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Imprimitive symmetric graphs with cyclic blocks

Cai Heng Li [a], Cheryl E. Praeger [a], Sanming Zhou [b]

[a] *School of Mathematics and Statistics, The University of Western Australia, 35 Stirling Highway, Crawley, WA 6009, Australia*

[b] *Department of Mathematics and Statistics, The University of Melbourne, Parkville, VIC 3010, Australia*

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ABSTRACT

Let Γ be a graph admitting an arc-transitive subgroup G of automorphisms that leaves invariant a vertex partition $\mathcal{B}$ with parts of size $v \geq 3$. In this paper we study such graphs where: for $B, C \in \mathcal{B}$ connected by some edge of Γ, exactly two vertices of B lie on no edge with a vertex of C; and as C runs over all parts of $\mathcal{B}$ connected to B these vertex pairs (ignoring multiplicities) form a cycle. We prove that this occurs if and only if $v = 3$ or 4, and moreover we give three geometric or group theoretic constructions of infinite families of such graphs.

1. Introduction

A graph $\Gamma = (V, E)$ is *G-symmetric* if $G \leq \mathrm{Aut}(\Gamma)$ is transitive on the set $\mathtt{Arc}(\Gamma)$ of arcs of Γ, where an *arc* is an ordered pair of adjacent vertices. For a G-symmetric graph Γ, a partition $\mathcal{B}$ of V is *G-invariant* if $B \in \mathcal{B}$ implies $B^g \in \mathcal{B}$ for all $g \in G$, where $B^g = \{\alpha^g : \alpha \in B\}$, and $\mathcal{B}$ is *nontrivial* if $1 < |B| < |V|$. Such a vertex partition gives rise to a *quotient graph* $\Gamma_{\mathcal{B}}$, namely the graph with vertex set $\mathcal{B}$ in which $B, C \in \mathcal{B}$ are adjacent if and only if there exists an edge of Γ joining a vertex of B to a vertex of C. Since Γ is G-symmetric and $\mathcal{B}$ is G-invariant, $\Gamma_{\mathcal{B}}$ is G-symmetric under the induced (not necessarily faithful) action of G on $\mathcal{B}$. Moreover, if Γ is connected, then $\Gamma_{\mathcal{B}}$ is connected and in particular all arcs join distinct parts of $\mathcal{B}$. For an arc (B, C) of $\Gamma_{\mathcal{B}}$, the subgraph $\Gamma[B, C]$ of Γ induced on $B \cup C$ with isolated vertices deleted is bipartite and, up to isomorphism, is independent of (B, C). In some examples, such as the case where Γ is a cover of $\Gamma_{\mathcal{B}}$, all vertices of B and C occur in $\Gamma[B, C]$, but many other possibilities also arise.

For an arc (B, C) of $\Gamma_{\mathcal{B}}$, let $\Gamma(C) = \bigcup_{\alpha \in C} \Gamma(\alpha)$, where $\Gamma(\alpha)$ denotes the set of vertices adjacent to α in Γ, and set

$$v := |B|, \qquad k := |\Gamma(C) \cap B|. \tag{1}$$

E-mail addresses: li@maths.uwa.edu.au (C.H. Li), praeger@maths.uwa.edu.au (C.E. Praeger), smzhou@ms.unimelb.edu.au (S. Zhou).

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An approach to understanding general G-symmetric graphs Γ in terms of $\Gamma_{\mathcal{B}}$, $\Gamma[B, C]$ and a 1-design induced on B was suggested in [3], and developed further in [5,8,9] in the case $k = v - 1$, where special additional structure on the parts B can be defined and exploited.

If $k = v - 2$ it turns out that we may also define additional structure on the parts. Since $\Gamma[B, C]$ consists of k vertices from each of B and C, in particular $v = k + 2 \geq 3$, and the set $B \setminus \Gamma(C)$ contains exactly two vertices. Thus we may define a multigraph Γ^B with vertex set B and an edge joining the two vertices of $B \setminus \Gamma(C)$ for each C in the set $\Gamma_{\mathcal{B}}(B)$ of parts of $\mathcal{B}$ adjacent to B in $\Gamma_{\mathcal{B}}$. Denote by $\mathtt{Simple}(\Gamma^B)$ the underlying simple graph of Γ^B. It was proved [4, Theorem 2.1] that $\mathtt{Simple}(\Gamma^B)$ is G_B-vertex-transitive and G_B-edge-transitive, and either Γ^B is connected or $\mathtt{Simple}(\Gamma^B)$ is a perfect matching $(v/2) \cdot K_2$, where G_B is the setwise stabiliser of B in G. In the latter case detailed information about Γ was obtained in [4, Theorem 1.3] when Γ^B is simple. However, no information about Γ was obtained in the case where Γ^B is connected. Here we considered the simplest possibility, namely $\mathtt{Simple}(\Gamma^B)$ has valency two. We find with surprise that the parts of $\mathcal{B}$ must have size 3 or 4 in this case. Our main result is Theorem 1.1 below. It involves the multiplicity m of the edges of the multigraph Γ^B, that is, for a pair $\{\alpha, \beta\}$ of adjacent vertices of Γ^B,

$$m = |\{C \in \Gamma_{\mathcal{B}}(B) : B \setminus \Gamma(C) = \{\alpha, \beta\}\}|.$$

Theorem 1.1. *Suppose Γ is a G-symmetric graph (where $G \leq \mathrm{Aut}(\Gamma)$) whose vertex set admits a nontrivial G-invariant partition $\mathcal{B}$ such that $k = v - 2 \geq 1$ with k, v as in (1), $\Gamma_{\mathcal{B}}$ is connected, and $\mathtt{Simple}(\Gamma^B)$ has valency two. Then $\mathtt{Simple}(\Gamma^B) = C_v$, $\Gamma_{\mathcal{B}}$ has valency mv, and one of the following* (a)–(c) *occurs for an arc (B, C) of $\Gamma_{\mathcal{B}}$.*

(a) $v = 3$ *and Γ has valency m;*
(b) $v = 4$, $\Gamma[B, C] = K_{2,2}$, *and Γ is connected of valency* $4m$*;*
(c) $v = 4$, $\Gamma[B, C] = 2 \cdot K_2$, *and Γ has valency* $2m$*.*

Remark 1.2. (1) In particular, if Γ^B is simple, then in case (a) we have $\Gamma = (|V(\Gamma)|/2) \cdot K_2$, and, in case (c), Γ has valency two and hence is a vertex-disjoint union of cycles of the same length. In Section 3 we construct an infinite family of graphs for each of these cases, and an infinite family of graphs for case (b) with Γ^B simple by using the coset graph construction.

(2) In cases (b) and (c) we prove that $G_B^B \cong D_8$, and for an arc (B, C) of $\Gamma_{\mathcal{B}}$, we prove that $G_{BC}^{B \cup C} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ in case (b), and $\mathbb{Z}_2$ in case (c).

(3) In case (a), Γ can be $(G, 2)$-arc-transitive even when $m > 1$; see [4, Example 4.6] for an infinite family of such graphs. In case (b) it is clear that Γ is not $(G, 2)$-arc-transitive. In case (c), if $m > 1$, then the stabiliser G_α of α in G is imprimitive on $\Gamma(\alpha)$ and hence Γ is not $(G, 2)$-arc-transitive. An example for case (c) such that Γ is $(G, 2)$-arc-transitive (hence $m = 1$) can be found in [4, Example 4.7].

Our construction for case (b) leads to an infinite family of connected 4-valent symmetric graphs Γ which have a 4-valent quotient not covered by Γ. To the best of our knowledge this is the first infinite family of symmetric graphs with these properties.

Corollary 1.3. *There exists an infinite family of connected symmetric graphs Γ of valency* 4 *which have a quotient graph $\Gamma_{\mathcal{B}}$ of valency* 4 *such that Γ is not a cover of $\Gamma_{\mathcal{B}}$.*

In the light of Theorem 1.1 we ask, for other connected graphs $\mathtt{Simple}(\Gamma^B)$:

Question 1.4. *In the case where $k = v - 2$ and Γ^B is connected, is v bounded by some function of the valency of $\mathtt{Simple}(\Gamma^B)$?*

We may also ask the following question.

Question 1.5. *Can Γ in Theorem* 1.1 *be determined for small values of m?*

The proof of Theorem 1.1 is given in Section 2 and the examples are constructed in Section 3. The reader is referred to [1] for group theoretic terminology used in the paper.

2. Proof of Theorem 1.1

Two parts $B, C \in \mathcal{B}$ are called *adjacent* if they are adjacent in the quotient graph $\Gamma_{\mathcal{B}}$, and if B, C are adjacent we write $G_{BC} = (G_B)_C$ and let $e(B, C)$ be the edge of $\mathtt{Simple}(\Gamma^B)$ joining the two vertices of $B \setminus \Gamma(C)$.

Proof of Theorem 1.1. Let $(\Gamma, G, \mathcal{B})$ satisfy the conditions of Theorem 1.1, and let B, C be adjacent parts. Then $\mathtt{Simple}(\Gamma^B) \neq (v/2) \cdot K_2$ since it is of valency two by our assumption. Thus Γ^B, and hence also $\mathtt{Simple}(\Gamma^B)$, is connected [4, Theorem 2.1] and so $\mathtt{Simple}(\Gamma^B) = C_v$. Thus, by the definition of Γ^B, the valency of $\Gamma_{\mathcal{B}}$ is mv.

Case 1: v odd. Since v is odd, there exists a unique vertex $\alpha \in B$ which is 'antipodal' to the edge $e(B, C)$ of $\mathtt{Simple}(\Gamma^B)$, that is, α is the unique vertex equi-distant in $\mathtt{Simple}(\Gamma^B)$ from the two vertices of $e(B, C)$. Now each element of G_{BC} fixes $B \setminus \Gamma(C)$ setwise and hence fixes α. (In the case where $m > 1$, an element of G_{BC} may permute the m edges of Γ^B joining the two vertices of $B \setminus \Gamma(C)$.) Thus, $G_{BC} \leq G_\alpha$. Since $\alpha \notin B \setminus \Gamma(C)$, there exists $\beta \in C$ adjacent to α in Γ. Suppose $v \geq 5$. Then there exists a vertex $\gamma \in B$ such that $\gamma \notin \{\alpha\} \cup (B \setminus \Gamma(C))$ and so γ is adjacent to a vertex $\delta \in C$. Since Γ is G-symmetric, there exists $g \in G$ such that $(\alpha, \beta)^g = (\gamma, \delta)$. Since g maps $\alpha \in B$ to $\gamma \in B$ and $\beta \in C$ to $\delta \in C$, it fixes B and C setwise. Thus, $g \in G_{BC} \leq G_\alpha$, which is a contradiction since $\alpha^g = \gamma \neq \alpha$. Therefore, $v = 3$ and consequently Γ has valency m.

Case 2: v even. Since v is even, there exists a unique edge of $\mathtt{Simple}(\Gamma^B)$, say, $e = \{\alpha, \beta\}$, which is 'antipodal' to $e(B, C)$ in $\mathtt{Simple}(\Gamma^B)$, that is, α and β are both at maximum distance $v/2$ from some vertex of $e(B, C)$. Note that $\alpha, \beta \in B \cap \Gamma(C)$. Each vertex $\gamma \in B \cap \Gamma(C)$ is adjacent to some vertex $\delta_\gamma \in C$. Since Γ is G-symmetric, for each such γ there exists $g_\gamma \in G$ such that $(\alpha, \delta_\alpha)^{g_\gamma} = (\gamma, \delta_\gamma)$. Since g_γ maps $\alpha \in B$ to $\gamma \in B$ and $\delta_\alpha \in C$ to $\delta_\gamma \in C$, we have $g_\gamma \in G_{BC}$. Thus for each $\gamma \in B \cap \Gamma(C)$, g_γ fixes $e(B, C)$ setwise and hence fixes $e = \{\alpha, \beta\}$ setwise also. Thus $\alpha^{g_\gamma} = \gamma \in \{\alpha, \beta\}$ and in particular $v = 4$ and $B \cap \Gamma(C) = \{\alpha, \beta\}$. Since g_β fixes e setwise, it interchanges α and β. Since G_B^B is transitive on B, it follows that $G_B^B \cong D_8$. Therefore, $1 \neq G_{BC}^{B \cup C} \leq \langle x^B \rangle \times \langle x^C \rangle$, where x^B is the reflection of Γ^B in $e(B, C)$ and x^C is the reflection of Γ^C in $e(C, B)$. Note that x^B interchanges α and β since it interchanges the two vertices of $e(B, C)$. Thus $g_\beta^B = x^B$. Similarly, x^C interchanges the two vertices of $C \setminus e(C, B) = \{\eta, \zeta\}$, say, and G_{BC} contains an element h such that $h^C = x^C$. Thus either $G_{BC}^{B \cup C} = \langle x^B \rangle \times \langle x^C \rangle$ or $G_{BC}^{B \cup C} = \langle x^B x^C \rangle \cong \mathbb{Z}_2$. Since G_{BC} preserves the adjacency of Γ, the first possibility occurs if and only if α is adjacent to both of the vertices of $C \setminus e(C, B)$ and hence $\Gamma[B, C] = K_{2,2}$ is the 4-cycle $(\alpha, \eta, \beta, \zeta, \alpha)$. Since $\mathtt{Simple}(\Gamma^B) = C_4$, in this case α is at distance 2 in Γ from β and from one of the vertices of $e(B, C)$, and at distance 3 or 4 from the other vertex of $e(B, C)$. Since $\Gamma_{\mathcal{B}}$ is connected, it follows that Γ is connected of valency $4m$ in this case. Suppose now that $G_{BC}^{B \cup C} = \langle x^B x^C \rangle \cong \mathbb{Z}_2$. Then the bipartite graph $\Gamma[B, C]$ consists of two edges only, namely, $\{\alpha, \delta_\alpha\}$ and $\{\beta, \delta_\beta\}$. Hence $\Gamma[B, C] = 2 \cdot K_2$ and Γ has valency $2m$. $\square$

3. Constructions

In this section we present several constructions of infinite families of graphs that satisfy the conditions of Theorem 1.1 in the case where the multigraph Γ^B is simple, that is $\Gamma^B = \mathtt{Simple}(\Gamma^B)$ or equivalently $m = 1$. The first two constructions involve regular maps on surfaces. Here and in what follows our use of the term 'regular map' agrees with that of [2], that is, a regular map is a 2-cell embedding of a connected (multi)graph on a closed surface such that its automorphism group is regular on incident vertex–edge–face triples.

3.1. Truncations of trivalent symmetric graphs

The construction below produces all graphs that arise in case (a) of Theorem 1.1 with $m = 1$.

Construction 3.1. Let Σ be a trivalent G-symmetric graph with n edges. Define $\Gamma(\Sigma)$ to be the graph with vertex set $\mathtt{Arc}(\Sigma)$ and edges $\{(\sigma, \tau), (\tau, \sigma)\}$ for $(\sigma, \tau) \in \mathtt{Arc}(\Sigma)$ [4, Example 2.4]. Then $\Gamma(\Sigma) = n \cdot K_2$, $\Gamma(\Sigma)$ is G-symmetric, and its vertex set admits the G-invariant partition

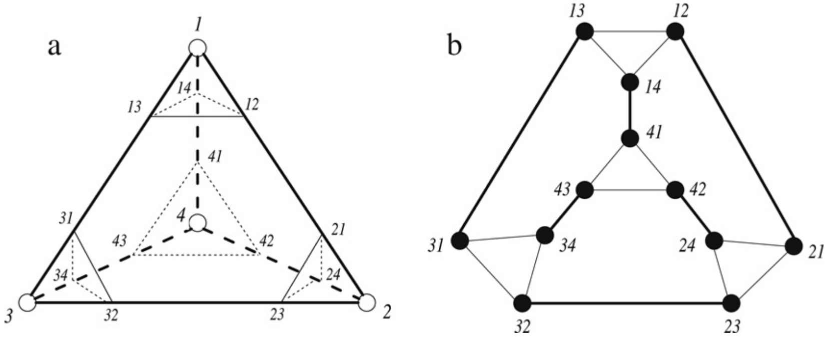

Fig. 1. Obtaining $\Gamma = 6 \cdot K_2$ (heavy edges in (b)) by truncating the tetrahedron as in (a).

$\mathcal{B}(\Sigma) = \{B(\sigma) : \sigma \in V(\Sigma)\}$ with parts of size $v = 3$, where $B(\sigma)$ is the set of arcs of Σ with first vertex σ. For this partition we have $k = v - 2 = 1$, $\Gamma(\Sigma)^{B(\sigma)}$ is the simple graph C_3, and Σ is isomorphic to the quotient graph $\Gamma(\Sigma)_{\mathcal{B}(\Sigma)}$ via the bijection $\sigma \mapsto B(\sigma)$.

As explained in [4, Example 2.4] this construction produces all imprimitive G-symmetric graphs $(\Gamma, \mathcal{B})$ such that $k = v - 2 = 1$ and $\Gamma^B = C_3$ is simple.

In case (a) of Theorem 1.1, if $m = 1$, then $G_B^B \cong \mathbb{Z}_3$ or D_6. From [2, Theorem 1.1] the former occurs if and only if $\Gamma_{\mathcal{B}}$ admits an embedding as an orientably-regular (rotary) map M on a closed orientable surface. In fact, $\Gamma_{\mathcal{B}}$ admits[1] two such embeddings which are mirror images of each other such that their automorphism groups are isomorphic to G. In this case we may view Γ as obtained from M by truncation: cutting off each corner and then removing the edges in the triangles thus produced. In particular, let M be the tetrahedron and let $G = A_4$ act on the vertices of M in its natural action. Then Construction 3.1 applied with Σ the underlying graph of M gives rise to $\Gamma(\Sigma) = 6 \cdot K_2$ as shown in Fig. 1.

3.2. *Flag graphs of 4-valent regular maps*

Next we construct four infinite families of graphs that arise in case (c) of Theorem 1.1 with $m = 1$. The constructions take as input a 4-valent regular map M with automorphism group $G = \mathrm{Aut}(M)$ so that the underlying graph Σ of M is G-symmetric and, for $\sigma \in V(\Sigma)$, $G_\sigma \cong G_\sigma^{\Sigma(\sigma)} = D_8$. The output of Construction 3.2 involves incident vertex–face pairs of M of the form (σ, h) where σ is a vertex and h is a face incident with σ.

Construction 3.2. Let M be a regular map on a closed surface such that its underlying graph Σ has valency four, and let $G = \mathrm{Aut}(M)$. For each edge $\{\sigma, \sigma'\}$ of Σ, let f, f' denote the faces of M such that $\{\sigma, \sigma'\}$ is on the boundary of both f and f'. Let $\mathrm{opp}_\sigma(f)$ and $\mathrm{opp}_\sigma(f')$ be the other two faces of M incident with σ and opposite to f and f' respectively, and define $\mathrm{opp}_{\sigma'}(f)$ and $\mathrm{opp}_{\sigma'}(f')$ similarly. Define four graphs $\Gamma_1(M), \Gamma_2(M), \Gamma_3(M), \Gamma_4(M)$ with vertices the incident vertex–face pairs of M and adjacency defined as follows (where $\sim$ means adjacency): for each edge $\{\sigma, \sigma'\}$ of Σ, $(\sigma, f) \sim (\sigma', f)$ and $(\sigma, f') \sim (\sigma', f')$ in $\Gamma_1(M)$; $(\sigma, f) \sim (\sigma', f')$ and $(\sigma, f') \sim (\sigma', f)$ in $\Gamma_2(M)$; $(\sigma, \mathrm{opp}_\sigma(f)) \sim (\sigma', \mathrm{opp}_{\sigma'}(f))$ and $(\sigma, \mathrm{opp}_\sigma(f')) \sim (\sigma', \mathrm{opp}_{\sigma'}(f'))$ in $\Gamma_3(M)$; $(\sigma, \mathrm{opp}_\sigma(f)) \sim (\sigma', \mathrm{opp}_{\sigma'}(f'))$ and $(\sigma, \mathrm{opp}_\sigma(f')) \sim (\sigma', \mathrm{opp}_{\sigma'}(f))$ in $\Gamma_4(M)$.

Let $\mathcal{B}(M) = \{B(\sigma) : \sigma \in V(\Sigma)\}$, where $B(\sigma) = \{(\sigma, f) : \sigma \text{ incident with } f\}$. The following lemma shows that the graphs produced by Construction 3.2 have the required properties.

Lemma 3.3. *Let M, Σ, G be as in Construction 3.2 and let $\Gamma = \Gamma_i(M)$ be as defined there, where $1 \le i \le 4$. Then Γ is a G-symmetric graph of valency two whose vertex set admits $\mathcal{B}(M)$ as a G-invariant*

[1] Details may be obtained from the authors.

partition such that $k = v - 2 = 2$, $\Gamma_{\mathcal{B}} \cong \Sigma$, *and* $\Gamma^{B(\sigma)} = C_4$ *is simple. Moreover, for adjacent blocks* $B(\sigma), B(\tau) \in \mathcal{B}(M)$, $\Gamma[B(\sigma), B(\tau)] = 2 \cdot K_2$.

Proof. Since M is a regular map, $G = \mathrm{Aut}(M)$ is transitive on the vertices of Γ and $\mathcal{B}(M)$ is a G-invariant partition of the vertex set of Γ. Since the underlying graph Σ of M is of valency four, the parts of $\mathcal{B}(M)$ have size $v = 4$ and a typical part is of the form $B(\sigma) = \{(\sigma, f_i) : 1 \le i \le 4\}$, where f_1, f_2, f_3, f_4 are the faces of M surrounding σ. Let τ_i, $1 \le i \le 4$ be the vertices of M adjacent to σ such that τ_{i-1} and τ_i are incident with the face f_i, where subscripts are taken modulo 4. If $\Gamma = \Gamma_1(M)$ then (σ, f_i) is adjacent to (τ_{i-1}, f_i) and (τ_i, f_i) only, and hence Γ has valency two. Similarly if $\Gamma = \Gamma_2(M)$ then (σ, f_i) is adjacent to (τ_{i-1}, f_{i-1}) and (τ_i, f_{i+1}) only, and again Γ has valency two. In either case $\Gamma[B(\sigma), B(\tau_1)]$ consists of two edges, namely $\{(\sigma, f_1), (\tau_1, f_1)\}$ and $\{(\sigma, f_2), (\tau_1, f_2)\}$ for $\Gamma_1(M)$, and $\{(\sigma, f_2), (\tau_1, f_1)\}$ and $\{(\sigma, f_1), (\tau_1, f_2)\}$ for $\Gamma_2(M)$, and hence $k = 2$ and $\Gamma[B(\sigma), B(\tau_1)] = 2 \cdot K_2$. Moreover, $\Gamma^{B(\sigma)}$ is a cycle C_4, namely $((\sigma, f_1), (\sigma, f_2), (\sigma, f_3), (\sigma, f_4), (\sigma, f_1))$ in both cases, and $\Gamma_{\mathcal{B}} \cong \Sigma$ via the mapping $B(\sigma) \mapsto \sigma$. Since M is a regular map, there exists $g \in G_\sigma$ which fixes f_1, interchanges τ_1 and τ_4, and interchanges f_2 and f_4. Thus g interchanges the two vertices adjacent to (σ, f_1) in both cases, so Γ is G-symmetric.

Similarly one can verify that all statements hold for $\Gamma = \Gamma_3(M)$ or $\Gamma_4(M)$. □

Each of $\Gamma_1(M)$, $\Gamma_2(M)$, $\Gamma_3(M)$ and $\Gamma_4(M)$ in Construction 3.2 is a union of cycles since it has valency two. For example, $\Gamma_1(M) \cong s \cdot C_t$ and each face of M gives rise to a cycle of $\Gamma_1(M)$, where t is the face length and s the number of faces of M. For the octahedron M one can check that $\Gamma_1(M) \cong 8 \cdot C_3$, $\Gamma_2(M) \cong 4 \cdot C_6$, $\Gamma_3(M) \cong 6 \cdot C_4$ and $\Gamma_4(M) \cong 4 \cdot C_6$.

3.3. *An explicit group theoretic construction*

Finally, we give a Sabidussi coset graph construction (see e.g. [6]) for an infinite family of graphs that satisfy part (b) of Theorem 1.1 with $m = 1$. Given a group G, a core-free subgroup H of G and a 2-element g such that $g \notin \mathbf{N}_G(H)$ and $g^2 \in H \cap H^g$, the coset graph $\mathrm{Cos}(G, H, HgH)$ is defined to have vertex set $[G : H] = \{Hx : x \in G\}$ such that Hx, Hy are adjacent if and only if $xy^{-1} \in HgH$. It is known, see for example [6], that $\mathrm{Cos}(G, H, HgH)$ is G-symmetric and is connected if and only if $\langle H, g\rangle = G$. For a subgroup $L < H$, let $B = [H : L] = \{Lh \mid h \in H\}$. For $x \in G$, let $B^x = \{Lhx \mid h \in H\}$, and let $\mathcal{B} = \{B^x \mid x \in G\}$. Then $\mathcal{B}$ is a G-invariant partition of $[G : L]$. Further, we have the following link between the two coset graphs.

Lemma 3.4. *Let* $\Gamma = \mathrm{Cos}(G, L, LgL)$ *and* $\Sigma = \mathrm{Cos}(G, H, HgH)$. *Then* $\Sigma \cong \Gamma_{\mathcal{B}}$.

Proof. Define a one-to-one correspondence between $[G : H]$ and $\mathcal{B}$ by:

$$\varphi : Hx \mapsto B^x, \quad x \in G.$$

We claim that φ induces an isomorphism between Σ and $\Gamma_{\mathcal{B}}$. For any $x, y \in G$, we have (where $\sim$ means adjacency):

$$\begin{aligned}
Hx \sim Hy \quad \text{in } \Sigma &\implies yx^{-1} \in HgH \\
&\implies yx^{-1} = h_1 g h_2 \quad \text{for some } h_1, h_2 \in H \\
&\implies h_1^{-1} y (h_2 x)^{-1} = g \in LgL \\
&\implies Lh_2 x \sim Lh_1^{-1} y \quad \text{in } \Gamma \\
&\implies B^x \sim B^y \quad \text{in } \Gamma_{\mathcal{B}}. \\
B^x \sim B^y \quad \text{in } \Gamma_{\mathcal{B}} &\implies Lh_1 x \sim Lh_2 y \quad \text{in } \Gamma, \text{ for some } h_1, h_2 \in H \\
&\implies h_2 y (h_1 x)^{-1} \in LgL \\
&\implies yx^{-1} \in h_2^{-1} LgL h_1 \subset HgH \\
&\implies Hx \sim Hy \quad \text{in } \Sigma.
\end{aligned}$$

Thus $\Sigma \cong \Gamma_{\mathcal{B}}$, as claimed. □

Now we construct examples satisfying part (b) of Theorem 1.1.

Construction 3.5. Let p be a prime such that $p \equiv 1 \pmod{16}$, and let $G = \mathrm{PSL}(2, p)$. Let H be a Sylow 2-subgroup of G. Then $H = \langle a \rangle : \langle b \rangle \cong D_{16}$, $\langle a^4, b \rangle \cong \mathbb{Z}_2^2$, and $\mathbf{N}_G(\langle a^4, b \rangle) = S_4$. There exists an involution $g \in \mathbf{N}_G(\langle a^4, b \rangle) \setminus \langle a^2, b \rangle$ such that g interchanges a^4 and b. Let $L = \langle a^4, ba \rangle \cong \mathbb{Z}_2^2$, and define

$$\Sigma = \mathrm{Cos}(G, H, HgH), \qquad \Gamma = \mathrm{Cos}(G, L, LgL).$$

Lemma 3.6. *Using the notation defined above, the following all hold:*

(a) *both Γ and Σ are G-symmetric, connected and of valency* 4*;*
(b) *$\mathcal{B}$ is a G-invariant partition of $V(\Gamma)$ such that $k = v - 2 = 2$, $\Gamma^B = C_4$ and $\Sigma \cong \Gamma_{\mathcal{B}}$;*
(c) *for $B = [H : L]$ and $C = B^g \in \Gamma_{\mathcal{B}}(B)$, the induced subgraph $\Gamma[B, C] = K_{2,2}$.*

Proof. It follows from the classification of the subgroups of G, see for example [7, pp. 417], that $\langle H, g \rangle$ is contained in no maximal subgroup of G. Thus $\langle H, g \rangle = G$, and so Σ is connected. Moreover, since $(a^4)^g = b$, it follows that $b, a \in \langle a^4, ba, g \rangle$. Thus $\langle L, g \rangle = G$, and so Γ is connected.

By the definition, $\langle a^4, b, g \rangle \cong D_8$, and $H \cap H^g = \langle a^4, b \rangle \cong \mathbb{Z}_2^2$. Hence Σ has valency 4. Since L is abelian, $L \cap L^g \lhd L$. Also $L \cap L^g$ is normalised by the involution g, and hence $L \cap L^g$ is normal in $\langle L, g \rangle = G$. As G is simple, $L \cap L^g = 1$, and so Γ is of valency 4. Part (a) now follows by Lemma 3.4.

As above $\mathcal{B}$ is a G-invariant partition of $V(\Gamma)$ with parts of size $v = |H : L| = 4$. The stabiliser $G_B = H$, and for $C = B^g$, we have $G_{BC} = G_B \cap G_C = H \cap H^g = \langle a^4, b \rangle$. Label the vertex L of Γ as α. Then $\alpha \in B$, $G_\alpha = L = \langle a^4, ba \rangle$, and $G_\alpha \cap G_{BC} = \langle a^4 \rangle$. The vertex $\beta = \alpha^g = Lg$ lies in $C \cap \Gamma(\alpha)$, and so $\beta^{a^4} \in C \cap \Gamma(\alpha)$ and $\{\beta, \beta^{a^4}\} \subseteq C \cap \Gamma(\alpha)$. Also, since $G_{\alpha\beta} = L \cap L^g = 1$, a^4 does not fix β and hence $\beta \neq \beta^{a^4}$. Counting the numbers of edge of Σ and Γ, we conclude that there are exactly 4 edges of Γ between B and C. It follows that $\Gamma[B, C] = K_{2,2}$ and $k = 2$. This together with the fact that both Γ and Σ have valency 4 forces Γ^B to be simple and isomorphic to C_4. Finally by Lemma 3.4, $\Sigma \cong \Gamma_{\mathcal{B}}$. This completes the proof of parts (b) and (c). □

Corollary 1.3 follows from Lemma 3.6 immediately.

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