Applied Mathematics Letters 25 (2012) 268–274

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

A first-order differential double subordination with applications[☆]

Rosihan M. Ali[a,*], Nak Eun Cho[b], Oh Sang Kwon[c], V. Ravichandran[a,d]

[a] School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Penang, Malaysia
[b] Department of Applied Mathematics, Pukyong National University, Busan 608-737, South Korea
[c] Department of Mathematics, Kyungsung University, Busan 608-736, South Korea
[d] Department of Mathematics, University of Delhi, Delhi-110 007, India

ARTICLE INFO

Article history:
Received 22 October 2010
Received in revised form 3 May 2011
Accepted 26 August 2011

Keywords:
Differential subordination
Differential superordination
Best subordinant
Best dominant
Φ-like functions

ABSTRACT

Let q_1 and q_2 belong to a certain class of normalized analytic univalent functions in the open unit disk of the complex plane. Sufficient conditions are obtained for normalized analytic functions p to satisfy the double subordination chain $q_1(z) \prec p(z) \prec q_2(z)$. The differential sandwich-type result obtained is applied to normalized univalent functions and to Φ-like functions.

1. Introduction

Let $\mathscr{H}$ be the class consisting of analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$. For $a \in \mathbb{C}$, let $\mathscr{H}[a, n] := \{f \in \mathscr{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}$, and $\mathscr{A} := \{f \in \mathscr{H} : f(0) = 0, f'(0) = 1\}$. A function $f \in \mathscr{H}$ is said to be subordinate to an analytic function $g \in \mathscr{H}$, or g superordinates f, written as $f(z) \prec g(z) (z \in \mathbb{D})$, if there exists a Schwarz function w, analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$, satisfying $f(z) = g(w(z))$. If the function g is univalent in $\mathbb{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. An exposition on the widely used theory of differential subordination, developed in the main by Miller and Mocanu, with numerous applications to univalent functions can be found in their monograph [1]. Miller and Mocanu [2] also introduced the dual concept of differential superordination. Let $p, h \in \mathscr{H}$ and $\phi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and p satisfies the second-order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \tag{1}$$

then p is a solution of the differential superordination (1). An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1). A univalent subordinant $\widetilde{q}$ satisfying $q \prec \widetilde{q}$ for all subordinants q of (1) is said to be the best subordinant. Miller and Mocanu [2] obtained conditions on h, q and ϕ for which the following differential implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

☆ The works by the first and the fourth authors were supported in part by grants from the Universiti Sains Malaysia and the University of Delhi respectively. The second author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2010-0017111). This work was completed during the visit of the first and the fourth authors to Pukyong National University. The authors are thankful to the referees for their suggestions that helped improve the presentation of this manuscript.

* Corresponding author.
E-mail addresses: rosihan@cs.usm.my (R.M. Ali), necho@pknu.ac.kr (N.E. Cho), oskwon@ks.ac.kr (O.S. Kwon), vravi68@gmail.com (V. Ravichandran).

0893-9659/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2011.08.025

Using these results, Bulboacă gave a treatment on certain classes of first-order differential superordinations [3,4], as well as superordination-preserving integral operators [5]. Ali et al. [6] gave several applications of first-order differential subordination and superordination to obtain sufficient conditions for normalized analytic functions f to satisfy $q_1(z) \prec zf'(z)/f(z) \prec q_2(z)$, where q_1 and q_2 are given univalent analytic functions in $\mathbb{D}$. In [7], they have also applied differential superordination to functions defined by means of linear operators. Recently Ali and Ravichandran [8] investigated first-order superordination to a class of meromorphic α-convex functions. Several differential subordination and superordination associated with various linear operators were also investigated in [9].

Generalizing the familiar starlike and convex functions, Lewandowski et al. [10] introduced γ-starlike functions consisting of $f \in \mathscr{A}$ satisfying the inequality

$$\operatorname{Re}\left(\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\right) > 0.$$

These functions are starlike. With $p(z) := zf'(z)/f(z)$, to show that γ-starlike functions are indeed starlike, is to analytically make the implication

$$\operatorname{Re}\left(p(z)\left(1+\frac{zp'(z)}{p^2(z)}\right)^{\gamma}\right) > 0 \Rightarrow \operatorname{Re} p(z) > 0.$$

Following the work of Lewandowski et al. [10,11], Kanas et al. [12] determined conditions on p and h satisfying

$$p(z)\left(1+\frac{zp'(z)}{p(z)}\right)^{\alpha} \prec h(z) \Rightarrow p(z) \prec h(z)$$

for a fixed $\alpha \in [0, 1]$. Lecko [13] (see [12] for a symmetric version) investigated the more general subordination

$$p(z)\left(1+\frac{zp'(z)}{p(z)}\varphi(p(z))\right)^{\alpha} \prec h(z) \Rightarrow p(z) \prec h(z).$$

Singh and Gupta [14] subsequently investigated the following first-order differential subordination that included the important Briot–Bouquet differential subordination.

$$(p(z))^{\alpha}\left(p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\right)^{\mu} \prec (q(z))^{\alpha}\left(q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}\right)^{\mu} \Rightarrow p(z) \prec q(z).$$

For a closely related class, see [15].

The present paper investigates differential subordination and superordination implications of expressions similar to the form considered above by Singh and Gupta [14]. Special cases of the results obtained include one involving the expression $\alpha p^2(z) + (1-\alpha)p(z) + \alpha zp'(z)$, a result which cannot be deduced from the work of Singh and Gupta [14]. The sandwich-type results obtained in our present investigation are then applied to normalized analytic univalent functions and to Φ-like functions.

The following definition and results will be required.

Lemma 1.1 (*cf. Miller and Mocanu [1, Theorem 3. 4h, p. 132]*)**.** *Let q be univalent in the unit disk $\mathbb{D}$, and let ϑ and φ be analytic in a domain $D \supset q(\mathbb{D})$ with $\varphi(w) \neq 0$, $w \in q(\mathbb{D})$. With $Q(z) := zq'(z)\varphi(q(z))$, let $h(z) := \vartheta(q(z)) + Q(z)$. Suppose that Q is starlike univalent in $\mathbb{D}$ and*

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

If p is analytic in $\mathbb{D}$ with $p(0) = q(0)$, $p(\mathbb{D}) \subset D$ and

$$\vartheta(p(z)) + zp'(z)\varphi(p(z)) \prec \vartheta(q(z)) + zq'(z)\varphi(q(z)),$$

then $p(z) \prec q(z)$, and q is the best dominant.

Definition 1.2 (*[2, Definition 2, p. 817]*)**.** Denote by $\mathscr{Q}$ the set of all functions f that are analytic and injective on $\overline{\mathbb{D}} - E(f)$, where

$$E(f) = \left\{\zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} f(z) = \infty\right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} - E(f)$.

Lemma 1.3 (*[4]*)**.** *Let q be univalent in the unit disk $\mathbb{D}$, ϑ and φ be analytic in a domain D containing $q(\mathbb{D})$. Suppose that $\operatorname{Re}[\vartheta'(q(z))/\varphi(q(z))] > 0$ for $z \in \mathbb{D}$ and $zq'(z)\varphi(q(z))$ is starlike univalent in $\mathbb{D}$. If $p \in \mathscr{H}[q(0), 1] \cap \mathscr{Q}$ with $p(\mathbb{D}) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in $\mathbb{D}$, then*

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z))$$

implies $q(z) \prec p(z)$, and q is the best subordinant.

2. A sandwich theorem

Our main result involves the following class of functions.

Definition 2.1. Let α and μ be fixed numbers with $0 < \mu \leq 1$, $\alpha + \mu \geq 0$. Also let β, γ and δ be complex numbers with $\beta \neq 0$. The class $\mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$ consists of analytic functions p with $p(0) = 1$, $p(z) \neq 0$ in $\mathbb{D}$, and are such that the functions

$$P(z) := (p(z))^{\alpha} \left(p(z) + \delta + \frac{zp'(z)}{\beta p(z) + \gamma} \right)^{\mu} \quad (z \in \mathbb{D})$$

are well-defined in $\mathbb{D}$. (Here the powers are principal values.)

By making use of Lemma 1.1, the following result is derived.

Theorem 2.2. *Let $q \in \mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$ be analytic and univalent in $\mathbb{D}$. Set*

$$R(z) := \frac{zq'(z)}{\beta q(z) + \gamma} \quad (z \in \mathbb{D}). \tag{2}$$

Assume that

$$\operatorname{Re}\left((\beta q(z) + \gamma) \left(1 + \frac{\alpha}{\mu} + \frac{\alpha\delta}{\mu q(z)} \right) \right) > 0 \quad (z \in \mathbb{D}), \tag{3}$$

and

$$\operatorname{Re}\left(\frac{\alpha}{\mu} \frac{zq'(z)}{q(z)} + \frac{zR'(z)}{R(z)} \right) > 0 \quad (z \in \mathbb{D}). \tag{4}$$

If $p \in \mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$ satisfies

$$(p(z))^{\alpha} \left(p(z) + \delta + \frac{zp'(z)}{\beta p(z) + \gamma} \right)^{\mu} \prec (q(z))^{\alpha} \left(q(z) + \delta + \frac{zq'(z)}{\beta q(z) + \gamma} \right)^{\mu}, \tag{5}$$

then $p(z) \prec q(z)$, and q is the best dominant.

Proof. We first write the differential subordination (5) as

$$(p(z))^{\frac{\alpha}{\mu}+1} + \delta (p(z))^{\frac{\alpha}{\mu}} + (p(z))^{\frac{\alpha}{\mu}} \frac{zp'(z)}{\beta p(z) + \gamma} \prec (q(z))^{\frac{\alpha}{\mu}+1} + \delta (q(z))^{\frac{\alpha}{\mu}} + (q(z))^{\frac{\alpha}{\mu}} \frac{zq'(z)}{\beta q(z) + \gamma}.$$

Define the functions ϑ and φ by

$$\vartheta(w) := w^{\frac{\alpha}{\mu}+1} + \delta w^{\frac{\alpha}{\mu}} \quad \text{and} \quad \varphi(w) := \frac{w^{\frac{\alpha}{\mu}}}{\beta w + \gamma}.$$

Since $q \in \mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$, then $q(z) \neq 0$ and therefore $\varphi(w) \neq 0$ when $w \in q(\mathbb{D})$. Also φ and ϑ are analytic in a domain containing $q(\mathbb{D})$. Define the function

$$Q(z) := zq'(z)\varphi(q(z)) = (q(z))^{\frac{\alpha}{\mu}} \frac{zq'(z)}{\beta q(z) + \gamma} = (q(z))^{\frac{\alpha}{\mu}} R(z),$$

where R is given by (2). It follows from (4) that

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = \Re\left(\frac{\alpha}{\mu} \frac{zq'(z)}{q(z)} + \frac{zR'(z)}{R(z)} \right) > 0,$$

and so Q is a starlike function. Now define h by

$$h(z) := \vartheta(q(z)) + Q(z) = (q(z))^{\frac{\alpha}{\mu}+1} + \delta (q(z))^{\frac{\alpha}{\mu}} + Q(z).$$

In view of the assumptions (3) and (4), it follows that

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re}\left\{ (\beta q(z) + \gamma) \left(1 + \frac{\alpha}{\mu} + \frac{\alpha\delta}{\mu q(z)} \right) + \frac{\alpha}{\mu} \frac{zq'(z)}{q(z)} + \frac{zR'(z)}{R(z)} \right\} > 0 \quad (z \in \mathbb{D}).$$

The result is now deduced from Lemma 1.1. □

Example 2.3. Let $q : \mathbb{D} \to \mathbb{C}$ be defined by $q(z) = (1 + Az)/(1 + Bz)$ with $-1 < B < A \leq 1$. It is evident that $q \in \mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$ whenever

$$\delta + \frac{1-A}{1-B} > \frac{A-B}{(1-B)|\,|\beta+\gamma| - |\beta A + \gamma B|\,|}.$$

With additional constraints on the parameters, there exist functions q satisfying the hypothesis of Theorem 2.2. For instance, in addition to the above condition, assuming that all the parameters α, β, γ, δ, and μ are positive with

$$\frac{1-2A}{1-A} > \frac{|\beta A + \gamma B|}{|\beta + \gamma - |\beta A + \gamma B|},$$

then q satisfies the conditions of Theorem 2.2.

By a similar application of Lemma 1.3, the following result can be established, which we state without proof.

Theorem 2.4. *Let $q \in \mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$ be as in Theorem 2.2. Let $p \in \mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$ satisfies $p \in \mathscr{H} \cap \mathscr{Q}$ and $(p(z))^{\frac{\alpha}{\mu}+1} + \delta(p(z))^{\frac{\alpha}{\mu}} + (p(z))^{\frac{\alpha}{\mu}} \frac{zp'(z)}{\beta p(z)+\gamma}$ be univalent. If p satisfies*

$$(q(z))^{\alpha}\left(q(z) + \delta + \frac{zq'(z)}{\beta q(z) + \gamma}\right)^{\mu} \prec (p(z))^{\alpha}\left(p(z) + \delta + \frac{zp'(z)}{\beta p(z) + \gamma}\right)^{\mu},$$

then $q(z) \prec p(z)$, and q is the best subordinant.

Combining Theorems 2.2 and 2.4, the following "sandwich theorem" is obtained.

Theorem 2.5. *Let $q_i \in \mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$ $(i = 1, 2)$ be analytic and univalent in $\mathbb{D}$. Set*

$$R_i(z) := \frac{zq_i'(z)}{\beta q_i(z) + \gamma} \quad (i = 1, 2; z \in \mathbb{D}),$$

$$h_i(z) := (q_i(z))^{\alpha}\left(q_i(z) + \delta + \frac{zq_i'(z)}{\beta q_i(z) + \gamma}\right)^{\mu} \quad (i = 1, 2).$$

Assume that

$$\mathrm{Re}\left((\beta q_i(z) + \gamma)\left(1 + \frac{\alpha}{\mu} + \frac{\alpha\delta}{\mu q_i(z)}\right)\right) > 0 \quad (z \in \mathbb{D})$$

and

$$\mathrm{Re}\left(\frac{\alpha}{\mu}\frac{zq_i'(z)}{q_i(z)} + \frac{zR_i'(z)}{R_i(z)}\right) > 0 \quad (i = 1, 2; z \in \mathbb{D}).$$

If $p \in \mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$ satisfies $p \in \mathscr{H} \cap \mathscr{Q}$ and $(p(z))^{\frac{\alpha}{\mu}+1} + \delta(p(z))^{\frac{\alpha}{\mu}} + (p(z))^{\frac{\alpha}{\mu}} \frac{zp'(z)}{\beta p(z)+\gamma}$ is univalent, then

$$h_1(z) \prec (p(z))^{\alpha}\left(p(z) + \delta + \frac{zp'(z)}{\beta p(z) + \gamma}\right)^{\mu} \prec h_2(z) \tag{6}$$

implies $q_1(z) \prec p(z) \prec q_2(z)$. Further q_1 and q_2 are the best subordinant and the best dominant respectively.

3. Applications to univalent functions

By use of Theorem 2.5, the following result is obtained.

Theorem 3.1. *Let α, μ be fixed numbers with $0 < \mu \leq 1$, $\alpha + \mu > 0$, and $\lambda \in \mathbb{C}$. Let $f, g \in \mathscr{A}$, and $q_i(z) = zg_i'(z)/g_i(z)$ $(i = 1, 2)$ be univalent in $\mathbb{D}$ satisfying*

$$\mathrm{Re}\left(\frac{1}{\lambda}q_i(z)\right) > 0$$

and

$$\mathrm{Re}\left(\left(\frac{\alpha}{\mu} - 1\right)\frac{zq_i'(z)}{q_i(z)} + 1 + \frac{zq_i''(z)}{q_i'(z)}\right) > 0.$$

Let

$$h_i(z) := \left(\frac{zg_i'(z)}{g_i(z)}\right)^{\alpha}\left((1-\lambda)\frac{zg_i'(z)}{g_i(z)} + \lambda\left(1+\frac{zg_i''(z)}{g_i'(z)}\right)\right)^{\mu} \quad (i=1,2).$$

If $f \in \mathscr{A}$ *satisfies* $0 \neq \frac{zf'(z)}{f(z)} \in \mathscr{H}[1,1] \cap \mathscr{Q}$ *and* $(\frac{zf'(z)}{f(z)})^{\alpha}((1-\lambda)\frac{zf'(z)}{f(z)} + \lambda(1+\frac{zf''(z)}{f'(z)}))^{\mu}$ *is univalent in* $\mathbb{D}$, *then*

$$h_1(z) \prec \left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left((1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right)^{\mu} \prec h_2(z)$$

implies

$$\frac{zg_1'(z)}{g_1(z)} \prec \frac{zf'(z)}{f(z)} \prec \frac{zg_2'(z)}{g_2(z)}.$$

Proof. The result follows from Theorem 2.5 by taking $\gamma = \delta = 0$, $\beta = 1/\lambda$, and

$$p(z) := \frac{zf'(z)}{f(z)} \quad \text{and} \quad q_i(z) := \frac{zg_i'(z)}{g_i(z)} \quad (i=1,2). \quad \square$$

The following two corollaries are immediate consequences of Theorem 2.5 (or Theorem 3.1).

Corollary 3.2 ([6]). *Let* $\alpha \in \mathbb{C}$, *and* $q_i(z) \neq 0$ $(i=1,2)$ *be univalent in* $\mathbb{D}$. *Assume that* $\mathrm{Re}[\overline{\alpha}q_i(z)] > 0$ *for* $i=1,2$ *and* $zq_i'(z)/q_i(z)$ $(i=1,2)$ *is starlike univalent in* $\mathbb{D}$. *If* $f \in \mathscr{A}$, $0 \neq zf'(z)/f(z) \in \mathscr{H}[1,1] \cap \mathscr{Q}$, $(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(1+\frac{zf''(z)}{f'(z)})$ *is univalent in* $\mathbb{D}$, *then*

$$q_1(z) + \alpha\frac{zq_1'(z)}{q_1(z)} \prec (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec q_2(z) + \alpha\frac{zq_2'(z)}{q_2(z)}$$

implies

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z).$$

Further q_1 *and* q_2 *are the best subordinant and best dominant respectively.*

Corollary 3.3 ([6]). *Let* $q_i(z) \neq 0$ *be univalent in* $\mathbb{D}$ *with* $\mathrm{Re}\, q_i(z) > 0$. *Let* $zq_i'(z)/q_i^2(z)$ *be starlike univalent in* $\mathbb{D}$ *for* $i=1,2$. *If* $f \in \mathscr{A}$, $0 \neq zf'(z)/f(z) \in \mathscr{H}[1,1] \cap Q$, $\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}$ *is univalent in* $\mathbb{D}$, *then*

$$1 + \frac{zq_1'(z)}{q_1^2(z)} \prec \frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \frac{zq_2'(z)}{q_2^2(z)}$$

implies $q_1(z) \prec zf'(z)/f(z) \prec q_2(z)$. *Further* q_1 *and* q_2 *are the best subordinant and best dominant respectively.*

Another application of Theorem 2.5 yields the following result.

Corollary 3.4 ([6]). *Let* q_1 *and* q_2 *be convex univalent in* $\mathbb{D}$. *Let* $0 \neq \alpha \in \mathbb{C}$, *and assume that* $\mathrm{Re}\, q_i(z) > \mathrm{Re}\frac{\alpha-1}{2\alpha}$ *for* $i=1,2$. *If* $f \in \mathscr{A}$, $zf'(z)/f(z) \in \mathscr{H}[1,1] \cap \mathscr{Q}$, $\frac{zf'(z)}{f(z)} + \alpha\frac{z^2f''(z)}{f(z)}$ *is univalent in* $\mathbb{D}$, *then*

$$(1-\alpha)q_1(z) + \alpha q_1^2(z) + \alpha zq_1'(z) \prec \frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \prec (1-\alpha)q_2(z) + \alpha q_2^2(z) + \alpha zq_2'(z)$$

implies

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z).$$

Further q_1 *and* q_2 *are the best subordinant and best dominant respectively.*

4. Application to Φ-like functions

Let Φ be an analytic function in a domain containing $f(\mathbb{D})$, $\Phi(0) = 0$ and $\Phi'(0) > 0$. A function $f \in \mathscr{A}$ is called *Φ-like* if

$$\operatorname{Re}\frac{zf'(z)}{\Phi(f(z))} > 0 \quad (z \in \mathbb{D}).$$

This concept was introduced by Brickman [16] and it was shown that an analytic function $f \in \mathscr{A}$ is univalent if and only if f is Φ-like for some Φ. When $\Phi(w) = w$ and $\Phi(w) = \lambda w$, the Φ-like function f is respectively starlike and spiral-like of type $\arg\lambda$. Ruscheweyh [17] introduced and studied the following general class of Φ-like functions.

Definition 4.1. Let Φ be analytic in a domain containing $f(\mathbb{D})$, $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(\omega) \neq 0$ for $\omega \in f(\mathbb{D}) - \{0\}$. Let q be a fixed analytic function in $\mathbb{D}$, with $q(0) = 1$. A function $f \in \mathscr{A}$ is called Φ-like with respect to q if

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z) \quad (z \in \mathbb{D}).$$

Theorem 4.2. *Let $\alpha \neq 0$ be a complex number and q_i $(i = 1, 2)$ be convex univalent in $\mathbb{D}$. Define h_i by*

$$h_i(z) := \alpha q_i^2(z) + (1-\alpha)q_i(z) + \alpha z q_i'(z) \quad (i = 1, 2),$$

and suppose that

$$\operatorname{Re}\left(\frac{1-\alpha}{\alpha} + 2q_i(z)\right) > 0 \quad (i = 1, 2; z \in \mathbb{D}).$$

If $f \in \mathscr{A}$ satisfies $f \in \mathscr{H}[1, 1] \cap \mathscr{Q}$ and $\frac{zf'(z)}{\Phi(f(z))}(1 + \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha z(f'(z) - (\Phi(f(z)))')}{\Phi(f(z))})$ is univalent in $\mathbb{D}$, then

$$h_1(z) \prec \frac{zf'(z)}{\Phi(f(z))}\left(1 + \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha z(f'(z) - (\Phi(f(z)))')}{\Phi(f(z))}\right) \prec h_2(z) \tag{7}$$

implies

$$q_1(z) \prec \frac{zf'(z)}{\Phi(f(z))} \prec q_2(z).$$

Further q_1 and q_2 are the best subordinant and the best dominant respectively.

Proof. Define the function p by

$$p(z) := \frac{zf'(z)}{\Phi(f(z))} \quad (z \in \mathbb{D}). \tag{8}$$

Then the function p is analytic in $\mathbb{D}$ with $p(0) = 1$. From (8), it follows that

$$\begin{aligned}\frac{zf'(z)}{\Phi(f(z))}\left(1 + \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha z(f'(z) - (\Phi(f(z)))')}{\Phi(f(z))}\right) &= p(z)\left(1 + \alpha\left(\frac{zp'(z)}{p(z)} - 1\right) + \alpha p(z)\right) \\ &= \alpha p^2(z) + (1-\alpha)p(z) + \alpha z p'(z).\end{aligned} \tag{9}$$

Substituting (9) in the subordination (7) yields

$$h_1(z) \prec \alpha p^2(z) + (1-\alpha)p(z) + \alpha z p'(z) \prec h_2(z).$$

The result now follows from Theorem 2.5. □

Remark 1. When $\Phi(w) = w$, Theorem 4.2 reduces to Corollary 3.4.

References

[1] S.S. Miller, P.T. Mocanu, Differential Subordinations: Theory and Applications, in: Series in Pure and Applied Mathematics, vol. 225, Marcel Dekker, New York, 2000.
[2] S.S. Miller, P.T. Mocanu, Subordinants of differential superordinations, Complex Variables 48 (10) (2003) 815–826.
[3] T. Bulboacă, Sandwich-type theorems for a class of integral operators, Bull. Belg. Math. Soc. Simon Stevin 13 (3) (2006) 537–550.
[4] T. Bulboacă, Classes of first-order differential superordinations, Demonstratio Math. 35 (2) (2002) 287–292.
[5] T. Bulboacă, A class of superordination-preserving integral operators, Indag. Math. (N.S.) 13 (3) (2002) 301–311.
[6] R.M. Ali, V. Ravichandran, M. Hussain Khan, K.G. Subramanian, Differential sandwich theorems for certain analytic functions, Far East J. Math. Sci. 15 (1) (2004) 87–94.

[7] R.M. Ali, V. Ravichandran, M. Hussain Khan, K.G. Subramanian, Applications of first order differential superordinations to certain linear operators, Southeast Asian Bull. Math. 30 (5) (2006) 799–810.
[8] R.M. Ali, V. Ravichandran, Classes of meromorphic α-convex functions, Taiwanese J. Math. 14 (4) (2010) 1479–1490.
[9] R.M. Ali, V. Ravichandran, N. Seenivasagan, Differential subordination and superordination of analytic functions defined by the Dziok–Srivastava linear operator, J. Franklin Inst. 347 (2010) 1762–1781.
[10] Z. Lewandowski, S. Miller, E.Zł otkiewicz, Gamma-starlike functions, Ann. Univ. Mariae Curie-Skłodowska Sect. A 28 (1974) 53–58. 1976.
[11] Z. Lewandowski, S. Miller, E.Zł otkiewicz, Generating functions for some classes of univalent functions, Proc. Amer. Math. Soc. 56 (1976) 111–117.
[12] S. Kanas, A. Lecko, J. Stankiewicz, Differential subordinations and geometric means, Complex Variables Theory Appl. 28 (3) (1996) 201–209.
[13] A. Lecko, On differential subordinations and inclusion relation between classes of analytic functions, Complex Variables Theory Appl. 40 (4) (2000) 371–385.
[14] S. Singh, S. Gupta, Some applications of a first order differential subordination, J. Inequal. Pure Appl. Math. 5 (3) (2004) Article 78.
[15] S. Kanas, J. Kowalczyk, A note on Briot–Bouquet–Bernoulli differential subordination, Comment. Math. Univ. Carolin. 46 (2) (2005) 339–347.
[16] L. Brickman, Φ-like analytic functions, I, Bull. Amer. Math. Soc. 79 (1973) 555–558.
[17] St. Ruscheweyh, A subordination theorem for Φ-like functions, J. London Math. Soc. 13 (2) (1976) 275–280.