinian rings, we prove that, given a right semi-Artinian ring \( R \), if \( R \) is either right semihereditary, or \( R \) is a piecewise domain, or \( \text{Soc}_{\xi}(RR) \) is a piecewise domain (where \( \xi + 1 \) is the Loewy length of \( RR \)), then \( R \) is nice.

Finally, in the sixth and last section we are faced with the problem of determining when a poset \( P \) is admissible, in the sense that \( P \) is order isomorphic to \( \text{Simp}_R \) (with its natural partial order) for some right semi-Artinian ring \( R \). As we stated previously, necessary conditions for the admissibility of \( P \) are that \( P \) is Artinian and all maximal chains of \( P \) have a maximum. Presently we are neither able to give a general characterization of all admissible posets, nor to decide if the above necessary conditions are sufficient. However, by mean of a construction we had presented in [9], we can give a wide class of admissible posets, which includes all Artinian posets which have a finite cofinal subset, in particular all finite posets.

Throughout this paper the term “ring” means ring with identity, unless otherwise specified, and all modules are unital. However, in order to avoid iterated specifications, we often refer to a two-sided ideal \( I \) of some ring (with identity) \( R \) by considering it as a ring, even if it need not have an identity (it has one if and only if it is generated by a central idempotent). For every ring \( R \) we denote by \( S_R \) the class of all simple right \( R \)-modules, while \( \text{Simp}_R \) will be a chosen irredundant and representative subset of \( S_R \); however, if \( A \subset S_R \), we shall write \( A = \text{Simp}_R \) meaning that every \( U \in S_R \) is isomorphic to exactly one member of \( A \). If \( I \) is an ideal of \( R \), then we take \( \text{Simp}_{R/I} \) as the subset of \( \text{Simp}_R \) of those members annihilated by \( I \). If \( M \) is any right \( R \)-module, \( \text{Soc}(M) \) and \( J(M) \) denote respectively the socle and the Jacobson radical of \( M \); the notation \( N \leq_e M \) means that \( N \) is an essential submodule of \( M \). If \( U \in \text{Simp}_R \), the \( U \)-homogeneous component of \( \text{Soc}(M) \) is the trace \( \text{Tr}_M(U) \) of \( U \) and we say that \( \text{Soc}(M) \) is \( U \)-homogeneous when it coincides with the trace of \( U \).

We say that an ideal \( I \) of a ring \( R \) is left pure if \( I \) is pure as a submodule of \( _RR \), that is \( _RR/I \) is flat; recall that this happens if and only if \( \text{Mod}_{R/I} \) is a TTF class (i.e., a torsion and torsion-free class) which is closed by \( R \)-injective envelopes, if and only if \( \{ M_R \mid M_1 = M \} \) is a hereditary torsion class. Often we shall make use of the following result from [5]: given a subset \( A \subset \text{Simp}_R \), each member
of \( A \) is projective if and only if \( \text{Tr}_R(A) \) is idempotent, if and only if \( \text{Tr}_R(A) \) is left pure, if and only if \( \text{Tr}_M(A) = M \text{Tr}_R(A) \) for every module \( M_R \); moreover the idempotent subideals of \( \text{Soc}(R_R) \) are precisely the direct sums of projective and homogeneous components. Finally, \( \mathbb{I}_2(R) \) denotes the lattice of all ideals of \( R \).

1. Right semi-Artinian rings are exchange rings

By a classical result (due to Herstein; see [1, Proposition 27.1]), if \( I \) is a nil two-sided ideal of a ring \( R \) with identity and \( x \) is an element of \( R \) such that \( x + I \) is idempotent in \( R/I \), then there is an idempotent \( e = e^2 \in xI \) such that \( x - e \in I \). We shall make use of the fact that, consequently, if \( L \) is a two-sided ideal of \( R \) such that \( I \subset L \), then idempotents of the ring \( L/I \) lift to idempotents of \( L \).

Let \( I \) be a ring without unit and let \( R \) be a (unital) ring containing \( I \) as an ideal. We say that \( I \) is an exchange ring if for every \( x \in I \) there exists \( e = e^2 \in xI \) such that \( 1 - e \in (1 - x)R \) (see [2, Lemma 1.1]; this is the extension to nonunital rings of the usual notion of an unital exchange ring). Also, we say that \( I \) is (Von Neumann) regular if for each \( x \in I \) there is some \( y \in I \) such that \( x = xyx \). Note that if \( I \) is regular, then it is an exchange ring.

Lemma 1.1. If \( I \) is a regular ideal of a ring \( R \), then idempotents lift modulo \( I \).

Proof. Let \( x + I \) be an idempotent of \( R/I \). Then \( x^2 - x \in I \) and hence \( x^2 - x = (x^2 - x)y(x^2 - x) \) for some \( y \in I \). By taking \( z = 1 - (1 - x)y(1 - x) \), we obtain that \( x = xzx \) and so \( xz \) is an idempotent of \( R \). Since \( z - 1 \in I \), we conclude that \( x + I = xz + I \).  

Lemma 1.2. If \( R \) is any ring, then \( \text{Soc}(R_R) \) is an exchange ring (without identity if \( R \) is not semisimple).

Proof. Set \( I = \text{Soc}(R_R) \cap J(R) \). Then \( S = \text{Soc}(R_R)/I \) and \( (\text{Soc}(R_R) + J(R))/J(R) \) are isomorphic as rings and the latter is contained in the socle of the semiprimitive ring \( R/J(R) \); consequently \( \text{Soc}(R_R)/I \) is regular and hence is an exchange ring. Inasmuch as \( I^2 = 0 \), then \( I \) also is an exchange ring and idempotents of \( \text{Soc}(R_R)/I \) lift to idempotents of \( \text{Soc}(R_R) \), therefore the thesis follows from [2, Theorem 2.2].  

Lemma 1.3. Given a ring \( R \), if \( L \) is a two-sided ideal contained in \( \text{Soc}(R_R) \), then idempotents lift modulo \( L \).

Proof. Let \( x \in R \), suppose that \( x - x^2 \in L \) and set \( M = L \cap J(R) \). If \( x - x^2 \in M \) then, since \( M^2 = 0 \), there is an idempotent \( e \in R \) such that \( x - e \in M \subset L \). Assume that \( x - x^2 \notin M \). We observe that the two-sided ideal \( H = L/M \) of
the ring $S = R/M$ is contained in $\text{Soc}(S_S)$ and $H \cap J(S) = 0$. Consequently, it follows from [5, Proposition 2.1] that $H$ is a sum of homogeneous projective components of $\text{Soc}(S_S)$ and every finitely generated right subideal of $H$ is generated by an idempotent, so that $H$ is regular. If we consider the element $\bar{x} = x + M \in S$, then $\bar{x} - \bar{x}^2 = x - x^2 + M \in H$. As a result, according to Lemma 1.1 there is an idempotent $\bar{e} \in S$ such that $\bar{x} - \bar{e} \in H$. Again, since $M^2 = 0$, we may assume that $e$ is an idempotent of $R$ and we get $\bar{x} - \bar{e} = \bar{y}$ for some $y \in L$. It follows that $x - e - y = m$ for some $m \in M$ and, finally, $x - e = y + m \in L$, as wanted.  

We are now in the position to prove the exchange property for semi-Artinian rings.

**Theorem 1.4.** Given a ring $R$ with right Loewy chain $(L_\alpha)_{\alpha \geq 0}$, for every ordinal $\alpha$ the following statement is true:

1. $L_\alpha$ is an exchange ring and if $x \in R$ is such that $x - x^2 \in L_\alpha$, then there exists an idempotent $e \in R$ such that $x - e \in L_\alpha$.

In particular, if $R$ is right semi-Artinian, then $R$ is an exchange ring.

**Proof.** Obviously $(0)$ is true. Let $\alpha > 0$ and assume that $(\beta)$ holds for all $\beta < \alpha$. If $\alpha$ is a limit ordinal, then it is clear that $(\alpha)$ holds. Thus, assume that $\alpha = \beta + 1$ for some $\beta$ and let $x \in R$ be such that $x - x^2 \in L_\alpha$. Set $\overline{R} = R/L_\beta$ and $\overline{a} = a + L_\beta$ for all $a \in R$. We have $\text{Soc}(\overline{R_R}) = L_\alpha/L_\beta$ and, since $\overline{x} + L_\alpha/L_\beta$ is an idempotent of $\overline{R}/\text{Soc}(\overline{R_R}) \simeq R/L_\alpha$, it follows from Lemma 1.3 that there exists $y \in R$ such that $\overline{y} = \overline{y}^2$ and $\overline{x} - \overline{y} \in L_\alpha/L_\beta$, from which $x - y \in L_\alpha$. By the inductive hypothesis there is $e \in L_\beta$ such that $x - y \in L_\alpha$. In order to show that $L_\alpha$ is an exchange ring, we first observe that, by the above, idempotents of $L_\alpha/L_\beta$ lift to idempotents of $L_\alpha$. On the other hand, $L_\alpha/L_\beta$ is an exchange ring by Lemma 1.2, thus, using the inductive hypothesis and [2, Theorem 2.2] we may conclude that $L_\alpha$ is an exchange ring.  

Let $R$ be a right semi-Artinian ring with Loewy length $\xi + 1$ and Loewy chain $(L_\alpha)_{\alpha \leq \xi + 1}$. If $M$ is a right $R$-module, we define the ordinal $h(M) = \min\{\alpha \leq \xi \mid ML_\alpha = M\}$; clearly, it cannot be a limit ordinal if $M$ is finitely generated. If $e$ is an idempotent of $R$, then it is easy to see that

$$h(eR) = \min\{\alpha \leq \xi \mid e \in L_\alpha\};$$

we write $h(e)$ for $h(eR)$. We remember that, since $\text{Soc}(\quad)$ is a left exact preradical in $\text{Mod}_R$, then for every ordinal $\alpha$ the preradical $\text{Soc}_\alpha(\quad)$ is left exact too, as it
can be easily proved by standard arguments. In particular, for every module \(M_R\) and submodule \(L \leq M\) we have that \(\text{Soc}_\alpha(L) = L \cap \text{Soc}_\alpha(M)\).

**Proposition 1.5.** Given a right \(R\)-module \(M\), we have

\[
ML_\alpha \subset \text{Soc}_\alpha(M) \quad \text{for every ordinal } \alpha.
\]

If \(P\) is a finitely generated projective right \(R\)-module, then

\[
P L_\alpha = \text{Soc}_\alpha(P) \quad \text{for every ordinal } \alpha;
\]

moreover, \(\text{End}(P_R)\) is a right semi-Artinian ring whose Loewy length is at most \(h(P)\).

**Proof.** The inclusion (1.1) is clear when \(\alpha \leq 1\). Suppose that \(0 < \alpha \leq \xi + 1\) and assume that \(ML_\beta \subset \text{Soc}_\beta(M)\) for all \(\beta < \alpha\). If \(\alpha\) is a limit ordinal, then

\[
ML_\alpha = M\left(\bigcup_{\beta < \alpha} L_\beta\right) = \bigcup_{\beta < \alpha} ML_\beta \subset \bigcup_{\beta < \alpha} \text{Soc}_\beta(M) = \text{Soc}_\alpha(M).
\]

Assume that \(\alpha = \beta + 1\) for some \(\beta\) and note that

\[
\left(ML_\alpha + \text{Soc}_\beta(M)\right)/\text{Soc}_\beta(M) \cong ML_\alpha/(ML_\alpha \cap \text{Soc}_\beta(M)).
\]

By the inductive hypothesis the right-hand term is a factor module of \(ML_\alpha//ML_\beta = (M/ML_\beta)(L_\alpha/L_\beta)\), which is in turn semisimple. We conclude that

\[
ML_\alpha \subset ML_\alpha + \text{Soc}_\beta(M) \subset \text{Soc}_\alpha(M).
\]

Let \(P\) be a finitely generated projective right \(R\)-module. Then \(P\) is a direct sum of right ideals generated by idempotents, because \(R\) is an exchange ring. On the other hand, if \(e = e^2 \in R\), then \(\text{Soc}_\alpha(eR) = L_\alpha \cap eR = eL_\alpha\) and so (1.2) follows.

Next, in order to prove the last statement, since the property of being a right semi-Artinian ring is Morita invariant it is sufficient to consider again the case in which \(P = eR\) for some idempotent \(e \in R\). Let \((K_\alpha)_{0 \leq \alpha}\) be the Loewy chain of \(eRe\); we claim that for each ordinal \(\alpha \leq \xi + 1\) we have

\[
eL_\alpha e \subset K_\alpha.
\]

This is trivial if \(\alpha = 0\). Let \(0 < \alpha \leq \xi + 1\) and assume that \(eL_\beta e \subset K_\beta\) for all \(\beta < \alpha\). If \(\alpha = \beta + 1\) for some \(\beta\), then \((eL_\beta + 1 + L_\beta)/L_\beta\) is a semisimple right \(R\)-module and consequently the right \(eRe\)-module

\[
B = eL_\beta + 1 e / eL_\beta e = eL_\beta + 1 e / (eL_\beta + 1 e \cap L_\beta)
\]

\[
\cong (eL_\beta + 1 e + L_\beta)/L_\beta = ((eL_\beta + 1 + L_\beta)/L_\beta)e
\]

is semisimple. Since \(eL_\beta e \subset K_\beta\), we may consider the canonical epimorphism \(\gamma: eRe/eL_\beta e \to eRe/K_\beta\) defined by \(\gamma(ere + eL_\beta e) = ere + K_\beta\); by the above \(\gamma(B) = (eL_\beta + 1 e + K_\beta)/K_\beta\) is contained in \(\text{Soc}(eRe/K_\beta) = K_\beta + 1 / K_\beta\) and so
If $\alpha$ is a limit ordinal, then $eL_\alpha e = \bigcup_{\beta < \alpha} eL_\beta e \subset \bigcup_{\beta < \alpha} K_\beta = K_\alpha$ and our claim is proved.

Finally, since $e \in L_{h(e)}$, we infer that $eRe \subset eL_{h(e)} e \subset eR_{h(e)}$ and therefore $K_{h(e)} = eRe$. This shows that $eRe$ is right semi-Artinian and has Loewy length at most $h(e)$. \hfill \Box

A couple of remarks are in order. Firstly, the inclusion (1.1) may really be proper. In fact, if $R = \mathbb{Z}_{p^n}$ for some prime $p$ and $n \geq 2$, then $L_i = p^{n-i} \mathbb{Z}/p^n\mathbb{Z}$ for $0 \leq i \leq n$. If $i < n$ and $A = L_i$, then $0 = AL_i \subset L_i = \text{Soc}_i(A) \neq 0$. Next, it is possible that, for some right semi-Artinian ring $R$ and a finitely generated projective right $R$-module $P$, the Loewy length of $\text{End}(PR)$ is strictly less than $h(P)$. For example, given an integer $n > 2$, let $R$ be the (Artinian) ring of upper triangular $n \times n$ matrices over some division ring $D$ and, for $1 < i \leq n$, let $e_i$ be the idempotent matrix with 1 at the $(i,i)$ entry and 0 elsewhere. Then $h(e_i) = i$ and, according to Proposition 1.5, $e_i R_R$ has Loewy length $i > 1$. But $e_i R_{e_i} \simeq D$ has Loewy length 1.

We recall that an abelian monoid $M$ is separative if, for every $x, y \in M$, the equalities $2x = x + y = 2y$ imply $x = y$, while $M$ is strongly separative if, for every $x, y \in M$, the equality $2x = x + y$ implies $x = y$. The ring $R$ is separative (respectively strongly separative) if the monoid $V(R)$ of isoclasses of finitely generated projective right $R$-modules is separative (respectively strongly separative).

**Theorem 1.6.** If $R$ is a right semi-Artinian ring, then $R$ is strongly separative.

**Proof.** Let us prove first that $R$ is separative, by showing that each corner ring of $R$ is separative (see [3, Lemma 4.1]). If $(L_\alpha)_{\alpha \leq \xi + 1}$ is the Loewy chain of $R$, we must prove that for every ordinal $\alpha \leq \xi + 1$ and $e = e^2 \in L_\alpha$ the corner ring $eRe$ is separative. This is obvious when $\alpha = 0$. Assume $0 < \alpha$ and suppose that if $\beta < \alpha$, then all corner rings of $L_\beta$ are separative. Clearly, the same is true for $L_\alpha$ in case $\alpha$ is a limit ordinal. Let $\alpha = \beta + 1$ for some $\beta$, let us consider an idempotent $e \in L_\alpha \setminus L_\beta$ and note that $eRe$ is right semi-Artinian by Proposition 1.5, thus is an exchange ring by Theorem 1.4. If $f$ is an idempotent of $eL_\beta e$, then $fRf$ is separative by the inductive hypothesis; consequently the ideal $eL_\beta e$ of $fRf$ is separative by [3, Theorem 4.2]. Since the ring $eRe/eL_\beta e = eL_\alpha e/eL_\beta e$ is semisimple (see the proof of Proposition 1.5), again from [3, Theorem 4.2] we infer that $eRe$ is separative, as wanted. Inasmuch as each corner ring $eRe$ is right semi-Artinian, then all simple factor rings of $eRe$ are Artinian and hence directly finite. Consequently we may conclude from [3, Proposition 5.6] that $R$ is strongly separative. \hfill \Box
2. Preordering the class of simple modules

In this section, unless otherwise specified, $R$ will be a given right semi-Artinian ring with Loewy length $\xi + 1$ and Loewy chain $(L_\alpha)_{\alpha \leq \xi + 1}$. Following Simson [23], we say that $R$ is a right peak ring if $\text{Soc}(RR)$ is projective, homogeneous and essential as a right ideal; we say that $R$ is $U$-peak when we need to emphasize that $\text{Soc}(RR)$ is $U$-homogeneous for some specific simple and projective module $U_R$. A $U$-peak ideal is an ideal $I$ of $R$ such that the ring $R/I$ is $U$-peak.

Given $U \in \text{Simp}_R$, we want to associate to $U$ a particular $U$-peak ideal. By setting $\alpha + 1 = h(U)$, we have that $U$ is $R/L_\alpha$-projective by [9, Lemma 1.2], hence, there is an element $e_U \in L_{\alpha + 1} \setminus L_\alpha$ such that $e_U + L_\alpha$ is an idempotent of $R/L_\alpha$ and $U \simeq e_U R/e_U L_\alpha$; moreover, if $L(U)$ denotes the unique ideal of $R$, containing $L_\alpha$, such that $L(U)/L_{\alpha}$ is the trace of $U$ in $R/L_{\alpha}$, then $L(U) = Re_U R + L_\alpha$. Inasmuch as $R$ is an exchange ring, we may assume that $e_U$ itself is an idempotent. The reader will easily recognize that, in case $R$ is left perfect, the idempotents $e_U$ can be further chosen in such a way that $\{e_U \mid U \in \text{Simp}_R\}$ is a (finite) basic set of pairwise orthogonal and primitive idempotents. Let us consider the set $I(U)$ of ideals of $R$ defined by

$$I(U) = \{ J \in \mathbb{L}_2(R) \mid L_\alpha \subset J \text{ and } J \cap L(U) = L_\alpha \}$$

$$= \{ J \in \mathbb{L}_2(R) \mid L_\alpha \subset J \text{ and } \text{Hom}_R(U, J/L_\alpha) = 0 \}$$

$$= \{ J \in \mathbb{L}_2(R) \mid L_\alpha \subset J \text{ and } J \cdot L(U) = L_\alpha \}$$

(recall that $L(U)/L_\alpha$ is a pure left ideal of $R/L_\alpha$). Note that the last equality implies that $I(Y)$ is upward directed. Set

$$I(U) =: \sum I(U) = \bigcup I(U).$$

By the above $I(U) = l_R(L(U)/L_\alpha)$ and, given $x \in R$, we have

$$x \in I(U) \quad \text{if and only if} \quad \text{Hom}_R(U, (xR + L_\alpha)/L_\alpha) = 0. \quad (2.1)$$

We observe that the ideal $I(U)$ is $U$-peak. In fact the right socle of the ring $R/I(U)$ is canonically isomorphic to $L(U)/L_\alpha$ and so is $U$-homogeneous and projective; it is also essential as a right ideal, simply because $R/I(U)$ is right semi-Artinian.

It is worth remarking that we have the inclusion $I(U) \subset r_R(L(U)/L_\alpha) = r_R(U)$, due to the left purity of $L(U)/L_\alpha$ in $R/L_\alpha$. Concerning the reverse inclusion we have the following result, which is a particular case of [6, Theorem 1.3].

**Proposition 2.1.** Let $R$ be a right semi-Artinian ring, let $U \in \text{Simp}_R$ and set $\alpha + 1 = h(U)$. Then, with the above notations, the following conditions are equivalent:
(1) $I(U) = r_R(U)$.
(2) $(L(U)/L_\alpha) \cap J(R/L_\alpha) = 0$.
(3) $L(U)/L_\alpha$ is a regular ideal.
(4) Every minimal right ideal of $R/L_\alpha$ isomorphic to $U$ is generated by an idempotent.

Given two simple modules $U_R$ and $V_R$, let us write $U \preceq V$ if and only if $I(U) \subset I(V)$. Then the relation $\preceq$ defines a preorder in the class of all simple right $R$-modules, which we call the natural preorder. The following theorem will be the key tool for investigating the properties of this preorder.

**Theorem 2.2.** Let $R$ be a right semi-Artinian ring with Loewy chain $(L_\alpha)_{\alpha \leq \xi + 1}$, let $U_R$ and $V_R$ be two simple modules, set $\alpha + 1 = h(U)$ and $\beta + 1 = h(V)$ and let us consider the following conditions:

(1) $U \preceq V$.
(2) $\alpha \leq \beta$ and if $x \in R$ is such that $(xR + L_\beta)/L_\beta \simeq V$, then $U \preceq (xR + L_\alpha)/L_\alpha$.
(3) $V : I(U) = 0$.
(4) $\alpha \leq \beta$ and if $e = e^2 \in L_{\beta + 1} \setminus L_\beta$ is such that $(eR + L_\beta)/L_\beta \simeq V$, then $U \preceq (eR + L_\alpha)/L_\alpha$.
(5) $\alpha \leq \beta$ and if $P_R$ is a finitely generated and projective module such that $P/PL_\beta \simeq V$, then $U \preceq P/PL_\alpha$.

Then the following implications are true:

(1) $\iff$ (2) $\iff$ (3) $\iff$ (4) $\iff$ (5).

If every minimal right ideal of $R/L_\beta$ isomorphic to $V$ is generated by an idempotent, then the five conditions are equivalent.

**Proof.** (1) $\Rightarrow$ (2) Assume that $I(U) \subset I(V)$. Inasmuch as $V$ is $R/L_\beta$-projective, then $\text{Hom}_R(V, L_{\beta + 1}/L_\beta) \neq 0$ and so $L_{\beta + 1} \not\subset I(V)$, consequently $\alpha \leq \beta$. If $x \in R$ is such that $(xR + L_\beta)/L_\beta \simeq V$, then $x \notin I(V)$ and hence $x \notin I(U)$, therefore $U \preceq (xR + L_\alpha)/L_\alpha$.

(2) $\Rightarrow$ (1) Suppose that (2) holds, let $x \in I(U)$ and assume that $x \notin I(V)$, namely that $xL(V) \not\subset L_\beta$. By recalling that $L(V)/L_\beta = \text{Tr}_{R/L_\beta}(V)$, we infer that there is some $y \in L(V)$ such that $(xyR + L_\beta)/L_\beta \simeq V$ and, from the assumption, we have that $(xyzR + L_\alpha)/L_\alpha \simeq U$ for some $z \in R$. On the other hand, $xyz \in I(U)$ and so $\text{Hom}_R(U, (xyzR + L_\alpha)/L_\alpha) = 0$, a contradiction. We conclude that $x \in I(V)$.

(2) $\Rightarrow$ (4) is obvious.
(4) $\Rightarrow$ (5) Suppose (4) and let $P$ be a finitely generated projective right $R$-module such that $P/PL_\beta \simeq V$. Since $R$ has the exchange property, there are some idempotents $e_1, \ldots, e_n \in R$ such that $P \simeq e_1 R \oplus \cdots \oplus e_n R$. Consequently

$$V \simeq P/PL_\beta \simeq (e_1 R/e_1 L_\beta) \oplus \cdots \oplus (e_n R/e_n L_\beta)$$

and, without loss of generality, we may assume that $e_1 \in R \setminus L_\beta$ and $e_2, \ldots, e_n \in L_\beta$. As a result $V \simeq e_1 R/e_1 L_\beta$ and therefore, by the assumption, $U \lesssim e_1 R/e_1 L_\alpha$. This shows that $U \lesssim P/PL_\alpha$.

(5) $\Rightarrow$ (3) Assume that (5) holds, choose an idempotent $e \in R$ such that $eR/eL_\beta \simeq V$ and let us prove that $ex \in I(U)$ for every $x \in I(U)$. Assume, on the contrary, that there is some $x \in I(U)$ such that $ex \notin L_\beta$. Since $(exR + L_\beta)/L_\beta = (eR + L_\beta)/L_\beta$, we infer that

$$\psi : RexR/(RexR \cap L_\beta) \longrightarrow ReR/(ReR \cap L_\beta)$$

of (nonunital) rings defined by

$$\varphi(rexs + RexR \cap L_\beta) = rexs + ReR \cap L_\beta.$$ 

Now $RexR$ is an ideal of the exchange ring $R$, therefore it is itself an exchange ring (see [2, Example (1) after Theorem 1.2]), consequently there is an idempotent $f \in RexR$ such that $f + RexR \cap L_\beta = \varphi^{-1}(e + ReR \cap L_\beta)$. Necessarily $f \notin L_\beta$ and from the above we infer that $(fR + L_\beta)/L_\beta = (eR + L_\beta)/L_\beta \simeq V$. By the assumption (5) there is some $y \in R$ such that $(fyR + L_\alpha)/L_\alpha \simeq U$; on the other hand, since $f \in RexR \subset I(U)$ we have that $\text{Hom}_R(U, (fyR + L_\alpha)/L_\alpha) = 0$, thus a contradiction. We conclude that $V \cdot I(U) = 0$.

(3) $\Rightarrow$ (4) Assuming (3), since $L_\alpha \subset I(U)$ we have that $VL_\alpha = 0$ and so $\alpha \leq \beta$. Let $e$ be an idempotent of $R$ such that $eR/eL_\beta \simeq V$ and note that $e \notin I(U)$, otherwise we would get $(eR/eL_\beta)e = 0$ and, hence, $e \in L_\beta$, a contradiction. We infer that $eR/eI(U)$ is isomorphic to a nonzero right ideal of $R/I(U)$ and, since this latter has essential and $U$-homogeneous right socle, we have that $U \lesssim eR/eI(U)$. Inasmuch as $U$ is $R/L_\alpha$-projective and $eR/eI(U)$ is an epimorphic image of $eR/eL_\alpha$, it follows that $U \lesssim eR/eL_\alpha$, as wanted.

Finally, assume that every minimal right ideal of $R/L_\beta$ isomorphic to $V$ is generated by an idempotent and suppose that (5) holds. Then, using Proposition 2.1, we obtain that $I(U) \subset rR(V) = I(V)$ and therefore $U \ll V$. $\square$

It follows from the above theorem that $I(U) = I(V)$ if and only if $U \simeq V$; thus the natural preorder $\ll$ induces a partial order in $\text{Simp}_R$; we call it the natural partial order and, from now on, we shall consistently consider $\text{Simp}_R$ as a poset with its natural partial order.
Remark 2.3. There are two main instances in which every \( U \in \text{Simp}_R \) satisfies the conditions of Proposition 2.1. The first one is, of course, when \( R \) is commutative; in this case \( \{ I(U) \mid U \in \text{Simp}_R \} \) is just the set of all maximal ideals and the natural partial order of \( \text{Simp}_R \) is the discrete one. Thus our investigations will not add any insight to commutative semi-Artinian ring theory. The second instance is when \( R \) is regular and is, on the contrary, of a great interest; in this case \( \{ I(U) \mid U \in \text{Simp}_R \} \) is the set of all primitive ideals. The study of the poset \( \text{Simp}_R \) for a regular right semi-Artinian ring \( R \) will be the objective of a forthcoming paper.

Let us write \( L_R(U) \) and \( I_R(U) \), instead of \( L(U) \) and \( I(U) \), if we need to emphasize that these ideals are considered in the specific ring \( R \). From the definition it is clear that if \( U \in \text{Simp}_R \) and \( \alpha \) is an ordinal such that \( U L_\alpha = 0 \), then
\[
I_R(U)/L_\alpha = I_R/L_\alpha(U).
\]
Consequently we have:

**Proposition 2.4.** With the same notations as above, for every ordinal \( \alpha \) the natural partial order of \( \text{Simp}_R/L_\alpha \) is induced by the natural partial order of \( \text{Simp}_R \).

With the next result we give a couple of basic features of the natural partial order.

**Proposition 2.5.** If \( R \) is a right semi-Artinian ring then, with respect to the natural partial order, \( \text{Simp}_R \) is an Artinian poset in which every maximal chain has a maximum.

**Proof.** If \( C \) is any chain of \( \text{Simp}_R \) and \( U, V \in C \), then it follows from the definition and Theorem 2.2 that \( U \not\leq V \) if and only if \( h(U) \not\leq h(V) \). Consequently \( C \) is well ordered and hence \( \text{Simp}_R \) is Artinian. Assume now that \( C \) is a maximal chain, let us consider the corresponding chain \( C = \{ I(U) \mid U \in C \} \) of ideals, set \( I = \bigcup C \) and note that \( I \) is proper, because each \( I(U) \) is proper. If \( \xi + 1 \) is the Loewy length of \( R_R \) and we consider the ordinal \( \alpha_0 = \min\{ \alpha \leq \xi + 1 \mid L_\alpha \not\subset I \} \), then necessarily \( \alpha_0 = \beta_0 + 1 \) for some \( \beta_0 \), therefore \( L_{\beta_0} \subset I \) but \( L_{\beta_0+1} \not\subset I \). It is not the case that \( L_{\beta_0} = I \) otherwise, by taking any \( V \in \text{Simp}_R/L_\xi \), we have that \( h(V) = \xi + 1 \) and \( U \not\leq V \) for all \( U \in C \), but \( V \notin C \), contradicting the maximality of \( C \). Consequently there exists \( U \in C \) such that \( L_{\beta_0} \not\subset I(U) \), that is \( h(U) \geq \alpha_0 \). By the definition of \( \alpha_0 \) we must have \( h(U) = \alpha_0 \) and \( U \) is the maximum of \( C \).

Another main feature of the natural preorder is that it is a Morita invariant. Recall that, given two rings \( R \) and \( S \) and an equivalence \( F : \text{Mod}_R \to \text{Mod}_S \), for every module \( A_R \) there is an induced isomorphism \( \Lambda_A \) from the lattice of
submodules of $A_R$ to the lattice of submodules of $F(A)_S$ which is defined as follows: if $B \leq A$ and $i : B \hookrightarrow A$ is the inclusion, then

$$\Lambda_A(B) = \text{Im}(F(i)) \simeq F(B);$$

(2.2)
m更要 there is a canonical isomorphism $F(A/B) \simeq F(B)/\Lambda_A(B)$. Furthermore, there is an isomorphism $\Phi$ from the lattice of all two-sided ideals of $R$ to the lattice of all two-sided ideals of $S$ defined by the rule

$$\Phi(I) = r_S(F(R/I))$$

(2.3)
and $F$ induces an equivalence between the categories $\text{Mod}_R$ and $\text{Mod}_S/\Phi(I)$. It was shown in [9, Proposition 4.1 and Theorem 4.3] that, for every module $A_R$,

$$\Lambda_A(\text{Soc}_\alpha(A)) = \text{Soc}_\alpha(F(A)) \quad \text{for each ordinal } \alpha$$

(2.4)
and

$$\Phi(\text{Soc}_\alpha(R_R)) = \text{Soc}_\alpha(S_S) \quad \text{for each ordinal } \alpha.$$  

(2.5)
In particular $R$ is right semi-Artinian if and only if so is $S$ with the same Loewy length as $R$.

**Theorem 2.6.** Let $R$ and $S$ be right semi-Artinian rings which are Morita equivalent via an equivalence $F : \text{Mod}_R \rightarrow \text{Mod}_S$. If $U$ is any simple right $R$-module, then

$$\Phi(I_R(U)) = I_S(F(U)).$$

(2.6)
Consequently, if $U, V$ are simple right $R$-modules and $U \preceq V$, then $F(U) \preceq F(V)$, so that $F$ induces an order isomorphism between the posets $\text{Simp}_R$ and $\text{Simp}_S$.

**Proof.** Let $U$ be a simple right $R$-module and let $\alpha + 1 = h(U)$. It follows from (2.5) that $R/L_\alpha$ and $S/\text{Soc}_\alpha(S_S)$ are Morita equivalent rings, therefore $h(F(U)) = \alpha + 1$ as well. Our first goal is to prove that

$$\Phi(L_R(U)) = L_S(F(U)).$$

(2.7)
Assume that $\alpha = 0$, let us consider in $S$ the ideal

$$J = \Phi(L_R(U)) = \{ s \in S \mid F(R)s \subset \text{Tr}_{F(R)}(F(U)) \}$$

and note that $L_S(F(U)) = \text{Tr}_S(F(U)) \subset J$. In order to prove the opposite inclusion, it will be sufficient to show that $J_S$ is semisimple and $F(U)$-homogeneous. Since $F(R)_S$ is a finitely generated projective generator, there is a positive integer $n$ and an exact sequence

$$F(R)^n \longrightarrow S \longrightarrow 0.$$
On the other hand, we have $F(R)^n J \simeq F(R)^n \otimes_S J$ by the flatness of $F(R)_S$. As a result we get an exact sequence

$$(F(R)J)^n = F(R)^n J \simeq F(R)^n \otimes_S J \rightarrow J \rightarrow 0$$

and, since $F(R)J \subset \text{Tr}_{F(R)}(F(U))$, we conclude that $J_S$ is semisimple and $F(U)$-homogeneous, as wanted. Next, suppose that $\alpha > 0$. Then $F$ induces a Morita equivalence between the rings $\hat{R} = R/L_\alpha$ and $\hat{S} = S/\text{Soc}_\alpha(S_S)$ and $\text{h}(U_{\hat{R}}) = 1$. Denoting by $\overline{\Phi}$ the corresponding lattice isomorphism between $\mathbb{L}_2(\hat{R})$ and $\mathbb{L}_2(\hat{S})$, it follows from the above that $\overline{\Phi}(L_{\hat{R}}(U)) = L_{\hat{S}}(F(U))$. As a result we obtain:

$$\Phi(L_R(U))/\text{Soc}_\alpha(S_S) = \Phi(L_R(U))/\Phi(\text{L}_\alpha) = \overline{\Phi}(L_{\hat{R}}(U)) = L_{\hat{S}}(F(U))$$

and, hence, (2.7) follows. Finally, given an ideal $J$ of $R$, using the fact that $\Phi$ is a lattice isomorphism and (2.5) we have that $J \cap L_R(U) = L_\alpha$ if and only if $\Phi(J) \cap L_S(F(U)) = \text{Soc}_\alpha(S_S)$; from this it is an easy matter to show that (2.6) holds. □

3. Idempotent ideals

Michler proved in [20] that every idempotent ideal of a left perfect ring is generated by idempotents, hence it is the trace of a projective module. With the help of the exchange property we can extend that result to any right semi-Artinian ring.

If $I$ is an ideal of a ring $R$, then $I = I^2$ if and only if $\text{Mod}_{R/I}$ is a TTF-class. As it is well known, under the present assumption that $R$ right semi-Artinian every hereditary torsion class $T$ of $\text{Mod}_R$ is generated by simple modules, in the sense that $T$ is the smallest hereditary torsion class containing $T \cap \text{Simp}_R$. In particular, if $I$ is an idempotent ideal of $R$, then $\text{Mod}_{R/I}$ is generated by $\text{Simp}_{R/I}$.

**Theorem 3.1.** Let $R$ be a right semi-Artinian ring with Loewy chain $(L_\alpha)_{0 \leq \alpha}$, assume that $I$ is an idempotent ideal of $R$ and set

$$A = \{U \in \text{Simp}_R \mid UI = U\}.$$

Then, for every $U \in A$, one can choose an idempotent $e_U \in I \cap L_{h(U)}$ such that $e_U R/e_U L_{h(U)} = U$ and

$$I = \sum \{Re_U R \mid U \in A\}. \quad (3.1)$$

**Proof.** Let $U \in A$ and take the idempotent $e_U$ as described at the beginning of Section 3. From $UI = U$ we infer that $I \not\subset L_\alpha$ and

$$(Re_U R + L_\alpha)/L_\alpha = [(Re_U R + L_\alpha)/L_\alpha][(I + L_\alpha)/L_\alpha] \subset (I + L_\alpha)/L_\alpha.$$
As a result $e_U + L_\alpha \in (I + L_\alpha)/L_\alpha$ and, inasmuch as $R$ is an exchange ring, the idempotent $e_U$ can be chosen in $I$. Let us denote by $J$ the second member of (3.1) and note that $J \subset I$, therefore $\text{Simp}_{R/I} \subset \text{Simp}_{R/J}$. On the other hand, for every $U \in A$ we have that $U e_U \neq 0$ and hence $U J = U$. We infer that $\text{Simp}_{R/I} = \text{Simp}_{R/J}$. Since $I$ and $J$ are idempotent, it follows that $\text{Simp}_{R/I}$ and $\text{Simp}_{R/J}$ generate respectively $\text{Mod}_{R/I}$ and $\text{Mod}_{R/J}$ as torsion classes, hence these categories must coincide. We conclude that $I = J$. □

We say that $R$ is a right NLF ring if $R$ is right semi-Artinian and all Loewy factor rings of $R$ are right nonsingular (see [9]), that is, for every $\alpha$ the ring $R/L_\alpha$ has projective right socle. In Section 4 we shall concentrate our attention to a wide class of right NLF-rings. In any case, as we are going to see, when $R$ is a right NLF-ring and $U \in \text{Simp}_R$ we can be more specific on the nature of the ideal $I(U)$. For our purposes it is convenient to have at disposal a notion which is more general than the notion of the Loewy chain. Let us call a right socular chain for a ring $R$ any nondecreasing chain $C = (H_\alpha)_{\alpha \leq \eta}$ of right ideals, parameterized over a nonzero ordinal $\eta$, such that:

$$H_0 = 0 \quad \text{and} \quad H_\eta = R,$$

$$H_{\alpha+1}/H_\alpha \subset \text{Soc}((R/H_\alpha)_{R/H_\alpha}) \quad \text{for every} \ \alpha < \eta,$$

$$H_\alpha = \bigcup_{\beta < \alpha} H_\beta \quad \text{for every limit ordinal} \ \alpha$$

(some authors call a Loewy chain what we call a socular chain and call the lower Loewy chain what we call simply the Loewy chain). It is not difficult to prove that $H_\alpha \subset \text{Soc}_\alpha(R_R)$ for every $\alpha < \eta$;

hence, if $R$ has a right socular chain, then $R$ must be right semi-Artinian with Loewy length no more than $\eta$. For every module $M_R$ we shall consider the ordinal $h_C(M) =: \min\{\beta \mid MH_\beta = M\}$.

**Theorem 3.2.** Let $R$ be a right semi-Artinian ring and let $C = (H_\alpha)_{\alpha \leq \eta}$ be a right socular chain consisting of ideals. Then the following conditions are equivalent:

1. $R/H_\alpha$ is right nonsingular for each $\alpha$.
2. $H_\alpha = H^2_\alpha$ for each $\alpha$.
3. $H_\alpha$ is pure in $R R$ for each $\alpha$.
4. For every module $M_R$ and simple module $S_R$ one has
   $$\text{Hom}_{R}(S, M/\text{Soc}_\alpha(M)) \neq 0 \quad \text{for at most one ordinal} \ \alpha.$$  
5. For each $\alpha$ and for every simple module $S_R$ one has
   $$\text{Hom}_{R}(S, R/H_\alpha) \neq 0 \quad \text{if and only if} \quad h_C(S) = \alpha + 1.$$
(6) For every \( x \in R \), \( h_C(xR) = \min\{\alpha \leq \xi + 1 \mid x \in H_\alpha\} \).

(7) \( R \) is a right NLF-ring.

**Proof.** The proof of the equivalence between each of the first five conditions and the seventh is the same as the proof of [9, Theorem 1.3], by replacing \( \text{Soc}_\alpha(RR) \) with \( H_\alpha \) everywhere; also note that [9, Lemma 1.2], on which the quoted theorem bears, remains valid in the present setting. \( \square \)

**Proposition 3.3.** Assume that \( R \) is a right NLF-ring and let \( U \in \text{Simp}_R \). Then, with the above notations, \( I(U) \) is the smallest \( U \)-peak ideal and the following equality holds:

\[
I(U) = \sum \{ ReR \mid e = e^2 \in R \text{ and } \text{Hom}_R(U, eR/eL_\alpha) = 0 \},
\]

(3.2)

where \( \alpha = h(U) - 1 \); in particular \( I(U) \) is idempotent.

**Proof.** Let \( J \) be any \( U \)-peak ideal of \( R \). Since \( J \) is proper, we may consider the smallest ordinal \( \beta \) such that \( L_\beta \subset J \) but \( L_\beta + 1 \not\subset J \); consequently the composition \( L_\beta + 1/L_\beta \hookrightarrow (L_\beta + 1 + J)/L_\beta \twoheadrightarrow (L_\beta + 1 + J)/J \) of canonical maps is not zero. As a result, it follows from our assumptions that \( \text{Hom}_R(U, L_\beta + 1/L_\beta) \neq 0 \) and, by using [9, Theorem 1.3], we infer that \( \beta = \alpha \). Consequently, in order to prove that \( I(U) \subset J \) there is no loss of generality in assuming that \( \alpha = 0 \), i.e., that \( U_R \) is projective. Suppose that \( x \in R \setminus J \). Then \( xR/(xR \cap J) \simeq (xR + J)/J \neq 0 \) and hence \( \text{Hom}_R(U, xR/(xR \cap J)) \neq 0 \); the projectivity of \( U_R \) implies then \( \text{Hom}_R(U, xR) \neq 0 \) and so \( x \notin I(U) \). This shows that \( I(U) \subset J \), as wanted.

Next, let us denote by \( K \) the second member of (3.2). Then \( K \) is idempotent and it follows from (2.1) that \( K \subset I(U) \). Inasmuch as \( R \) is right NLF, it follows from Theorem 3.2 that, for every \( \beta \), the ideal \( L_\beta \) is idempotent, hence it is generated by idempotents by Theorem 3.1. On the other hand, by Theorem 3.2 we have that \( \text{Hom}_R(U, eR/eL_\beta) \neq 0 \) if and only if \( \beta = \alpha \), therefore \( L_\alpha \subset K \). In order to prove that \( I(U) \subset K \), in view of the first part of the proposition it will be sufficient to show that \( K \) is a \( U \)-peak ideal. Let us prove firstly that \( R/K \) is right NLF. For each ordinal \( \beta \geq \alpha \) let us consider the ideal \( H_\beta = (L_\beta + K)/K \) of \( R/K \) and note that, since \( L_\beta \) and \( K \) are idempotent, then \( H_\beta \) is idempotent as well. There are two canonical isomorphisms

\[
H_{\beta+1}/H_\beta \simeq (L_{\beta+1} + K)/(L_\beta + K) \simeq L_{\beta+1}/[L_{\beta+1} \cap (L_\beta + K)]
\]

and, since the last term is a factor module of \( L_{\beta+1}/L_\beta \), we infer that \( H_{\beta+1}/H_\beta \) is semisimple and it is easy to see that \( (H_\beta)_{\beta \geq \alpha} \) is a right socular chain for \( R/K \). It follows then from Theorem 3.2 that \( R/K \) is a right NLF-ring. Next, suppose that \( V \) is a minimal right ideal of \( R/K \). Inasmuch as \( V_{R/K} \) is projective, then there is an idempotent \( e + K \) of \( R/K \) such that \( V \simeq (e + K)(R/K) \) and, since
\( R \) is an exchange ring, we may assume that \( e \) is an idempotent of \( R \). We have a canonical epimorphism \( \varphi: (eR + L_\alpha)/L_\alpha \to (eR + K)/K \simeq V \) whose kernel is \((K \cap (eR + L_\alpha))/L_\alpha\). As \( e \notin K \), we have that \((eR + L_\alpha)/L_\alpha\) contains a simple submodule \( W \simeq U \). But since \( K \subset I(U) \), then \( W \cap (K/L_\alpha) = 0 \) and consequently \( U \simeq \varphi(W) \simeq V \), as wanted. 

\[ \square \]

**Remark 3.4.** If \( R \) is not NLF, for some \( U \in \text{Simp}_R \) the ideal \( I(U) \) may fail either to be the smallest \( U \)-peak ideal or to be idempotent. Indeed, given a field \( F \), let us consider the \( F \)-algebra

\[ R = \begin{pmatrix} F & F & 0 \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix} \]

with zero multiplication between the first row and the third column. Then \( R \) is neither right nor left NLF; by taking

\[ e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

and setting \( U_i = e_i R/J(R) \) for \( i = 1, 2, 3 \), then \( \{U_1, U_2, U_3\} = \text{Simp}_R \) and we can compute that

\[ I(U_1) = \begin{pmatrix} F & F & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I(U_2) = \begin{pmatrix} F & F & 0 \\ 0 & 0 & F \\ 0 & 0 & F \end{pmatrix}, \]

\[ I(U_3) = \begin{pmatrix} 0 & F & 0 \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}. \]

Thus \( U_1 \prec U_2 \) and the element \( U_3 \) is isolated. The ideal

\[ K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & F \\ 0 & 0 & F \end{pmatrix} \]

is \( U_2 \)-peak, but \( I(U_2) \not\subset K \). Note that \( I(U_1), I(U_2) \) and \( I(U_3) \) are all idempotent.

Next, let \( S \) be the subalgebra of \( R \) of those matrices of the form

\[ \begin{pmatrix} a & b & 0 \\ 0 & a & c \\ 0 & 0 & d \end{pmatrix} \]

and note that \( S \) too is not NLF. Up to an isomorphism, there are only two simple right \( S \)-modules, namely

\[ U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} S/J(S), \quad V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} S/J(S). \]
We see that

\[ I(U) = \begin{pmatrix} 0 & F & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad I(V) = \begin{pmatrix} 0 & F & 0 \\ 0 & 0 & F \\ 0 & 0 & F \end{pmatrix}, \]

therefore \( U < V \); moreover, \( I(V) \) is idempotent but \( I(U) \) is not.

4. Nice semi-Artinian rings and upper subsets of \( \text{Simp}_R \)

Let \( R \) be a right semi-Artinian ring. Given two simple right \( R \)-modules \( U \) and \( V \), let us write \( U \triangleright V \) if any (and hence the remaining) of the equivalent conditions (3)–(5) listed in Theorem 2.2 holds. Since the same proposition states that \( U \preceq V \) implies \( U \triangleright V \), it appears quite natural to ask whether the reverse implication holds. With the next example we show that this is not always the case. Note that if \( U \triangleright V \) and \( V \triangleright U \), then it is easy to see that \( U \simeq V \) and one might wonder if the relation “\( \triangleright \)” is a preorder, i.e., transitive, in the class of all simple right \( R \)-modules. Again, the same example we are going to discuss shows that this is not always the case.

**Example 4.1.** Given a field \( F \), there exists an Artinian \( F \)-algebra \( R \), which is right NLF, such that \( \text{Simp}_R \) has four elements \( U_1, U_2, U_3, U_4 \) with a Hasse diagram

\[ \begin{align*}
& U_4 \\
& \quad \quad U_3 \\
& \quad \quad \quad U_1 \\
& \quad \quad \quad \quad U_2 \\
& \end{align*} \quad \text{(4.1)}
\]

and the relation “\( \triangleright \)” is not a preorder, hence is not equivalent to “\( \preceq \)”.

**Proof.** Let \( S \) be the incidence \( F \)-algebra of the poset

\[ \begin{align*}
& 1 \\
& \quad \quad 2 \\
& \quad \quad \quad 3 \\
& \quad \quad \quad \quad 4 \\
& \end{align*} \]
that is
\[ S = \begin{pmatrix} F & F & F & F \\ 0 & F & F & F \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{pmatrix}, \]
and let us consider the factor ring
\[ R = S / \begin{pmatrix} 0 & 0 & 0 & F \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]
Then we can identify \( R \) with the matrix ring
\[ \begin{pmatrix} F & F & F & 0 \\ 0 & F & F & F \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{pmatrix} \]
with trivial multiplication between the first row and the fourth column. The ideals
\[ M_1 = \begin{pmatrix} F & F & F & 0 \\ 0 & F & F & F \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{pmatrix}, \quad M_2 = \begin{pmatrix} F & F & F & 0 \\ 0 & F & F & F \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F \end{pmatrix}, \]
\[ M_3 = \begin{pmatrix} F & F & F & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{pmatrix}, \quad M_4 = \begin{pmatrix} F & F & F & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{pmatrix} \]
are the only maximal right ideals of \( R \). Thus, by setting
\[ U_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F \end{pmatrix} \simeq R/M_1, \quad U_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \simeq R/M_1, \]
\[ U_3 = R/M_3, \quad U_4 = R/M_4, \]
we have that \( \text{Simp}_R = \{U_1, U_2, U_3, U_4\} \) and easy computations show that the following equalities hold:
\[ I(U_1) = \begin{pmatrix} F & F & F & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I(U_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F \end{pmatrix}, \]
\[ I(U_3) = \begin{pmatrix} 0 & 0 & F & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{pmatrix} = L_1. \]
$I(U_4) = \begin{pmatrix} 0 & F & F & 0 \\ 0 & F & F & F \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{pmatrix} = M_4 = L_2.$

As a result the Hasse diagram of $\text{Simp}_R$ is (4.1). On the other hand, we see that $U_1 \nmid U_3$ but $U_1 \not\approx U_3$. In addition $U_3 \nmid U_4$ but $U_1 \not\nmid U_4$, hence “$\nmid$” is not a preorder. $\Box$

Let us say that a right semi-Artinian ring $R$ is *nice* if $R$ satisfies the following conditions:

(1) $R$ is right NLF,

(2) the relations “$\preceq$” and “$\nmid$” in $\text{Simp}_R$ coincide, that is

$$\{ V \in \text{Simp}_R \mid U \preceq V \} = \text{Simp}_{R/I(U)} \quad \text{for all } U \in \text{Simp}_R.$$ 

These two conditions are actually independent, indeed the algebra of Example 4.1 satisfies the first but not the second, while the two algebras $R$ and $S$ described in Remark 3.4 satisfy the second but not the first. As we shall see in the remaining two sections the class of nice semi-Artinian rings is wide enough to deserve our interest and the good behavior of these rings justify the name we have chosen for them.

**Proposition 4.2.** Given a right semi-Artinian ring $R$, the following conditions are equivalent:

(1) $R$ is nice.

(2) $I(U)$ is left pure for all $U \in \text{Simp}_R$.

**Proof.** As previously, we set $L_\alpha = \text{Soc}_\alpha(R_R)$ for all $\alpha$.

$(1) \Rightarrow (2)$ Assume that $R$ is nice, take any $U \in \text{Simp}_{R/I}$, set $I = I(U)$ and remember that $I = I^2$ by Proposition 3.3. Consequently $C = \{ M_R \mid MI = M \}$ is a torsion class and we aim to prove that $C$ is hereditary, from which it will follow that $R(R/I)$ is flat. As a first step, we claim that if $M \in C$ and $S$ is any simple submodule of $M$, then $S \in C$ too. If not, then $SI = 0$ and hence $I \subset I(S)$ by the assumption. Set $\gamma + 1 = h(S)$ and observe that $S \not\in ML_\gamma$, otherwise we would get $S = SL_\gamma$ by the left purity of $L_\gamma$; consequently $S$ imbeds into $M/ML_\gamma$. On the other hand, by using the projectivity of $S_R/L_\gamma$ and the fact that $C$ is closed by factor modules we obtain that

$$\text{Tr}_{M/ML_\gamma}(S) = (M/ML_\gamma)(L(S)/L_\gamma) = [(M/ML_\gamma)I](L(S)/L_\gamma) \subset (M/ML_\gamma)(I(S)/L_\gamma)(L(S)/L_\gamma) = 0.$$
This is a contradiction and therefore $S = SI$, as claimed. Next, suppose again that $M \in C$, let $N$ be any submodule of $M$ and let us prove that

$$
\text{Soc}_\alpha(N) = \text{Soc}_\alpha(N)I
$$

(4.2)

for every ordinal $\alpha$. This is the case when $\alpha = 1$, as it follows from the above argument, taking into account that $\text{Soc}(N) = N \cap \text{Soc}(M)$. Given an ordinal $\alpha > 1$, assume that $\text{Soc}_\beta(N) = \text{Soc}_\beta(N)I$ whenever $\beta < \alpha$. If $\alpha$ is a limit ordinal, then (4.2) is straightforward. Suppose that $\alpha = \beta + 1$ for some $\beta$ and consider the exact sequence

$$
0 \longrightarrow \text{Soc}_\beta(N) \longrightarrow \text{Soc}_\alpha(N) \longrightarrow \text{Soc}(N/\text{Soc}_\beta(N)) \longrightarrow 0.
$$

Using the fact that $\text{Soc}_\beta(N) = N \cap \text{Soc}_\beta(M)$, it is not difficult to see that $N/\text{Soc}_\beta(N)$ is isomorphic to a submodule of $M/\text{Soc}_\beta(M)$; consequently, since this latter belongs to $C$, it follows from our first claim that $\text{Soc}(N/\text{Soc}_\beta(N))$ belongs to $C$. Finally, by using the inductive assumption, the fact that $C$ is closed by extensions and $N$ is a semi-Artinian module, we conclude that $N \in C$.

(2) $\Rightarrow$ (1) Suppose (2), let $\alpha$ be any ordinal, let $U$ be a minimal right ideal of $R/L_\alpha$ and let us show that $U$ is $R/L_\alpha$-projective; it will follow that $L_{\alpha+1} = L_{\alpha+1}^2$. If $U$ were not $R/L_\alpha$-projective, then $h(U) > \alpha + 1$, consequently $L_{\alpha+1} \subset I(U)$ and so $U \subset I(U)/L_\alpha$. On the other hand, $C = \{M_R \mid M \cdot I(U) = M\}$ is a hereditary torsion class by the left purity of $I(U)$; since $I(U)/L_\alpha \in C$, it follows that $U \in C$ and this leads to a contradiction, because $U \cdot I(U) = 0$. Using induction we can easily conclude that $L_\alpha = L_\alpha^2$ for every ordinal $\alpha$ and therefore $R$ is right NLF by Theorem 3.2.

Next, given $U, V \in \text{Simp}_R$ and set $I = I(U)$ and $J = I(V)$, we must prove that if $VI = 0$, then $I \subset J$. First, if $\alpha + 1 = h(U)$ and $\beta + 1 = h(V)$, then $VI = 0$ implies that $\alpha \leq \beta$. Inasmuch as $I$ is left pure, then it is idempotent and so $[(I + L_\beta)/L_\beta]I = (I + L_\beta)/L_\beta$, therefore $\text{Hom}_R(R/I, (I + L_\beta)/L_\beta) = 0$. Since $VI = 0$, it follows that $\text{Hom}_R(V, (I + L_\beta)/L_\beta) = 0$ as well and, according to (2.1), we conclude that $I \subset J$ as wanted. \(\square\)

Given a poset $P$, a subset $A \subset P$ is an upper subset if $x \in A$, $y \in P$ and $x \leq y$ together imply $y \in A$. For a given $x \in P$ we shall denote by $\{x \leq\}$ the upper subset $\{y \in P \mid x \leq y\}$. A lower subset of $P$ is defined dually, as well as the lower subset $\{\leq x\}$. The sets $\uparrow P$ of all upper subsets and $\downarrow P$ of all lower subsets of $P$, ordered by inclusion, are complete lattices.

Let us denote by $\mathbb{LP}_2(R)$ the set of all left pure ideals of $R$. If $R$ is a right semi-Artinian ring, then there is a strict connection between left pure ideals of $R$ and certain upper subsets of $\text{Simp}_R$. 

Proposition 4.3. Suppose that $R$ is a right semi-Artinian ring with Loewy chain $(L_\alpha)_{0 \leq \alpha}$. Then the assignment $I \mapsto \text{Simp}_{R/I}$ defines an injective and strictly decreasing map

$$\Phi : \mathbb{LP}_2(R) \rightarrow \uparrow \text{Simp}_R.$$

Proof. Assume that $R(R/I)$ is flat and let $U, V \in \text{Simp}_R$ be such that $U \preceq V$. We must prove that if $UI = 0$, then $VI = 0$ as well. Suppose, on the contrary, that $VI = V$ and set $\alpha + 1 = h(U)$, $\beta + 1 = h(V)$. As in the proof of Theorem 3.1 we can choose an idempotent $e \in (I \cap L_{\beta+1}) \setminus L_\beta$ such that $eR/eL_\beta \simeq V$. By the assumption the torsion class $C = \{M_R \mid MI = M\}$ is hereditary and consequently, since $eR \in C$ and $U$ imbeds into $eR/eL_\alpha$, we infer that $U \in C$ and hence $UI = U$, a contradiction. Thus $\text{Simp}_{R/I}$ is an upper subset of $\text{Simp}_R$. Since $I = I^2$, then $\text{Mod}_{R/I}$ is the TTF class generated by $\text{Simp}_{R/I}$; this is enough to conclude that the map $\Phi$ is injective and strictly decreasing. \[\square\]

An antichain of a poset $P$ is a subset $A \subset P$ such that if $x, y \in A$ and $x \leq y$, then $x = y$. The following result is part of [10, Proposition 2.21]; however, Gary Brookfield kindly noticed us that the earliest reference seems to be [19, Proposition 2.1].

Proposition 4.4. Given a poset $P$, the following conditions are equivalent:

1. Every nonempty upper subset of $P$ has finitely many minimal elements.
2. $\downarrow P$ is Artinian.
3. $\uparrow P$ is Noetherian.
4. $P$ is Artinian and contains no infinite antichains.

Theorem 4.5. Assume that $R$ is a nice right semi-Artinian ring and let $A$ be an upper subset of $\text{Simp}_R$ which has only finitely many minimal elements $U_1, \ldots, U_n$. If $I = I(U_1) \cap \cdots \cap I(U_n)$, then $I$ is left pure and $A = \text{Simp}_{R/I}$. Consequently, if $\text{Simp}_R$ has no infinite antichains, then the map $\Phi : \mathbb{LP}_2(R) \rightarrow \uparrow \text{Simp}_R$ is an anti-isomorphism. In particular $\mathbb{LP}_2(R)$ is a complete Artinian lattice.

Proof. Set $I_i = I(U_i)$ for $i = 1, \ldots, n$ and note that each $I_i$ is left pure by Proposition 4.2, therefore

$$I = I_1 \cap \cdots \cap I_n = I_1 \cdots I_n. \quad (4.3)$$

If $J$ is any right ideal of $R$, then $J \cap I_1 = JJ_1$ and hence, by assuming inductively that $J \cap (I_1 \cdots I_{n-1}) = J(I_1 \cdots I_{n-1})$ and using (4.3), we infer easily that $J \cap (I_1 \cdots I_n) = J(I_1 \cdots I_n)$. This shows that $I$ is left pure. It is clear that
$A \subset \text{Simp}_{R/I}$. On the other hand, if $S \in \text{Simp}_{R/I}$ then necessarily $SI_i = 0$ for at least one $i$, otherwise we would get

$$SI = S(I_1 \cdots I_n) = S(I_2 \cdots I_n) = \cdots = SI_n = S,$$

contradicting the assumption. This shows that $A = \text{Simp}_{R/I}$. $\blacksquare$

The following is an easy example of a nice semi-Artinian ring $R$ such that $\text{Simp}_R$ has an upper subset $A$ with infinitely many minimal elements, but $A$ is not of the form $\text{Simp}_{R/I}$ for any idempotent ideal $I$.

**Example 4.6.** Given a field $F$, let us consider the regular subring $R = F^{(N)} + SF$ of $S = F^N$. Note that $F^{(N)} = \text{Soc}(R) = V_1 \oplus V_2 \oplus \cdots$, where $V_i = \{a \in R \mid a_j = 0 \text{ for } j \neq i\}$ for all $i$ and the poset $\text{Simp}_R = \{R/\text{Soc}(R), V_1, V_2, \ldots\}$ has the trivial ordering. If $A = \{V_1, V_2, \ldots\}$, for no ideal $I$ of $R$ we have that $A = \text{Simp}_{R/I}$, yet $A$ is an upper subset of $\text{Simp}_R$.

The next example exhibits a semi-Artinian ring $R$ with a proper nonzero left pure ideal $I$ such that the upper subset $\text{Simp}_{R/I}$ has infinitely many minimal elements.

**Example 4.7.** Let $F$ be a field, let $X$ be an infinite set and let $P$ be a partition of $X$ such that $|P| = |X| = |Y|$ for all $Y \in P$. Next, let us consider the ring $Q = \text{CFM}_X(F)$ of all column-finite $X$ by $X$ matrices with entries in $F$, set $I = \text{Soc}(Q)$ and let $K$ be the (nonunital) subring of $Q$ of those diagonal matrices $a \in Q$ such that $a_{yy} = a_{xy}$ when $x, y$ belong to the same $Y \in P$ (here $a_{xy}$ denotes the $(x,y)$-entry of the matrix $a$) and there are at most finitely many $Y \in P$ with $a_{yy} \neq 0$ for $y \in Y$. Finally, let us consider the regular ring

$$R = I \oplus K \oplus 1_Q F.$$

Then $R$ is primitive and is a nice semi-Artinian ring with Loewy length 3. We have that $\text{Soc}(R) = I$ and $\text{Soc}_2(R) = I + K$. Let $e$ be a primitive idempotent of $R$ and, for each $Y \in P$, let $f_Y$ be the idempotent of $K$ such that $(f_Y)_{yy}$ is 1 if $y \in Y$ and is 0 otherwise. Then it is not difficult to see that

$$\text{Simp}_R = \{eR\} \cup \{f_Y R/f_Y I \mid Y \in P\} \cup \{R/(I + K)\};$$

moreover $eR$ is the smallest element of $\text{Simp}_R$ and the others are pairwise incomparable. Now $R(R/I)$ is flat and $\text{Simp}_{R/I} = \text{Simp}_R \setminus \{eR\}$ is an upper class with infinitely many minimal elements.

The last result of this section concerns maximal elements of $\text{Simp}_R$. Recall that an ideal $I$ of a ring $R$ is left pure if and only if every injective right $R/I$-module is injective as an $R$-module.
Corollary 4.8. Let $R$ be a right semi-Artinian ring, let $U \in \text{Simp}_R$ and let us consider the following conditions:

1. $UR$ is injective and the vector space $\text{End}(UR)U$ is finite dimensional.
2. $U$ is a maximal element of $\text{Simp}_R$.

Then (1) implies (2), while if $R$ is nice, then (2) implies (1) as well.

Proof. Assuming (1) and by setting $\alpha + 1 = h(U)$, it follows that the trace of $U$ in $R/L_\alpha$, namely $L(U)/L_\alpha$, is generated by a central idempotent (see [5, Theorem 2.7]). As a result $I(U)/L_\alpha = I_{R/L_\alpha}(L(U)/L_\alpha) = r_{R/L_\alpha}(L(U)/L_\alpha) = r_R(U)/L_\alpha$, hence $I(U) = r_R(U)$ and the ring $R/I(U)$ is simple Artinian. If $V \in \{U \preceq\}$, then $V \cdot I(U) = 0$ by Theorem 2.2 and so $V = U$. Assume now that $R$ is nice and suppose that (2) holds. Since $\{U \preceq\} = \text{Simp}_{R/I(U)}$ by Theorem 4.5, the maximality of $U$ implies that $R/I(U)$ is simple Artinian. As $I(U) \subset r_R(U)$ by Proposition 2.2, it follows that $I(U) = r_R(U)$ and therefore $\text{End}(UR)U$ is finite dimensional. Finally, since $R(R/I(U))$ is flat by (3) of Proposition 4.2, then $UR$ is injective being an injective right $R/r_R(U)$-module.

Remark 4.9. Neither the finite dimensionality of $\text{End}(UR)U$, nor the injectivity of $UR$, implies alone the maximality of $U$. In fact, as we shall illustrate in the last section, every finite poset is order isomorphic to $\text{Simp}_R$ for some nice Artinian ring $R$ and, of course, every simple right $R$-module is finite dimensional over its endomorphism division ring. On the other hand, given any ordinal $\xi$, the construction we set up in [8, Example 4.3] produces a semi-Artinian Von Neumann regular $R$ having all simple right $R$-modules injective and $\text{Simp}_R$ is order isomorphic to $\xi + 1$. Finally, if we look at the Artinian ring $R$ of Example 4.1, we see that the minimal right ideal $U_1$ is maximal in $\text{Simp}_R$, but it is not injective.

5. The class of nice right semi-Artinian rings is large

Our next objective is to show that the class of nice right semi-Artinian rings contains three large and important classes of rings. The first one consists of right semi-Artinian rings which are right semihereditary (in particular, the regular ones), the second consists of all right semi-Artinian piecewise domains and the third consists of those right semi-Artinian rings $R$ such that if $\xi + 1 = L(R_R)$, then $\text{Soc}_\xi(R_R)$ is a right piecewise domain in a sense we are going to explain.

We say that an ideal $L$ of a ring $R$ is a right piecewise domain (rPWD) with respect to a set $E$ of idempotents of $R$ if:

1. $L = \sum\{ReR \mid e \in E\}$,
(2) $L$ is faithful and pure as a left ideal
(3) for every $e \in \mathbb{E}$, every nonzero $R$-homomorphism $eR \to R$ is a monomorphism.

The terminology is justified by the fact that if the above conditions hold then, for every $e, f, g \in \mathbb{E}$, $a \in eRf$, and $b \in fRg$, one has $ab = 0$ if and only if either $a = 0$ or $b = 0$. Consequently $eRe$ is a domain for all $e \in \mathbb{E}$ and, hence, all the idempotents in $\mathbb{E}$ are primitive; moreover, if $e, f \in \mathbb{E}$ and $eRf \neq 0$, then $eRf$ is torsion-free both as a left $eRe$-module and a right $fRf$-module. Our concept of a piecewise domain is an extension to nonunital rings of the concept originally introduced by Gordon and Small in [18] (see also [17] and Section 3 of [7]) for the special case in which $L = R$ and $\mathbb{E}$ is a (finite) complete set of orthogonal idempotents. We remark that if $L$ is faithful also as a right ideal, in particular if $L = R$, then the property of being a PWD is right/left symmetric; in fact we have the following result.

**Proposition 5.1.** Suppose that the ring $R$ contains a set $\mathbb{E}$ of nonzero idempotents such that the ideal $L = \sum \{ ReR \mid e \in \mathbb{E} \}$ is faithful both as a right and a left ideal. Then the following conditions are equivalent:

1. If $e, f, g \in \mathbb{E}$, $a \in eRf$, and $b \in fRg$, then $ab = 0$ if and only if either $a = 0$ or $b = 0$.
2. For every $e \in \mathbb{E}$, every nonzero $R$-homomorphism $eR \to R$ is a monomorphism.
3. For every $e, f \in \mathbb{E}$, every nonzero $R$-homomorphism $eR \to fR$ is a monomorphism.
4. For every $e \in \mathbb{E}$, every nonzero $R$-homomorphism $Re \to R$ is a monomorphism.
5. For every $e, f \in \mathbb{E}$, every nonzero $R$-homomorphism $Re \to Rf$ is a monomorphism.

**Proof.** We prove only the implication (1) $\Rightarrow$ (2), the others being straightforward. Assume (1), let $e \in \mathbb{E}$, suppose that $0 \neq \varphi \in \text{Hom}_R(eR, R)$ and let $a \in R$ be such that $ea \neq 0$. Then, by the hypothesis, there are some $f \in \mathbb{E}$ and $b \in R$ such that $eabf \neq 0$; similarly, since $\varphi(e) \neq 0$, there are $g \in \mathbb{E}$ and $c \in R$ such that $gc\varphi(e) \neq 0$. Consequently, we get $gc\varphi(ea)b = gc\varphi(e)ebaf \neq 0$, therefore $\varphi(ea) \neq 0$ and, hence, $\text{Ker}(\varphi) = 0$. □

**Lemma 5.2.** Let $\mathbb{E}$ be a set of idempotents of a ring $R$ and let $M$ be a right $R$-module such that if $0 \neq x \in M$, then $xRe \neq 0$ for some $e \in \mathbb{E}$. If for all $e \in \mathbb{E}$ every nonzero $R$-homomorphism $eR \to M$ is a monomorphism, then $M$ is nonsingular. Consequently, if $L = \sum \{ ReR \mid e \in \mathbb{E} \}$ is a rPWD with respect to $\mathbb{E}$, then $R$ is right nonsingular.
Proof. Assume that $0 \neq x \in Z(M)$ and take $e \in \overline{E}$, $b \in R$ such that $xeb \neq 0$. Then $xb \in Z(M)$ and so, if $\varphi : R \to M$ is the $R$-homomorphism defined by $\varphi(a) = xba$, we have that $\varphi(eR) \neq 0$ and $\text{Ker}(\varphi) = r_R(xb) \leq_r R_R$. On the other hand, we have $eR \cap r_R(xb) \neq 0$, so that the restriction of $f$ to $eR$ is not zero and is not a monomorphism, a contradiction. Thus $Z(M) = 0$. □

Proposition 5.3. Let $R$ be a right semi-Artinian ring, with Loewy length $\xi + 1$ and Loewy chain $(L_\alpha)_{\alpha \leq \xi + 1}$, let $\eta \leq \xi + 1$, assume that $L_\eta$ is a rPWD with respect to a set $E$ of idempotents and, for every successor ordinal $\alpha \leq \eta$, set $E_\alpha = \{ e \in E \mid e \in L_\alpha \setminus L_{\alpha - 1}\}$. Then the following properties hold:

1. If $e \in E_{\alpha + 1}$ for some $\alpha < \eta$, then $eR/eL_\alpha$ is a simple right $R$-module. Conversely, for every $U \in \text{Simp}_R$ with $h(U) \leq \eta$ we can choose an idempotent $e_U \in \overline{E}_{h(U)}$ such that $U \simeq e_U R/eU L_{h(U) - 1}$.
2. For all $\alpha < \eta$, $L_\eta / L_\alpha$ is a PWD with respect to $\overline{E} = \bigcup\{ e + L_\alpha \mid e \in E \setminus L_\alpha \}$.
3. Given $U, V \in \text{Simp}_R$ with $h(U), h(V) \leq \eta$, we have that $U \not\preceq V$ if and only if $e_V R e_U \neq 0$.
4. If $\eta = \xi$, then $R$ is right NLF.

Proof. (1) Take any $\alpha < \eta$, let us write $\overline{R} = R / L_\alpha$ and $\overline{x} = x + L_\alpha$ for every $x \in R$ and let us first prove that, given $\overline{\varphi} \in \overline{E}$, every nonzero homomorphism $\varphi : \overline{eR} \to \overline{R}$ is a monomorphism. Assume, on the contrary, that $\overline{\varphi}$ is not a monomorphism, let $U$ be any simple submodule of $\text{Ker}(\overline{\varphi})$ and note that $h(U) = \alpha + 1$. It follows that $uR / uL_\alpha \simeq U$ for some idempotent $u \in L_{\alpha + 1} \setminus L_\alpha$. Thus, by the assumption, there is some $f \in E$ such that $uRf \neq 0$; this gives rise to a nonzero homomorphism $\theta : fR \to uR$ which must be a monomorphism and, since $f \notin L_\alpha$, we infer that $\theta(fR) \not\subseteq L_\alpha$. As a result there is a homomorphism $\overline{\psi} : fR \to \overline{\varphi}$ whose image is $U$ and hence $\overline{\theta} \overline{\psi} = 0$. Now both $\overline{\varphi}$ and $\overline{\psi}$ lift, modulo $L_\alpha$, to nonzero homomorphisms $\varphi : eR \to R$ and $\psi : fR \to eR$, respectively. By the hypothesis $\varphi$ and $\psi$ must be monomorphisms and, since $\text{Im}(\varphi \psi) \subset L_\alpha$, it follows that $fR$ embeds into $L_\alpha$ and we reach a contradiction, because $f \notin L_\alpha$. We conclude that $\overline{\varphi}$ must be a monomorphism.

Let $e \in \overline{E}$ and note that $eRe$ is a division ring, because it is a semi-Artinian domain (Proposition 1.5). Consequently, if $e \in L_{\alpha + 1} \setminus L_\alpha$, then $eL_\alpha e = 0$ and so $\text{End}(eR/eL_\alpha) \simeq eRe$. Since $eR/eL_\alpha$ is semisimple, it follows that it must be simple. Conversely, assume that $U_R$ is simple and suppose that $\alpha + 1 = h(U) \leq \eta$. Then $U \simeq uR/uL_\alpha$ for some idempotent $u \in L_{\alpha + 1} \setminus L_\alpha$ and, by the assumptions, there is some $e_U \in E$ such that $uRe_U \neq 0$. Hence, there is a nonzero homomorphism $\varphi : e_U R \to uR$ which is a monomorphism by the assumption on $L_{\eta}$. Since $eU \notin L_\alpha$, it follows that $\text{Im}(\varphi) \not\subseteq L_\alpha$ and consequently $\varphi$ induces a nonzero homomorphism $\overline{\varphi} : e_U R/eU L_\alpha \to uR/uL_\alpha$. As we have shown previously $\overline{\varphi}$ must be a monomorphism and so it is an isomorphism and necessarily $e_U \in E_{\alpha + 1}$. 

(2) Inasmuch as $L_\eta$ is left pure and $L_\alpha \subseteq L_\eta$, then $L_\eta = L_\eta/L_\alpha$ is left pure in $\overline{R}$ and all remains to prove is that $L_\eta$ is faithful as a left ideal of $\overline{R}$. Since the right socle of $\overline{R}$ is essential, it will be sufficient to show that it is $\overline{R}$-projective. Let $U$ be a minimal right ideal of $\overline{R}$, set $\beta + 1 = h(U)$ and note that $U \subseteq L_{\alpha+1} \subseteq L_\eta$. From the left purity of $L_\eta$ we have that $U = UL_\eta$, namely $U = UL_\eta$ and so $\beta < \eta$. According to (1) there is $e \in E_{\beta + 1}$ such that $eR/eL_\beta \simeq U$ and we infer that there is homomorphism $\varphi: e\overline{R} \rightarrow \overline{R}$ whose image is $U$. Since $\varphi$ must be a monomorphism, we conclude that $U$ is $\overline{R}$-projective, as wanted.

(3) Let $U,V \in \text{Simp}_R$, write $\alpha + 1 = h(U)$, $\beta + 1 = h(V)$ and assume that $U \preceq V$. Then $\text{Hom}_R(eUR/eUL_\alpha,eVR/eVL_\alpha) \neq 0$ and therefore $eV ReU \neq 0$. Conversely, assume that $eV ReU \neq 0$, i.e., there is a nonzero homomorphism $\phi: eUR \rightarrow eVR$, which must be a monomorphism because $L_\eta$ is a PWD. Since $eU \subseteq L_{\alpha+1}$, $L_\alpha = \text{Soc}_{\alpha+1}(eUR) \setminus \text{Soc}_\alpha(eUR)$ (see Proposition 1.5), it follows that $\phi(eU) \notin \text{Soc}_\alpha(eVR) = eVL_\alpha$. As a result $U \simeq eUR/eUL_\alpha$ imbeds into $eVR/eVL_\alpha$. Next, let us consider any element $x \in L_{\beta+1} \setminus L_\beta$ such that $(xR + L_\beta)/L_\beta \simeq V \simeq (eVR + L_\beta)/L_\beta$. Using the $R/L_\alpha$-projectivity of $(eVR + L_\alpha)/L_\alpha \simeq eVR/eVL_\alpha$ we get a nonzero homomorphism $\psi: eVR/eVL_\alpha \rightarrow (xR + L_\alpha)/L_\alpha$. But $\psi$ must be a monomorphism since $R/L_\alpha$ is a PWD by the property (2), therefore $U$ imbeds into $(xR + L_\alpha)/L_\alpha$. Thus $U \preceq V$ by Theorem 2.2.

(4) If $\eta = \xi$, then by (2) and Lemma 5.2 each $R/L_\alpha$ (for $\alpha \leq \xi$) is right nonsingular. □

Let $R$ be a semiprimary ring with Jacobson radical $N$. An ideal $I$ of $R$ is called an heredity if $I^2 = I$, $I_R$ is projective and $INI = 0$; $R$ is said to be quasi-hereditary if it admits a heredity chain $0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_m = R$, that is a chain such that $I_k/I_{k-1}$ is an heredity in $R/I_{k-1}$ for all $k = 1, \ldots, m$ (see [11] and [14] for details). We characterized in [9, Theorem 1.4] the semiprimary right NLF rings; the right Loewy chain of any such ring is an heredity chain, therefore these rings are quasi-hereditary. However, the algebra $R$ of Remark 3.4 is quasi-hereditary but it is neither right nor left NLF. As we are going to show, the class of semiprimary right NLF rings includes the class of those right semi-Artinian rings which are PWD with respect to a finite complete set of orthogonal idempotents. Recall that a ring $R$ is right $\ell$-hereditary if every local (i.e., having a unique maximal submodule) right ideal is projective.

**Proposition 5.4.** Given a right semi-Artinian ring $R$, the following conditions are equivalent:

1. $R$ is a PWD with respect to a complete set $\{e_1, \ldots, e_n\}$ of orthogonal idempotents.
2. $R$ is left perfect and right $\ell$-hereditary.
If these conditions hold, then \( R \) is a semiprimary right NLF ring.

**Proof.** (1) \( \Rightarrow \) (2) Suppose that (1) holds. Then \( R \) is right NLF by Proposition 5.3. According to [9, Main Theorem] there is a partition \( X_1 \cup \cdots \cup X_r \) of the set \( \{1, \ldots, n\} \) such that, by setting \( f_h = \sum_{i \in X_h} e_i \) for all \( h = 1, \ldots, r \) and after a possible renumbering of the idempotents \( f_h \), we have that \( f_h R f_k = 0 \) if \( h > k \); moreover, by setting \( P_{hk} = f_h R f_k \) and \( P_h = f_h R f_h \), each \( P_h \) is a prime PWD with respect to \( X_h \) and we may assume that

\[
R = \begin{pmatrix}
P_1 & P_{12} & \cdots & P_{1n} \\
0 & P_2 & \cdots & P_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & P_n
\end{pmatrix}.
\]

By Proposition 1.5 all the rings \( P_h \) are right semi-Artinian, because so is \( R \). As a consequence the \( P_h \) must be simple Artinian and this implies that \( R \) is semiprimary. Let \( H \) be any local right ideal of \( R \) and let \( K \) be the unique maximal submodule of \( H \). For some \( i \) there is an epimorphism \( p : e_i R \to H/K \) which is a projective cover for the simple module \( H/K \) and, if \( q : H \to H/K \) is the canonical epimorphism, then there is an homomorphism \( p' : e_i R \to H \) such that \( p = qp' \). Inasmuch as \( R \) is (right and left) perfect, then \( K = J(H) \) is small in \( H \) and consequently \( p' \) must be an epimorphism. On the other hand, by the assumption (1) \( p' \) is a monomorphism and so it is an isomorphism, therefore \( H \) is projective.

(2) \( \Rightarrow \) (1) If (2) holds and \( \{e_1, \ldots, e_n\} \) is a complete set of primitive and orthogonal idempotents, each \( e_i R \) is local and indecomposable and every nonzero homomorphic image of \( e_i R \) is local. This is enough to conclude that every nonzero homomorphism from \( e_1 R \) to \( RR \) must be a monomorphism.

\[\blacksquare\]

**Theorem 5.5.** Let \( R \) be a right semi-Artinian ring, with Loewy length \( \xi + 1 \) and Loewy chain \((L_\alpha)_{\alpha \leq \xi + 1} \). Assume that either \( R \) is right semihereditary or there is a set \( \mathcal{E} \) of idempotents of \( R \) such that one of the following conditions holds:

1. \( \mathcal{E} \) is a finite complete set of orthogonal idempotents and \( R \) is a PWD with respect to \( \mathcal{E} \);
2. \( L_\xi \) is a rPWD with respect to \( \mathcal{E} \).

Then \( R \) is nice.

**Proof.** Given \( U, V \in \text{Simp}_R \), assume that \( V \cdot I(U) = 0 \) and let us prove that \( U \not\approx V \). Set \( \alpha + 1 = h(U) \) and \( \beta + 1 = h(V) \); given \( x \in R \setminus L_\beta \) such that \( xR/xR \cap L_\beta \simeq V \), in view of Theorem 2.2 it will be sufficient to show that
$U$ imbeds into $xR/xR \cap L_\alpha$. By choosing an idempotent $e \in R \setminus L_\beta$ such that $V \simeq eR/eL_\beta$, we have a commutative diagram

$$
\begin{array}{ccc}
e R & \xrightarrow{u} & V \xrightarrow{0} \\
\downarrow{\varphi} & & \downarrow{=} \\
x R & \xrightarrow{v} & V \xrightarrow{0}
\end{array}
$$

(5.1)

with exact rows, where $u, v$ are the canonical projections and $\varphi$ is induced by the projectivity of $eR$. If $R$ is right semihereditary, then $\varphi(eR)$ is projective and therefore $eR = e'R \oplus e''R$ for a pair $e', e''$ of idempotents of $R$, where $\varphi|_{e'R} : e'R \rightarrow \varphi(eR)$ is an isomorphism and $e''R = \text{Ker}(\varphi)$. We infer that $u(e'R) = V$, therefore we may assume that $e = e'$ and $\varphi$ is a monomorphism. As a consequence, given $r \in R$, we have from Proposition 1.5 that $er \in eL_\alpha = \text{Soc}_\alpha(eR)$ if and only if $\varphi(er) \in xR \cap L_\alpha = \text{Soc}_\alpha(xR)$, therefore $\varphi$ induces a monomorphism $eR/eL_\alpha \rightarrow xR/xR \cap L_\alpha$. By the assumption and Theorem 2.2 we have that $U \lesssim eR/eL_\alpha$ and hence $U \lesssim xR/xR \cap L_\alpha$, as wanted. Next, suppose that $R$ satisfies (1) or (2) and assume first that $\alpha < \xi$. According to Proposition 5.3, we may assume that $e \in E$, so that $\varphi$ is a monomorphism since $\varphi \neq 0$ and we may repeat the previous argument. If $\alpha = \xi$, then every right ideal of $R/L_\xi$ is generated by an idempotent since $R/L_\xi$ is semisimple, consequently it follows from the last statement of Theorem 2.2 that $U \lesssim V$.

Finally, using [9, Corollary 1.6] if $R$ is right semihereditary, Proposition 5.4 if $R$ satisfies (1) and Proposition 5.3 if $R$ satisfies (2) we conclude that $R$ is right NLF.

A right semi-Artinian and right semihereditary ring need not be a PWD. For example, let $R$ be a regular and semi-Artinian ring with Loewy length at least three (see Section 4 of [8]) and take any idempotent $e \in R \setminus L_1$ which is primitive modulo $L_1$. Then $eR_K$ is not simple, yet contains a simple direct summand and hence there is a nonzero $R$-homomorphism $eR \rightarrow R$ which is not a monomorphism. Thus $R$ cannot be a PWD. On the other hand, given a field $F$ with an extension field $G$ such that $\dim(G_F) = 2$, the Artinian ring

$$
\begin{pmatrix}
F & G & G \\
0 & F & G \\
0 & 0 & F
\end{pmatrix}
$$

is a PWD with respect to the three obvious primitive and basic idempotents, but it is neither right nor left hereditary (this is probably known since long; see however [9, Example 2.1] for a proof).
6. Admissible partial orders

Let us say that a poset $P$ is \textit{admissible} when $P$ is order isomorphic to $\text{Simp}_R$ for some right semi-Artinian ring $R$. Of course, the obvious question is: which posets are admissible? At present we are not able to give a complete answer, nonetheless we can give some partial results. Firstly, Proposition 2.5 tells us that necessary conditions in order that a poset $P$ is admissible are that $P$ is Artinian and all maximal chains of $I$ have a maximum. With the example we are going to discuss we see that, conversely, every finite poset is admissible and every Artinian infinite poset becomes admissible by just adding a suitable maximal element.

We recall that the \textit{dual classical Krull filtration} of a poset $P$ is the ascending chain $(P_\alpha)^{\geq 0}$ of subsets of $P$ defined as follows (we denote by $m(X)$ the set of all minimal elements of any subset $X$ of $P$):

\begin{align*}
P_0 &= \emptyset, \\
P_{\alpha+1} &= P_\alpha \cup m(P \setminus P_\alpha) \quad \text{for every ordinal } \alpha, \\
P_\alpha &= \bigcup_{\beta < \alpha} P_\beta \quad \text{if } \alpha \text{ is a limit ordinal.}
\end{align*}

Clearly there exists a smallest ordinal $\xi$ such that $P_{\xi+1} = P_\xi$; moreover, $P$ is Artinian if and only if $P = P_\xi$ and, in this case, the ordinal $\xi$ is called the \textit{dual classical Krull dimension} of $P$. For every $i \in P$ we denote by $\lambda(i)$ the unique ordinal $\alpha + 1$ such that $i \in P_{\alpha+1} \setminus P_\alpha$.

\textbf{Example 6.1.} Given an Artinian poset $P$, there exists a left NLF-ring $R$ satisfying the following conditions:

1. If $P$ is finite, then $R$ is Artinian and $\text{Simp}_R$ is order isomorphic to $P$.
2. If $P$ is infinite, then $\text{Simp}_R$ has a maximal element $V$ such that $\text{Simp}_R \setminus \{V\}$ is order isomorphic to $P$ and, given $U \in \text{Simp}_R$, one has $U \preceq V$ if and only if $\{U \preceq\}$ is infinite.

Moreover, $R$ has a set $E = \{e(i) \mid i \in P\}$ of pairwise orthogonal idempotents such that $R$ is a PWD (respectively $\text{Soc}_{L(R)}^{-(R)}$ is a rPWD) with respect to $E$ if $P$ is finite (respectively infinite).

\textbf{Proof.} Given any division ring $D$, let us consider the ring $R$ of all $P \times P$-matrices with entries in $D$ such that $a_{ii} \neq a_{jj}$ only for a finite number of $i, j \in I$ and, off the diagonal, $a_{ij} \neq 0$ only for a finite number of $i, j$ with $i < j$ (we denote with $a_{ij}$ the $(i, j)$-entry of the matrix $a$). For each $i \in P$ let $e_{(i)}$ be the matrix whose $(i, i)$-entry is 1 and all others are zero. When $P$ is finite $R$ is clearly right and left Artinian and is a PWD with respect to $E = \{e_{(i)} \mid i \in P\}$; it is the \textit{incidence algebra} of $I$ with coefficients in $D$ and is extensively investigated by many authors working
in Representations Theory of Algebras (see Simson’s book [24] for details). For every subset \( X \) of \( P \) let us consider the right ideal \( H(X) = \bigoplus \{ e(i)R \mid i \in X \} \). If \( X \) is an upper subset of \( P \), then \( H(X) \) is an ideal of \( R \). We proved in [9, Proposition 5.4] that \( R \) is left semi-Artinian and is left NLF; moreover, if \( \xi \) is the dual classical Krull dimension of \( P \), then

\[
\text{Soc}_\alpha(RR) = \begin{cases} H(P_\alpha) & \text{if } \alpha \leq \alpha_0, \\ (1_R - f_\alpha)R & \text{if } \alpha_0 < \alpha \leq \xi, \end{cases}
\]

where \( \alpha_0 = \min \{ \alpha \leq \xi \mid P \setminus P_\alpha \text{ is finite} \} \) and \( f_\alpha = \sum \{ e(i) \mid i \in P \setminus P_\alpha \} \); moreover,

\[
L(RR) = \begin{cases} \xi + 1 & \text{if } \alpha_0 = \xi, \\ \xi & \text{if } \alpha_0 < \xi. \end{cases}
\]

Let us write \( L_\alpha \) for \( \text{Soc}_\alpha(RR) \) and observe that for all \( \alpha \geq 0 \) we have the inclusion

\[
L_\alpha \subset \{ a \in R \mid a_{ik} = 0 \text{ if } i \in P \setminus P_\alpha \},
\]

which is an equality when \( \alpha_0 < \alpha \leq \xi \). If \( P \) is infinite, then \( LL(RR)-1 \) is faithful as a right and a left ideal and we infer from Proposition 5.1 that it is a PWD with respect to \( E \).

For each \( i \in P \) let us consider the ideal

\[
M_i = R(1 - e(i))R = \{ a \in R \mid a_{ii} = 0 \}
\]

and note that \( M_i \) is maximal as a right and a left ideal; also, let us consider the simple left \( R \)-module \( U_i = R/M_i \). By setting \( H = H(P) \), if \( P \) is finite it is clear that \( H = R \); if \( P \) is infinite, then \( R/H \cong D \) as rings and hence \( H \) is another maximal right and maximal left ideal. Set \( V = R/H \). Next, we observe that the ideal \( N = \{ a \in R \mid a_{ii} = 0 \text{ for all } i \in P \} \) is nil and \( R/N \cong D(P) + 1_R D \) is regular, therefore \( N = J(R) \) and we may conclude that

\[
\mathcal{R}\text{Simp} = \begin{cases} \{ U_i \mid i \in P \} & \text{if } P \text{ is finite}, \\ \{ U_i \mid i \in P \} \cup \{ V \} & \text{if } P \text{ is infinite}. \end{cases}
\]

Let \( i \in P \) and let \( \alpha + 1 = \lambda(i) \). Since \( M_i e(i) = L_\alpha e(i) \), then \( Re(i)/L_\alpha e(i) \cong U_i \) and we have the equalities

\[
L(U_i) = Re(i)R + L_\alpha e(i)R + L_\alpha.
\]

We claim that

\[
I(U_i) = \{ a \in R \mid a_{jk} = 0 \text{ if } i \leq j \}.
\]

Indeed, if \( a \in R \) is such that \( a_{jk} = 0 \) when \( j \geq i \) and \( b \in e(i)R \), that is \( b_{rs} = 0 \) if \( r \neq i \), we can see easily that \( ba = 0 \) and hence \( L(U_i)a \subset L_\alpha \) by (6.2). Conversely, suppose that \( L(U_i)a \subset L_\alpha \) and assume that \( a_{jk} \neq 0 \) for some \( j \geq i \). If we consider the matrix \( b \) defined by

\[
b_{rs} = \begin{cases} 1 & \text{if } r = i \text{ and } s = j, \\ 0 & \text{otherwise}, \end{cases}
\]

then...
then $b \in e(i)R \subset L(U_i)$, but $(ba)_{ik} = a_{jk} \neq 0$ and this implies that $ba \notin L_{\alpha}$ by (6.1), a contradiction. This establishes the equality (6.3). Now, if $i \leq j$ and $a \in I(U_i)$, then $a_{jj} = 0$; consequently $I(U_i) \subset M_j$ and, hence, $U_i \preceq U_j$ by Theorem 5.5. Conversely, assume that $U_i < U_j$, that is $I(U_i) \subsetneq I(U_j)$. In particular, each matrix in $I(U_i)$ has zero $j$th row and consequently $i \leq j$, otherwise it would be true that $e_{(j)} \in I(U_i) \subset I(U_j)$, in contradiction with the fact that $e_{(j)}I(U_j) = 0$. We conclude that

$$i \leq j \quad \text{if and only if} \quad U_i \preceq U_j.$$ 

Assume now that $P$ is infinite. If $I(U_i) \not\subset H$, then there is some $a \in I(U_i)$ and a finite subset $X \subset P$ such that $a_{jj} = a_{kk} \neq 0$ for $j, k \in P \setminus X$. It follows then from (6.3) that $\{i \leq j\} \subset X$ and so $\{i \leq j\}$ is finite. Conversely, if $\{i \leq j\}$ is finite, by taking the idempotent $e = \sum \{e_{(j)} \mid j \geq i\}$ we have that $1 - e \in I(U_i) \setminus H$. As a result, since $U_i \preceq V$ if and only if $I(U_i) \subset H$, for every $i \in P$ we have that

$$U_i \preceq V \quad \text{if and only if} \quad \{i \leq j\} \text{ is infinite.}$$

Finally, we observe that $L(V) = R$ and hence $I(V) = H$, therefore, $V$ is a maximal element of $\_R\text{Simp}$. $\square$

**Final Remarks 6.2.** Several problems arise from the investigations we have accomplished so far. We limit us to list the following ones.

1. Give characterizations of admissible posets.
2. If $U$ is a maximal element of $\text{Simp}_R$, where $R$ is any right semi-Artinian ring, is it always true that the vector space $\text{End}(U_R)U$ is finite dimensional?
3. As we mentioned in the introduction, it was proven by Camillo and Fuller that if $R$ is a right semi-Artinian ring with finite (right) Loewy length, then $R$ is left semi-Artinian as well. Of course $\text{Simp}_R$ and $\_R\text{Simp}$ have the same cardinality and it would be interesting to detect any relationship between the respective natural partial orders.

**References**