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The Combinatorics of Meixner Polynomials: Linearization Coefficients

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We describe various aspects of the Meixner polynomials. These include combinatorial descriptions of the moments, the orthogonality relation, and the linearization coefficients.

1. INTRODUCTION

The Meixner polynomials $m_n(x; \beta, c)$ are analytically well known [2], and have been studied combinatorially by various authors [1, 9, 12, 13, 16]. The moments for the measure of these orthogonal polynomials are

$$\mu_n(\beta, C) = (1-c)^{-\beta} \sum_{k \geq 0} k^n c^k \frac{(\beta)_k}{k!}. \tag{1.1}$$

Viennot [16] gave a unified combinatorial approach to the moments μ_n of the Sheffer orthogonal polynomials (Hermite, Charlier, Laguerre, Meixner and Meixner–Pollaczek) in terms of special sets of weighted permutations on a n-elements set.

Given a sequence $\{p_n(x)\}_{n\geq 0}$ of orthogonal polynomials, let L be the linear functional on polynomials that corresponds to integrating with respect to their measure. We define the *linearization coefficients* of $p_n(x)$ to be

$$a(n_1, n_2, \ldots, n_l) = L\left(\prod_{i=1}^{l} p_{n_i}(x)\right). \tag{1.2}$$

The problem of evaluating $a(n_1, n_2, \ldots, n_l)$ is equivalent to determining the coefficient $b(n_1, n_2, \ldots, n_l)$ in the expansion

$$p_{n_2}(x) p_{n_3}(x) \ldots p_{n_l}(x) = \sum_{n_1=0}^{n_2+n_3+\cdots+n_l} b(n_1, n_2, \ldots, n_l) p_{n_1}(x). \tag{1.3}$$

Zeng [18] gave a combinatorial interpretation for the linearization coefficients of the five classes of monic Sheffer orthogonal polynomials. More precisely, let $n_1, n_2, \ldots, n_l$ be non-negative integers and consider the set of pairs $(i, j) \in \cup_{k=1}^{l} \{k\} \times [n_k]$, where $[n] = \{1, 2, \ldots, n\}$ and $[0] = \emptyset$. We say that such a pair (i, j) is *of color* i and we write $\text{color}(i, j) = i$. A permutation σ on $\cup_{k=1}^{l} \{k\} \times [n_k]$ is called a *colored derangement* iff $\text{color}(\sigma(i, j)) \neq \text{color}(i, j)$ for all pairs (i, j). Zeng's theorem states that the linearization coefficients for the monic Sheffer orthogonal polynomials are given by Viennot's [16] combinatorial model for their corresponding moments on the set $\cup_{k=1}^{l} \{k\} \times [n_k]$, with the additional condition that all permutations considered must also be colored derangements. This combinatorial representation of (1.2) easily shows the non-negativity of the linearization coefficients for some range of values of their parameters, since Viennot's weights for the structures are positive monomials in terms of these parameters.

Zeng proved his theorem by computing the generating functions for the linearization coefficients of the Sheffer orthogonal polynomials, using their measure. Then he showed that these correspond to the generating functions of the proposed combinatorial interpretations. Our approach differs. Our main result (Theorem 4) is a totally combinatorial proof of the linearization coefficients for the Meixner polynomials. From the combinatorial interpretations of the

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polynomials and their moments, in terms of weighted permutations and endofunctions, we deduce a combinatorial interpretation for the linearization coefficients of a product of Meixner polynomials. We then apply a weight-preserving sign-reversing involution defined in three steps. Theorem 4 is obtained by enumerating the remaining fixed points, which reduces the matter to Zeng's interpretation.

Similar approaches to the linearization coefficients problem have been used by de Saint-Catherine and Viennot [6] for Hermite polynomials, by de Médicis [3] for Charlier polynomials, by de Sainte-Catherine and Viennot [6] for Laguerre polynomials $L_n^{(\alpha)}(x)$ with $\alpha = 0$, and by Foata and Zeilberger [10] for general Laguerre polynomials. So far, no such proof exists for the Meixner–Pollaczek polynomials. For the q-analogs of the Sheffer orthogonal polynomials, Ismail, Stanton and Viennot [11] solved the linearization coefficients problem for q-Hermite polynomials, and found some remarkable consequences. Also, de Médicis, Stanton and White [4] studied the linearization coefficients problem for q-Charlier polynomials, which uses some deep results on the combinatorics of q-Stirling numbers. The combinatorics of q-Laguerre polynomials has been studied by de Médicis and Viennot [5], Simion and Stanton [15], and Zeng [19], but no-one has addressed the linearization coefficients problem yet in that case.

The basic combinatorial interpretation of the Meixner polynomials is given in Theorem 1. Several facts about the polynomials can be proven combinatorially. The statistics for the moments is given in Theorem 2. In Section 3, we state our main theorem, Theorem 4, giving a combinatorial interpretation for the linearization coefficient for a product of l Meixner polynomials. The three steps of the weight-preserving sign-reversing involution proving Theorem 4 follow in Section 4.

Let us recall that a *weight-preserving sign-reversing involution* Φ on a set E with weight function ω is an involution such that for any $e \notin \mathrm{Fix}\Phi$, $\omega(\Phi(e)) = -\omega(e)$. Obviously,

$$\sum_{e \in E} \omega(e) = \sum_{e \in \mathrm{Fix}\Phi} \omega(e), \tag{1.4}$$

where $\mathrm{Fix}\Phi$ denotes the set of fixed points of Φ.

2. The Meixner Polynomials and their Moments

We define the Meixner polynomials by the following generating function:

$$\sum_{n=0}^{\infty} m_n(x;\beta,c)\frac{t^n}{n!} = \left(1-\frac{t}{c}\right)^x (1-t)^{-x-\beta}. \tag{2.1}$$

This gives the explicit formula

$$\begin{aligned} m_n(x;\beta,c) &= (\beta)_n\, {}_2F_1\left(\begin{matrix}-n,-x\\ \beta\end{matrix}; 1-\frac{1}{c}\right),\\ &= \sum_{k=0}^{n}\binom{n}{k}(-x)_k(\beta+k)_{n-k}\left(-1+\frac{1}{c}\right)^k, \end{aligned} \tag{2.2}$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)\ldots(a+n-1), n \geq 1$.

Note that these polynomials are not monic. Their three-term recurrence relation is given by

$$\begin{aligned} m_{n+1}(x;\beta,c) = {} & \left(\left(1-\frac{1}{c}\right)x + \left(1+\frac{1}{c}\right)n+\beta\right)m_n(x;\beta,c)\\ & -\frac{n}{c}(n+\beta-1)m_{n-1}(x;\beta,c), \end{aligned} \tag{2.3}$$

where $m_{-1}(x;\beta,c) = 0$ and $m_0(x;\beta,c) = 1$.

For the combinatorial interpretation of these polynomials, we use the terminology introduced by Foata and Labelle [9], with some slight modifications.

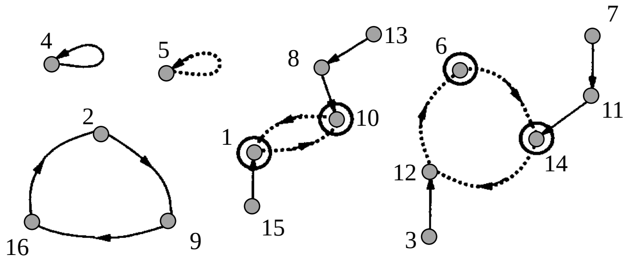

FIGURE 1. A Meixner endofunction $\tau = ((A, B), f, C)$.

DEFINITION. *A Meixner endofunction* on a finite set S is a triple $\tau = ((A, B), f, C)$ where:

(a) (A, B) is an ordered partition of S(A or B can be empty);
(b) $f : S \to S$ is an endofunction on S such that its restriction to A, $f|_A : A \to S$, is injective and its restriction to B, $f|_B : B \to S$ is a permutation of the set B;
(c) C is a subset (possibly empty) of B.

For example, if $S = [16]$, $A = \{2, 3, 4, 7, 8, 9, 11, 13, 15, 16\}$, $B = \{1, 5, 6, 10, 12, 14\}$, $C = \{1, 6, 10, 14\}$ and the graph of f is given in Figure 1, $\tau = ((A, B), f, C)$ is a Meixner endofunction.

Note that for convenience we have drawn the edges corresponding to $f|_A$ (respectively $f|_B$) as solid (respectively dotted) lines, and that we have circled the elements of C. This way, we have given a complete graphic representation for $\tau = ((A, B), f, C)$. From this representation, it is easy to see that Meixner endofunctions can be decomposed into connected components of two types, which are:

(i) cycles from the set A (solid lines in Figure 1). We will refer to all these cycles as the *polynomial cycles* and let cycle (τ) be their number;
(ii) cycles from the set B (dotted lines in Figure 1) and the successive pre-images from the set A attached to them (that is any $a \in A$ such that $f^i(a)$ lands on B, for some $i \geq 1$). Such a component is called an *octopus*, the cycle in B being its *body*, and the chains from A attached to it being the *legs*. We let octopi(τ) be the number of octopi in τ.

Finally, we will refer to the elements of C (respectively $B - C$) as *circled elements* (respectively *uncircled elements*) and let $\text{circ}(\tau) = |C|$ and $\text{uncirc}(\tau) = |B - C|$.

For example, τ in figure 1 has $\text{cycle}(\tau) = 2$, $\text{octopi}(\tau) = 3$, $\text{circ}(\tau) = 4$ and $\text{uncirc}(\tau) = 2$.

THEOREM 1. *The Meixner polynomials are given by*

$$m_n(x; \beta, c) = \sum_{\tau \in M(S)} \beta^{\text{cycle}(\tau)} (-1)^{\text{octopi}(\tau)+\text{uncirc}(\tau)} (1/c)^{\text{circ}(\tau)} x^{\text{octopi}(\tau)},$$
$$= \sum_{\tau \in M(S)} \omega(\tau) x^{\text{octopi}(\tau)},$$

where S is a set of cardinality n and $M(S)$ denotes the set of all Meixner endofunctions on S.

PROOF. This is easily seen from the explicit formula (2.2), knowing that $(a)_n$ (respectively $(a + n - k)_k$) is the generating polynomial for permutations σ of $[n]$ (respectively injections f from $[k]$ into $[n]$) weighted by $a^{\text{cycle}(\sigma)}$ (respectively $a^{\text{cycle}(f)}$). We use circling so that the points on the bodies of octopi have monomial weight (either (-1) or (l/c), as opposed to $(1/c - 1)$). For more details, see [9].

A combinatorial proof of the three-term recurrence relation (2.3) can be given using Theorem 1. An involution is necessary.

For the moments of the polynomials, we need two statistics on permutations. Let $\mathfrak{S}(U)$ denote the set of permutations on a totally ordered set U and let $\sigma \in \mathfrak{S}(U)$. We denote by $\mathrm{cycle}(\sigma)$ the number of cycles of σ and $\mathrm{nexc}(\sigma)$ the number of *non-excedances* of σ, that is the number of elements $u \in U$ such that $u \geq \sigma(u)$.

THEOREM 2. *The nth moment for the Meixner polynomials is given by*

$$\mu_n(\beta, c) = (1-c)^{-n} \sum_{\sigma \in \mathfrak{S}(U)} \beta^{\mathrm{cycle}(\sigma)} c^{\mathrm{nexc}}(\sigma),$$

where U is a totally ordered set of cardinality n.

We will refer to the cycles of σ as the *moment cycles* and to σ as the *moment permutation*. A proof of Theorem 2 using generating functions can be found in [18]. Note that this combinatorial interpretation for the moments is slightly different from Viennot's interpretation [16], using number of left–right minima and number of descents on permutations. However, one can recover Theorem 2 by applying Foata's [7] fundamental transformation to Viennot's interpretation.

3. THE ORTHOGONALITY RELATION AND THE LINEARIZATION OF PRODUCTS

Let L be the linear functional on polynomials that corresponds to integrating with respect to the measure for the Meixner polynomials. The orthogonality relation is

$$L(m_n(x;\beta,c)m_p(x;\beta,c)) = c^{-n}(\beta)_n n!\delta_{n,p}. \tag{3.1}$$

More generally, the generating function for the linearization coefficients of Meixner polynomials is given by

$$\sum_{n_1,\dots,n_l=0}^{\infty} L(m_{n_1}(x;\beta,c)\dots m_{n_l}(x;\beta,c))\frac{x_1^{n_1}}{n_1!}\cdots\frac{x_l^{n_l}}{n_l!} = \left(1 - \sum_{k=2}^{l}(-1)^k\left(\frac{1}{c}+\frac{1}{c^2}+\cdots+\frac{1}{c^{k-1}}\right)e_k\right)^{-\beta}, \tag{3.2}$$

where $e_k = e_k(x_1, x_2, \dots, x_l)$ denotes the elementary symmetric function of degree k in l variables [14]. This generating function can be evaluated directly using the measure [2].

Since the polynomials $m_n(x;\beta,c)$ and L have combinatorial definitions from Theorems 1 and 2, it is possible to restate (3.1) and (3.2) as combinatorial problems. We will give an involution that then proves (3.1) and (3.2) in this framework.

Hereafter, we will consider combinatorial structures ξ on pairs $(i, j) \in \cup_{k=1}^{l}\{k\} \times [n_k]$ (these structures may be octopi or cycles or more complicated structures). The subset of $\cup_{k=1}^{l}\{k\} \times [n_k]$ on which ξ is constructed, denoted $\mathrm{Supp}(\xi)$, is called the *support* of ξ. If $\mathrm{Supp}(\xi) \subseteq \{i\} \times [n_i]$, we say that ξ is of *color i* and we write $\mathrm{color}(\xi) = i$.

The set $\cup_{k=1}^{l}\{k\} \times [n_k]$ is naturally ordered by lexicographic order ($(i, j) \leq (i', j') \Leftrightarrow i < i'$ or $i = i'$ and $j \leq j'$). Denote by $\min(\xi) = \min(\mathrm{Supp}(\xi))$ the minimum of the support of a structure ξ. The lexicographic order naturally induces an order on structures with disjoint support sets, by increasing minima, i.e. if $\mathrm{Supp}(\xi) \cap \mathrm{Supp}(\xi') = \emptyset, \xi \leq \xi' \Leftrightarrow \min(\xi) \leq (\xi')$. We will use this relation whenever an ordering of our combinatorial structures is needed.

Let $\tau = ((A, B), f, C)$ be a Meixner endofunction on $\{i\} \times [n_i]$, we will need the following notations. We will denote by $\mathrm{Cyc}(\tau)$ the set of polynomial cycles of τ and by $\mathrm{Oct}(\tau)$ the set of octopi of τ. Let $\Omega \in \mathrm{Oct}(\tau)$, we denote by $\mathrm{Body}(\Omega) = \mathrm{Supp}(\Omega) \cap B$ (respectively

$\mathrm{Circ}(\Omega) = \mathrm{Supp}(\Omega) \cap C$ and $\mathrm{Uncirc}(\Omega) = \mathrm{Body}(\Omega) - \mathrm{Circ}(\Omega))$ the set of points (respectively circled and uncircled) on the body of Ω. For $b \in \mathrm{Body}(\Omega)$, we let $\mathrm{Leg}(b)$ (respectively $\widetilde{\mathrm{Leg}}(b)$) be the support (respectively ordered support) of the leg attached to b, that is

$$\mathrm{Leg}(b) = \{a \in A \mid \exists i \geq 1 \text{ such that } f^i(a) = b \text{ and } f^j(a) \notin B, \text{ for } 1 \leq j < i\},$$

and $\widetilde{\mathrm{Leg}}(b) = (a_1, \ldots, a_s)$ such that $a_i \in \mathrm{Leg}(b)$ and $f^i(a_i) = b$. We denote by $\mathrm{attmin}(\Omega)$ the attachment point of $\min(\Omega)$ to the body of the octopus, that is the unique $b \in \mathrm{Body}(\Omega)$ such that $\min(\Omega) \in \{b\} \cup \mathrm{Leg}(b)$.

Finally, for technical reasons, we define the *associated minimum cycle* $\mathrm{mcyc}(\Omega)$ of an octopus Ω to be the cycle obtained from Ω by replacing every element b on the body of Ω by $\min(\{b\} \cup \mathrm{Leg}(b))$. Again, $\mathrm{Supp}(\mathrm{mcyc}(\Omega))$ can be partitioned into two sets $\widetilde{\mathrm{Circ}}(\Omega)$ and $\widetilde{\mathrm{Uncirc}}(\Omega)$, according to the circling of the corresponding points b on the body of Ω.

For example, if Ω is the rightmost octopus in Figure 1, $\mathrm{Body}(\Omega) = \{6, 12, 14\}$, $\mathrm{Circ}(\Omega) = \{6, 14\}$, and $\mathrm{Uncirc}(\Omega) = \{12\}$. Moreover, $\mathrm{Leg}(14) = \{7, 11\}$, $\widetilde{\mathrm{Leg}}(14) = (11, 7)$, $\min(\Omega) = 3$, and $\mathrm{attmin}(\Omega) = 12$. Finally, $\mathrm{mcyc}(\Omega) = (3, 6, 7)$, $\widetilde{\mathrm{Circ}}(\Omega) = \{6, 7\}$, and $\widetilde{\mathrm{Uncirc}}(\Omega) = \{3\}$.

We now give our first combinatorial interpretation for the Meixner linearization coefficients. Define

$$\begin{aligned} L(\mathbf{n}) = L(n_1, n_2, \ldots, n_l) = \{(\tau_1, \tau_2, \ldots, \tau_l; \sigma) | \tau_i &= ((A_i, B_i,), f_i, C_i) \\ &\text{is a Meixner endofunction on the set } \{i\} \times [n_i] \text{ such that} \\ &\mathrm{attmin}(\Omega) \in C_i \text{ for all } \Omega \in \mathrm{Oct}(\tau_i), \text{ and } \sigma \text{ is a permutation} \\ &\text{on the set of all octopi } \cup_{k=1}^{l} \mathrm{Oct}(\tau_k)\}. \end{aligned}$$

LEMMA 3. *Let $n_1, n_2, \ldots, n_l \geq 0$. The linearization coefficient for Meixner polynomials is given by*

$$\begin{aligned} &L(m_{n_1}(x; \beta, c) m_{n_2}(x; \beta, c) \ldots m_{n_1}(x; \beta, c)) \\ &\qquad = \sum_{(\tau_1, \tau_2, \ldots, \tau_l; \sigma) \in L(\mathbf{n})} \omega(\tau_1)\omega(\tau_2) \ldots \omega(\tau_l) \beta^{\mathrm{cycle}(\sigma)} c^{\mathrm{nexc}(\sigma)}, \end{aligned}$$

where the weight ω was defined in Theorem 1.

PROOF. From Theorem 1, the power of x associated to a Meixner endofunction corresponds to its number of octopi. So applying the linear functional L to a product of l Meixner polynomials corresponds to adding a permutation σ on the set of all octopi in their combinatorial representation. Notice that each octopus gets weighted $(1 - c)^{-1}$ from the moments. If we combine this weight and one factor $(1/c - 1)$ from the weight of the points on the body of the octopus (circled or uncircled), we obtain $(1 - c)^{-1}(1/c - 1) = 1/c$. We translate this combinatorially by requiring that the specific point $\mathrm{attmin}(\Omega)$ on the body of each octopus has to be circled.

THEOREM 4. *Let $n_1, n_2, \ldots, n_l \geq 0$. There exists a weight-preserving sign-reversing involution Φ on the set $L(n_1, n_2, \ldots, n_l)$ such that its set of fixed points* $\mathrm{Fix}\Phi$ *is in bijection with the set* $\mathrm{CD}(n_1, n_2, \ldots, n_l)$ *of all colored derangements on the set $\cup_{i=1}^{l} \{i\} \times [n_i]$. Moreover, the linearization coefficient for Meixner polynomials is given by*

$$\begin{aligned} &L(m_{n_1}(x; \beta, c) m_{n_2}(x; \beta, c) \ldots m_{n_l}(x; \beta, c)) \\ &\qquad = (-1/c)^{n_1 + n_2 + \cdots + n_l} \sum_{\sigma \in \mathrm{CD}(n_1, n_2, \ldots, n_l)} \beta^{\mathrm{cycle}(\sigma)} c^{\mathrm{nexc}(\sigma)}. \end{aligned}$$

PROOF. The involution Φ consists of three successive weight-preserving sign-reversing involutions, each one acting on the fixed points of the preceding one. These involutions Φ_i and their respective fixed points sets $\mathrm{Fix}\Phi_i$ are given in the next section. The final set of fixed points $\mathrm{Fix}\Phi = \mathrm{Fix}\Phi_3$ is given by

$$\begin{aligned}\mathrm{Fix}\Phi = \{(\tau_1, \tau_2, \ldots, \tau_l; \sigma) \in L(\mathbf{n}) | \tau_i = ((\emptyset, \{i\} \times [n_i]), \mathrm{Id}, \{i\} \times [n_i]) \text{and color}(\sigma(\Omega)) \\ \neq \text{color}(\Omega), \forall \Omega \in \cup_{k=1}^{l} \mathrm{Oct}(\tau_k)\},\end{aligned}$$

where Id denotes the identity function on appropriate sets.

So for $1 \geq i \geq l$, the only Meixner endofunction τ_i that survives is formed by n_i octopi, with the only point on the body circled, and therefore its weight is $\omega(\tau_i) = (-1/c)^{n_i}$. This and Lemma 3 give

$$\begin{aligned}&L(m_{n_1}(x;\beta,c)m_{n_2}(x;\beta,c)\ldots m_{n_l}(x;\beta,c)) \\ &\quad= \sum_{(\tau_1,\tau_2,\ldots,\tau_l;\sigma)\in \mathrm{Fix}\Phi} \omega(\tau_1)\omega(\tau_2)\ldots\omega(\tau_1)\beta^{\mathrm{cycle}(\sigma)}c^{\mathrm{nexc}(\sigma)}, \\ &\quad= (-1/c)^{n_1+n_2+\cdots+n_l} \sum_{\sigma\in \mathrm{CD}(n_1,n_2,\ldots,n_l)} \beta^{\mathrm{cycle}(\sigma)}c^{\mathrm{nexc}(\sigma)}.\end{aligned}$$

REMARKS. For $l = 2$ it is easy to see that $\mathrm{CD}(n_1, n_2)$ is empty unless $n_1 = n_2$, in which case all colored derangements will have exactly n_2 non-excedances (one for each pair $(2, j)$). Let $\sigma \in \mathrm{CD}(n, n)$, we can associate bijectively to σ a pair (σ_1, σ_2) where $\sigma_1 = \sigma \circ \sigma|_{\{1\}\times[n]}$ is a permutation of $\{1\} \times [n]$ and $\sigma_2 = \sigma|_{\{1\}} \times [n]$ is a bijection from $\{1\} \times [n]$ to $\{2\} \times [n]$, with $\mathrm{cycle}(\sigma) = \mathrm{cycle}(\sigma_1)$. This decomposition gives the orthogonality relation (3.1).

The generating function for the linearization coefficients (3.2) can be obtained from Theorem 4 by computing the generating function of colored derangements according to the number of cycles and non-excedances. This was done by Zeng [17], using the β-extension of MacMahon's Master Theorem.

Note that considering the monic Meixner polynomials,

$$\hat{m}_n(x;\beta,c) = \left(\frac{c}{c-1}\right)^n m_n(x;\beta,c),$$

gives Zeng's Theorem [18] as was stated in the introduction:

$$\begin{aligned}&L(\hat{m}_{n_1}(x;\beta,c)\hat{m}_{n_2}(x;\beta,c)\ldots\hat{m}_{n_l}(x;\beta,c)) \\ &\quad= (1-c)^{-(n_1+n_2+\cdots+n_l)} \sum_{\sigma\in \mathrm{Cd}(n_1,n_2,\ldots,n_l)} \beta^{\mathrm{cycle}(\sigma)}c^{\mathrm{nexc}(\sigma)}.\end{aligned}$$

4. THE WEIGHT-PRESERVING SIGN-REVERSING INVOLUTIONS Φ_i

Let $(\tau_1, \tau_2, \ldots, \tau_l; \sigma) \in L(\mathbf{n})$. From Lemma 3, the weight of such a structure is given by

$$\begin{aligned}\omega(\tau_1,\tau_2,\ldots,\tau_l;\sigma) = (-1)^{\Sigma_{i=1}^{l}\mathrm{octopi}(\tau_i)} + \Sigma_{i=1}^{l}\mathrm{uncirc}(\tau_i) \\ \times\beta^{\mathrm{cycle}(\sigma)+\Sigma_{i=1}^{l}\mathrm{cycle}(\tau_i)}c^{\mathrm{nexc}(\sigma)+\Sigma_{i=1}^{l}\mathrm{circ}(\tau_i)}.\end{aligned} \tag{4.1}$$

4.1. Involution Φ_1. The principle of this involution is to replace an octopus Ω by a chain of octopi in the moment permutation σ. We achieve this by cutting the body of the octopus into smaller parts, the legs always following their attachment points on the body. So to determine the chain of octopi, we need only specify how to cut the associate minimum cycle $\mathrm{mcyc}(\Omega)$ into smaller cycles.

More precisely, let $(\tau_1, \tau_2, \ldots, \tau_l; \sigma) \in L(\mathbf{n})$ and let $\Omega \in \cup_{k=1}^{l} \mathrm{Oct}(\tau_k)$ such that $|\mathrm{Circ}(\Omega)| \geq 2$. So $\mathrm{mcyc}(\Omega) = (b_1, b_2, \ldots, b_q)$ has at least two circled points, one of them being $b_1 = \min(\Omega)$. Let $1 = i_1 < i_2 < \cdots < i_r \leq q, r \geq 2$ such that:

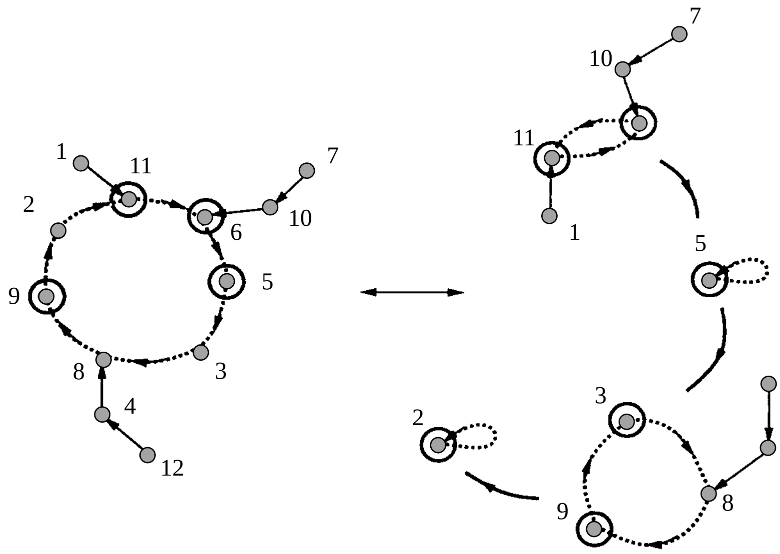

FIGURE 2. The division chain of an octopus $\hat{\Omega}$.

(i) $b_{i_2} = \min\left(\widetilde{\text{Circ}}(\Omega) - \{b_1\}\right)$,

(ii) $b_{i_3}, \ldots, b_{i_r}$ are the left–right minima that lie to the right of b_{i_2} in $\text{mcyc}(\Omega)$ and are among uncircled points less than b_{i_2}, i.e.

$$b_{i_j} = \min\left(\{b_{i_2}, b_{i_{2+1}}, \ldots, b_{i_j}\} \cap \left[\widetilde{\text{Uncirc}}(\Omega) \cup \{b_{i_2}\}\right]\right).$$

We define the *division chain* of Ω to be $\text{Div}(\Omega) = (\hat{\Omega}_1, \ldots, \hat{\Omega}_r)$, where for $1 \le j \le r$, $\hat{\Omega}_j$ is the octopus with

$$\text{mcyc}(\hat{\Omega}_j) = (b_{i_j}, b_{i_j+1}, \ldots, b_{i_{j+1}-1})$$

and

$$\widetilde{\text{Circ}}(\hat{\Omega}_j) = \{b_{i_j}\} \cup \left(\widetilde{\text{Circ}}(\Omega) \cap \{b_{i_j}, b_{i_{j+1}}, \ldots, b_{i_{j+1}-1}\}\right).$$

By convention, $b_{i_{r+1}} = b_1$ and $b_{i_{r+1}-1} = b_q$.

This operation is illustrated in Figure 2, with $\text{mcyc}(\Omega) = (1, 6, 5, 3, 4, 9, 2)$, $\widetilde{\text{Circ}}(\Omega) = \{1, 5, 6, 9\}$, $\widetilde{\text{Uncirc}}(\Omega) = \{2, 3, 4\}$, $b_{i_1} = 1$, $b_{i_2} = 5$, $b_{i_3} = 3$ and $b_{i_4} = 2$.

The division chain $(\hat{\Omega}_1, \ldots, \hat{\Omega}_r)$ of Ω has the following properties:

Div. 1 $\min(\hat{\Omega}_j) = b_{i_j}$ and its corresponding attachment point $\text{attmin}(\hat{\Omega}_j)$ is always circled.

Div. 2 $\min(\hat{\Omega}_1) < \min(\hat{\Omega}_2)$ and $\min(\hat{\Omega}_2) > \min(\hat{\Omega}_3) > \cdots > \min(\hat{\Omega}_r) > \min(\hat{\Omega}_1)$.

Div. 3 If $|\widetilde{\text{Circ}}(\hat{\Omega}_1| \ge 2$, then $\min(\widetilde{\text{Circ}}(\hat{\Omega}_1) - \{\min(\hat{\Omega}_1)\}) > \min(\hat{\Omega}_2)$.

Div. 4 Given $(\hat{\Omega}_1, \ldots, \hat{\Omega}_r)$ with properties Div. 1–Div. 3, there is a unique way to glue together $\hat{\Omega}_1, \ldots, \hat{\Omega}_r$ to recover the octopus Ω such that $\text{Div}(\Omega) = (\hat{\Omega}_1, \ldots, \hat{\Omega}_r)$. We write $\Omega = \text{Div}^{-1}(\hat{\Omega}_1, \ldots, \hat{\Omega}_r)$.

We can now define Φ_1. Let $(\tau_1, \tau_2, \ldots, \tau_l; \sigma) \in L(\mathbf{n})$. Basically, to make Φ_1 an involution, we need to determine uniquely an octopus or its division chain (corresponding to Cases 1 and 2 below) in the moment permutation σ. Let $\rho = (\Omega_1, \Omega_2, \ldots, \Omega_p)$ be the smallest cycle of σ such that for some $1 \le i \le p$, either:

(a) $|\mathrm{Circ}(\Omega_i)| \geq 2$, or
(b) $\mathrm{color}(\sigma(\Omega_i)) = \mathrm{color}(\Omega_i)$ and $\Omega_i < \sigma(\Omega_i)$.

If ρ does not exist then $(\tau_1, \tau_2, \dots, \tau_l; \sigma) \in \mathrm{Fix}\Phi_1$. Otherwise, suppose that $\min(\{\Omega_1, \Omega_2, \dots, \Omega_p\}) = \Omega_1$ and let i_0 be the smallest index such that (a) or (b) occur, $1 \leq i_0 \leq p$.

Case 1. If Ω_{i_0} satisfies the conditions:

(1.1) $|\mathrm{Circ}(\Omega_{i_0})| \geq 2$, and
(1.2) if $\mathrm{color}(\sigma(\Omega_{i_0})) = \mathrm{color}(\Omega_{i_0})$, then either $\Omega_{i_0} \geq \sigma(\Omega_{i_0})$ or $\min(\widetilde{\mathrm{Circ}}(\Omega_{i_0}) - \{\min(\Omega_{i_0})\}) < \min\,(\sigma(\Omega_{i_0}))$,

then $\Phi_1(\tau_1, \tau_2, \dots, \tau_l; \sigma)$ is obtained by replacing Ω_{i_0} by its division chain in ρ. More precisely, if $\mathrm{Div}(\Omega_{i_0}) = (\hat{\Omega}_1, \dots, \hat{\Omega}_r)$, then we replace ρ by $\rho' = (\Omega_1, \dots, \Omega_{i_0-1}, \hat{\Omega}_1, \hat{\Omega}_2, \dots, \hat{\Omega}_r, \Omega_{i_0+1}, \dots, \Omega_p)$.

Case 2. If Ω_{i_0} satisfies the conditions:

(2.1) $|\mathrm{Circ}(\Omega_{i_0})| \geq 1$, and
(2.2) $\mathrm{color}(\sigma(\Omega_{i_0})) = \mathrm{color}(\Omega_{i_0}), \Omega_{i_0} < \sigma(\Omega_{i_0}$, and $\min(\widetilde{\mathrm{Circ}}(\Omega_{i_0}) - \{\min(\Omega_{i_0})\}) > \min(\sigma(\Omega_{i_0}))$,

then to obtain $\Phi_1(\tau_1, \tau_2, \dots, \tau_l; \sigma)$, replace by $\mathrm{Div}^{-1}(\Omega_{i_0}, \Omega_{i_0+1}, \dots, \Omega_{i_0+r-1})$ the longest string $\Omega_{i_0}, \Omega_{i_0+1}, \dots, \Omega_{i_0+r-1}$ in the cycle ρ such that $r \geq 2$, $\mathrm{color}(\Omega_{i_0}) = \mathrm{color}(\Omega_{i_0+1}) = \cdots = \mathrm{color}(\Omega_{i_0+r-1})$, and $\Omega_{i_0} < \Omega_{i_0+1} > \Omega_{i_0+2} > \cdots > \Omega_{i_0+r-1} > \Omega_{i_0}$.

Details that Φ_1 is a well-defined involution, mapping Case 1 to Case 2 and preserving the value i_0 are left to the reader. Note that if $\Phi_1(\tau_1, \tau_2, \dots, \tau_l; \sigma) = (\tau_1', \tau_2', \dots, \tau_l'; \sigma')$ was obtained from Case 1 by replacing Ω_{i_0} by $\mathrm{Div}(\Omega_{i_0}) = (\hat{\Omega}_1 < \hat{\Omega}_2 > \hat{\Omega}_3 > \cdots > \hat{\Omega}_r)$, $r \geq 2$, then

$$\begin{aligned}
\sum_{i=1}^{l} \mathrm{cycle}(\tau_i') &= \sum_{i=1}^{l} \mathrm{cycle}(\tau_i),\\
\sum_{i=1}^{l} \mathrm{octopi}(\tau_i') &= \sum_{i=1}^{l} \mathrm{octopi}(\tau_i) + r - 1,\\
\sum_{i=1}^{l} \mathrm{circ}(\tau_i') &= \sum_{i=1}^{l} \mathrm{circ}(\tau_i) + r - 2,\\
\sum_{i=1}^{l} \mathrm{uncirc}(\tau_i') &= \sum_{i=1}^{l} \mathrm{uncirc}(\tau_i) - (r - 2),\\
\mathrm{cycle}(\sigma') &= \mathrm{cycle}(\sigma),\\
\mathrm{nexc}(\sigma') &= \mathrm{nexc}(\sigma) + r - 2.
\end{aligned}$$

From (4.1), this means that $\omega(\tau_1, \tau_2, \dots, \tau_l; \sigma) = -\omega(\Phi_1(\tau_1, \tau_2, \dots, \tau_l; \sigma))$, and Φ_1 is sign-reversing. Moreover, we have

$$\begin{aligned}
\mathrm{Fix}\Phi_1 = \{&(\tau_1, \tau_2, \dots, \tau_l; \sigma) \in L(\mathbf{n}) \mid \forall \Omega \in \cup_{k=1}^{l} \mathrm{Oct}(\tau_k), |\mathrm{Circ}(\Omega)| = 1,\\
&\text{and if } \mathrm{color}(\sigma(\Omega)) = \mathrm{color}(\Omega), \text{ then } \Omega \geq \sigma(\Omega)\}.
\end{aligned}$$

4.2. Involution Φ_2 This involution acts locally on octopi Ω to eliminate those such that $\mathrm{Body}(\Omega) \neq \{\min(\Omega)\}$. Note that since $\mathrm{Circ}(\Omega) = \{\mathrm{attmin}(\Omega)\}$ for all octopi Ω coming from the fixed points of Φ_1, we need not specify the circling of Ω.

Let $(\tau_1, \tau_2, \dots, \tau_l; \sigma) \in \mathrm{Fix}\Phi_1$ and let Ω_0 be the smallest octopus in $\cup_{k=1}^{l} \mathrm{Oct}(\tau_k)$ such that $\mathrm{Body}(\Omega_0) \neq \{\min(\Omega_0)\}$. If Ω_0 does not exist then $(\tau_1, \tau_2, \dots, \tau_l; \sigma) \in \mathrm{Fix}\Phi_2$. Otherwise, suppose that $\mathrm{color}(\Omega_0) = i$.

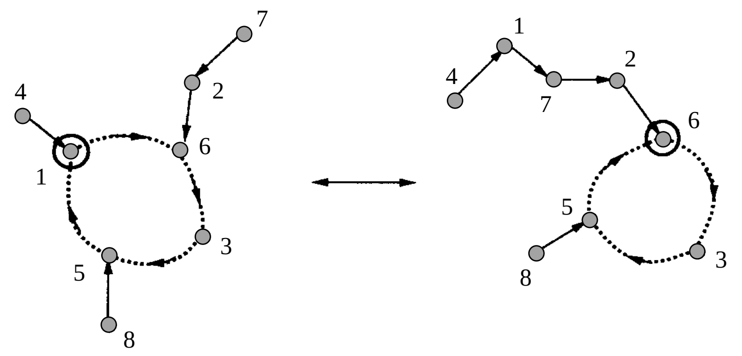

FIGURE 3. Local action of involution Φ_2 on octopi.

Case 1. If Ω_0 satisfies the conditions:

(1.1) $|\text{Body}(\Omega_0)| \geq 2$, and
(1.2) $\min(\Omega_0) \in \text{Body}(\Omega_0)$,

then we construct a new octopus Ω_0' from Ω_0 by appending $b_0 = \min(\Omega_0)$ and the leg attached to b_0 to the extremity of the leg of its successor $f_i(b_0)$ on the body of Ω_0. More precisely, if $\widetilde{\text{Leg}}(b_0) = (a_1, a_2, \ldots, a_r)$ and $\widetilde{\text{Leg}}(f_i(b_0)) = (e_1, e_2, \ldots, e_s)$ in Ω_0, then the new leg attached to $f_i(b_0)$ in Ω_0' would be $\widetilde{\text{Leg}}(f_i(b_0)) = (e_1, e_2, \ldots, e_s, b_0, a_1, a_2, \ldots, a_r)$. This process is illustrated in Figure 3. $\Phi_2(\tau_1, \tau_2, \ldots, \tau_l; \sigma)$ is obtained from $(\tau_1, \tau_2, \ldots, \tau_l; \sigma)$ by simply replacing Ω_0 by Ω_0'.

Case 2. Conversely, if Ω_0 satisfies the conditions:

(2.1) $|\text{Body}(\Omega_0)| \geq 1$, and
(2.2) $\min(\Omega_0) \notin \text{Body}(\Omega_0)$,

$\Phi_2(\tau_1, \tau_2, \ldots, \tau_l; \sigma)$ is defined in the obvious way so that Φ_2 is an involution.

Note that if $\Phi_2(\tau_1, \tau_2, \ldots, \tau_l; \sigma) = (\tau_1', \tau_2', \ldots, \tau_l', \sigma')$ was obtained using Case 1, then

$$\begin{aligned}
\sum_{i=1}^{l} \text{cycle}(\tau_i') &= \sum_{i=1}^{l} \text{cycle}(\tau_i),\\
\sum_{i=1}^{l} \text{octopi}(\tau_i') &= \sum_{i=1}^{l} \text{octopi}(\tau_i),\\
\sum_{i=1}^{l} \text{circ}(\tau_i') &= \sum_{i=1}^{l} \text{circ}(\tau_i),\\
\sum_{i=1}^{l} \text{uncirc}(\tau_i') &= \sum_{i=1}^{l} \text{uncirc}(\tau_i) - 1,\\
\text{cycle}(\sigma') &= \text{cycle}(\sigma),\\
\text{nexc}(\sigma') &= \text{nexc}(\sigma).
\end{aligned}$$

From (4.1), this means that $\omega(\tau_1, \tau_2, \ldots, \tau_l; \sigma) = -\omega(\Phi_2(\tau_1, \tau_2, \ldots, \tau_l; \sigma))$, and Φ_2 is sign-reversing. Moreover, we have

$$\text{Fix}\Phi_2 = \{(\tau_1, \tau_2, \ldots, \tau_1; \sigma) \in \text{Fix}\Phi_1 | \forall \Omega \in \cup_{k=1}^{l} \text{Oct}(\tau_k), \text{Body}(\Omega) = \{\min(\Omega)\}\}.$$

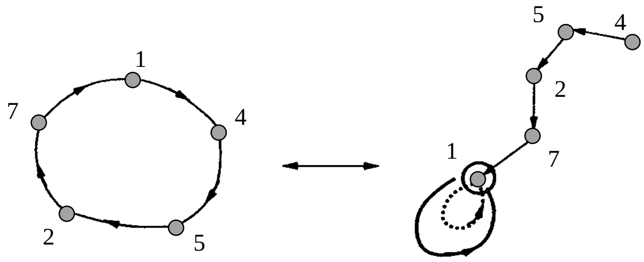

FIGURE 4. Cases 1 and 2 of involution Φ_3.

4.3. Involution Φ_3. The purpose of this involution is to make the moment permutation σ a colored derangement on octopi, eliminate all octopi with non-empty legs, and eliminate all the polynomial cycles.

Let $(\tau_1, \tau_2, \ldots, \tau_l; \sigma) \in \mathrm{Fix}\Phi_2$, let ρ_0 be the smallest polynomial cycle in the set $\cup_{k=1}^{l}\mathrm{Cyc}(\tau_k)$, and let Ω_0 be the smallest octopus in $\cup_{k=1}^{l}\mathrm{Oct}(\tau_k)$, $\mathrm{Body}(\Omega_0) = \{b_0\}$ such that either:

(a) $\mathrm{color}(\Omega_0) = \mathrm{color}(\sigma^{-1}(\Omega_0))$, or
(b) $\mathrm{Leg}(b_0) \neq \emptyset$.

If neither ρ_0 nor Ω_0 exist then $(\tau_1, \tau_2, \ldots, \tau_l; \sigma) \in \mathrm{Fix}\Phi_3$. Note that in case (a), we have necessarily $\sigma^{-1}(\Omega_0) \geq \Omega_0$, from $\mathrm{Fix}\Phi_1$. The involution Φ_3 has four cases. Cases 1 and 2 correspond to $\rho_0 < \Omega_0$ and $\sigma^{-1}(\Omega_0) = \Omega_0$ respectively, while Cases 3 and 4 cover $\sigma^{-1}(\Omega_0) \neq \Omega_0$.

Case 1. If $\rho_0 = (a_1, a_2, \ldots, a_r)$ (with $a_1 = \min\{a_1, \ldots, a_r\}$) exists and satisfies the condition:

(1.1) If Ω_0 exists, then $\rho_0 < \Omega_0$ (i.e. $a_1 < \min(\Omega_0)$),

then $\Phi_3(\tau_1, \tau_2, \ldots, \tau_l; \sigma) = (\tau_1', \tau_2', \ldots, \tau_l'; \sigma')$ is obtained from $(\tau_1, \tau_2, \ldots, \tau_l; \sigma)$ by replacing ρ_0 by the octopus Ω such that $\mathrm{Body}(\Omega) = \{a_1\}$ (a_1 is circled) and $\widetilde{\mathrm{Leg}}(a_1) = (a_2, a_3, \ldots, a_r)$, and setting $\sigma'(\Omega) = \Omega$. This process is illustrated in Figure 4.

Case 2. If Ω_0 exists, with $\mathrm{Body}(\Omega_0) = \{b_0\}$ and $\widetilde{\mathrm{Leg}}(b_0) = (a_1, a_2, \ldots, a_{r-1})$, and satisfies the conditions:

(2.1) If ρ_0 exists, then $\Omega_0 < \rho_0$, and
(2.2) $\sigma(\Omega_0) = \Omega_0$,

then $\Phi_3(\tau_1, \tau_2, \ldots, \tau_l; \sigma)$ is obtained from $(\tau_1, \tau_2, \ldots, \tau_l; \sigma)$ by replacing Ω_0 by the polynomial cycle $\rho = (b_0, a_1, a_2, \ldots, a_{r-1})$.

Case 3. If Ω_0 exists, with $\mathrm{Body}(\Omega_0) = \{b_0\}$, and satisfies the conditions:

(3.1) If ρ_0 exists, then $\Omega_0 < \rho_0$;
(3.2) $|\mathrm{Leg}(b_0)| \geq 1$, and
(3.3) If $\mathrm{color}(\Omega) = \mathrm{color}(\sigma^{-1}(\Omega_0))$, then $\min(\mathrm{Leg}(b_0)) < \min(\sigma^{-1}(\Omega_0))$,

then let $a_0 = \min(\mathrm{Leg}(b_0))$ and $\widetilde{\mathrm{Leg}}(b_0) = (e_1, e_2, \ldots, e_r, a_0, a_1, \ldots, a_s)$. To obtain $\Phi_3(\tau_1, \tau_2, \ldots, \tau_l; \sigma) = (\tau_1', \tau_2', \ldots, \tau_l'; \sigma')$, we replace Ω_0 by the two octopi Ω_1 and Ω_2, such that $\mathrm{Body}(\Omega_1) = \{a_0\}$, $\mathrm{Body}(\Omega_2) = \{b_0\}$ (a_0 and b_0 are both circled), $\widetilde{\mathrm{Leg}}(a_0) = (a_1, a_2, \ldots, a_s)$,

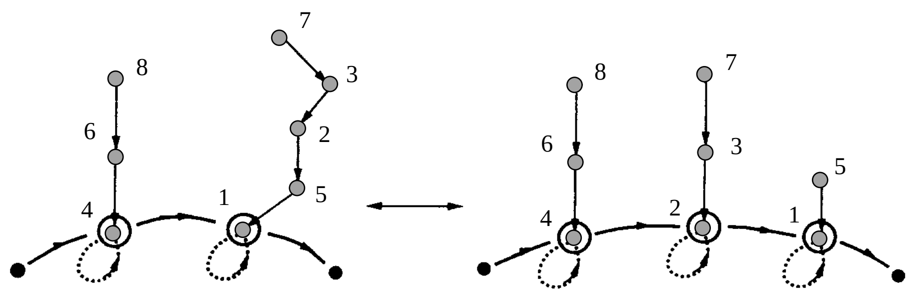

FIGURE 5. Cases 3 and 4 involution Φ_3.

and $\widetilde{\mathrm{Leg}}(b_0) = (e_1, e_2, \ldots, e_r)$. Moreover, if $\Omega \in \cup_{k=1}^{l} \mathrm{Oct}(\tau_k') = \cup_{k=1}^{l} \mathrm{Oct}(\tau_k) \cup \{\Omega_1, \Omega_2\} - \{\Omega_0\}$, the new moment permutation σ' is given by

$$\sigma'(\Omega) = \begin{cases} \Omega_1, & \text{if } \Omega = \sigma^{-1}(\Omega_0), \\ \Omega_2, & \text{if } \Omega = \Omega_1, \\ \sigma(\Omega_0), & \text{if } \Omega = \Omega_2, \\ \sigma(\Omega), & \text{otherwise.} \end{cases}$$

This process is illustrated in Figure 5.

Case 4. If Ω_0 exists, with $\mathrm{Body}(\Omega_0) = \{b_0\}$, and satisfies the conditions:

(4.1) If ρ_0 exists, then $\Omega_0 < \rho_0$;
(4.2) $\mathrm{color}(\Omega_0) = \mathrm{color}(\sigma^{-1}(\Omega_0))$, and
(4.3) If $|\mathrm{Leg}(b_0)| \geq 1$, then $\min(\sigma^{-1}(\Omega_0)) < \min(\mathrm{Leg}(b_0))$,

then we obtain $\Phi_3(\tau_1, \tau_2, \ldots, \tau_l; \sigma)$ from $(\tau_1, \tau_2, \ldots, \tau_l; \sigma)$ by appending the octopus $\sigma^{-1}(\Omega_0)$ to the extremity of the leg of Ω_0 and contracting the moment permutation σ as to reverse the process described in Case 3.

Details that Φ_3 is a well-defined involution on $\mathrm{Fix}\Phi_2$, mapping Case 1 to Case 2, and Case 3 to Case 4, are left to the reader. Note that if $\Phi_3(\tau_1, \tau_2, \ldots, \tau_l; \sigma) = (\tau_1', \tau_2', \ldots, \tau_l'; \sigma')$ was obtained using Case 1, then

$$\begin{aligned} \sum_{i=1}^{l} \mathrm{cycle}(\tau_i') &= \sum_{i=1}^{l} \mathrm{cycle}(\tau_i) - 1, \\ \sum_{i=1}^{l} \mathrm{octopi}(\tau_i') &= \sum_{i=1}^{l} \mathrm{octopi}(\tau_i) + 1, \\ \sum_{i=1}^{l} \mathrm{circ}(\tau_i') &= \sum_{i=1}^{l} \mathrm{circ}(\tau_i) + 1, \\ \sum_{i=1}^{l} \mathrm{uncirc}(\tau_i') &= \sum_{i=1}^{l} \mathrm{uncirc}(\tau_i), \\ \mathrm{cycle}(\sigma') &= \mathrm{cycle}(\sigma) + 1, \\ \mathrm{nexc}(\sigma') &= \mathrm{nexc}(\sigma) + 1. \end{aligned}$$

Note also that if $\Phi_3(\tau_1, \tau_2, \dots, \tau_l; \sigma) = (\tau_1', \tau_2', \dots, \tau_l'; \sigma')$ was obtained using Case 3 instead, then

$$\begin{aligned}\sum_{i=1}^{l} \operatorname{cycle}(\tau_i') &= \sum_{i=1}^{l} \operatorname{cycle}(\tau_i),\\ \sum_{i=1}^{l} \operatorname{octopi}(\tau_i') &= \sum_{i=1}^{l} \operatorname{octopi}(\tau_i) + 1,\\ \sum_{i=1}^{l} \operatorname{circ}(\tau_i') &= \sum_{i=1}^{l} \operatorname{circ}(\tau_i) + 1,\\ \sum_{i=1}^{l} \operatorname{uncirc}(\tau_i') &= \sum_{i=1}^{l} \operatorname{uncirc}(\tau_i),\\ \operatorname{cycle}(\sigma') &= \operatorname{cycle}(\sigma),\\ \operatorname{nexc}(\sigma') &= \operatorname{nexc}(\sigma) + 1.\end{aligned}$$

In either case, from (4.1), we have $\omega(\tau_1, \tau_2, \dots, \tau_l; \sigma) = -\omega(\Phi_3(\tau_1, \tau_2, \dots, \tau_l; \sigma))$ and Φ_3 is sign-reversing. Moreover, we have

$$\begin{aligned}\operatorname{Fix}\Phi_3 = \{(\tau_1, \tau_2, \dots, \tau_l; \sigma) \in \operatorname{Fix}\Phi_2 | \operatorname{Cyc}(\tau_i) = \emptyset, \text{ and } \forall \Omega \in \cup_{k=1}^{l} \operatorname{Oct}(\tau_k),\\ \forall b \in \operatorname{body}(\Omega), \operatorname{Leg}(b) = \emptyset, \text{ and } \operatorname{color}(\sigma(\Omega)) \neq \operatorname{color}(\Omega)\}.\end{aligned}$$

Let $(\tau_1, \tau_2, \dots, \tau_l; \sigma) \in \operatorname{Fix}\Phi_3$. In terms of Meixner endofunctions $\tau_i = ((A_i, B_i), f_i, C_i)$, requiring no polynomial cycles and empty legs corresponds to the ordered partition $A_i = \emptyset$ and $B_i = \{i\} \times [n_i]$. The fact that the body of each octopus is reduced to one point (from Φ_2) then corresponds to f_i being the identity function on $\{i\} \times [n_i]$. Finally, the minimum and only point on each octopus being circled translates to $C_i = \{i\} \times [n_i]$. So $\operatorname{Fix}\Phi_3 = \operatorname{Fix}\Phi$ as was given in the proof of Theorem 4.

REMARK. Involution Φ_2 and Cases 1 and 2 of involution Φ_3 are reminiscent of Foata and Zeilberger's [10] combinatorial proof of the combinatorial interpretation for the linearization coefficients of Laguerre polynomials. They use a weight-preserving sign-reversing involution on a special set of marked colored permutations, which is in natural correspondence with assemblies of colored octopi and cycles. Expressed in these terms, their construction corresponds to creating a new octopus by cutting part of the leg of another octopus, or transforming an octopus into a cycle.

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