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Note

P-selectivity: Intersections and indices

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Abstract

The P-selective sets (Selman, 1979) are those sets for which there is a polynomial-time algorithm that, given any two strings, determines which is "more likely" to belong to the set: if either of the strings is in the set, the algorithm chooses one that is in the set. We prove that, for each k, the k-ary Boolean connectives under which the P-selective sets are closed are exactly those that are either completely degenerate or almost-completely degenerate. We determine the complexity of the index set of the r.e. P-selective sets – Σ_3^0-complete.

1. Introduction and definitions

Selman [17] defined the P-selective sets, the complexity-theoretic analogs of Jockush's [11] semi-recursive sets. A set is P-selective if it has a polynomial-time "selector" ("semi-decision") function that determines, given any two strings, one that is logically no less likely to belong to the set.

Definition 1.1 (*Selman* [17]). A set L is P-selective if there is a (total) polynomial-time function $f(\cdot,\cdot)$ such that $(\forall x, y \in \Sigma^*)$ $[f(x,y) = x \vee f(x,y) = y]$ and $(\forall x, y \in \Sigma^*)$ $[(x \in L \vee y \in L) \Rightarrow f(x,y) \in L]$. We will refer to $f(\cdot,\cdot)$ as a *selector function* for L. We will use P-Sel to refer to the class of P-selective sets.

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After the P-selective sets were defined in 1979, for just over half a decade rapid progress was made in their study [12, 13, 17–19]. There followed, for no particularly clear reason, a half decade during which little progress was made in the study of P-selectivity. However, during the past few years, brisk progress has resumed, and many open issues regarding the P-selective sets have been settled. For example, resolving issues open since Selman's seminal papers, Buhrman et al. [3] proved that a set is in P if and only if it is both P-selective and Turing self-reducible, and Buhrman et al. [2] established that the P-selective sets are closed downwards under positive Turing reductions. Many other recent papers (see the survey [4]) have studied the properties of the P-selective sets and of such related classes as the NP-selective sets, and have shown, for example, that NP cannot have a bounded-truth-table-complete P-selective set unless P = NP ([1], [15] and the paper by Agrawal and Arvind in the same proceedings as [1]), and that if multivalued NP functions have single-valued refinements then the polynomial hierarchy collapses [10].

In this paper, we study two issues regarding the P-selective sets: Boolean closure properties and index set complexity. A *k-ary Boolean connective* is a function $I:(2^{\Sigma^*})^k \to 2^{\Sigma^*}$ satisfying $(\exists f_I:\{0,1\}^k \to \{0,1\})$ $(\forall B_1,\dots,B_k \subseteq \Sigma^*)$ $(\forall x \in \Sigma^*)$ $[x \in I(B_1,\dots,B_k) \Leftrightarrow 1 = f_I(\chi_{B_1}(x),\dots,\chi_{B_k}(x))]$, where χ_C represents, for each set C, the characteristic function of C. For example, intersection $\bigcap(B_1,B_2) =_{\mathrm{def}} B_1 \cap B_2$, is one of the sixteen 2-ary Boolean connectives. Since each I has a unique f_I, we may speak of the function f_I as if it were the connective itself. We say a k-ary Boolean connective I is *completely degenerate* if f_I is a constant function, and we say a k-ary Boolean connective is *almost-completely degenerate* if there is a j, $1 \leqslant j \leqslant k$, such that the two $(k-1)$-ary Boolean connectives $f_I(x_1,x_2,\dots,x_{j-1},0,x_{j+1},\dots,x_k)$ and $f_I(x_1,x_2,\dots,x_{j-1},1,x_{j+1},\dots,x_k)$ are both completely degenerate. The P-selective sets are said to be *closed under k-ary Boolean connective I* if I is a k-ary Boolean connective and $(\forall B_1,\dots,B_k \in \text{P-Sel})$ $[I(B_1,\dots,B_k) \in \text{P-Sel}]$. For each k, we prove that, among the 2^{2^k} k-ary Boolean connectives, the P-selective sets are closed under exactly the $2k+2$ that are completely or almost-completely degenerate.

An important issue in recursive function theory is the index complexity of classes [16, 20]. We determine that the index set of the r.e. P-selective sets, $I_{\text{P-Sel}} =_{\mathrm{def}} \{M \mid L(M) \text{ is P-selective}\}$, is Σ_3^0-complete. That is, determining whether Turing machines' languages are recursive and determining whether Turing machines' languages are P-selective are, in recursion-theoretic terms, equally hard.

2. Results

2.1. Intersections and other Boolean closures

For each k, we completely characterize the k-ary Boolean connectives under which the P-selective sets are closed. Our construction uses the technique of spacing a set so widely that smaller strings can be brute-forced, a technique dating back to an early paper of Kurtz [14].

Lemma 2.1. *The* P-*selective sets are not closed under intersection.*

Proof. Define $\mu(0) = 2$, $\mu(i+1) = 2^{2^{\mu(i)}}$ for each $i \geqslant 0$, and $R_k = \{i \mid i \in N \wedge \mu(k) \leqslant i < \mu(k+1)\}$. We will implicitly use the standard correspondence between Σ^* and N. We define two special classes of languages:

$$\mathscr{C}_1 = \{A \subseteq N \mid (\forall j \geqslant 0)[R_{2j} \cap A = \emptyset] \wedge (\forall j \geqslant 0)(\forall x, y \in R_{2j+1}) [(x \leqslant y \wedge x \in A) \Rightarrow y \in A]\}.$$

$$\mathscr{C}_2 = \{A \subseteq N \mid (\forall j \geqslant 0)[R_{2j} \cap A = \emptyset] \wedge (\forall j \geqslant 0)(\forall x, y \in R_{2j+1}) [(x \leqslant y \wedge y \in A) \Rightarrow x \in A]\}.$$

As is standard, E will denote $\bigcup_{c \geqslant 0} \mathrm{DTIME}[2^{cn}]$. We claim that

$$\mathscr{C}_1 \cap \mathrm{E} \subseteq \text{P-Sel}. \qquad (*)$$

To prove $(*)$, consider an arbitrary set A in $\mathscr{C}_1 \cap \mathrm{E}$. We define a selector function f as follows: (i) if $x, y \in R_{2j+1}$ for some j, then let $f(x,y) = \max\{x, y\}$; (ii) if x (or y, or both) is in R_{2j} for some j, then let $f(x,y) = y$ (or x, or x); (iii) if $x \in R_{2j+1}$, $y \in R_{2j'+1}$ for some j, j' such that $j < j'$, then decide whether $x \in A$, and if $x \in A$ then let $f(x,y) = x$ else let $f(x,y) = y$; (iv) if $y \in R_{2j+1}$, $x \in R_{2j'+1}$, for some j, j' such that $j < j'$ then decide whether $y \in A$, and if $y \in A$ then let $f(x,y) = y$ else let $f(x,y) = x$. Note that, as $A \in \mathrm{E}$ and – in cases (iii) and (iv) – $|y| \geqslant 2^{2^{|x|}}$ (or $|x| \geqslant 2^{2^{|y|}}$), in these cases we indeed can easily decide by brute force whether $x \in A$ (or $y \in A$). So it is not hard to see that f is computable in time polynomial in $\max\{|x|, |y|\}$.

For the same reason, $\mathscr{C}_2 \cap \mathrm{E} \subseteq$ P-Sel.

Let $\{M_i\}_{i \in N}$ be a standard enumeration of deterministic polynomial-time Turing machines. Without loss of generality, we may assume this enumeration has the property that for each $j \in N$ the running time of $M_j(z_1, z_2)$ is less than $2^{\max\{|z_1|, |z_2|\}}$ for all $z_1, z_2 \in R_{2j+1}$.

For each $j \geqslant 0$, define $w_{j,a} = \mu(2j+1)$, $w_{j,b} = 1 + \mu(2j+1)$, and

$$B = \{i \mid (\exists j)[i \in R_{2j+1} \wedge (i \geqslant w_{j,b} \vee (i \geqslant w_{j,a} \wedge M_j(w_{j,a}, w_{j,b}) = w_{j,b}))]\};$$

$$D = \{i \mid (\exists j)[i \in R_{2j+1} \wedge (i \leqslant w_{j,a} \vee (i \leqslant w_{j,b} \wedge M_j(w_{j,a}, w_{j,b}) \neq w_{j,b}))]\}.$$

Clearly $B \in \mathscr{C}_1 \cap \mathrm{E}$ and $D \in \mathscr{C}_2 \cap \mathrm{E}$, so B and D are P-selective. Now we prove that $B \cap D$ is not a P-selective set. By the way of contradiction, suppose that $B \cap D$ is P-selective, and suppose M_{j_0} computes a P-selector function for $B \cap D$. By the definition of B and D, for each j, $\{w_{j,a}, w_{j,b}\} \cap (B \cap D) \neq \emptyset$. However, if $w_{j_0,a} \in B \cap D$, then by the definition of B, $M_{j_0}(w_{j_0,a}, w_{j_0,b})$ must be $w_{j_0,b}$, but by the definition of D, this implies $w_{j_0,b} \notin D$. So $w_{j_0,b} \notin B \cap D$. Thus we have $w_{j_0,a} \in B \cap D$, $w_{j_0,b} \notin B \cap D$, and $M_{j_0}(w_{j_0,a}, w_{j_0,b}) = w_{j_0,b}$, which contradicts the assumption that M_{j_0} is a P-selector function for $B \cap D$. On the other hand, if $w_{j_0,b} \in B \cap D$, then by the definition of D, $M_{j_0}(w_{j_0,a}, w_{j_0,b})$ must be $w_{j_0,a}$, but by the definition of B, this implies $w_{j_0,a} \notin B$. And so

$w_{j_0,a} \notin B \cap D$, which similarly is a contradiction. Hence $B \cap D$ is not a P-selective set. □

Corollary 2.2. *The* P-*selective sets are not closed under the following eight* 2-*ary Boolean connectives:* $L_1 \cap L_2$, $\overline{L_1} \cap L_2$, $L_1 \cap \overline{L_2}$, $\overline{L_1} \cap \overline{L_2}$, $L_1 \cup L_2$, $\overline{L_1} \cup L_2$, $L_1 \cup \overline{L_2}$, $\overline{L_1} \cup \overline{L_2}$.

Proof. Follows easily from the standard fact that a language L is P-selective if and only if its complement is P-selective. □

Lemma 2.3. *The* P-*selective sets are not closed under* $NXOR$ *or* XOR (i.e., $(L_1 \cap L_2) \cup (\overline{L_1} \cap \overline{L_2})$ or $(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2)$).

Proof. Let B and D be the same as in Lemma 2.1. Note that $BNXORD = (B \cap D) \dot{\cup} (\bigcup_{k \geqslant 0} R_{2k})$, where $\dot{\cup}$ denotes that the two operands being unioned happen to be disjoint. By a proof analogous to that of Lemma 2.1 (except that all R_{2j} strings will be in all $\mathscr{C}_1$ and $\mathscr{C}_2$ sets), it is easy to see $BNXORD$ is not P-selective either. The XOR case follows immediately from this. □

From the above proofs, we easily have the following claim.

Theorem 2.4. *The* P-*selective sets are closed under exactly six* 2-*ary Boolean connectives, namely, the completely degenerate and almost-completely degenerate connectives.*

This concludes our discussion of the closure properties of the P-selective sets under 2-ary Boolean connectives. Now we discuss the general case – the closure properties of the P-selective sets under k-ary Boolean connectives.

By the definition of k-ary Boolean connective, we know for k languages $L_1, L_2, \ldots, L_k$, $I(L_1, \ldots, L_k) = \{x \mid (\exists \alpha_1, \ldots, \alpha_k)\ [f_I(\alpha_1, \ldots, \alpha_k) = 1 \wedge x \in L_1^{\alpha_1} \cap \cdots \cap L_k^{\alpha_k}]\}$, where $\alpha_1, \ldots, \alpha_k$ are Boolean valued, $L^\alpha = L$ if $\alpha = 1$, and $L^\alpha = \bar{L}$ if $\alpha = 0$.

Theorem 2.5. *If I is a k-ary Boolean connective, then the* P-*selective sets are closed under I if and only if I is either completely degenerate or almost-completely degenerate.*

Proof. By the definition of being completely degenerate and being almost-completely degenerate, we just need to prove the following claim.

Claim. *The* P-*selective sets are closed under a k-ary Boolean connective f if and only if one of the following conditions holds:*

(1) $(\forall \alpha_1 \cdots \alpha_k)\ [f(\alpha_1, \ldots, \alpha_k) = 1]$;

(2) $(\forall \alpha_1 \cdots \alpha_k)\ [f(\alpha_1, \ldots, \alpha_k) = 0]$;

(3) $(\exists i)\,(\forall \alpha_1 \cdots \alpha_{i-1}, \alpha_{i+1} \cdots \alpha_k)\,[f(\alpha_1, \ldots, \alpha_k) = \alpha_i]$;
(4) $(\exists i)\,(\forall \alpha_1 \cdots \alpha_{i-1}, \alpha_{i+1} \cdots \alpha_k)\,[f(\alpha_1, \ldots, \alpha_k) = \bar{\alpha}_i]$.

For $k = 1$, we know P-selective sets are closed under identity and complement, so the claim holds. For $k = 2$, the above claim is proven as Theorem 2.4.

Suppose f is a $(k + 1)$-ary Boolean connective. The "if" direction of the claim is trivial. Now we prove that the "only if" direction holds. Assume f does not satisfy any of the following conditions:

(5) $(\forall \alpha_1 \cdots \alpha_{k+1})\,[f(\alpha_1, \ldots, \alpha_{k+1}) = 1]$;
(6) $(\forall \alpha_1 \cdots \alpha_{k+1})\,[f(\alpha_1, \ldots, \alpha_{k+1}) = 0]$;
(7) $(\exists i)\,(\forall \alpha_1 \cdots \alpha_{i-1}, \alpha_{i+1} \cdots \alpha_{k+1})\,[f(\alpha_1, \ldots, \alpha_{k+1}) = \alpha_i]$;
(8) $(\exists i)\,(\forall \alpha_1 \cdots \alpha_{i-1}, \alpha_{i+1} \cdots \alpha_{k+1})\,[f(\alpha_1, \ldots, \alpha_{k+1}) = \bar{\alpha}_i]$.

Define $f_{i,\alpha_i}(\alpha_1 \cdots \alpha_{i-1}, \alpha_{i+1} \cdots \alpha_{k+1}) = f(\alpha_1, \ldots, \alpha_{k+1})$, and note that for each i and α_i, it holds that f_{i,α_i} is a k-ary Boolean connective. We will prove the claim via the following two cases:

Case 1: $(\forall i, \alpha_i)\,[f_{i,\alpha_i}$ *satisfies one of* (1)–(4)].

Subcase 1: For some i, $f_{i,0}$ satisfies (1) or (2) and $f_{i,1}$ satisfies (1) or (2).

It is easy to see that f becomes completely degenerate or almost-completely degenerate, and therefore satisfies one of (5)–(8). This contradicts our assumption.

Subcase 2: For some i, one of $f_{i,0}$ and $f_{i,1}$ satisfies (1) or (2), and the other one of $f_{i,0}$ and $f_{i,1}$ satisfies (3) or (4). In this case, f does not satisfy any of (5)–(8). We must prove that the P-selective sets are not closed under this f. Without loss of generality, we assume $i = 1$. Throughout the rest of this proof, let B and D be the specific P-selective sets constructed in Lemma 2.1.

Consider $f(0, \alpha_2, \ldots, \alpha_{k+1}) = 0$ and $f(1, \alpha_2, \ldots, \alpha_{k+1}) = \alpha_j$ for some j such that $2 \leqslant j \leqslant k + 1$. Let $L_1 = B$, $L_j = D$, and $L_i = \Sigma^*$ for each i such that $1 < i \leqslant k + 1$ and $i \neq j$. Then we know $I(L_1, \ldots, L_{k+1}) = B \cap D$, and $B \cap D$ is not a P-selective set. So the P-selective sets are not closed under this f.

Consider $f(0, \alpha_2, \ldots, \alpha_{k+1}) = 1$ and $f(1, \alpha_2, \ldots, \alpha_{k+1}) = \alpha_j$ for some j such that $2 \leqslant j \leqslant k + 1$. Let $L_1 = \bar{B}$, $L_j = D$, and $L_i = \Sigma^*$ for each i such that $1 < i \leqslant k + 1$ and $i \neq j$. Then we have $I(L_1, \ldots, L_{k+1}) = B \cap D$ which is not a P-selective set.

Next, consider $f(0, \alpha_2, \ldots, \alpha_{k+1}) = 0$ and $f(1, \alpha_2, \ldots, \alpha_{k+1}) = \bar{\alpha}_j$ for some j such that $2 \leqslant j \leqslant k + 1$. Set $L_1 = B$, $L_j = \bar{D}$, and $L_i = \Sigma^*$ for each i such that $1 < i \leqslant k + 1$ and $i \neq j$. Again we have $I(L_1, \ldots, L_{k+1}) = B \cap D$ which is not a P-selective set.

Finally consider $f(0, \alpha_2, \ldots, \alpha_{k+1}) = 1$ and $f(1, \alpha_2, \ldots, \alpha_{k+1}) = \bar{\alpha}_j$ for some j such that $2 \leqslant j \leqslant k + 1$. Let $L_1 = \bar{B}$, $L_j = \bar{D}$, and $L_i = \Sigma^*$ for each i such that $1 < i \leqslant k + 1$ and $i \neq j$. Again we have $I(L_1, \ldots, L_{k+1}) = B \cap D$ which is not a P-selective set.

Subcase 3: For some i, $f_{i,0}$ satisfies (3) or (4) and $f_{i,1}$ satisfies (3) or (4). Without loss of generality, we assume $i = 1$.

Consider $f(0, \alpha_2, \ldots, \alpha_{k+1}) = \alpha_j$ and $f(1, \alpha_2, \ldots, \alpha_{k+1}) = \alpha_{j'}$ for some j and j' such that $2 \leqslant j$ and $2 \leqslant j' \leqslant k + 1$ (the case $j' = 1$ is included in Subcase 2). If $j = j'$, then (7) holds. This contradicts the assumption. If $j \neq j'$, then f indeed does not satisfy any of (5)–(8). Set $L_1 = B$, $L_j = B$, $L_{j'} = D$, and $L_i = \Sigma^*$ for $i \neq 1, j, j'$. Then

$I(L_1, \ldots, L_{k+1}) = (\bar{L}_1 \cap L_j) \cup (L_1 \cap L_{j'}) = (\bar{B} \cap B) \cup (B \cap D) = B \cap D$. So in this case also the P-selective sets are not closed under f.

Consider $f(0, \alpha_2, \ldots, \alpha_{k+1}) = \alpha_j$ and $f(1, \alpha_2, \ldots, \alpha_{k+1}) = \bar{\alpha}_{j'}$ for some j and j' such that $2 \leqslant j$ and $2 \leqslant j' \leqslant k+1$. If $j = j'$, then let $L_i = \Sigma^*$ for $i \neq 1, j$. Now we have $I(L_1, \ldots, L_{k+1}) = (\overline{L_1} \cap L_j) \cup (L_1 \cap \overline{L_j})$, which is the XOR operator. By Lemma 2.3, the P-selective sets are not closed under this operator. If $j \neq j'$, then let $L_1 = B$, $L_j = B$, $L_{j'} = \overline{D}$, we have $I(L_1, \ldots, L_{k+1}) = B \cap D$ which is not a P-selective set.

By a similar argument, we can also prove that the P-selective sets are not closed under the operator f such that $f(0, \alpha_2, \ldots, \alpha_{k+1}) = \bar{\alpha}_j$ and $f(1, \alpha_2, \ldots, \alpha_{k+1}) = \alpha_{j'}$ for some j and j' such that $2 \leqslant j$ and $2 \leqslant j' \leqslant k+1$, and that the P-selective sets are not closed under the operator f such that $f(0, \alpha_2, \ldots, \alpha_{k+1}) = \bar{\alpha}_j$ and $f(1, \alpha_2, \ldots, \alpha_{k+1}) = \bar{\alpha}_{j'}$ for some j and j' such that $2 \leqslant j$ and $2 \leqslant j' \leqslant k+1$.

Case 2: There exists an i and α_i such that f_{i,α_i} does not satisfy any of (1)–(4). Without loss of generality, suppose $i = k+1$. Consider $\alpha_i = 0$. Let $L_{k+1} = \emptyset$. Then $I_f(L_1, \ldots, L_{k+1}) = I_{f_{k+1,\alpha_{k+1}}}(L_1, \ldots, L_k)$ (by the definition of I). By our inductive assumption, we know that under $f_{k+1,\alpha_{k+1}}$, the P-selective sets are not closed, and thus they are not closed under this f.

Now consider $\alpha_i = 1$. Let $L_{k+1} = \Sigma^*$. Then $I_f(L_1, \ldots, L_{k+1}) = I_{f_{k+1,\alpha_{k+1}}}(L_1, \ldots, L_k)$. Again, by our inductive assumption, we know that the P-selective sets are not closed under this f. □

Note that there are exactly $2k+2$ completely degenerate or almost-completely degenerate k-ary connectives (2 of the former and $2k$ of the latter).

Corollary 2.6. *For each $k \geqslant 0$, the* P*-selective sets are closed under exactly $2k+2$ k-ary Boolean connectives.*

It is not hard to see, from the approach of the proof of Theorem 2.5, not just that the P-selective sets are not closed under any k-ary Boolean connective that is neither completely degenerate nor almost-completely degenerate, but also that for any such connective I and any reasonable time (or space) class (e.g., DTIME $[2^{2^{2^n}}]$), there are P-selective sets such that applying the operator to them yields a set not selective via any selector function from the time (or space) class.

Finally, let $\oplus$ denote the join operation, which is also variously known as "marked union" and "disjoint union". In particular, for any sets A and B, let $A \oplus B = \{0x \mid x \in A\} \cup \{1y \mid y \in B\}$. Note that almost all known complexity classes are closed under the join. Indeed, even such a badly behaved class as UP – which does not robustly possess complete sets [5, 9], positive relativizations [8], or upward separation [7] – is clearly closed under the join. However, as the P-selective sets are closed downwards under positive truth-table reductions [18] and $A \cap B$ positive truth-table reduces to $A \oplus B$, Lemma 2.1 implies the following result.

Theorem 2.7. *The* P*-selective sets are not closed under the join.*

2.2. Indices

We now show that the complexity of determining whether a set (specified by a Turing machine) is P-selective – that is, the index set of the (r.e.) P-selective sets – is Σ_3^0-complete. Let $M_1, M_2, \ldots$ be a standard enumeration of deterministic, clocked, polynomial-time Turing machines. Let $\hat{M}_1, \hat{M}_2, \ldots$ be a standard enumeration of Turing machines. For each i, let $W_i = L(\hat{M}_i)$. For each i and t, let $W_{i,t}$ denote the set of all strings accepted by $\hat{M}_i$ within t steps.

Definition 2.8. We define w_{2n} and w_{2n+1} (which will be the pair of elements witnessing the failure of some P-selector) by: $w_0 = 2$, and for each $n \geqslant 0$, $w_{2n+1} = 1 + w_{2n}$ and $w_{2(n+1)} = 2^{2^{2^{w_{2n}}}}$.

Definition 2.9. We define a language L as follows: For all i and m, if $M_i(w_{2\langle i,m\rangle}, w_{2\langle i,m\rangle+1}) = w_{2\langle i,m\rangle}$, then put $w_{2\langle i,m\rangle+1}$ into L and keep $w_{2\langle i,m\rangle}$ out of L. If $M_i(w_{2\langle i,m\rangle}, w_{2\langle i,m\rangle+1}) = w_{2\langle i,m\rangle+1}$, then put $w_{2\langle i,m\rangle}$ into L and keep $w_{2\langle i,m\rangle+1}$ out of L. Otherwise do nothing.

Lemma 2.10. *The language L defined above is recursive but not* P*-selective.*

Proof. Since each machine M_i is polynomial-time clocked, L is recursive.

Suppose that L is a P-selective set and suppose M_{i_0} is a polynomial-time selector. Fix any m. Then if $M_{i_0}(w_{2\langle i_0,m\rangle}, w_{2\langle i_0,m\rangle+1}) = w_{2\langle i_0,m\rangle}$, by the definition of L, $w_{2\langle i_0,m\rangle+1}$ is in L but $w_{2\langle i_0,m\rangle}$ is not in L, so M_{i_0} selects the one that is not in L. Thus M_{i_0} certainly does not compute a P-selector function for L. If $M_{i_0}(w_{2\langle i_0,m\rangle}, w_{2\langle i_0,m\rangle+1}) = w_{2\langle i_0,m\rangle+1}$, in the same way we again get a contradiction. Hence L is not a P-selective set. □

Lemma 2.11. *Let $L^{[i]} = \{w_{2\langle i,m\rangle}, w_{2\langle i,m\rangle+1} \mid m \in N\} \cap L$. Then $L^{[i]}$ is a polynomial-time computable set, i.e., $L^{[i]} \in$ P.*

Proof. It is clear that $L^{[i]}$ can be recognized by simulating the machine M_i, which is polynomial-time clocked, on the appropriate elements. □

Theorem 2.12. $\{\hat{M}_i \mid L(\hat{M}_i) \in \text{P-Sel}\}$ *is Σ_3^0-complete.*

Proof. (i) (*Counting the quantifiers*) $i \in I_{\text{P-Sel}} \Leftrightarrow W_i$ is P-selective $\Leftrightarrow (\exists f{:}f$ is polynomial-time computable function) $(\forall x, y \in \Sigma^*)$ $[((x \in W_i \vee y \in W_i) \Rightarrow f(x,y) \in W_i) \wedge (f(x,y) = x \vee f(x,y) = y)]$.

By the Tarski–Kuratowski algorithm (see [16]), it is easy to see that $I_{\text{P-Sel}} \in \Sigma_3^0$, since the predicate $(z \in W_i)$ is Σ_1^0.

(ii) (*Construction of the reduction*) Fix a set $A \in \Sigma_3^0$. We will describe a recursive reduction h such that $(\forall x)\,[x \in A \Leftrightarrow h(x) \in I_{\text{P-Sel}}]$. In particular, we will construct sets B_x. We arrange the construction so that $[x \in A \Rightarrow B_x$ is P-selective$]$ and $[x \notin A \Rightarrow B_x$ is not P-selective$]$. The instructions for B_x will depend effectively on x. Hence we shall have a recursive function h such that $B_x = W_{h(x)}$, and thus, $A \leqslant_m I_{\text{P-Sel}}$ via reduction h.

From basic recursion theory ([20], see also [6]), we know there is a recursive function $g(x, y)$ such that

$$x \in A \Leftrightarrow (\exists y)[W_{g(x,y)} \text{ is infinite}].$$

For every x, we describe below the effective construction of a set C_x by induction on stages. Note that C_x is a recursively enumerable set an index of which, by our construction, will be recursive in x. Define $C_x = \bigcup_{s \geqslant 0} C_{x,s}$.

Stage 0: Set $C_{x,0} = \emptyset$.

Stage $s+1$: During the following construction, the set L is defined as in Definition 2.9. For each $y \leqslant s$, if

$$(W_{g(x,y),s} \cap \Sigma^{\leqslant s}) \neq (W_{g(x,y),s+1} \cap \Sigma^{\leqslant s+1}),$$

then we add to $C_{x,s+1}$ all $w_{2\langle i,m\rangle}$ and $w_{2\langle i,m\rangle+1}$ that (a) are in $L^{[i]}$ and (b) satisfy $y \leqslant i \leqslant s$ and $m \leqslant s$. This ends the construction of C_x.

Finally, we define $B_x = \bar{L} \cup C_x$. The construction just given yields, keeping in mind that L is recursive and g is recursive, that C_x is a recursively enumerable set an index of which is recursive in x, that is, there is a recursive function k so that $(\forall x)\,[W_{k(x)} = C_x]$. So, as $\bar{L}$ is recursive, there indeed is a recursive reduction h such that $(\forall x)$ $[W_{h(x)} = B_x]$.

(iii) (*Verification*) If $x \in A$, then there is a y_0 (for specificity, consider the least one) such that $W_{g(x,y_0)}$ is infinite. Then by our construction, $(W_{g(x,y_0),s} \cap \Sigma^{\leqslant s}) \neq (W_{g(x,y_0),s+1} \cap \Sigma^{\leqslant s+1})$ will hold at infinitely many stages, and so the elements in each $L^{[i]}$ with $i \geqslant y_0$ will eventually be enumerated into C_x, and will eventually be removed from $\overline{C_x}$. Since $\overline{B_x} = L \cap \overline{C_x}$, it follows that $\overline{B_x}$ is the finite union of $L^{[i]}$ where $i < y_0$. By Lemma 2.11, this implies that $\overline{B_x}$ is the finite union of P sets, and thus is in P, and so B_x is also in P, and thus is P-selective.

If $x \notin A$, then $W_{g(x,y)}$ is finite for each y. By our construction, only finitely many elements in each $L^{[i]}$ will be enumerated into C_x, and thus be removed from $\overline{C_x}$, that is, regarding the set $L \cap \overline{C_x}$, for each i it holds that in $L^{[i]} \cap \overline{C_x}$ there are still infinitely many elements available to witness the failure of M_i to be a P-selector function for $L \cap \overline{C_x}$ (i.e., $\overline{B_x}$), and thus for no i is M_i a P-selector for B_x (essentially by the same argument as in the proof of Lemma 2.10, except choosing in that proof the m now to be some m not in the finite number chopped out of $\overline{C_x}$ by the construction). So $\overline{B_x}$ is not a P-selective set, and thus neither is B_x.

Now we have achieved the equivalence,

$$x \in A \Leftrightarrow W_{h(x)} \text{ is P-selective} \Leftrightarrow h(x) \in I_{\text{P-Sel}},$$

which completes our proof. □

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