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Matrix orthogonal polynomials satisfying second-order differential equations: Coping without help from group representation theory☆

Antonio J. Durán[a,*], F. Alberto Grünbaum[b]

[a]Departamento de Análisis Matemático, Universidad de Sevilla, Apdo (P.O. Box) 1160, 41080 Sevilla, Spain
[b]Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720, USA

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Abstract

The method developed in Duran and Grünbaum [Orthogonal matrix polynomials satisfying second order differential equations, Internat. Math. Res. Notices 10 (2004) 461–484] led us to consider polynomials that are orthogonal with respect to weight matrices $W(t)$ of the form $e^{-t^2}T(t)T^*(t)$, $t^\alpha e^{-t}T(t)T^*(t)$ and $t^\alpha(1-t)^\beta T(t)T^*(t)$, with T satisfying $T' = (2Bt + A)T$, $T(0) = I$, $T' = (A + B/t)T$, $T(1) = I$ and $T'(t) = (A/t + B/(1-t))T$, $T(1/2) = I$, respectively. Here A and B are in general two non-commuting matrices. To proceed further and find situations where these polynomials satisfied second-order differential equations, we needed to impose commutativity assumptions on the pair of matrices A, B. In fact, we only dealt with the case when one of the matrices vanishes.

The only exception to this arose as a gift from group representation theory: one automatically gets a situation where A and B do not commute, see Grünbaum et al. [Matrix valued orthogonal polynomials of the Jacobi type: the role of group representation theory, Ann. Inst. Fourier Grenoble 55 (6) (2005) 2051–2068]. This corresponds to the last of the three cases mentioned above.

The purpose of this paper is to consider the other two situations and since now we do not get any assistance from representation theory we make a direct attack on certain differential equations in Duran and Grünbaum [Orthogonal matrix polynomials satisfying second order differential equations, Internat. Math. Res. Notices

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*Corresponding author.
E-mail addresses: duran@us.es (A.J. Durán), grunbaum@math.berkeley.edu (F.A. Grünbaum).

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10 (2004) 461–484]. By solving these equations we get the appropriate weight matrices $W(t)$, where the matrices A, B give rise to a solvable Lie algebra.

Keywords: Matrix orthogonality; Differential operator

1. Introduction

In the last 3 years a large class of families of $N \times N$ weight matrices W connected with a second-order differential operator of the form

$$D^2 A_2(t) + D^1 A_1(t) + D^0 A_0(t), \tag{1.1}$$

which is symmetric with respect to the inner product $\int P(t)\, dW(t) Q^*(t)$ defined by W, has appeared (see [5,6,13–17]); the differential coefficients A_2, A_1 and A_0 are matrix polynomials (which do not depend on n) of degrees less than or equal to 2, 1 and 0, respectively. If we write $(P_n)_n$ for a family of orthonormal matrix polynomials with respect to W, the symmetry of the second-order differential operator is equivalent to the second-order differential equation

$$P_n''(t) A_2(t) + P_n'(t) A_1(t) + P_n(t) A_0 = \Gamma_n P_n(t) \tag{1.2}$$

for the orthonormal polynomials where Γ_n are Hermitian matrices. These families of orthonormal polynomials are very likely going to play in the case of matrix orthogonality the crucial role that the classical families of Hermite, Laguerre and Jacobi play in the scalar one.

The basic idea for the study of the symmetry of a second-order differential operator ℓ_2 as in (1.1) with respect to the inner product defined by the weight matrix W is to convert this condition of symmetry into a set of differential equations relating W and the differential coefficient of ℓ_2. This is given in the following theorem:

Theorem 1.1 (*Theorem 3.1 of Duran and Grünbaum [6], see also Grünbaum et al. [15]*). *The following conditions are equivalent*

1. *The operator ℓ_2 is symmetric with respect to W.*
2. *The expressions*

$$A_2(t) W(t) \quad \text{and} \quad (A_2(t) W(t))' - A_1(t) W(t) \tag{1.3}$$

vanish at each of the endpoints of the support of $W(t)$, and the weight matrix W satisfies

$$A_2 W = W A_2^* \tag{1.4}$$

as well as

$$2(A_2 W)' = W A_1^* + A_1 W \tag{1.5}$$

and

$$(A_2 W)'' - (A_1 W)' + A_0 W = W A_0^*. \tag{1.6}$$

Assuming that $A_2(t)$ is a scalar, it has been proved in [6] that the differential equation (1.5) for W is equivalent to the fact that W can be factorized in the form $W(t) = \rho(t)T(t)T^*(t)$, where ρ is a scalar function and T is a matrix function satisfying a first-order differential equation (for details see the next section; for the case when A_2 is properly a matrix polynomial, see [5]). When ρ is one of the classical scalar weights of Hermite, Laguerre or Jacobi, this first-order differential equation for T takes the form

$$\begin{cases} T'(t) = (2Bt + A)T(t) & \text{if } \rho(t) = e^{-t^2}, \\ T'(t) = \left(A + \dfrac{B}{t}\right) T(t) & \text{if } \rho(t) = t^\alpha e^{-t}, \\ T'(t) = \left(\dfrac{A}{t} + \dfrac{B}{1-t}\right) T(t) & \text{if } \rho(t) = t^\alpha (1-t)^\beta. \end{cases}$$

Using this approach, all the situations corresponding to the case when in the first-order differential equation for T either A or B vanish have been classified in [6,7].

A completely different approach to the problem of finding examples of orthogonal matrix polynomials satisfying second-order differential equations has been used in [15–17]; these examples grew out of a study of matrix valued spherical functions initiated in [20]. This development is an extension to the matrix case of fundamental work by E. Cartan and H. Weyl that allowed them to put under one roof a number of isolated results for several families of special functions, including Jacobi polynomials. Using this approach, Grünbaum, Pacharoni and Tirao have found examples where, in the situation described above, the matrices A and B do not commute, see for instance [13,15,17]. These examples correspond with the case $\rho(t) = t^\alpha(1-t)^\beta$.

The purpose of this note is to provide examples of this sort (where A and B do not commute) for $\rho(t) = e^{-t^2}$ and $\rho(t) = t^\alpha e^{-t}$. This will be carried out in Section 3, where the examples are found by using the same approach as in [6] (actually, we find in Section 3 all the examples corresponding to $A_2 = Id$ and with A and B simultaneously triangularizable). All the examples have size 2×2, and it appears that substantially new ideas will be needed to find examples of an arbitrary size.

We go on to show that these families of orthogonal matrix polynomials can be given by Rodrigues' formulas of the same type we have already found for the families introduced in [6,7] (see [8,10]).

It is worth noticing that the matrix Rodrigues' formulas differ from the corresponding scalar ones: compare Rodrigues' formula for the Hermite polynomials $(-1)^n (e^{-t^2})^{(n)} e^{t^2}$ with that of our polynomials given in Theorem 1.2 below where the perturbation $X(t)$, which appears there, cannot be disposed of. There is again very good evidence to suspect that scalar-type Rodrigues' formulas $(\phi^n w)^{(n)} w^{-1}$, ϕ a polynomial of degree not bigger that 2, are not going to play in the matrix orthogonality case the important role that they played in the scalar orthogonality case (see [9]). Instead, Rodrigues' formulas like (1.7) are very likely going to be more typical.

Using Rodrigues' formula, we find the three-term recurrence formula satisfied by our families.

To make this introduction more useful to the reader, we display here one of the examples in full detail. This example is of the form $e^{-t^2} T(t)T^*(t)$, where the matrix function $T(t) = \begin{pmatrix} e^t & te^{-t} \\ 0 & e^{-t} \end{pmatrix}$

satisfies

$$T'(t) = \left(2\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} t + \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\right) T(t), \qquad T(0) = I.$$

Theorem 1.2. *A sequence of orthogonal matrix polynomials with respect to the weight matrix*

$$W(t) = e^{-t^2} \begin{pmatrix} e^{2t} + t^2 e^{-2t} & t\, e^{-2t} \\ t\, e^{-2t} & e^{-2t} \end{pmatrix}$$

can be defined by using Rodrigues' formula, valid for $n = 1, 2, 3, \ldots$

$$\mathcal{P}_n(t) = \left[e^{-t^2-2t} \left(\begin{pmatrix} e^{4t} + t^2 & t \\ t & 1 \end{pmatrix} - X(t) \right) \right]^{(n)} \begin{pmatrix} e^{-4t} & -t\, e^{-4t} \\ -t\, e^{-4t} & 1 + e^{-4t} t^2 \end{pmatrix} e^{t^2+2t}, \quad (1.7)$$

where

$$X(t) = \begin{pmatrix} -t - \dfrac{n}{2} & -1 \\ n\phi(t)\, e^{t^2+2t} & 0 \end{pmatrix},$$

where $\phi(t) = \int_{-\infty}^{t} e^{-x^2-2x}\, dx - \int_{-\infty}^{t} e^{-x^2+2x}\, dx$.

For $n = 0$ we take $\mathcal{P}_0(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

The polynomials $(\mathcal{P}_n)_n$ satisfy the second-order differential equation

$$(\mathcal{P}_n)''(t) + (\mathcal{P}_n)'(t) \begin{pmatrix} -2t+2 & -4t+2 \\ 0 & -2t-2 \end{pmatrix} + \mathcal{P}_n(t) \begin{pmatrix} -2 & -2 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -2n-2 & \\ 0 & -2n \end{pmatrix} \mathcal{P}_n(t).$$

The polynomials constructed in this fashion satisfy the three-term recursion

$$t\mathcal{P}_n(t) = D_{n+1}\mathcal{P}_{n+1}(t) + E_n\mathcal{P}_n(t) + F_n\mathcal{P}_{n-1}(t)$$

with the matrices D_{n+1}, E_n, $n \geqslant 0$, and F_n, $n \geqslant 1$, given explicitly by the expressions

$$D_{n+1} = \begin{pmatrix} -\dfrac{1}{2} & \dfrac{2}{n+3} \\ 0 & \dfrac{-(n+2)}{2(n+3)} \end{pmatrix}, \quad E_n = \begin{pmatrix} \dfrac{-(n-1)}{n+3} & \dfrac{1}{n+2} \\ \dfrac{1}{n+3} & \dfrac{n-2}{n+2} \end{pmatrix},$$

$$F_n = \begin{pmatrix} \dfrac{-n(n+3)}{n+2} & 0 \\ \dfrac{4n}{n+2} & -n \end{pmatrix}.$$

2. Preliminary results

We start by recalling the usual definition for a weight matrix:

Definition 2.1. We say that an $N \times N$ matrix of measures supported in the real line is a weight matrix if

1. $W(A)$ is positive semidefinite for any Borel set $A \subset \mathbb{R}$;
2. W has finite moments of every order, and
3. $\int P(t)\, dW(t) P^*(t)$ is nonsingular if the leading coefficient of the matrix polynomial P is nonsingular.

Condition (3) is necessary and sufficient to guarantee the existence of a sequence $(P_n)_n$ of matrix polynomials orthogonal with respect to W, P_n of degree n and with nonsingular leading coefficient. In this paper, we always consider weight matrices W that have a smooth absolutely continuous derivative W' with respect to the Lebesgue measure; if we assume that this matrix derivative W' is positive definite at infinitely many real numbers, then condition (3) holds automatically. For other basic definitions and results on matrix orthogonality, see for instance [1,3,4,11,12,18,19].

Using (1.5), the second of the boundary conditions (1.3) can be simplified to the requirement that $WA_1^* - A_1 W$ should vanish at each of the endpoints of the support of $W(t)$.

In Section 4 of [6], assuming the differential coefficient A_2 to be scalar (that is, of the form $A_2(t) = a_2(t)I$, with a_2 a real polynomial of degree not bigger than 2), we gave a *description* of all solutions for Eqs. (1.4)–(1.6). Eq. (1.5) implies that any solution W must have the form

$$W(t) = \frac{\rho(t)}{\rho(a)} T(t) W(a) T^*(t),$$

where a is certain real number, ρ is a scalar function and T is a matrix function satisfying the differential equation

$$T'(t) = F(t)T(t), \quad T(a) = I,$$

where the matrix function F is defined as

$$F(t) = \frac{1}{2a_2(t)} \left(A_1(t) - \frac{(\rho(t)a_2(t))'}{\rho(t)} I \right). \tag{2.1}$$

In addition, Eq. (1.6) implies that the following matrix function has to be Hermitian:

$$\chi(t)W(a) = T^{-1}(t) \left(a_2(t)F'(t) + a_2(t)F^2(t) + \frac{(\rho(t)a_2(t))'}{\rho(t)} F(t) - A_0 \right) T(t)W(a). \tag{2.2}$$

The converse is also true, i.e. if the expression in (2.2) is Hermitian then (1.6) holds.

When $\rho(t) = e^{-t^2}$, the function F is a polynomial of degree 1: $Bt + C$; and for $\rho(t) = t^\alpha e^{-t}$ the function F is of the form: $A + B/t$.

In the case $\rho(t) = t^\alpha (1-t)^\beta$ the function F is given by $A/t + B/(1-t)$. We will not be concerned with this case since, as mentioned before, this case is one where the presence of a group representation scenario makes these considerations unnecessary.

We now list the explicit expression of the matrix function χ in the two cases that concern us:

Lemma 2.2. *Assuming that the weight matrix W satisfies the boundary conditions* (1.3), *the symmetry of the second-order differential operator* (1.1) (A_2 *scalar*) *with respect to the inner product defined by each of the following weight matrices is equivalent to the requirement that the corresponding matrix function χ should be Hermitian for all t*:

1. *If* $W(t) = e^{-t^2} T(t)T^*(t)$ *then*

$$\chi(t) = T^{-1}(t)(2B + A^2 - A_0 + (2AB + 2BA - 2A)t + (4B^2 - 4B)t^2)T(t) \tag{2.3}$$

and $T'(t) = (2Bt + A)T(t)$, $T(0) = I$.

2. *If* $W(t) = t^{\alpha} e^{-t} T(t)T^*(t)$, *then*

$$\chi(t) = T^{-1}(t)\left((\alpha B + B^2)\frac{1}{t} + AB + BA + (\alpha + 1)A - A_0 - B + (A^2 - A)t\right)T(t)$$

and $T'(t) = (A + B/t)T(t)$, $T(1) = I$.

3. The examples

3.1. $\rho(t) = e^{-t^2}$

We first consider the weight matrix $W = e^{-t^2} TT^*$, where T satisfies $T' = (2Bt + A)T$, $T(0) = I$, where A and B are 2×2 matrices. We are interested here in the case when A and B are simultaneously triangularizable. We also assume that A and B have real entries.

Writing ξ and η for an eigenvalue of B and A, respectively, a linear change of variable $x = at+b$, where $a = 1/\sqrt{1 - 2\xi}$ and $b = \eta/(1 - 2\xi)$, shows that we can assume A and B to be singular matrices ($\xi < 1/2$, because otherwise the weight matrix W should not be integrable in $\mathbb{R}$).

Hence, we assume that A and B have the form

$$A = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ 0 & 0 \end{pmatrix}. \tag{3.1}$$

We study now the case when $a_1 \neq 0$, $b_1 = 0$ and $b_2 \neq 0$. The matrix A_0, which appears in the differential equation for the polynomials $(P_n)_n$ (see 1.2), can always be taken with a diagonal entry equal to 0:

$$A_0 = \begin{pmatrix} u & v \\ x & 0 \end{pmatrix}.$$

Since T has to satisfy $T'(t) = (2Bt + A)T(t)$, $T(0) = I$, we find that

$$T(t) = \begin{pmatrix} e^{a_1 t} & q(t) - q(0)\, e^{a_1 t} \\ 0 & 1 \end{pmatrix}, \tag{3.2}$$

where q is the following polynomial (of degree 1)

$$q(t) = -\frac{2b_2}{a_1}t - \frac{1}{a_1}\left(a_2 + \frac{2b_2}{a_1}\right).$$

Writing

$$2B + A^2 - A_0 + (2AB + 2BA - 2A)t + (4B^2 - 4B)t^2 = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}, \tag{3.3}$$

a straightforward computation shows that the Hermitian condition for the function χ defined by (2.3) is

$$(T_{11}^2 + T_{12}^2)p_3 - T_{22}^2 p_2 + T_{12}T_{22}(p_4 - p_1) = 0, \tag{3.4}$$

where T_{ij} denotes the entry ij of the matrix function T.

Taking into account the formulas for T_{ij} (see (3.2)), and since $e^{2a_1 t}$, 1 and $e^{a_1 t}$ are linearly independent in the linear space of polynomials ($a_1 \neq 0$), we have from (3.4) that

$$\begin{aligned} &p_3(t) = 0, \\ &q(0)(p_4(t) - p_1(t)) = 0, \\ &- p_2(t) + q(t)(p_4(t) - p_1(t)) = 0. \end{aligned} \tag{3.5}$$

But, we find from (3.3) that

$$\begin{aligned} &p_1(t) = -2a_1 t + a_1^2 - u, \\ &p_2(t) = -4b_2 t^2 + 2(b_2 a_1 - a_2)t + a_2 a_1 + 2b_2 - v, \\ &p_3(t) = -x, \\ &p_4(t) = 0. \end{aligned}$$

Solving now Eqs. (3.5), we find that the weight matrix $e^{-t^2}T(t)T^*(t)$ has a symmetric second-order differential operator of the form $D^2 + D^1 A_1(t) + D^0 A_0(t)$ if and only if A, B and A_0 have the form

$$A = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -a_2 a_1/2 \\ 0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Using this approach, we can actually find all the examples of 2×2 weight matrices of the form $e^{-t^2}T(t)T^*(t)$ with $T'(t) = (2Bt + A)T(t)$, A and B simultaneously triangularizable having a symmetric second-order differential operator of the form

$$D^2 + D^1 A_1 + D^0 A_0.$$

Indeed, assume now that the matrices A and B in (3.1) satisfy $b_1 \neq 0$. Taking into account that $T' = (2Bt + A)T$, we find for the entries of T the expressions

$$\begin{aligned} &T_{11}(t) = e^{b_1 t^2 + a_1 t}, \quad T_{22}(t) = 1, \quad T_{21}(t) = 0, \\ &T_{12}(t) = c - c\, e^{b_1 t^2 + a_1 t} + \psi(t) c\, e^{b_1 t^2 + a_1 t}, \end{aligned}$$

where $\psi(t) = d \int_0^t c\, e^{-b_1 x^2 - a_1 x}\, dx$, and c, d are complex numbers. Using now that $T_{11}^2 + T_{12}^2$, T_{22}^2, $T_{12}T_{22}$ are linearly independent functions in the linear space of polynomials we deduce that in order that $W(t) = e^{-t^2}T(t)T^*(t)$ should satisfy the differential equation (1.6) it is necessary that $b_1 = 1$ and $a_1 = 0$. But the condition $b_1 = 1$ implies that $W(t)$ is not integrable in $\mathbb{R}$. That means that the case $b_1 \neq 0$ should be ignored.

When $b_1 = 0$, we have two cases, $a_1 \neq 0$ and $a_1 = 0$.

The first case is the one we have studied above.

Finally, the second case, $b_1 = 0$ and $a_1 = 0$, can be managed as follows. Integrating the differential equation for T gives

$$T(t) = \begin{pmatrix} 1 & b_2t^2 + a_2t \\ 0 & 1 \end{pmatrix}.$$

By solving the Hermitian condition for the function χ defined by (2.3), one then has that

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} -4 & 2b_2 \\ 0 & 0 \end{pmatrix}.$$

We summarize all these results in the following theorem:

Theorem 3.1. *Let W be a 2×2 weight matrix of the form $W(t) = e^{-t^2}T(t)T^*(t)$, where T satisfies the differential equation $T'(t) = (2Bt + A)T(t)$ (we remind that this is a necessary condition for W to have a symmetric second-order differential operator of the form $D^2 + D^1A_1 + D^0A_0$). Assume that A and B are real singular upper triangular matrices*:

$$A = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ 0 & 0 \end{pmatrix}.$$

The following conditions are then equivalent:

1. *The second-order differential operator* (1.1), *with $A_2(t) = I$, is symmetric for the inner product defined by W.*
2. $b_1 = 0$ *and then*
   (a) *if $b_2 = 0$, then $a_1 = 0$ and*
   $$A_0 = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix};$$
   (b) *if $b_2 \neq 0$ and $a_1 \neq 0$, then $b_2 = -a_2a_1/2$ and*
   $$A_0 = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix};$$
   (c) *if $b_2 \neq 0$ and $a_1 = 0$, then $a_2 = 0$ and*
   $$A_0 = 2B + \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix}.$$

The coefficient A_1 of the second-order differential operator is then $A_1(t) = 2(2B - 2I)t + 2A$.

3.2. $\rho(t) = t^{\alpha} e^{-t}$

In a similar way, one can find examples of weight matrices of the form $W = t^{\alpha} e^{-t} T T^*$, where T satisfies $T' = (A + B/t)T$, $T(1) = I$, and A and B are 2×2 non-commuting matrices.

We just show here a three-parameter family: define the real matrices A and B by

$$A = \begin{pmatrix} 0 & -b_2(b_3 - b_1 + 1)/(b_3 - b_1) \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix}. \tag{3.6}$$

This gives for $T' = (A + B/t)T$, $T(1) = I$, the expression

$$T(t) = \begin{pmatrix} t^{b_1} & -b_2(t^{b_3+1} - t^{b_3})/(b_3 - b_1) \\ 0 & t^{b_3} \end{pmatrix}.$$

Then, the weight matrix $W = t^{\alpha} e^{-t} T T^*$ goes along with the symmetric second-order differential operator

$$tD^2 + D^1[t(2B - I) + 2A + (1 + \alpha)I] + D^0 A_0,$$

where

$$A_0 = \begin{pmatrix} -1 & b_2(\alpha - 2b_3)/(b_3 - b_1) \\ 0 & 0 \end{pmatrix}.$$

4. A Rodrigues-type formula

In this section, we find a Rodrigues' formula for the orthogonal polynomials with respect to the weight matrices found earlier.

For the sake of clarity, we find a Rodrigues' formula for a single choice of a pair A, B; the general case can be found in a similar way. In exchange for dealing with only one example, and for the benefit of the reader, we indicate in great detail the steps that led us, finally, to a correct Rodrigues-type formula.

We start from a natural (but inappropriate) choice and after a few failures we arrive at the desired expression. We feel that this display of a sequence of missteps may be instructive and we display them in full view. This also reveals the trail blazing nature of our approach.

4.1. $\rho(t) = e^{-t^2}$

For purely sentimental reasons, we consider the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$

and define the weight matrix $W = e^{-t^2} T(t) T^*(t)$, where the matrix function $T(t) = \begin{pmatrix} e^t & t\,e^{-t} \\ 0 & e^{-t} \end{pmatrix}$ satisfies

$$T'(t) = (2Bt + A)T(t), \quad T(0) = I.$$

This gives for W the expression

$$W(t) = e^{-t^2} \begin{pmatrix} e^{2t} + t^2 e^{-2t} & t\, e^{-2t} \\ t\, e^{-2t} & e^{-2t} \end{pmatrix}.$$

(This case can be put in the framework of Theorem 3.1 by performing the change of variable $x = t - 1$, and then it goes with the matrices

$$\tilde{A} = \begin{pmatrix} 2 & e^2 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -e^2 \\ 0 & 0 \end{pmatrix}.\Bigg)$$

Taking into account Rodrigues' formulas found in [8,10], we look for a matrix function $X(t)$ (possibly depending on n) for which

$$\left[e^{-t^2-2t} \left(\begin{pmatrix} e^{4t} + t^2 & t \\ t & 1 \end{pmatrix} - X(t) \right) \right]^{(n)} \begin{pmatrix} e^{-4t} & -t\, e^{-4t} \\ -t\, e^{-4t} & 1 + e^{-4t} t^2 \end{pmatrix} e^{t^2+2t} \tag{4.1}$$

is a matrix polynomial of degree n with nonsingular leading coefficient. We also need that $X(t)\, e^{-t^2-2t}$ should vanish at $\pm\infty$, otherwise the integration by parts will not guarantee the orthogonality of the polynomials with respect to W. And, most importantly, we need the function X to be as simple as possible, otherwise the usefulness of Rodrigues' formula would be seriously compromised.

Let us notice that there is an important difference between the weight matrix

$$W(t) = e^{-t^2} \begin{pmatrix} e^{2t} + t^2 e^{-2t} & t\, e^{-2t} \\ t\, e^{-2t} & e^{-2t} \end{pmatrix}$$

and the weight matrices considered in [8]; there the weight matrices have the form $e^{-t^2} P(t)$, P being a certain matrix polynomial (of degree 2 and 4, respectively). Despite the fact that $\begin{pmatrix} e^{2t} + t^2 e^{-2t} & t\, e^{-2t} \\ t\, e^{-2t} & e^{-2t} \end{pmatrix}$ is not a matrix polynomial, we still look for a matrix polynomial X (of degree 1 to be more precise) such that the expression (4.1) is a matrix polynomial of degree n.

By forcing the expression (4.1) to have polynomial entries, namely, p_{11}, p_{12}, p_{21} and p_{22} (to simplify the notation we are removing the dependence of n), we find a differential equation for each of the entries of $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$:

$$\begin{aligned} &(e^{4t} + t^2)h_n - h_n x_1 + n h_{n-1}(4\, e^{4t} + 2t) - n h_{n-1} x_1' \\ &\quad + \frac{n}{2}(16\, e^{4t} + 2)h_{n-2} + \sum_{k=3}^{n} 4^k \frac{n}{k} e^{4t} h_{n-k} = (e^{4t} + t^2) p_{21} + t p_{22}, \end{aligned}$$

$$t h_n - h_n x_2 + n h_{n-1} - n h_{n-1} x_2' = t p_{21} + p_{22},$$

$$t h_n - h_n x_3 + n h_{n-1} - n h_n n - 1 x_3' = (e^{4t} + t^2) p_{21} + t p_{22},$$

$$h_n - h_n x_4 - n h_{n-1} x_4' = t p_{21} + p_{22},$$

where h_n denotes the polynomial of Hermite-type (degree n) given by

$$h_n(t) = (e^{-t^2-2t})^{(n)}\, e^{t^2+2t}.$$

A careful computation shows that x_i, $i = 1, 2, 3, 4$, can be taken to be polynomials of degree at most one; more precisely, the entries of the matrix polynomial of degree 1

$$X(t) = \begin{pmatrix} -t - n/2 & -1 \\ t & 1 \end{pmatrix}$$

satisfy the above differential equations and so $X(t)$ makes the expression (4.1) a matrix polynomial of degree n.

Unfortunately, the leading coefficient of this polynomial is a singular matrix (since the polynomial is orthogonal with respect to t^k, $k = 0, \dots, n-1$, that means that its norm is also singular). The problem is that for our choice of X, the entries 12 and 22 in the expression defined by (4.1) are both polynomials of degree $n-1$.

The kind of differential equations one is solving for suggests a slight modification of the function X which consists in allowing x_3 to be of the form $\gamma e^{4t} + r$ with γ a complex number and r a polynomial of degree 2; the differential equations for x_3 has now the form:

$$th_n - h_n r + nh_{n-1} - nh_{n-1}r' - \frac{n}{2}h_{n-2}r'' - \gamma e^{4t} g_n = (e^{4t} + t^2)p_{21} + tp_{22},$$

where g_n denotes the polynomial of Hermite-type (degree n) given by

$$g_n(t) = (e^{-t^2+2t})^{(n)} e^{t^2-2t}.$$

A careful computation shows now that the matrix function

$$X(t) = \begin{pmatrix} -t - \dfrac{n}{2} & -1 \\ \dfrac{2c}{n}t^2 + \left(\dfrac{2c}{n} + 1\right)t + c + \dfrac{2c}{n}e^{4t} & \dfrac{2c}{n}t + \dfrac{2c}{n} + 1 \end{pmatrix},$$

(whatever the number c is) makes also the expression (4.1) a matrix polynomial of degree n. But, unfortunately, once again with singular matrix leading coefficient $\left(\text{equal to } \begin{pmatrix} -2 & -6 \\ 4c & 12c \end{pmatrix}\right)$.

But we are now not far from the solution: the key (again suggested by a careful glance on the differential equation for x_3) is to take x_3 in the form $\gamma e^{4t} + \phi e^{t^2+2t} + r$, where $\phi(t) = \int_{-\infty}^t e^{-x^2-2x}\,dx - \int_{-\infty}^t e^{-x^2+2x}\,dx$ (so that $\phi(-\infty) = \phi(+\infty) = 0$). The differential equation for x_3 is now

$$th_n - h_n r + nh_{n-1} - nh_{n-1}r' - \frac{n}{2}h_{n-2}r'' - \gamma e^{4t} g_n + g_{n-1} e^{4t} - h_{n-1}$$
$$= (e^{4t} + t^2)p_{21} + tp_{22}.$$

A careful computation shows that the matrix function

$$X(t) = \begin{pmatrix} -t - \dfrac{n}{2} & -1 \\ e^{4t} + \phi(t)\, e^{t^2+2t} + t^2 + \left(2 - \dfrac{1}{n}\right)t + \dfrac{n}{2} & t + 2 - \dfrac{1}{n} \end{pmatrix}$$

makes also the expression (4.1) a matrix polynomial of degree n but now with nonsingular leading coefficient equal to $\begin{pmatrix} (-2)^n & (-2)^n(1+2n) \\ -(-2)^n & (-2)^{n-1}(1+4n-2/n) \end{pmatrix}$. Rodrigues' formula given in Theorem 1.2 can be obtained by multiplying (4.1) on the left by the normalization matrices

$$\Gamma_n = \begin{pmatrix} 1 & 0 \\ n & n \end{pmatrix}.$$

Recall that, earlier in the paper (Theorem 1.2), we used this Rodrigues-type formula to derive an explicit form of the three-term recursion satisfied by the polynomials defined by the (normalized) Rodrigues' formula (4.1).

4.2. $\rho(t) = t^\alpha e^{-t}$

Proceeding in the same way, one can find Rodrigues' formula for the orthogonal polynomials with respect to the weight matrices related to the weight above. Here is an example.

Take the matrices

$$A = \begin{pmatrix} 0 & 3/4 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1/4 & 1/4 \\ 0 & 0 \end{pmatrix}. \tag{4.2}$$

This gives for $T' = (A + B/t)T$, $T(1) = I$, the expression

$$T(t) = \begin{pmatrix} t^{1/4} & t-1 \\ 0 & 1 \end{pmatrix}.$$

Then, a sequence of orthogonal polynomials for the weight matrix

$$W = t^\alpha e^{-t} T T^* = t^\alpha e^{-t} \begin{pmatrix} t^{1/2} + (t-1)^2 & t-1 \\ t-1 & 1 \end{pmatrix}$$

can be defined by means of the following Rodrigues' formula:

$$\begin{aligned} \mathcal{P}_n(t) = & \left[t^{\alpha+n} e^{-t} \left(\begin{pmatrix} t^{1/2} + (t-1)^2 & t-1 \\ t-1 & 1 \end{pmatrix} - X(t) \right) \right]^{(n)} \\ & \times t^{-\alpha} e^{t} \begin{pmatrix} t^{1/2} + (t-1)^2 & t-1 \\ t-1 & 1 \end{pmatrix}^{-1}, \end{aligned} \tag{4.3}$$

where

$$X(t) = \begin{pmatrix} \alpha t - n - \alpha & n+\alpha \\ -\dfrac{n}{\xi_n}\phi_n(t) t^{-\alpha-n} e^t - \dfrac{n}{\xi_n}\sqrt{t} & 0 \end{pmatrix},$$

$\phi_n(t) = \int_0^t x^{\alpha+n+1/2} e^{-x}\,dx - \xi_n \int_0^t x^{\alpha+n} e^{-x}\,dx$ and $\xi_n = \Gamma(\alpha+n+3/2)/\Gamma(\alpha+n+1)$ (so that $\phi_n(+\infty) = 0$). We take $P_0 = I$.

For the resulting polynomials the leading coefficient of $\mathcal{P}_n(x)$ is given by

$$\begin{pmatrix} (-1)^n & (-1)^{n+1}(3n+2)/2 \\ 0 & \dfrac{(-1)^n n\,(n+1/2+\xi_n/n)}{\xi_n} \end{pmatrix}$$

and this guarantees that the leading coefficient is nonsingular.

Using this Rodrigues formula we can obtain the following three-term recursion relation satisfied by the polynomials $(\mathcal{P}_n)_n$ defined by (4.3):

$$t\mathcal{P}_n(t) = D_{n+1}\mathcal{P}_{n+1}(t) + E_n\mathcal{P}_n(t) + F_n\mathcal{P}_{n-1}(t)$$

with the matrices D_{n+1}, E_n, $n \geqslant 0$, and F_n, $n \geqslant 1$, given explicitly by the expressions

$$D_{n+1} = \begin{pmatrix} -1 & -\dfrac{3}{2}\dfrac{\xi_n}{(n+1)^2+\xi_n} \\ 0 & -\dfrac{n(n+1/2)+\xi_n}{(n+1)^2+\xi_n} \end{pmatrix},$$

$$F_n = \begin{pmatrix} -\dfrac{(2n^2+n)((n+1)^2+\xi_n)}{2n^2+n+2\xi_n} & 0 \\ -\dfrac{6n^2+3n}{2(2n^2+n+2\xi_n)} & -n^2 \end{pmatrix},$$

$$E_n = \begin{pmatrix} \dfrac{(n+1)^2(2n+3)+(2n+3/2)\xi_n}{(n+1)^2+\xi_n} & \dfrac{(3n+2)\xi_n}{2n(n+1/2)+2\xi_n} \\ \dfrac{(n+1)^2-n(n+1/2)}{(n+1)^2+\xi_n} & \dfrac{8(n+1/2)(n(n-1/4)+\xi_n)}{4n(n+1/2)+4\xi_n} \end{pmatrix}.$$

5. The algebras of differential operators

The main goal of this brief section is to indicate some results pertaining to structure of the following set:

$$\mathcal{D} = \{D : D(P_n) = \Lambda_n(D)P_n, n = 0, 1, 2, \ldots\}$$

for the two examples considered in detail above.

Here, D is a differential operator (with coefficients acting on the right) of arbitrary order and not necessarily symmetric with respect to the weight matrix $W(t)$.

A similar analysis was given for a set of five examples in [2]. One of the examples here corresponds to the last example there.

Our main experimental result in that extensive computational work seems to support the claim that both in the case of the examples studied in Sections 4.1 and 4.2 the corresponding algebras are generated by one second-order differential operator (and of course the identity).

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