Asymptotic Behavior of Solutions of Second Order Parabolic Partial Differential Equations with Unbounded Coefficients

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Classical solutions of a second order parabolic partial differential equation are considered in unbounded domains. The coefficients are allowed to have unbounded growth as the space variables tend to infinity. A Phragmén-Lindelöf principle is proved for such equations. That principle is used together with comparison functions to derive sufficient conditions for the asymptotic decay in time of solutions.

This paper is concerned with the asymptotic behavior of solutions of certain second order parabolic partial differential equations in unbounded domains. A Phragmén-Lindelöf principle is derived, and then used together with comparison functions to study the asymptotic behavior of solutions.

Let \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). Let \( |x| \) be the usual Euclidean norm of \( x \); we will often denote \( |x| \) by \( r \). Let \( D = \Omega \times (0, T) \), where \( \Omega \) is an unbounded domain in \( \mathbb{R}^n \). We consider equations of the form

\[
L[u] = \sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x, t) \frac{\partial u}{\partial x_j} + c(x, t)u - u_t = 0 \tag{1.1}
\]

in \( D \). Let \( \sum = (\partial \Omega \times \{0\}) \cup (\partial \Omega \times (0, T)) \).

We assume throughout this paper that the coefficients of (1) are continuous in \( D \) and bounded on the closure of every bounded subdomain of \( D \). We assume that the matrix \( \{(a_{ij})\} \) is symmetric for all \( (x, t) \in D \), and that

\[
\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geq 0. \tag{1.2}
\]

We consider only classical solutions to (1.1); thus we require \( u \in C^0(D) \cap C^2(D) \). For some results we will take \( T = \infty \).

407
The first major result is a Phragmèn–Lindelöf principle, which provides conditions on the growth rates of the coefficients of $L$ and the function $u$ as $|x| \to \infty$ that imply that if $L[u] \geq 0$ in $D$ and $u \leq 0$ on $\Sigma$, then $u \leq 0$ in $D$. Phragmèn–Lindelöf principles for parabolic equations have been studied by various authors, in particular Bodanko [2], Chabrowski [3], Friedman [10], Il'In, Kalasnikov and Oleinik [11], Kusano, Kuroda and Chen [16], [17], and Protter and Weinberger [19]. The Phragmèn–Lindelöf principle proved here generalizes all the results listed except those obtained in [3], by allowing a wider range of growth conditions on the coefficients of $L$. The result in [3] is similar to the one proved here, differing in some details of the growth conditions.

The Phragmèn–Lindelöf principle and comparison functions are used to study the behavior of solutions of (1.1). Conditions are found which imply that if $u(x, 0)$ is bounded for $x \in \Omega$, and $u(x, t)$ satisfies (1.1), then $u(x, t)$ must decay as $t \to \infty$. We also find conditions under which solutions of (1.1), which are not necessarily bounded as $|x| \to \infty$ for $t = 0$, must become bounded in $x$ for $t = T_0 > 0$. Finally, we note that solutions of certain elliptic equations can be considered as time independent solutions of an associated parabolic equation, and use this fact to study solutions of those elliptic equations. The results presented here can be generalized to systems of the form

$$L^\alpha[u^\alpha] + \sum_{\beta=1}^N c^\alpha_{\beta} u^\beta = 0, \quad \beta = 1, \ldots, N,$$

where

$$L^\alpha[u] \equiv \sum_{i,j=1}^n a_{ij}^\alpha u_{x_i x_j} + \sum_{i=1}^n b_i^\alpha u_{x_i} - u_t.$$ (1.3)

This is done in [7].

Results on the asymptotic behavior of parabolic equations and systems with unbounded coefficients have been obtained by various authors, including Chen [4], [5], [6], Eidel'man and Porper [8], [9], Kuroda [12] Kuroda and Chen [13], Kusano [14], [15], and Kusano, Kuroda, and Chen [16], [17]. The results obtained here generalize or augment existing results by allowing a wider range of growth conditions on the coefficients of (1.1). In particular, all the existing results require that the coefficients of (1.1) satisfy polynomial growth conditions of some fairly specific form, or satisfy a sign condition. The results proved here allow more rapid growth of the coefficient $c(x, t)$ in (1.1) and allow more general forms of the growth conditions on the other coefficients. However, some of the previous results are sharper in certain specific cases, or require conditions not strictly comparable with those imposed here. The techniques used in the present article are primarily adapted from those used in the work of Kusano, Kuroda, and Chen, especially [16] and [17].

The maximum principle for parabolic equations is the main tool used in this article. Suppose that $\omega \subseteq \mathbb{R}^n$ is a bounded domain; let $D = \omega \times (0, T)$ and let $\Sigma = (\partial \omega \times (0, T)) \cup (\bar{\omega} \times \{0\})$. 
LEMMA 1. Suppose that $L$ is as in (1.1), with (1.2) satisfied in $D$. Suppose that

$$Lu \geq 0 \quad \text{in } D \quad \text{with} \quad u \in C^0(D) \cap C^2(D), \quad \text{and}$$

$$u \leq 0 \quad \text{on } \Sigma. \quad \text{(1.4)}$$

Then $u \leq 0$ on $\bar{D}$.

Lemma 1 is a version of the standard weak maximum principle for parabolic equations; forms of that maximum principle are discussed in [10], [11], and [19]. Lemma 1 can be generalized in two ways. First, the single inequality (1.4) may be replaced with a weakly coupled system of the form (1.3); then (1.5) and the conclusion of the lemma should be taken componentwise. The generalization of the maximum principle to such systems is discussed in [19]. Second, under certain conditions, the condition (1.5) may be omitted on parts of $\partial \omega \times (0, T)$ and replaced with the requirement that (1.4) hold there. Specifically, suppose that $\nu = (\nu_1, \ldots, \nu_n)$ is the inward normal to $\omega$. Note that (1.2) allows $\sum_{i,j} a_{ij} \xi_i \xi_j = 0$. If there is a region of $\partial \omega \times (0, T)$ for which $\sum_{i,j} a_{ij} \nu_i \nu_j = 0$ and $\sum_{i,j} [b_i - a_{ij} \nu_j] \nu_i \geq 0$, then on that region (1.5) may be replaced with (1.4). This generalization of the maximum principle is due to Fichera; it is discussed in the context of general second order equations with nonnegative characteristic form in [18].

The results obtained here depend essentially on lemma 1. They can be generalized in the ways mentioned above by replacing lemma 1 with the appropriate generalization of the maximum principle. The extension of the results obtained here to those situations is straightforward; it is done in detail in [7].

2

The object of this section is to prove a maximum principle for parabolic equations in unbounded domains, which will be used as a tool in later sections. To extend the maximum principle to unbounded domains, it is necessary to impose growth restrictions on the coefficients of $L$. To state these conditions it will be convenient to introduce the following notation: for $\eta \in [1, \infty)$, let $\phi(\eta)$ be a positive $C^1$ function of $\eta$. Let $\Theta(\eta)$ be an antiderivative of $1/\phi(\eta)$ such that $\Theta(\eta) \geq 1$ for $\eta \in [1, \infty)$. Note that $\Theta(\eta)$ is necessarily strictly increasing. The growth conditions on the coefficients of $L$ are as follows (we adopt the notation $r$ for $|x|$):

**Condition I.** There exist positive constants $\mu$, $K_1$, $K_2$, and $K_3$ such that for $\xi \in \mathbb{R}^n$,

$$\sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \leq K_3 \phi(1 + r^2) |\xi|^2$$

and

$$|b_i(x, t)| \leq K_2 \phi(1 + r^2) \Theta(1 + r^2)(1 + r^2)^{-1/2}, \quad \text{for } i = 1, \ldots, n$$
and
\[ c(x, t) \leq K_0 [\Theta(1 + r^2)]^\mu \quad \text{for} \quad (x, t) \in D. \]

Associated with condition I is the following condition on \( \Theta(\eta) \):

**Condition II.** There exist nonnegative constants \( m_1 \) and \( m_2 \) such that for \( n \geq 1 \),
\[
\eta \Theta''(\eta) \leq m_1 \Theta(\eta) \Theta'(\eta)
\]
\[
\eta \Theta'(\eta) \leq m_2 [\Theta(\eta)]^{2-\mu}
\]
where \( \mu \) is the same positive constant as in condition I.

**Remarks.** Condition II restricts the growth of the function \( \Theta(\eta) \) as \( \eta \to \infty \). For example, observe that in the case \( \mu = 1 \), the second inequality of condition II implies that \( \Theta(\eta) \) is of at most polynomial growth as \( \eta \to \infty \). In the case \( 0 < \mu < 1 \), it is still true that \( \Theta(\mu) = \eta^p, p > 0 \), satisfies condition II. Furthermore, it is easily seen that if \( \Theta(\eta) \) is increasing and greater than or equal to 1 for \( 1 \leq \eta < \infty \), and \( \Theta(\eta) \) satisfies condition II \( 0 < \mu < 1 \), then the function \( \exp[\Theta(\eta)] \) will also satisfy condition II. (It may be necessary to change the constants \( m_1 \) and \( m_2 \), but \( \mu \) will remain the same.) Consequently when \( 0 < \mu < 1 \), we can consider cases where \( \Theta(\eta) \) has exponential or faster growth. If, however, \( \Theta(\eta) = e^\eta \), we must have \( \phi(\eta) = e^{-\eta} \); so if \( \Theta(\eta) \) grows rapidly, \( \phi(\eta) \) has to decay. In the case \( \mu > 1 \), \( \Theta(\eta) \) can only grow on the order of \( [\ln(\eta)]^{1/(\mu-1)} \). For example, if \( \phi(\eta) = \eta \) then we can take \( \theta(\eta) = [1 + \ln(\eta)] \). In this case, \( \Theta(\eta) \) satisfies condition II for any choice of \( \mu \) with \( 0 < \mu \leq 2 \), but not for \( \mu > 2 \).

To extend the maximum principle to unbounded domains, it is necessary to construct a function with certain special properties. Let \( D(T_1, \tau) = \Omega \times (T_1, T_1 + \tau) \). Then we have the following:

**Lemma 2.** Suppose that the coefficients of \( L \) satisfy condition I in \( D \), with \( \Theta(\eta) \) satisfying condition II. Then for any \( T_1 \in [0, T) \), there exists a function \( H(x, t, T_1, k) \) (where \( k > 1 \) is a parameter) such that
\[
L[H] \leq -H \leq -1 \tag{2.1}
\]
in \( D \cap D(T_1, \tau) \), where \( \tau > 0 \) depends on \( k \) and the constants in conditions I and II, but not on \( T_1 \).

**Proof.** Define \( H(x, t, T_1, k) = \exp\{k[\Theta(1 + r^2)]^\mu e^{\gamma(t-T_1)}\} \) where \( \mu > 0 \) is the constant appearing in conditions I and II, and \( \gamma > 0 \) will be chosen later. Computing \( L[H] \) and estimating the resulting terms using conditions I and II yields the following estimate:
\[
L[H] \leq [Mk^\mu e^{\gamma(t-T_1)} - k\gamma e^{\gamma(t-T_1)}]H
\]
\[
\leq k e^{\gamma(t-T_1)} \theta^\mu [Me^{\gamma(t-T_1)} - \gamma]H. \tag{2.2}
\]
ASYMPTOTIC BEHAVIOR

(Details of the calculation are given in [7].) If we choose \( \gamma = (Me + 1) \) and \( \tau = 1/\gamma \), then (2.2) implies that for \( T_1 < t < T_1 + \tau \) and \( x \in \Omega, H(x, t, T_1, k) \) satisfies

\[
L[H] \leq -ke^{\gamma(t - T_1)}\theta^\mu H.
\]  

(2.3)

Since \( k \geq 1, \Theta \geq 1, \) and \( H \geq 1 \), (2.3) implies (2.1) for \( H(x, t) \) with \( (x, t) \in D(T_1, \tau) \), which proves the lemma.

We can now state and prove the main result of this section. The following theorem is a type of maximum principle for solutions to the inequality \( L[u] \geq 0 \) in the unbounded domain \( D \). More precisely, since it will be necessary to impose restrictions on the behavior of the function \( u(x, t) \) as \( \tau = |x| \to \infty \), the theorem is a version of the Phragmén–Lindelöf principle for parabolic equations.

**Theorem 1** (Phragmén–Lindelöf Principle). Suppose that \( u(x, t) \in C^0(\overline{D}) \cap C^\alpha(D) \) satisfies the inequalities

\[
L[u] \geq 0 \quad \text{in } D
\]

\[
u \leq 0 \quad \text{on } \Sigma.
\]  

(2.4)

Suppose further that the coefficients of \( L \) satisfy condition I in \( D \), with \( \theta(\eta) \) satisfying condition II. Finally, suppose that if \( \mu > 0 \) is the constant appearing in conditions I and II, there is a constant \( k \geq 1 \) such that

\[
\liminf_{R \to \infty} \left( \max_{(x, t) \in \partial D} u(x, t) \exp\{-k[\theta(1 + R^2)]^\mu\} \right) \leq 0.
\]  

(2.5)

Then \( u(x, t) \leq 0 \) in \( D \).

**Proof.** Since conditions I and II are assumed to hold, lemma 2 applies. Let \( k \geq 1 \) be the same constant that appears in (2.5). Then for \( T_1 \) with \( 0 < T_1 < T \), we can choose \( H = H(x, t, T_1, k) \) as in lemma 2. By that lemma, we have \( L[H] \leq -1 \) in \( D(T_1, \tau) \), where \( \tau > 0 \) does not depend on \( T_1 \). Fixing this value of \( \tau \), for \( R > 0 \) define \( D(T_1, R) = D(T_1, \tau) \cap \{ (x, t) \in D : |x| < R \} \) and denote \( D(T_1, \tau) \) by \( D(T_1) \). Then define

\[
\Sigma(T_1, R) = [\Sigma \cup (\partial \Omega \times \{T_1\})] \cap \partial D(T_1, R).
\]

(2.6)

To prove Theorem 1, we show that \( u \leq 0 \) in \( D(0) \), then that \( u \leq 0 \) in \( \overline{D(\tau)} \), and so on. Since \( \tau > 0 \) is fixed, this allows us to conclude after finitely many steps that \( u \leq 0 \) in \( \overline{D} \).

To begin with, let \( H = H(x, t, 0, k) \). Suppose that \( (x_0, t_0) \in D(0) \). Given \( \epsilon > 0 \), hypothesis (2.5) implies that we can choose \( R \) large enough so that \( (x_0, t_0) \in D(T_0, R) \) and such that we have

\[
u(x, t) - \epsilon H(x, t, 0, k) \leq 0 \quad \text{for } |x| = R.
\]  

(2.7)
By assumption, \( u(x, t) \leq 0 \) on \( \Sigma(0, R) \), so \( u - \epsilon H \leq 0 \) there. Also, by lemma 2, we have \( L[H] \leq -1 \) in \( D(0) \), and hence in \( D(0, R) \).

Thus, (2.4) implies that
\[
L[u - \epsilon H] = L[u] - \epsilon L[H] \geq \epsilon > 0.
\]

Thus, by Theorem 1, \( u - \epsilon H \leq 0 \) or \( u \leq \epsilon H \) in \( D(0, R) \) and hence at \( (x_0, t_0) \).

Since \( \epsilon > 0 \) was arbitrary, \( u(x_0, t_0) \leq 0 \), Since \( (x_0, t_0) \) \( D(0) \) was arbitrary, \( u \leq 0 \) throughout \( D(0) \) and hence by continuity on \( D(0) \); in particular for \( t = \tau \).

The process can now be repeated in \( D(\tau) \), replacing \( H(x, t, 0, k) \) with \( H(x, t, \tau, k) \), and so on until \( D \) is exhausted, thus proving the theorem.

**Remarks.** If \( u \) satisfies
\[
I[u] \leq 0 \quad \text{in} \quad D
\]
\[
u \geq 0 \quad \text{on} \quad \Sigma.
\]

with
\[
\lim_{R \to \infty} \left( \inf_{R \to \infty} \left[ \inf_{(x, t) \in \bar{D}} \left( x^2(t) \right) \right] \exp\{ -k[\theta(1 + R^2)]^{1/n} \} \right) > 0,
\]
and the other hypotheses of Theorem 1 hold, then Theorem 1 can be applied to \( -u \) to show that \( u \geq 0 \) in \( \bar{D} \). Theorem 1 generalizes easily to weakly coupled systems. Also, if the characteristic form \( \sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \) degenerates in the proper way on part of \( \partial \Omega \times (0, T) \), then the requirement \( u \leq 0 \) of (2.4) may be replaced with the requirement \( L[u] \geq 0 \) there. The proof of Theorem 1 must then be modified by using the maximum principle of Fichera for general second order equations with nonnegative characteristic form (see [18]) in place of Lemma 1. If the hypotheses of Theorem 1 are satisfied in \( \Omega \times (0, T) \) for each \( T > 0 \), then the conclusion of the theorem holds in \( \Omega \times (0, T) \) for all \( T \) and thus in \( \Omega \times (0, \infty) \).

This section will be concerned with determining sufficient conditions so that solutions of
\[
L[u] = 0 \quad (3.1)
\]
decay as \( t \to \infty \). Let \( D = \Omega \times (0, \infty) \) where \( \Omega \) is an unbounded domain in \( \mathbb{R}^n \), and let \( \Sigma = \partial D \). The simplest case in which solutions of (3.1) with bounded initial data must decay occurs when the coefficient \( c(x, t) \) of \( L \) satisfies \( c(x, t) \leq -c_0 \) for some positive constant \( c_0 \).

**Proposition 3.** Suppose that the coefficients of \( L \) satisfy condition I in \( D \),
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with $\Theta(\eta)$ satisfying condition II, and $\Theta(\eta) \to \infty$ as $\eta \to \infty$. Suppose that there exists a positive constant $c_0$ such that $c(x, t) \leq -c_0$. If $u \in C^0(\bar{D}) \cap C^2(D)$ is a solution of (3.1) with

$$|u| \leq Me^{-c_0 t} \quad \text{on } \Sigma,$$  \hspace{1cm} (3.2)

such that for every $T > 0$ there exists a constant $k(T) \geq 1$ so that

$$\lim_{R \to \infty} \left( \max_{|x|=R, 0 \leq t \leq T} |u(x, t)| \exp\{-k(T)[\theta(1 + R^2)]^{1/2}\} \right) = 0,$$  \hspace{1cm} (3.3)

then $|u| \leq Me^{-c_0 t}$ in $\bar{D}$.

Proof. Let $v(x, t) = Me^{-c_0 t}$. Then $L[v] = (c_0 + c) v < 0$. Thus $L(u - v) \geq 0$ in $D$, by (3.1). Also, by (3.2), $u - v \leq 0$ on $\Sigma$. The hypotheses of Theorem 1 are met on each subdomain $\bar{Q} \times (0, T)$ of $D$, so by Theorem 1, $u - v \leq 0$ in each such subdomain and hence in $D$. Thus, $u(x, t) \leq v(x, t) = Me^{-c_0 t}$ in $D$. Applying the same argument to $u + v$, we find that $u(x, t) \geq -Me^{-c_0 t}$. Hence $|u| \leq Me^{-c_0 t}$.

Remarks. Hypotheses (3.2) will be satisfied if $u$ is bounded for $t = 0$ and $u = 0$ on $\partial Q \times (0, \infty)$.

The question now arises whether decay as $t \to \infty$ can be obtained for solutions of (3.1) if $c(x, t)$ is positive for some $(x, t) \in D$. This question has been investigated by Chen [5], [6], Kuroda [17], and Kusano, Kuroda, and Chen [16], [17]. The remainder of Section 3 will be devoted to a further study of this question, using techniques developed by those authors. It will be necessary to introduce the following conditions on the coefficients of the operator $L$:

**Condition IA.** There exist nonnegative constants $\mu, K_1, K_2, K_3,$ and $K_4$, with $\mu, K_1, K_3$ strictly positive, so that for $\xi \in \mathbb{R}^n$,

$$k_4 \phi(1 + r^2) |\xi|^2 \leq \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \leq K_3 \phi(1 + r^2) |\xi|^2$$

$$|b_i| \leq K_2 \phi(1 + r^2) \theta(1 + r^2)(1 + r^2)^{-1/2} \quad \text{for } i = 1, \ldots, n$$

and

$$c \leq K_4 - K_2[\theta(1 + r^2)]^{3/2} \quad \text{in } D.$$

The functions $\phi(\eta)$ and $\theta(\eta)$ are defined in the same way as in I of Section 2. Since we assume $\theta(\eta) \geq 1$ for $\eta \geq 1$, we have

$$c \leq K_4 \leq K_2[\theta(1 + r^2)]$$

if condition IA holds; so condition IA implies condition I. We also introduce the following condition:
Condition IIA. There exist nonnegative constants $m_1$, $m_2$, and $m_4$ such that for $\eta \geq 1$, we have
\[ \eta \theta''(\eta) \leq m_1 \theta(\eta) \theta'(\eta) \]
\[ \eta \theta''(\eta) \leq m_2 \theta(\eta) \theta'(\eta) \]
\[ m_4 \leq \theta'(\eta) \]
where $\mu > 0$ is the constant appearing in condition IA; and in addition, $\theta(\eta) \to \infty$ as $\eta \to \infty$.

Note that condition IIA implies condition II. We can always choose $m_4 = 0$ since by definition $\theta'(\eta) = 1/\phi(\eta) > 0$ for $\eta \geq 0$. Similarly, we have $\eta \theta'(\eta) = \eta/\phi(\eta) > 0$, so we must have $m_2 > 0$. The condition that $\theta(\eta) \to \infty$ is necessary because we will eventually want to apply the results of Section 3 to solutions $u$ of (3.1) which are bounded, but do not necessarily decay, when $t = 0$ and $|x| \to \infty$.

We will need the following lemma, due to Chen [6]:

**Lemma 4.** If $\gamma$ is the positive root of the quadratic equation $Ay^2 + By + C = 0$, with $A > 0$, $B > 0$, and $C < 0$, then the function $\rho(\tau) = \gamma \tanh(A\gamma \tau)$ satisfies the inequality $\rho'(\tau) + A\rho(\tau)^2 + B\rho(\tau) + C \leq 0$ for $\tau > 0$. Also, $\rho(\tau)$ and $\rho'(\tau)$ are nonnegative for $\tau \geq 0$.

This result is proved in [6].

We now state the main result of this section:

**Theorem 2.** Suppose

(i) that $u \in C^0(\bar{D}) \cap C^2(D)$ satisfies (3.1) in $D$,

(ii) that the coefficients of $L$ satisfy conditions IA, with $0(\eta)$ satisfying condition IIA,

(iii) that the constant $\mu$ in conditions IA and IIA satisfies $0 < \mu \leq 1$, and

(iv) that for each $T > 0$, there exists a constant $k(T) \geq 1$ such that

\[ \lim_{R \to \infty} \left( \max_{0 \leq t < T} \left| u \right| \exp \left\{-k(T)(\theta(1 + R^2))\mu \right\} \right) = 0. \]  

(3.4)

Let $\gamma$ be the positive root of the quadratic equation

\[ 4K_1m_2 \gamma^2 + 2K_1m_2 \gamma - K_3 = 0. \]  

(3.5)

Define $\rho(t) = \gamma \tanh(4K_1m_2yt)$. Suppose that either of the following is true:

Case 1. $\theta''(\eta) \geq 0$ for $\eta \geq 1$, with

\[ K_4 - 2K_1m_2 \gamma^2 < 0 \]  

(3.6)
ASYMPTOTIC BEHAVIOR

and

\[ |u| \leq M_0 \exp\{[K_4 - 2k_1ny - 4K_1m_\delta y^2] t - \rho(t) \Theta(1 + r^2)\} \]  

(3.7)
on \Sigma, for some constant \( M_0 \); or

Case 2. There exists a constant \( m_3 \geq 0 \) such that

\[-\eta \theta''(\eta) \leq m_3 \theta''(\eta) \quad \text{for} \quad \eta \geq 1 \]  

(3.8)

with

\[ K_4 + (4k_1m_3 - 2k_1n) \gamma - 4K_4y^2 < 0 \]  

(3.9)

\[ |u| \leq M_0 \exp\{[K_4 - (4k_1m_3 - 2k_1n) \gamma - 4K_1m_\delta y^2] t - \rho(t) \Theta(1 + r^2)\} \]  

(3.10)
on \Sigma, where \( M_0 \) is a constant.

Then in Case 1, \( |u| \leq \tilde{M}_0 \exp\{[K_4 - 2k_1ny - 4K_1m_\delta y^2] t\} \) in \( D \), and in Case 2,

\[ |u| \leq \tilde{M}_0 \exp\{[K_4 + (4k_1m_3 - 2k_1n) \gamma - 4K_1m_\delta y^2] t\} \]  

in \( D \)

for some constant \( \tilde{M}_0 \); so by hypothesis (3.6) or (3.9), the function \( u \) decays exponentially as \( t \to \infty \) in either case.

Proof. The proofs for Cases 1 and 2 are quite similar, so we will only discuss Case 1 in detail. Let \( v = ue^{-K_4t} \). Then \( v \) satisfies

\[ \tilde{L}[v] \equiv \sum_{i,j=1}^{n} a_{ij}v_{x_i x_j} + \sum_{t=1}^{h} b_{t}v_{x_t} + \tilde{c}v - v = 0 \]  

(3.11)

where

\[ \tilde{c} = c - K_4 \leq -K_8[\theta(1 + r^2)]^{2-\mu}. \]  

(3.12)

Now let \( w(x, t) = \exp\{-\rho(t) \Theta(1 + r^2) + \psi(t)\} \), where \( \rho(t) \) is the function defined in the statement of the theorem, and \( \psi(t) \) will be chosen later so that \( \psi(t) \leq 0 \). Computing \( \tilde{L}[w] \) and estimating the result using conditions IA and IIA, equation (3.12), the nonnegativity of \( \rho(t) \) and \( \rho'(t) \), the fact that \( \theta'' \geq 0 \), and the fact that since \( 0 < \mu < 1 \) and \( \theta \geq 1 \), we have \( \theta \leq \theta^{2-\mu} \), yields the inequality

\[ \tilde{L}[w] \leq ((4K_1m_\delta \rho(t)^2 + 2K_2n \rho(t) - K_3 + \rho'(t)) \theta^{2-\mu} - 4K_1m_\delta \rho(t)^2 - 2k_1n \rho(t) - \psi'(t)) w. \]

By the definition of \( \rho(t) \) and lemma 4, \( 4K_1m_\delta \rho(t)^2 + 2K_2n \rho(t) - K_3 + \rho'(t) \leq 0 \); hence

\[ \tilde{L}[w] \leq (-4K_1m_\delta \rho(t)^2 - 2k_1n \rho(t) - \psi'(t)) w. \]
If we now choose
\[ \psi(t) = -4K_1 m_4 \psi^2 + \frac{4K_1 m_4}{4K_1 m_2} \tanh(4K_1 m_2 t) - \frac{2k_1 \eta}{4K_1 m_2} \ln[\cosh(4K_1 m_2 t)] \]
(where \( \gamma \) is, as in the statement of the theorem, the positive root of the quadratic equation (3.5)), it follows that
\[ \psi'(t) = -4K_1 m_4 \psi^2 \tanh(4K_1 m_2 t) - 2k_1 \eta \tanh(4K_1 m_2 t) \]
\[ = -4K_1 m_4 \psi(t)^2 - 2k_1 \eta \psi(t). \]
Hence, for this choice of \( \psi(t) \), we find that
\[ \bar{\mathcal{L}}[\psi] < 0. \quad (3.13) \]
Furthermore,
\[ w(x, t) = \exp\{-\rho(t) \theta(1 + r^2) + \psi(t)\} \]
\[ \geq \exp\left[-\frac{2k_1 \eta}{4K_1 m_2} \ln[\cosh(4K_1 m_2 t)] - 4K_1 m_4 \psi^2 t - \rho(t) \theta(1 + r^2)\right] \]
\[ \geq \exp\left[-2k_1 \eta - 4K_1 m_4 \psi^2 t - \rho(t) \theta(1 + r^2)\right]. \]
By the definition of \( v \) and inequality (3.7),
\[ |v| \leq M_0 \exp\left[\left(-2k_1 \eta - 4K_1 m_4 \psi^2\right) t - \rho(t) \theta(1 + r^2)\right] \quad \text{on } \Sigma; \]

hence
\[ |v| \leq M_0 w \quad \text{on } \Sigma. \quad (3.14) \]
By (3.11), (3.13), and (3.14), we have
\[ \bar{\mathcal{L}}[v - M_0 w] \geq 0 \quad \text{in } D, \]
\[ v - M_0 w \leq 0 \quad \text{on } \Sigma. \]

It follows from (3.5) and the construction of \( w \) that \( v - M_0 w \) satisfies the growth condition (2.5) of Theorem 1 in each subdomain \( \Omega \times (0, T) \subseteq D \). Thus by Theorem 1, \( v - M_0 w \leq 0 \) or \( v \leq M_0 w \) in each such subdomain and hence in \( D \). Applying the remarks following Theorem 1 to \( v + M_0 w \) yields \( v \geq -M_0 w \) in \( D \), so that \( |v| \leq M_0 w \) in \( D \), and hence by continuity in \( \bar{D} \). By the definition of \( v \), we have
\[ |u| = |e^{K_1 t} v| \leq M_0 e^{K_1 t} w \quad \text{in } \bar{D}. \quad (3.15) \]
But
\[ w(x, t) = \exp\{-\rho(t) \theta(1 + r^2) + \psi(t)\} \]
\[ = \exp\{-4K_1m_4^2t + (m_4/m_3) \rho(t) - \rho(t) \Theta(1 + r^2)\} \]
\[ \times \exp\left[ -\frac{2k_{1n}}{4K_1m_2^2} \ln[\cosh(4K_1m_3^2t)] \right] \]
\[ \leq 2(k_{1n}/nK_{1n}m_3^2)^{1/2} \exp\{-2k_{1n} - 4K_1m_3^2t\} \]
\[ \times \exp\{(m_4/m_3 - \theta(1 + r^2) \rho(t)\}. \]

Now, \( m_4 \leq m_3 \) and \( \theta \geq 1 \), with \( \rho(t) \geq 0 \); so by (3.15) it follows that
\[ |u| \leq M_0 \exp\{(K_4 - 2k_{1n} - 4K_1m_3^2t)\} \]
\[ \times \exp\{(m_4/m_3 - \theta(1 + r^2) \rho(t)\} \]
\[ \leq M_0 \exp\{(K_4 - 2k_{1n} - 4K_1m_3^2t)\} \]

in \( D \), where \( M_0 = M_0 2^{(k_{1n}/2K_{1n}m_3^2)} \). Thus, if condition (3.6) is satisfied and \( K_4 - 2k_{1n} - 4K_1m_3^2 < 0 \), then (3.16) implies that \( u \) decays exponentially as \( t \to \infty \), which proves the theorem for Case 1. Note that since \( \rho(t) > 0 \) for \( t > 0 \), inequality (3.19) implies that \( u \) decays on the order of \( \exp\{-\rho(t) \theta(1 + r^2)\} \) as \( |x| = r \to \infty \) for \( t > 0 \).

The proof for Case 2 is essentially the same as for Case 1, but certain estimates must be made differently. In particular, we cannot simply drop the \( \theta^\sigma \) term when computing \( \bar{L}[u] \) since \( \theta^\sigma \) may be negative. However, noting that \( 4\rho(t) \sum_{i,j=1}^{n} a_{ij}x_i x_j \geq 0 \) and using inequality (3.8) allows us to make the estimate
\[ -4\rho(t) \theta^\sigma \sum_{i,j=1}^{n} a_{ij}x_i x_j \leq 4\rho(t) \frac{m_3 \theta^\sigma}{(1 + r^2)} \sum_{i,j=1}^{n} a_{ij}x_i x_j \leq 4\rho(t) K_1m_3. \]

The remaining terms in \( \bar{L}[w] \) can be estimated as for Case 1. Details are given in [7].

Remarks. Conditions (3.7) and (3.10) hold if \( |u(x, 0)| \leq M_0 \) and \( u = 0 \) on \( \partial \Omega \times (0, \infty) \). Theorem 2 generalizes decay results appearing in the articles of Chen [5], Kuroda [12, and Kusano, Kuroda, and Chen [16] and one of the decay results in Kusano, Kuroda, and Chen [17]. (The results in [17] are stated for systems, but the generalization of Theorem 2 to systems is straightforward; this is done in [7].) The result of Chen [6] and the other result of Kusano, Kuroda, and Chen [17] are not included in Theorem 2; these results treat equations such as equation (3.1); but the growth conditions on the coefficients are related to a special case of conditions I and II with \( \mu > 1 \). Theorem 2 requires \( \mu \leq 1 \), so these last results do not follow from Theorem 2.
All of these results by Kusano, Kuroda, and/or Chen require that \( \theta(\eta) \) have polynomial growth, at most. Theorem 2, however, allows us to deal with cases where, for example, \( \mu < 1 \), \( \phi(\eta) = e^{-\eta} \) and \( \theta(\eta) = e^\eta \). In this case, condition IA imposes rather strong restrictions on the coefficients of the operator \( L \); however, the growth condition (3.4) imposed on the function \( u \) becomes quite weak. Eidel’man and Porper [8] obtain estimates for solutions of an equation such as (3.1) with \( D = \mathbb{R}^n \times (0, \infty) \), under the conditions that the largest eigenvalue of the matrix \( ((a_{ij})) \) be bounded by a constant for all \( (x, t) \in D \), that

\[
\left( \sum_{i=1}^n b_i^2 \right)^{1/2} \leq K_2 f(x),
\]

and \( c \leq -\delta_0 [f(x)]^2 \) for some \( \delta_0 > 0 \), where \( f(x) > 1 \) and \( f(x) \to \infty \) as \( |x| \to \infty \). In certain cases these estimates yield decay results analogous to those of Theorem 2. However, the precise conditions for decay are rather different. In particular, if \( f(x) \) grows rapidly as \( |x| \to \infty \), the results of Eidel’man and Porper impose strong conditions on \( c \), but weak conditions on the growth of \( u \) as \( |x| \to \infty \). In their article [9], Eidel’man and Porper obtain decay results for those solutions of higher order parabolic systems which can be expressed in terms of a fundamental solution. In the second order case, these results are very similar to those obtained in the article [8].

To finish this section, we return briefly to proposition 4. Suppose that the coefficient \( c(x, t) \) of \( L \) satisfies \( c \leq m \) for some positive constant \( m \), and that \( u \) is a solution of (3.1) satisfying (3.3), with \( |u| \leq Me^{mt} \) on \( \Sigma \) for some \( M \). Let \( \epsilon \) be a positive number, and let \( v = u e^{-(m+c)t} \). Then \( L[v] = L[u] e^{-(m+c)t} + (m+c) v = (m+c) v \), so \( L[v] = 0 \) where \( L[v] = L[sv] - (m+c) sv \). Since \( c \leq m \), the coefficient of the undifferentiated term in \( \bar{L} \) is bounded above by \( -\epsilon < 0 \). Also, \( |v| \leq Me^{-\epsilon t} \) on \( \Sigma \). Thus \( v \) satisfies the hypotheses of proposition 4, so \( |v| \leq Me^{-\epsilon t} \) in \( D \); hence \( |u| \leq Me^{mt} \). Thus, no solution to (3.1) can grow faster than exponentially as \( t \to \infty \), provided \( c \) is bounded above and the boundary data grows no faster than exponentially as \( t \to \infty \).

In Section 3 we found certain sufficient conditions for the decay as \( t \to \infty \) of solutions of

\[
Lu = 0.
\]

All of these conditions required, among other things, that \( u \) be bounded in \( x \) for \( t = 0 \). In this section, we will find conditions insuring that solutions to (4.1) which may be unbounded in \( x \) for \( t = 0 \) will be bounded for \( t = T^* \) for some finite \( T^* \). As in Section 3, the approach used here is an adaptation of the techniques developed by Chen [4], Kuroda and Chen [13], Kusano [14],
ASYMPTOTIC BEHAVIOR

1151, and Kusano, Kuroda, and Chen [16], [17]. These authors investigate various properties of solutions to parabolic equations and systems as $|x| \to \infty$; here, however, we will only consider the question of boundedness for such solutions at finite values of $t$.

Let $D = \Omega \times (0, \infty)$, where $\Omega$ is an unbounded domain in $\mathbb{R}^n$. We will consider solutions of (4.1) in $D$. We introduce the following condition:

**Condition IB.** There exist nonnegative constants $\mu, K_1, K_2, K_3, K_4$, with $\mu$ and $K_3$ strictly positive, such that for $\xi \in \mathbb{R}^n$,

$$0 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i \xi_j \leq K_1(1 + r^2) |\xi|^2$$

$$|b_i| \leq K_2\phi(1 + r^2) \theta(1 + r^2) (1 + r^2)^{-1/2} \quad \text{for} \quad i = 1, \ldots, n$$

$$c \leq K_4 - K_3[\theta(1 + r^2)]^\mu \quad \text{in} \; D.$$ 

Here, $\phi(\eta)$ and $\theta(\eta)$ are defined in the same way and satisfy the same hypotheses as in conditions I and II of Section 2.

We will need the following lemma, due to Kusano [14]:

**Lemma 6.** Suppose that $a, b, c$ are nonnegative constants with $a - b - c > 0$, and that $\rho > 1$ is a parameter. Then the series

$$\sum_{\rho=1}^{\infty} \frac{\ln(\rho)}{a\rho^\mu - b - c\rho^{-\mu}}$$

converges for each $\rho > 1$, and

$$\lim_{\rho \to 1} \sum_{\rho=1}^{\infty} \frac{\ln(\rho)}{a\rho^\mu - b - c\rho^{-\mu}} = -\frac{1}{(b^2 + 4ac)^{1/2}} \ln \frac{2a - b + (b^2 + 4ac)^{1/2}}{2a - b - (b^2 + 4ac)^{1/2}}.$$

**Proof.** For any fixed $\rho > 1$, the convergence of the series

$$\sum_{\rho=1}^{\infty} \frac{\ln(\rho)}{a\rho^\mu - b - c\rho^{-\mu}}$$

follows immediately from comparison with the convergent geometric series

$$\frac{\ln(\rho)}{a - b - c} \sum_{\rho=1}^{\infty} \rho^{-1}.$$ 

For $\eta \geq 0, \rho > 1$, let $F(\eta, \rho) = \ln(\rho)/(a\rho^\mu - b - c\rho^{-\mu})$. We find by computation that

$$\lim_{\rho \to 1} \sum_{\rho=1}^{\infty} F(j, \rho) = \lim_{\rho \to 1} \int_{0}^{\infty} F(\eta, \rho) d\eta = \frac{1}{(b^2 + 4ac)^{1/2}} \ln \frac{2a - b + (b^2 + 4ac)^{1/2}}{2a - b - (b^2 + 4ac)^{1/2}},$$

which, by our definition of $F(\eta, \rho)$, is the desired result.
**Theorem 3.** Suppose that

(i) $u \in C^0(D) \cap C^0(B)$ is a solution of (4.1),

(ii) the coefficients of $L$ satisfy condition IB, with $\theta(\eta)$ satisfying condition II and $\theta(\eta) \to \infty$ as $\eta \to \infty$,

(iii) the constant $\mu$ in conditions IB and II satisfies $0 < \mu \leq 1$

(iv) there exists, for every $T > 0$, a constant $\bar{k}(T) > 1$ such that

$$\lim_{R \to \infty} \left( \frac{1}{|x|=R} \max_{0 \leq t \leq T} |u| \exp\{-\bar{k}(T)[\theta(1 + r^2)/\mu]\} \right) = 0$$

(v) there exist positive constants $M$ and $k$ such that

$$|u| \leq M \exp\{k[\theta(1 + r^2)/\mu]\} \text{ for } t = 0, \text{ and } |u| \leq M$$

on the remainder of $\Sigma$.

If the inequality

$$K_3 - 2kK_2\mu n - 4kK_1\mu m_1 - 4k^2K_1\mu^2m_2 > 0$$

is satisfied, then there exist constants $\bar{M}_0 > M$ and $T^* > 0$ such that

$$|u| \leq \bar{M}_0 \text{ for } t = T^*.$$  

**Proof.** As in the proof of Theorem 2, let $v = ue^{-K_4t}$. Then it follows from (4.1) that $v$ satisfies

$$\tilde{L}[v] = Lv - K_4v = 0 \text{ in } D.$$  

The coefficient of the undifferentiated term in $\tilde{L}$ is $c - K_4$; by condition IB, $c - K_4 \leq -K_4[\theta(1 + r^2)/\mu]$ in $D$. Let $w_0(x, t) = M \exp\{k[\theta(1 + r^2)/\mu] \rho^{-\tau_0} + [B(1 - \rho^{-\tau_0})/\tau_0 \ln(\rho)]\}$, where $\rho > 1$ is a parameter, $\tau_0 = \tau_0(\rho) > 0$ is a function of $\rho$ which will be chosen later, and $B$ is a constant independent of $\rho$ which will be chosen later. Computing $\tilde{L}[w_0]$ and estimating the result, using conditions IB and II and the fact that $0 < \mu - 1 < 0$, yields the inequality

$$\tilde{L}[w_0] \leq \left[ 4k^2K_1\mu^2m_2 + 4kK_1\mu m_1 + 2kK_2\mu n - K_3 + k\tau_0 \ln(\rho) \right] \rho^{-\tau_0}w_0.$$  

By hypothesis (4.4), $K_3 - 2kK_2\mu n - 4kK_1\mu m_1 - 4k^2K_1\mu^2m_2 > 0$, so we can choose $\tau_0 = \tau_0(\rho) = \ln(\rho)^{-1}[K_3K_1^{-1} - 2K_2\mu n - 4K_1\mu m_1 - 4K_1\mu^2m_2]$ and have $\tau_0(\rho) > 0$. If we then choose $B = (2kK_1\mu n)$, inequality (4.7) implies that

$$\tilde{L}[w_0] \leq 0 \text{ in } D.$$  

We also have, by the definitions of the functions \(w_0\) and \(v\) and hypothesis (4.3), that

\[
|v| = |u| \leq M \exp\{k[\theta(1 + r^2)]^u\} \leq w_0
\]  

(4.9)

for \(t = 0\) and \(|v| = e^{-\kappa t}u \leq M \leq w_0\) on the remainder of \(\Sigma\). Combining (4.6), (4.8) and (4.9) yields

\[
L[v - w_0] \geq 0 \quad \text{in } D
\]

\[
v - w_0 \leq 0 \quad \text{on } \Sigma.
\]

(4.10)

It follows from Theorem 1 that \(v - w_0 \leq 0\) in \(\Omega \times (0, T)\) for each fixed \(T\), so \(v - w_0 \leq 0\) in \(D\) and thus by continuity in \(\bar{D}\). We can apply the remarks following Theorem 1 to \(v + w_0\) in a similar way and conclude that \(v + w_0 \geq 0\) or \(v \geq -w_0\) in \(\bar{D}\). Thus \(|v| \leq w_0\) in \(D\). In particular, for \(t = [\tau_0(\rho)]^{-1}\), we have

\[
|v| \leq w_0 = M_2(\rho) \exp\{k\rho^{-1}[\theta(1 + r^2)]^u\}
\]

(4.11)

with \(M_2(\rho) = M \exp\{(B(1 - \rho^{-1}/\ln(\rho))[\tau_0(\rho)]^{-1}\}\). Note that \(M_2(\rho) \geq M\). Now define a function

\[
w_1(x, t) = M_2(\rho) \exp\left\{k\rho^{-1}[\theta(1 + r^2)]^u \rho^{-\tau_1(t-\tau_0^{-1})} + \frac{B\rho^{-1}(1 - \rho^{-\tau_1(t-\tau_0^{-1})})}{\tau_1 \ln(\rho)}\right\}
\]

where \(\tau_1 = \tau_1(\rho) > 0\) will be chosen later. By using conditions IB and II we can obtain estimates for \(w_1\) analogous to those made for \(w_0\). Doing so, we find that

\[
L[w_1] \leq \left\{4k^2K_1 \mu^2m_2 \rho^{-2} + 4kK_4 \mu m_1 \rho^{-1} + 2kK_2 \mu \rho^{-1} - K_3 + k\tau_1 \ln(\rho) \rho^{-1}\right\} \theta^u
\]

\[
+ \left(2kK_3 \mu \rho^{-1} - B\rho^{-1}\right) \rho^{-\tau_1(t-\tau_0^{-1})} w_1
\]

(4.12)

in \(D \cap \{t > \tau_0^{-1}\}\). If we now let

\[
\tau_1 = \tau_1(\rho) = \ln(\rho)^{-1}[K_5k^{-1}\rho - 2K_2 \mu - 4K_1 \mu m_1 - 4kK_4 \mu \rho^{-1}]
\]

then since \(\rho > 1\), we have \(\tau_1(\rho) > 0\) by hypothesis (4.4). By the choice of \(\tau_1\) and \(B\), we obtain from (4.12) the inequality

\[
\bar{L}[w_1] \leq 0 \quad \text{in } D \cap \{t > \tau_0^{-1}\}.
\]

(4.13)

By hypotheses (v) of Theorem 3 and (4.12), we have

\[
|v| \leq M_2(\rho) \exp\{k\rho^{-1}[\theta(1 + r^2)]^u\} = w_1
\]

(4.14)
for $t = \tau_0^{-1}$, and $|v| \leq M \leq M_1(\rho) \leq w_1$ on $\Sigma \cap \{t > \tau_0^{-1}\}$. From (4.6), (4.14), and (4.15) it follows that

$$L[v - w_1] \geq 0 \quad \text{in } D \cap \{t > \tau_0^{-1}\}$$

$$v - w_1 \leq 0 \quad \text{on } \Sigma \cap \{t \geq \tau_0^{-1}\} \quad \text{and for } t = \tau_0^{-1}.$$  

It follows by Theorem 1 that $v - w_1 \leq 0$ or $v \leq w_1$ in $D \cap \{t \geq \tau_0^{-1}\}$. Similarly, $v + w_1 > 0$ or $v > -w_1$ in $D \cap \{t \geq \tau_0^{-1}\}$. In particular for $t = [\tau_0(\rho)]^{-1} + [\tau_1(\rho)]^{-1}$, we have

$$|v| \leq M_0(\rho) \exp\{h_0^\rho \cdot \theta(1 + r)^u\} = w_1$$  

where

$$M_2(\rho) = M_1(\rho) \exp\left\{B_0 \left(1 - \rho^{-1}\right) \left[\tau_1(\rho)\right]^{-1}\right\}$$

$$= M \exp\left\{B(1 - \rho^{-1}) \left[\left[\tau_0(\rho)\right]^{-1} + \rho^{-1}[\tau_1(\rho)]^{-1}\right]\right\}.$$  

We may start with (4.16) instead of (4.11) and repeat the above process, replacing $w_1$ with

$$w_2(x, t) = M_2(\rho) \exp\left\{h_0^\rho \cdot \theta(1 + r)^u \cdot \rho^{-1}\left(t - \tau_0^{-1} - \tau_1^{-1}\right)\right\}$$

$$+ \frac{B_0^\rho \cdot \left(1 - \rho^{-1}\right)}{\tau_2 \ln(\rho)} \left[\left[\tau_0(\rho)\right]^{-1} + \rho^{-1}[\tau_1(\rho)]^{-1}\right].$$  

The process can be repeated as often as we wish. At the $q$th step, we find that

$$|v| \leq M_0(\rho) \exp\{h_0^\rho \cdot \theta(1 + r)^u\}$$  

in $D$ for $t = \sum_{l=0}^{q-1} [\tau_l(\rho)]^{-1}$, where for each $l$

$$\tau_l(\rho) = \ln(\rho)^{-1}[K_n^l \rho^{-1} - 2K_n^l \mu_n - 4K_n^l \mu_m - 4K_n \mu^2 m^2 \rho^{-1}] > 0$$

and $M_q(\rho) = M \exp\{B(1 - \rho^{-1})[\ln(\rho)] \sum_{l=0}^{q-1} \rho^{-1}[\tau_l(\rho)]^{-1}\}$. If we write $k_n^l \rho^{-1} = f$, $(2K_n^l \mu_n + 4K_n^l \mu_m) = g$, and $4K_n \mu^2 m^2 = h$, we have for each $l$ the inequality

$$[\tau_l(\rho)]^{-1} = \frac{\ln(\rho)}{f \rho - g - h} \leq \frac{\ln(\rho)}{f - g - h}$$  

since $\rho > 1$ and by hypothesis (4.4), $f - g - h > 0$. From (4.18) it follows that for each $q$, we have

$$\sum_{l=0}^{q-1} \rho^{-1}[\tau_l(\rho)]^{-1} \leq \frac{\ln(\rho)}{f - g - h} \sum_{l=0}^{q-1} \rho^{-1} \leq \frac{\ln(\rho)}{f - g - h} \sum_{l=0}^{\infty} \rho^{-1}$$

$$= \frac{\ln(\rho)}{f - g - h (1 - \rho^{-1})}.$$  

(4.19)
Applying (4.19) to the formula for $M_\delta(\rho)$, we obtain the bound

$$M_\delta(\rho) \leq M \exp \left\{ \frac{B(1 - \rho^{-1})}{\ln(\rho)} \cdot \frac{1}{f - g - h (1 - \rho^{-1})} \right\} = M \exp \left\{ \frac{B}{f - g - h} \right\}$$

(4.20)

Note that (4.20) gives a bound for $M_\delta(\rho)$ which has an upper bound independent of $q$ and $\rho$. Let $M_0 = M \exp[B/(f - g - h)]$. Then (4.17) and (4.20) imply that

$$|v| \leq M_0 \exp\{kp^{-q}[\theta(1 + r^2)]^a\}$$

(4.21)

in $\bar{D}$ for $t = \sum_{l=0}^{q-1}$. Since $[\tau_l(\rho)]^{-1} - \ln(\rho)/(f \rho^l - g - h \rho^{-1})$ with $f - g - h > 0$, we may apply lemma 6 and conclude that for each $\rho > 1$, the series $\sum_{l=0}^{\infty} [\tau_l(\rho)]^{-1}$ converges, and

$$\lim_{\rho \to 1} \sum_{l=0}^{\infty} [\tau_l(\rho)]^{-1} = \frac{1}{(g^2 + 4fh)^{1/2}} \ln \frac{2f - g + (g^2 + 4fh)^{1/2}}{2f - g - (g^2 + 4fh)^{1/2}}.$$  

(4.22)

Let

$$T^* = \frac{1}{(g^2 + 4fh)^{1/2}} \ln \frac{2f - g + (g^2 + 4fh)^{1/2}}{2f - g - (g^2 + 4fh)^{1/2}}.$$  

Suppose that $(x_0, T^*) \in \bar{D}$. Let $\epsilon > 0$ be any positive number. Since $v(x, t)$ is continuous in $\bar{D}$, we can choose $\rho_0 > 1$ such that if $1 < \rho < \rho_0$,

$$|v(x_0, T^*) - v(x_0, \sum_{l=0}^{\infty} [\tau_l(\rho)]^{-1})| < \epsilon/2.$$  

(4.23)

Fix $\rho$ so that $\rho_0 > \rho > 1$. Then since $v$ is continuous in $\bar{D}$ and $\sum_{l=0}^{\infty} [\tau_l(\rho)]^{-1}$ is convergent, there exists an integer $Q$ such that if $Q > Q$, $\sum_{l=0}^{Q} [\tau_l(\rho)]^{-1}$ is close enough to $\sum_{l=0}^{\infty} [\tau_l(\rho)]^{-1}$ to guarantee that

$$|v\left(x_0, \sum_{l=0}^{\infty} [\tau_l(\rho)]^{-1}\right) - v\left(x_0, \sum_{l=0}^{Q} [\tau_l(\rho)]^{-1}\right)| < \epsilon/2.$$  

(4.24)

Thus, combining (4.23) and (4.24) via the triangle inequality, we have

$$|v(x_0, T^*) - v\left(x_0, \sum_{l=0}^{Q} [\tau_l(\rho)]^{-1}\right)| < \epsilon.$$  

(4.25)

Hence,

$$|v(x_0, T^*)| \leq |v\left(x_0, \sum_{l=0}^{Q} [\tau_l(\rho)]^{-1}\right)| + \epsilon.$$  

(4.26)

Combining (4.21) with (4.26) yields

$$|v(x_0, T^*)| \leq M_0 \exp\{kp^{-q}[\theta(1 + |x_0|^2)]^a\} + \epsilon.$$  

(4.27)
for \( q > Q \). We can let \( q \to \infty \); then (4.27) yields
\[
|v(x_0, T^*)| \leq M_0 + \epsilon. \tag{4.28}
\]
Since \( \epsilon > 0 \) was arbitrary, (4.28) implies that
\[
|v(x_0, T^*)| \leq M_0. \tag{4.29}
\]
Since \( x_0 \) was arbitrary, (4.29) implies that
\[
|v| \leq M_0 \quad \text{for} \quad t = T^*. \tag{4.30}
\]

Let \( \tilde{M}_0 = M_0 e^{K_1 T^*} \). Then (4.3) and the definition of \( v \) imply that \( |u| \leq \tilde{M}_0 \) for \( t = T^* \), which is the desired result.

**Remarks.** Kusano [15] proves a theorem similar to Theorem 3, in the case \( \Omega = \mathbb{R}^n \). His result can be obtained from Theorem 3 by setting \( \mu = 1 \), \( \phi(\eta) = (\eta^{1-\mu})/\lambda \) and \( \phi(\eta) = \eta^\lambda \), \( 0 < \lambda < 1 \). Kusano, Kuroda, and Chen [16] prove a result that overlaps with Theorem 3. Kusano, Kuroda, and Chen [16] state their result for a weakly coupled system, but Theorem 3 can be generalized easily to weakly coupled systems; see [7]. Theorem 3 allows more general growth conditions than does the theorem in [16], but is not as sharp in some cases. In another article, Kusano, Kuroda, and Chen [17] prove a result similar to Theorem 3 which does not follow from Theorem 3; their result appears to be related to the case where \( \mu > 1 \), which Theorem 3 does not cover.

In [16] and [17], Kusano, Kuroda, and Chen combine pairs of theorems analogous to Theorems 2 and 3 into single theorems. This was not done here, since for \( 0 < \mu = 1 \), conditions IA and IB are distinct, with condition IB being weaker. However, when \( \mu = 1 \), conditions IA and IB are almost the same.

As was the case in previous sections, we can deal with situations where \( \theta(\eta) \) grows very rapidly by using \( 0 < \mu < 1 \); so Theorem 3 can be applied to cases in which \( \theta(\eta) \) has, for example, exponential growth as \( n \to \infty \).

Finally, note that hypothesis (iv) of Theorem 3 need only hold for \( 0 < T \leq T^* \), rather than for all \( T > 0 \). Since we can compute \( T^* \) explicitly, imposing hypothesis (iv) only for \( 0 < T \leq T^* \) truly weakens the hypotheses of Theorem 3, and thus strengthens the theorem.

In this section we apply the results of sections 3 and 4 to elliptic equations. We proceed by considering solutions of an elliptic equation as stationary or time-independent solutions of an associated parabolic equation, and applying the results obtained for parabolic equations in the previous sections. Let
\[
F[u] = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} h_i(x) u_{x_i} + r(x)u \tag{5.1}
\]
and let $L[u] \equiv F[u] - u_t$. Let $\Omega \subseteq \mathbb{R}^n$ be an unbounded domain. Assume that for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$, $\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq 0$. Suppose that $u(x)$ satisfies

$$F[u] = 0 \text{ in } \Omega. \quad (5.2)$$

Then if $D = \Omega \times (0, \infty)$, $u(x)$ satisfies

$$L[u] = F[u] - u_t = 0 \text{ in } D. \quad (5.3)$$

Note that conditions IA and II make sense for the coefficients of $F$. We have the following:

**Theorem 4.** Suppose that

(i) $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ is a solution of (5.2),
(ii) $u = 0$ on $\partial \Omega$
(iii) the coefficients of the operator $F$ defined in (5.1) satisfy condition IA of Section 3, with $\theta(\eta)$ satisfying condition IIA,
(iv) the constant $\mu$ in conditions IA and IIA satisfies $0 < \mu \leq 1$, and
(v) there exist positive constants $M$ and $k$ such that

$$|u| \leq M \exp\{k[\theta(1 + r^2)]^{1/2}\} \text{ in } \Omega, \quad (5.4)$$

with $k$ satisfying

$$K_3 - 2kK_2m_2 - 4kK_1m_1 - 4k^2K_1\mu^2m_2 > 0 \quad (5.5)$$

(where $K_1$, $K_2$, $K_3$, $\mu$, $m_1$, and $m_2$ are the constants in conditions IA and IIA).

Let $\gamma$ be the positive root of the quadratic equation

$$4K_1m_2\gamma^4 + 2K_1\gamma - K_3 = 0. \quad (5.6)$$

Suppose that either of the following is true:

**Case 1.** $\theta''(\eta) \geq 0$ for $\eta \geq 1$, with

$$K_4 - 2k_1\gamma - 4K_1m_2\gamma^2 < 0 \quad (5.7)$$

or

**Case 2.** there exists a constant $m_3 \geq 0$ such that

$$-\eta\theta''(\eta) \leq m_3 \theta'(\eta) \text{ for } \eta \geq 1 \quad (5.8)$$

with

$$K_4 + (4K_1m_3 - 2k_1n)\gamma - 4K_1m_2\gamma^2 < 0. \quad (5.9)$$

Then $u \equiv 0$ in $\overline{\Omega}$. 

505/35/3-11
Proof. Since $u(x)$ satisfies (5.2) in $Q$, it also satisfies (5.3) in $D = Q \times (0, \infty)$. Conditions IA and IIA imply conditions I, II, and IB for $0 < \mu < 1$, and condition IIA implies that $\theta(\eta) \to \infty$ as $\eta \to \infty$. Hence, it follows that $u(x)$ satisfies the hypotheses of Theorem 3. Thus, since $u(x)$ is time independent there exists a constant $M_0$ such that

$$\| u(x) \| \leq M_0. \tag{5.10}$$

But the remaining hypotheses of Theorem 4 imply that the hypotheses of Theorem 2 are satisfied, so since (5.10) holds, $| u(x) |$ must decay exponentially to zero as $t \to \infty$. Since $| u(x) |$ is independent of $t$, it follows that $u(x) = 0$ for any $x \in Q$, so $u \equiv 0$.

Remarks. Theorem 4 can be extended to weakly coupled systems of elliptic equations; see [7]. Besala and Ugowski [1] consider elliptic systems of the form

$$F^{\alpha}[u^2] + f^{\alpha}(x, u^1, \ldots, u^N) = 0, \quad \alpha = 1, \ldots, N$$

with the functions $f^{\alpha}$ satisfying a Lipschitz-type condition. They obtain results similar to Theorem 4 in the case where $\theta(\eta) = \eta^{1-\lambda}$, $0 \leq \lambda \leq 1$. However, they assume conditions on the functions $f^{\alpha}$ which reduce to $c(x) \leq -M < 0$ in the case of a single linear equation such as (5.2), whereas in Theorem 4 we require (by condition IA) that $c(x) \leq K_4 - K_5[\theta(1 + r^2)]^{2-\mu}$; thus in Theorem 4, we may have $c(x) > 0$ at some points of $Q$. Besala and Ugowski do not require conditions similar to (5.17) and (5.9), but they do require conditions analogous to (5.4) and (5.5). However, in the case $\phi(\eta) = \eta^{1-\lambda}$, $0 < \lambda < 1$, condition (5.4) with $\mu = 1$ requires that $| u | \leq M \exp[k(1 + r^2)]$ where $k$ satisfies (5.5); Besala and Ugowski, on the other hand, require that

$$\int_{\Omega} | u^\alpha | \exp\{-k_0(1 + r^2)^{1/2}\} < \infty,$$

where $k_0$ satisfies a condition similar to but distinct from (5.5). The results of Besala and Ugowski are obtained by methods different from ours; they have a similar general flavor, but differ considerably in details. Theorem 4 can be applied in situations where the growth conditions of (5.4) are rather weak; for example, if $\phi(\eta) = e^{-n}$ and $\mu = 1/2$, we need only require $| u | \leq M \exp[k(\exp(1 + r^2)^{1/2})$, assuming the other conditions of Theorem 4 are satisfied. Eidel'man and Porper [8] prove results similar to Theorems 2 and 3 and use them to obtain results analogous to Theorem 4; however, they treat the case of an equation where $| a_{ij} | < M_1 < \infty$, $| b_i | < M_2 f(x)$, and $c < -\delta[f(x)]^2$, where $f(x) \to \infty$ as $x \to \infty$, and $M_1$, $M_2$, and $\delta$ are positive constants. Their results overlap somewhat with Theorem 4, but are mainly concerned with situations where the solution and coefficients of the equation satisfy growth conditions different from those of Theorem 4. In another article, Eidel'man and Porper [9] obtain
results similar to those of [8] for those solutions of certain higher order parabolic systems which are representable in terms of a fundamental solution, which can also be used to obtain results analogous to Theorem 4.

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