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Solution of nonlinear Volterra–Fredholm–Hammerstein integral equations via a collocation method and rationalized Haar functions

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Abstract

Rationalized Haar functions are developed to approximate the solution of the nonlinear Volterra–Fredholm–Hammerstein integral equations. The properties of rationalized Haar functions are first presented. These properties together with the Newton–Cotes nodes and Newton–Cotes integration method are then utilized to reduce the solution of Volterra–Fredholm–Hammerstein integral equations to the solution of algebraic equations. The method is computationally attractive, and applications are demonstrated through illustrative examples.

Keywords: Rationalized Haar; Newton–Cotes; Collocation method; Volterra–Fredholm–Hammerstein; Nonlinear integral equations

1. Introduction

Several numerical methods for approximating the solution of Hammerstein integral equations are known. For Fredholm–Hammerstein integral equations, the classical method of successive approximations was introduced in [1]. A variation of the Nystrom method was presented in [2]. A collocation type method was developed in [3]. In [4], Brunner applied a collocation-type method to nonlinear Volterra–Hammerstein integral equations and integro-differential equations, and discussed its connection with the iterated collocation method. Guoqiang [5] introduced and discussed the asymptotic error expansion of a collocation-type method for Volterra–Hammerstein integral equations.

The orthogonal set of Haar functions is a group of square waves [6] with magnitude of $+2^{\frac{i}{2}}$, $-2^{\frac{i}{2}}$, and 0, $i = 0, 1, 2, \ldots$. The use of Haar functions comes from the rapid convergence feature of Haar series in expansions of functions compared with that of Walsh series [7]. In [8–10] the authors offered a numerical method for solving linear differential equations and its application to function evaluation. The method used was based on the stair step approximation using Haar functions and on mathematical manipulation using quasi-binary numbers. However, there are some difficulties for practical use in [8–10]. This is because of the magnitude of the Haar functions, and an

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operating system dealing with quasi-binary numbers is required for speedy manipulation. Lynch and Reis [11] have rationalized the Haar transform by deleting the irrational numbers and introducing the integral power of two. This modification results in what is called the rationalized Haar (RH) transform. The RH transform preserves all the properties of the original Haar transform and can be efficiently implemented using digital pipeline architecture [12]. The corresponding functions are known as RH functions. The RH functions are composed of only three amplitudes $+1$, -1 and 0. In [13–15], the authors applied the RH functions to solve differential equations.

In [16], Yalcinbas used Taylor series to the following nonlinear Volterra–Fredholm integral equation:

$$y(t) = f(t) + \lambda_1 \int_0^t \kappa_1(t,s)[y(s)]^p \mathrm{d}s + \lambda_2 \int_0^1 \kappa_2(t,s)[y(s)]^q \mathrm{d}s, \quad 0 \le t, s \le 1,$$

where p and q are nonnegative integers and λ_1 and λ_2 are constants. Moreover, $f(t)$ and the kernels $\kappa_1(t,s)$ and $\kappa_2(t,s)$ are assumed to have nth derivatives on the interval $0 \le t, s \le 1$. In the present work, we are concerned with the application of RH functions to the numerical solution of a nonlinear Volterra–Fredholm–Hammerstein integral equation of the form

$$y(t) = f(t) + \lambda_1 \int_0^t \kappa_1(t,s) g_1(s, y(s)) \mathrm{d}s + \lambda_2 \int_0^1 \kappa_2(t,s) g_2(s, y(s)) \mathrm{d}s, \quad 0 \le t, s \le 1, \tag{1}$$

where $f(t)$ and the kernels $\kappa_1(t,s)$ and $\kappa_2(t,s)$ are assumed to be in $L^2(R)$ on the interval $0 \le t, s \le 1$. We assume that Eq. (1) has a unique solution $y(t)$ to be determined. The method consists of expanding the solution in RH functions with unknown coefficients. The properties of RH functions together with the Newton–Cotes nodes and Newton–Cotes integration method [17] are then utilized to evaluate the unknown coefficients and find an approximate solution to Eq. (1). It is known that spectral projection methods provide highly accurate approximations for the solutions of operator equations in function spaces, provided that these solutions are sufficiently smooth [18]. Moreover, the uniform convergence under suitable conditions using the spectral methods is established in [19] and [20] for nonlinear Fredholm–Hammerstein [19] and nonlinear Volterra–Hammerstein [20] integral equations.

This work is organized as follows. In Section 2, we describe the basic formulation of RH functions required for our subsequent development. Section 3 is devoted to the solution of Eq. (1) by using RH functions. In Section 4, we report our numerical findings and demonstrate the accuracy of the proposed scheme by considering numerical examples.

2. Properties of rationalized Haar functions

2.1. Rationalized Haar functions

The RH function $\mathrm{RH}(r,t)$, $r = 1, 2, 3, \ldots$, are composed of three values $+1$, -1 and 0 and can be defined on the interval $[0, 1)$ as [15]

$$\mathrm{RH}(r,t) = \begin{cases} 1, & J_1 \le t < J_{1/2} \\ -1, & J_{1/2} \le t < J_0 \\ 0, & \text{otherwise} \end{cases}$$

where

$$J_u = \frac{j-u}{2^i}, \quad u = 0, \frac{1}{2}, 1.$$

The value of r is defined by two parameters i and j as

$$r = 2^i + j - 1, \quad i = 0, 1, 2, 3, \ldots \; j = 1, 2, 3, \ldots, 2^i.$$

$\mathrm{RH}(0,t)$ is defined for $i = j = 0$ and is given by

$$\mathrm{RH}(0,t) = 1, \quad 0 \le t < 1.$$

The orthogonality property is given by

$$\int_0^1 \mathrm{RH}(r,t)\mathrm{RH}(v,t)\mathrm{d}t = \begin{cases} 2^{-i}, & r = v \\ 0, & r \ne v \end{cases}$$

where

$$v = 2^n + m - 1, \quad n = 0, 1, 2, 3, \ldots, \; m = 1, 2, 3, \ldots, 2^n.$$

2.2. Function approximation

A function $f(t)$ defined over the interval $[0, 1)$ may be expanded in RH functions as

$$f(t) = \sum_{r=0}^{+\infty} a_r \mathrm{RH}(r, t), \tag{2}$$

where a_r are given by

$$a_r = 2^i \int_0^1 f(t)\mathrm{RH}(r, t)\mathrm{d}t, \quad r = 0, 1, 2, \ldots,$$

with $r = 2^i + j - 1, i = 0, 1, 2, 3, \ldots, j = 1, 2, 3, \ldots, 2^i$ and $r = 0$ for $i = j = 0$. The series in Eq. (2) contains infinite terms. If, we let $i = 0, 1, 2, \ldots, \alpha$, then the infinite series in Eq. (2) is truncated to its first k terms as

$$f(t) = \sum_{r=0}^{k-1} a_r \mathrm{RH}(r, t) = A^{\mathrm{T}} \Phi(t),$$

where

$$k = 2^{\alpha+1}, \quad \alpha = 0, 1, 2, 3, \ldots.$$

The RH function coefficient vector A and RH function vector $\Phi(t)$ are defined as

$$A = [a_0, a_1, \ldots, a_{k-1}]^{\mathrm{T}}, \qquad \Phi(t) = [\phi_0(t), \phi_1(t), \ldots, \phi_{k-1}(t)]^{\mathrm{T}}, \tag{3}$$

where

$$\phi_r(t) = \mathrm{RH}(r, t), \quad r = 0, 1, 2, \ldots, k-1.$$

3. Nonlinear Volterra–Fredholm–Hammerstein integral equations

Consider the nonlinear Volterra–Fredholm–Hammerstein integral equations given in Eq. (1). In order to use RH functions, we first approximate $y(t)$ as

$$y(t) = A^{\mathrm{T}} \Phi(t), \tag{4}$$

where A and $\Phi(t)$ are defined like in Eq. (3). Then from Eqs. (1) and (4) we have

$$A^{\mathrm{T}} \Phi(t) = f(t) + \lambda_1 \int_0^t \kappa_1(t, s) g_1(s, A^{\mathrm{T}} \Phi(s))\mathrm{d}s + \lambda_2 \int_0^1 \kappa_2(t, s) g_2(s, A^{\mathrm{T}} \Phi(s))\mathrm{d}s, \tag{5}$$

and we now collocate Eq. (5) at k points t_p as

$$A^{\mathrm{T}} \Phi(t_p) = f(t_p) + \lambda_1 \int_0^{t_p} \kappa_1(t_p, s) g_1(s, A^{\mathrm{T}} \Phi(s))\mathrm{d}s + \lambda_2 \int_0^1 \kappa_2(t_p, s) g_2(s, A^{\mathrm{T}} \Phi(s))\mathrm{d}s. \tag{6}$$

For a suitable collocation points we choose Newton–Cotes nodes as

$$t_p = \frac{2p - 1}{2k}, \quad p = 1, 2, 3, \ldots, k.$$

In order to use the Newton–Cotes integration formula for Eq. (6), we transfer the k interval $[0, t_p]$ to interval $[0, 1]$ by means of the transformation $s = t_p \tau$, letting

$$\zeta_1(t_p, s) = \kappa_1(t_p, s) g_1(s, A^{\mathrm{T}} \Phi(s)), \qquad \zeta_2(t_p, s) = \kappa_2(t_p, s) g_2(s, A^{\mathrm{T}} \Phi(s)).$$

Eq. (6) may then be restated as

$$A^{\mathrm{T}}\Phi(t_p) = f(t_p) + \lambda_1 t_p \int_0^1 \zeta_1(t_p, t_p\tau)\mathrm{d}\tau + \lambda_2 \int_0^1 \zeta_2(t_p, \tau)\mathrm{d}\tau.$$

By using the Newton–Cotes integration formula we get

$$A^{\mathrm{T}}\Phi(t_p) = f(t_p) + \lambda_1 t_p \sum_{j=1}^{k_1} \omega_{1j}\zeta_1(t_p, t_p\tau_{1j}) + \lambda_2 \sum_{j=1}^{k_2} \omega_{2j}\zeta_2(t_p, \tau_{2j}), \quad p = 1, 2, \ldots, k, \tag{7}$$

where τ_{1j} and τ_{2j} are k_1 and k_2 Newton–Cotes nodes respectively in interval $[0, 1)$, and w_{1j}, w_{2j} are the corresponding weights given in [17]. Eq. (7) gives k nonlinear equations which can be solved for the elements of A in Eq. (4) using Newton's iterative method.

4. Illustrative examples

4.1. Example 1

In this example RH function approximation is used to solve the integral equation reformulation of the nonlinear two-point boundary value problem

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - \mathrm{e}^{y(t)} = 0, \quad t \in [0, 1]; \; y(0) = y(1) = 0, \tag{8}$$

which is of great interest in hydrodynamics [21, p. 41]. This problem has a unique solution given in [8] as

$$y(t) = -\ln(2) + \ln\left(\frac{c}{\cos(\frac{1}{2}c(t - \frac{1}{2}))}\right).$$

Here, c is the root of $[\frac{c}{\cos(\frac{c}{4})}]^2 = 2$. Eq. (8) can be reformulated as the integral equation

$$y(t) = \int_0^1 \kappa(t, s)\mathrm{e}^{y(s)}\mathrm{d}s, \tag{9}$$

where

$$\kappa(t, s) = \begin{cases} -s(1 - t), & \text{for } s \le t \\ -t(1 - s), & \text{for } t \le s. \end{cases}$$

We applied the proposed method to approximate the solution of Eq. (9) with $\lambda_1 = 0, \lambda_2 = 1, k_1 = 8$, and $k = 8, 16$ and 32. In [3], the above problem was solved with the collocation points chosen to be $t_i = \frac{i-1}{N-1}, i = 1, \ldots, N$, and the basis functions as piecewise-linear functions, in which a rather large system of nonlinear equations have to be solved to obtain accuracy of comparable order. Table 1 represents the error estimates using the method of [3] together with the results obtained for maximum errors by the present method.

4.2. Example 2

Consider the nonlinear Volterra–Fredholm integral equation given in [16] by

$$y(t) = \exp(t) - \frac{1}{3}\exp(3t) + \frac{1}{3} + \int_0^t [y(s)]^3\mathrm{d}s, \tag{10}$$

which has the exact solution $y(t) = \exp(t)$. We applied the RH function approach and solved Eq. (10). Table 2 presents values of $y(t)$ obtained using the present method with $k_2 = 8$, and $k = 8$ and $k = 16$ together with the exact values.

Table 1
Error estimates for Example 1

Methods	$\|y - \hat{y}\|$
Method of [3]	
$N = 5$	7.81×10^{-3}
$N = 17$	5.61×10^{-4}
$N = 65$	3.66×10^{-5}
Present method,	
$k_1 = 8$, and	
$k = 8$	$<10^{-5}$
$k = 16$	$<10^{-6}$
$k = 32$	$<10^{-7}$

Table 2
Estimated and exact values of $y(t)$

t	Present $k_2 = 8$ and $k = 8$	Present $k_2 = 8$ and $k = 16$	Exact
0	1	1	1
0.2	1.221341	1.221400	1.221403
0.4	1.491758	1.491821	1.491825
0.6	1.822086	1.822116	1.822119
0.8	2.225467	2.225538	2.225541
1	2.718199	2.718278	2.718282

Table 3
Estimated and exact values of $y(t)$

t	Present $k_1 = k_2 = 8$ and $k = 8$	Present $k_1 = k_2 = 8$ and $k = 16$	Exact
0	−1.998431	−1.999992	−2
0.2	−1.958019	−1.959996	−1.96
0.4	−1.839764	−1.839989	−1.84
0.6	−1.640501	−1.640013	− 1.64
0.8	−1.360513	−1.360014	− 1.36
1	−1.005901	−1.000014	− 1

4.3. *Example 3*

Consider the nonlinear Volterra–Fredholm–Hammerstein integral equation given in [16] by

$$y(t) = \frac{-1}{30}t^6 + \frac{1}{3}t^4 - t^2 + \frac{5}{3}t - \frac{5}{4} + \int_0^t (t-s)[y(s)]^2 \mathrm{d}s + \int_0^1 (t+s)[y(s)]\mathrm{d}s, \quad 0 \le t, s \le 1. \tag{11}$$

We applied the RH function method and solved Eq. (11) with $k_1 = k_2 = 8$, and $k = 8$ and $k = 16$. The computational results together with the exact solution $y(t) = t^2 - 2$ are given in Table 3.

5. Conclusion

In the present work the RH functions are developed to solve the nonlinear Volterra–Fredholm–Hammerstein integral equations. The properties of the RH functions together with Newton–Cotes nodes and the Newton–Cotes integration method are used to reduce the problem to the solution of algebraic equations. Illustrative examples are given to demonstrate the validity and applicability of the proposed method.

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