JOURNAL OF ALGEBRA **180,** 483–512 (1996)
ARTICLE NO. 0078

Segre Product of Artin–Schelter Regular Algebras of Dimension 2 and Embeddings in Quantum $\mathbb{P}^3$'s

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Communicated by J. T. Stafford

Received February 20, 1995

The Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ as a smooth quadric Q in $\mathbb{P}^3$ corresponds to the surjection of the four-dimensional polynomial ring onto the Segre product S of two copies of the homogeneous coordinate ring of $\mathbb{P}^1$. We study Segre products of noncommutative algebras. If in particular A and B are two copies of a quantum $\mathbb{P}^1$ then $S = \bigoplus_i (A_i \otimes_k B_i)$ is a twisted homogeneous coordinate ring of the quadric Q. The main result of this paper is the classification of all embeddings of the Segre product of two quantum planes into so-called quantum $\mathbb{P}^3$'s. These are (the Proj of) Artin–Schelter regular algebras R of global dimension four with the Hilbert series of a commutative polynomial ring and which map onto S. If R is not a twist of a polynomial ring, then the point scheme of R either is the union of the quadric Q with a line or is only the quadric Q. In the first case, R is a central extension of a three-dimensional Artin–Schelter regular algebra and a twist of an algebra mapping onto the (commutative) homogeneous coordinate ring of Q; in the second case, such an algebra R is the first known example of a four-dimensional Artin–Schelter regular algebra which is not determined by its point scheme.

1. INTRODUCTION

On the level of projective spaces (over an algebraically closed field k), the Segre embedding is given by

$$\phi: \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{n+m+nm}$$
$$((x_0, \ldots, x_n), (y_0, \ldots, y_m)) \to (x_0 y_0, x_1 y_0, \ldots, x_n y_m).$$

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0021-8693/96 $18.00

The map ϕ is injective and its image is a subvariety of $\mathbb{P}^{nm+n+m}$ [10, Exercise I.2.14]. In this way one defines the Segre product of two projective varieties $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ as the image $X \circ Y$ of $X \times Y$ in $\mathbb{P}^{nm+n+m}$ under the map ϕ.

On the level of graded k-algebras, one has a Segre product in the following way. For A and B commutative $\mathbb{N}$-graded k-algebras with $A_0 = B_0 = k$, we define the Segre product (called the cartesian product in [10, Exercise II.5.11]) to be the graded k-algebra $A \circ B := \bigoplus_i (A_i \otimes_k B_i)$. Before we see how this relates to the Segre embedding, we recall the following definition.

Given a commutative graded k-algebra R such that $R_0 = k$, one can associate to it the commutative projective scheme $X = (\operatorname{proj} R, O_X)$ (cf. [10, Chap. II.2]). If R is finitely generated by homogeneous elements of degree one, then, by Serre's theorem [10, Prop. II.5.15], the geometry of X may be described in terms of the quotient category R-gr/tors since R-gr/tors $\cong$ coh X, where coh X is the category of coherent sheaves on X and R-gr/tors is the category of finitely generated left R-modules modulo torsion modules. A general definition of tors will be given later, but if R is noetherian, (which will be the case for all algebras we consider in the sequel), tors consists of the finite-dimensional left R-modules.

In the commutative setting the Segre product may be viewed as a product because the commutative scheme $\operatorname{proj}(A \circ B)$ is isomorphic to the fibred product $\operatorname{proj} A \times_k \operatorname{proj} B$; cf. [10, Exercise II.5.11]. The fibred product of two schemes is a scheme [10, Thm. 3.3] and there are projection homomorphisms from the fibred product to $\operatorname{proj} A$ and $\operatorname{proj} B$.

Let $A = k[x, y]$ and $B = k[u, v]$ be two polynomial rings in two variables (two copies of the homogeneous coordinate ring of the projective line $\mathbb{P}^1$). The Segre embedding ϕ embeds $\mathbb{P}^1 \times \mathbb{P}^1$ as a smooth quadric $Q = \mathscr{V}(XY - UZ)$ in $\mathbb{P}^3$ (where $U = x \otimes u$, $X = y \otimes u$, $Y = x \otimes v$, and $Z = y \otimes v$), while the Segre product $A \circ B$ is the (commutative) homogeneous coordinate ring of Q; that is, $A \circ B = k[U, X, Y, Z]/\langle XY - UZ \rangle$. Furthermore, the surjection $k[U, X, Y, Z] \to A \circ B$ implies that we have a projective embedding $\operatorname{proj}(A \circ B) \hookrightarrow \operatorname{proj} k[U, X, Y, Z] = \mathbb{P}^3_k$. The aim of this paper is to determine to what extent the previous constructions may be repeated for noncommutative graded algebras.

The definition of the Segre product of two noncommutative graded algebras will be the same as that in the commutative setting. Motivated by Serre's theorem, we take the noncommutative scheme $\operatorname{proj} R$ to be the quotient category R-gr/tors. If one considers R-modules that are not necessarily finitely generated then the quotient category is denoted $\operatorname{Proj} R$ as in [6]. We will show in Theorem 2.4 that (with some conditions on A and B), there is a category equivalence $\operatorname{Proj} A \circ B \cong \operatorname{Proj} A \otimes B$ and maps $\operatorname{Proj} A \circ B \to \operatorname{Proj} A$, $\operatorname{Proj} A \circ B \to \operatorname{Proj} B$ in the sense of [6, Sect. 2]. The same result holds for proj. The latter two maps are analogous to the projection maps in the case of the commutative fibred product described above; however, $\mathrm{Proj}(A \circ B)$ is not a categorical product of $\mathrm{Proj}\, A$ and $\mathrm{Proj}\, B$.

The main result of this paper is a "quantization" of the embedding of two copies of $\mathbb{P}^1$ into $\mathbb{P}^3$ as a smooth quadric. We take a quantum $\mathbb{P}^1$ to be the Proj of an Artin–Schelter regular algebra of global dimension two (quantum plane or Jordan plane). If A and B are such Artin–Schelter regular algebras, then $A \circ B$ is a twisted homogeneous coordinate ring [3, 5] of a smooth quadric Q (a quantum quadric) in $\mathbb{P}^3$; cf. Section 3. In the case in which A and B are quantum planes, we classify in Section 4 the graded algebras R which map onto $A \circ B$ with the property that R is Artin–Schelter regular of global dimension four with the Hilbert series of a polynomial ring in four variables. Since such properties of R are shared by the polynomial ring on four variables, we view $\mathrm{Proj}\, R$ as being a quantum $\mathbb{P}^3$ containing a copy of the Segre product of the two quantum $\mathbb{P}^1$'s. This classification relies on Schelter's computer program "Affine," and so relevant calculations are omitted from this paper (therefore a proof that *all* embeddings are classified cannot be provided here). Motivated by the current study, the authors consider a more general setting in [21], in which $A \circ B$ is replaced by any twisted homogeneous coordinate ring of a smooth quadric in $\mathbb{P}^3$ and different techniques are employed.

Our main result, Theorem 4.3, classifies the algebras R into the following categories.

- Twists [4, Sect. 8] of the four-dimensional polynomial ring.

- Twists of algebras R' which map onto the commutative homogeneous coordinate ring of Q. Such R' are determined by the geometric data $(Q \cup L, \sigma)$, where L is a line which either meets Q in two points (counted with multiplicity) or is embedded on Q, and $\sigma \in \mathrm{Aut}(Q \cup L)$ has the property that $\sigma|_Q$ is induced via the Segre embedding from automorphisms of the two copies of $\mathbb{P}^1$. In certain cases there are restrictions on σ, which are discussed in Theorem 4.3. In [20], the case in which L meets Q in two distinct points has been studied in detail and $O_q(M_2(\mathbb{C}))$, the coordinate ring of quantum 2×2 matrices, is such an algebra [20, Ex. 1.5]. However, the embedded line and tangent line cases are new. Furthermore such R' have two central elements in degree one (the linear terms determining the line L) and are therefore central extensions of three-dimensional Artin–Schelter regular algebras, which were studied in [11].

- Algebras R which have associated geometric data (Q, σ), where $\sigma \in \mathrm{Aut}(Q)$ is induced as before via the Segre embedding from automorphisms of the two copies of $\mathbb{P}^1$, but this time R is not determined by the geometric data (since these data determine $A \circ B$). The relations of such algebras R will be given explicitly in Theorem 4.3. The importance of these algebras is that they are the first known examples of four-dimensional Artin–Schelter regular algebras which are not determined by their point scheme and automorphism. It is also interesting that they have no normal elements in degree one (unlike the central extensions of the three-dimensional Artin–Schelter regular algebras above) and that they cannot be twisted to an algebra mapping onto the commutative homogeneous coordinate ring of Q. A more detailed study of these algebras is presented in [22].

To conclude, Theorem 4.7 describes the line modules over these algebras.

2. THE NONCOMMUTATIVE SEGRE PRODUCT

Let $A = k \oplus A_1 \oplus A_2 \oplus \dots$ and $B = k \oplus B_1 \oplus B_2 \oplus \dots$ be connected k-algebras ($\mathbb{N}$-graded with $k = A_0 = B_0$) over an algebraically closed field k.

DEFINITION 2.1. The Segre product $A \circ B$ of the $\mathbb{N}$-graded algebras A and B is the $\mathbb{N}$-graded k-algebra $A \circ B = \bigoplus_{i \geq 0}(A \circ B)_i$ with $(A \circ B)_i = A_i \otimes_k B_i$.

The Segre product $M \circ N$ of a graded left A-module M and a graded left B-module N is the $\mathbb{N}$-graded left $(A \circ B)$-module $M \circ N = \bigoplus_{i \geq 0}(M_i \otimes_k N_i)$, where the action of $a_i \otimes b_i \in (A \circ B)_i$ on $m_j \otimes n_j \in (M \circ N)_j$ is given by $a_i n_j \otimes b_i m_j$.

The notation $A \circ B$ was used in [15, Sect. 3.2]. Closely related to $A \circ B$ is the tensor product $A \otimes_k B$. Apart from the usual $\mathbb{N}$-grading, given by $(A \otimes_k B)_n = \bigoplus_{i+j=n} A_i \otimes B_j$, several other gradings can be put on $A \otimes_k B$. The following two are used in the sequel:

- an $\mathbb{N}^2$-bigrading given by $(A \otimes_k B)_{(i,j)} = A_i \otimes_k B_j$ and
- a $\mathbb{Z}$-grading given by $(A \otimes_k B)_n = \bigoplus_{m \geq 0}(A_{n+m} \otimes_k B_m)$.

For this $\mathbb{Z}$-grading, $A \circ B$ is the degree zero part of $A \otimes_k B$.

As mentioned in the Introduction, it was Serre's theorem [10, Prop. II.5.15] that led to the definition of a noncommutative scheme in [1, Def. 1.2; 6]. Following the notation of [6], one defines for R a left noetherian $\mathbb{N}$-graded k-algebra:

R-Gr $:=$ the category of $\mathbb{N}$-graded left R-modules and

R-gr $:=$ the category of finitely generated $\mathbb{N}$-graded left R-modules.

In both cases the morphisms in the category are the degree zero homomorphisms.

An element m of a graded module M is called torsion if $R_{\geq s}m = 0$ for some s in $\mathbb{N}$. Let $\tau(M)$, the torsion submodule of M, be the graded R-submodule formed by all torsion elements in M. A module M is then torsion if $M = \tau(M)$ and torsion-free if $\tau(M) = 0$. The torsion modules form a full subcategory in R-Gr (resp. R-gr), called Tors (resp. tors). As Tors and tors are dense subcategories of R-Gr and R-gr, we can construct:

$$\operatorname{Proj} R := \text{the quotient category } R\text{-Gr}/\text{Tors},$$

$$\operatorname{proj} R := \text{the quotient category } R\text{-gr}/\text{tors}.$$

Remark 2.2.

1. In [6, Sect. 2], $\operatorname{Proj} R$ is the pair $(R\text{-Gr}/\text{Tors}, \mathscr{R})$, where $\mathscr{R}$ is the image of ${}_R R$ in R-Gr/Tors. The canonical polarization, given by the shift operator s, $(s(M))_n = M_{n+1}$, is also omitted here.

2. If R is left noetherian and connected then R is locally finite ($\dim_k R_n < \infty$ for all n in $\mathbb{N}$ [6, Prop. 2.1]) and tors consists of all finite-dimensional modules.

3. If R is not noetherian, then $\tau(M)$ must be defined as the smallest submodule of M such that $M/\tau(M)$ is torsion free; cf. [19, Chap. 6].

4. By [6, Prop. 2.5], any $\mathbb{Z}$-graded left noetherian algebra R satisfies the category equivalences $\operatorname{Proj} R \simeq \operatorname{Proj} R_{\geq 0}$ and $\operatorname{proj} R \simeq \operatorname{proj} R_{\geq 0}$. Therefore we assumed from the beginning that R is $\mathbb{N}$-graded.

For this definition of Proj and proj to be applied to $A \otimes B$ and $A \circ B$, both algebras must be (left) noetherian. In general it is not known under what assumptions the tensor product of two noetherian algebras is again noetherian. However, from Section 3 on, $A \otimes B$ will be an iterated Ore extension, hence noetherian, and therefore $A \otimes B$ is assumed to be noetherian for the rest of this section. This implies that $A \circ B$ is also noetherian because $A \circ B = (A \otimes B)_0$, and by [16, Lemma II.3.2]. The foregoing may then be applied to the $\mathbb{N}$-graded $A \circ B$: if $(A \circ B)_{\geq s} = \bigoplus_{i \geq s}(A_i \otimes B_i)$ then an element n of a left graded $(A \circ B)$-module is torsion if $(A \circ B)_{\geq s} n = 0$ for some s in $\mathbb{N}$. For the bigraded $A \otimes_k B$ we define $(A \otimes_k B)$-BiGr (respectively $(A \otimes_k B)$-bigr) as the category of $\mathbb{N}^2$-bigraded left $A \otimes_k B$-modules (respectively the finitely generated ones). If $(A \otimes_k B)_{(\geq s, \geq s)} = \bigoplus_{i \geq s,\ j \geq s}(A_i \otimes_k B_j)$ then an element m of a left bigraded $(A \otimes_k B)$-module is torsion if $(A \otimes_k B)_{(\geq s, \geq s)} m = 0$ for some s in $\mathbb{N}$. In this way it is clear what is meant by Tors for both categories $(A \circ B)$-Gr and $(A \otimes B)$-BiGr, thus defining $\operatorname{Proj}(A \circ B)$ and $\operatorname{Proj}(A \otimes_k B)$ (similarly for tors and proj).

Notation. The Segre product $A \circ B$ will also be denoted by S and the tensor product $A \otimes_k B$ by T. We write $A_+ = \bigoplus_{i>0} A_i$, $B_+ = \bigoplus_{i>0} B_i$, $S_+ = (A \circ B)_+ = \bigoplus_{i>0} (A \circ B)_i$ (the positive parts w.r.t the $\mathbb{N}$-grading) and $T_{++} = (A \otimes_k B)_{++} = \bigoplus_{\alpha,\beta>0} (A \otimes_k B)_{(\alpha,\beta)} = A_+ \otimes_k B_+$ (the positive part of $A \otimes_k B$ w.r.t. the double grading).

LEMMA 2.3. *If A and B are connected k-algebras, generated by homogeneous elements of degree one, then*

1. $\forall n \in \mathbb{N}$: $T_{(\geq n, \geq n)} = (T_{++})^n = T(S_+)^n = (S_+)^n T$
2. $\forall n \in \mathbb{N}$: $T_n T_{-n} = T_{-n} T_n = (S_+)^n = S_{\geq n} = (T_0)_{\geq n}$.

Note that T is not strongly graded [16, Sect. I.3] because $T_n T_{-n} \neq T_0$.

THEOREM 2.4. *If A and B are connected k-algebras, left noetherian, generated in degree one, and such that $A \otimes_k B$ is left noetherian, then the following equivalences of categories hold*:

$$\operatorname{Proj}(A \otimes_k B) \approx \operatorname{Proj}(A \circ B) \qquad \textit{and} \qquad \operatorname{proj}(A \otimes_k B) \approx \operatorname{proj}(A \circ B).$$

Proof. As before, put $T = A \otimes_k B$ and $S = A \circ B$. Each doubly graded T-module $M = \bigoplus_{\alpha,\beta \geq 0} M_{(\alpha,\beta)}$ is also a $\mathbb{Z}$-graded $T = \bigoplus_{n\in\mathbb{Z}} T_n$-module by $M = \bigoplus_{n\in\mathbb{Z}} M_n = \bigoplus_{n\in\mathbb{Z}} (\bigoplus_{m\in\mathbb{N}} M_{(n+m,m)})$.

To obtain the category equivalence $\operatorname{Proj} T \approx \operatorname{Proj} S$, define two functors

$$F\colon T\text{-BiGr} \to S\text{-Gr}$$
$$M \to M_0$$

and

$$G\colon S\text{-Gr} \to T\text{-BiGr}$$
$$N \to T \otimes_S N.$$

Then $T \otimes_S N$ is a doubly graded left T-module by

$$\begin{aligned} T_{(\alpha,\beta)} \cdot (T \otimes_S N)_{(\gamma,\delta)} &= T_{(\alpha,\beta)} \cdot \Bigg(\bigoplus_{(j\in\mathbb{N},\, j\leq\gamma,\delta)} T_{(\gamma-j,\delta-j)} \otimes_S N_j \Bigg) \\ &\subseteq \bigoplus_{(j\in\mathbb{N},\, j\leq\gamma,\delta)} T_{(\alpha+\gamma-j,\beta+\delta-j)} \otimes_S N_j \\ &\subseteq (T \otimes_S N)_{(\alpha+\gamma,\beta+\delta)} \end{aligned}$$

as well as a $\mathbb{Z}$-graded T-module with $(T \otimes_S N)_n = T_n \otimes_S N$ and $T_m (T \otimes_S N)_n \subset (T \otimes_S N)_{n+m}$.

The first step is to show that F and G define functors on the quotient categories $\operatorname{Proj} T$ and $\operatorname{Proj} S$. Suppose, for example, that N is torsion in

S-Gr; then there exists for all n in N an m_n such that $S_{\geq m_n} n = 0$. Take an element $g = \sum_{i=1}^{l} t_i \otimes_S n_i$ in $G(N)$ and m_{n_i} such that $S_{\geq m_{n_i}} n_i = 0$ for all i in $\{1, \ldots, l\}$. If m is the maximum of those m_{n_i}, then by Lemma 2.3, $T_{(\geq m, \geq m)} g \subset T_{(\geq m, \geq m)} \otimes_S \sum n_i = TS_{\geq m} \otimes_S \sum n_i = T \otimes_S \sum S_{\geq m} n_i = 0$. Therefore every element of $G(N)$ is torsion and the functor G passes to the quotient categories. Clearly the same holds for the functor F.

The natural equivalence $F \circ G \approx \mathrm{Id}$ on $\mathrm{Proj}\, S$ follows from $(F \circ G)(N) = (T \otimes_S N)_0 = S \otimes_S N \approx N$.

It remains to show that $G \circ F \approx \mathrm{Id}$ on $\mathrm{Proj}\, T$ or that $T \otimes_S M_0 \approx M$ in $\mathrm{Proj}\, T$. (Note that if S were the degree zero part of a strongly graded ring, then the statement would be true in T-gr; cf. [16, Thm. I.3.4]. The weaker situation in Lemma 2.3 forces us to pass to the quotient category of T-Gr.)

Define the bigraded degree $(0, 0)$ and $\mathbb{Z}$-graded degree 0 map $\phi: T \otimes_S M_0 \to M$ by $\phi(t \otimes m_0) = t \cdot m_0$. If $K = \mathrm{Ker}\, \phi$ and $C = \mathrm{Coker}\, \phi$, then K and C are objects of T-BiGr as well as T-Gr. We show that K is torsion in $\mathrm{Proj}\, T$ (similarly for C), hence ϕ is an isomorphism in $\mathrm{Proj}\, T$. First of all, $K_0 = \mathrm{Ker}\, \phi_0 = 0$ because ϕ_0 is the isomorphism $T_0 \otimes_S M_0 \approx M_0$. Lemma 2.3 then implies that $S_{\geq l} K_l = T_l T_{-l} K_l \subseteq T_l K_0 = 0$ for all l in $\mathbb{Z}$, which means that all K_l are torsion in $\mathrm{Proj}\, S$, hence $K = \bigoplus_l K_l$ is torsion in $\mathrm{Proj} S$. Since for every element z of K there exists an m such that $S_{\geq m} z = 0$ and $T_{(\geq m, \geq m)} z = TS_{\geq m} z = 0$, K is torsion in $\mathrm{Proj}\, T$.

The same proof holds for proj, but in this case a torsion module N, for example over S, satisfies the (stronger) property that there exists an s in $\mathbb{N}$ such that $S_{\geq s} N = 0$. ∎

The algebra homomorphism from A to $A \otimes_k B$, sending an element a of A to $a \otimes 1$ in $A \otimes B$, induces a right exact functor from A-Gr to $(A \otimes B)$-BiGr, sending a graded A-module M to $(A \otimes_k B) \otimes_A M$. This functor sends A to $A \otimes_k B$ and passes to the quotient categories. In the terminology of [6, Sect. 2] this means there is a map $\mathrm{Proj}(A \otimes_k B) \to \mathrm{Proj}\, A$. As the following corollary of Theorem 2.4 shows, not all maps between projective schemes arise in this way from an algebra homomorphism.

COROLLARY 2.5. *For A and B as in Theorem* 2.4 *there exist maps* (*in the sense of* [6, *Sect*. 2])

$$\mathrm{Proj}(A \circ B) \to \mathrm{Proj}\, A \quad \textit{and} \quad \mathrm{Proj}(A \circ B) \to \mathrm{Proj}\, B,$$

$$\mathrm{proj}(A \circ B) \to \mathrm{proj}\, A \quad \textit{and} \quad \mathrm{proj}(A \circ B) \to \mathrm{proj}\, B.$$

PROPOSITION 2.6. *If A and B are connected and locally finite with Hilbert series $H_A(t) = \sum_{n \geq 0} (\dim_k A_n) t^n$ and $H_B(t) = \sum_{n \geq 0} (\dim_k B_n) t^n$, then*

1. $H_S(t) = \sum_{n \geq 0} (\dim_k A_n)(\dim_k B_n) t^n$,
2. $H_T(s, t) = \sum_{i, j \geq 0} (\dim_k A_i)(\dim_k B_j) s^i t^j$ *for the double grading.*

Henceforth A and B are quadratic algebras generated in degree one, written as

$$A = T(A_1)/\mathscr{R}_A \qquad \text{with } \mathscr{R}_A \subset A_1 \otimes A_1$$

and

$$B = T(B_1)/\mathscr{R}_B \qquad \text{with } \mathscr{R}_B \subset B_1 \otimes B_1,$$

where $T(\cdot)$ is the tensor algebra. In this case

$$A \circ B = T(A_1 \otimes_k B_1)/\mathscr{S}_{(23)}(\mathscr{R}_A \otimes B_1 \otimes B_1 + A_1 \otimes A_1 \otimes \mathscr{R}_B),$$

where $\mathscr{S}_{(23)}(a \otimes a' \otimes b \otimes b') = a \otimes b \otimes a' \otimes b'$ and

$$A \otimes_k B = T(A_1 \oplus B_1)/(\mathscr{R}_A \oplus [A_1, B_1] \oplus \mathscr{R}_B).$$

Suppose $R = T(R_1)/W$ is such a quadratic algebra with $\dim_k R_1 = n + 1$. Since $W \subset R_1 \otimes R_1$ and $R_1 \otimes R_1$ acts on $R_1^* \otimes R_1^*$, hence on $\mathbb{P}(R_1^*) \otimes \mathbb{P}(R_1^*)$, it makes sense to define the scheme $\mathscr{V}(W) = \{(p, q) \in \mathbb{P}^n \times \mathbb{P}^n | \forall f \in W\colon f(p, q) = 0\}$, where $\mathbb{P}(R_1^*)$ is identified with $\mathbb{P}^n$. Henceforth we restrict attention to algebras R for which $\mathscr{V}(W)$ is the graph Γ_R of an automorphism σ_R of a scheme $P_R \subseteq \mathbb{P}^n$, i.e., $\mathscr{V}(W) = \Gamma_R = \{(p, \sigma_R(p)) | p \in P_R\}$. Such an R is said to have associated geometric data (P_R, σ_R). One can also ask the converse to hold:

DEFINITION 2.7. Given geometric data (P_R, σ_R) consisting of a subscheme P_R of $\mathbb{P}^n$ and $\sigma_R \in \mathrm{Aut}(P_R)$ and a quadratic algebra $R = T(R_1)/W$ with $\dim_k R_1 = n + 1$, we say that R is determined by the geometric data (P_R, σ_R) exactly when $W = \{f \in R_1 \otimes R_1 | f(\Gamma_R) = 0\}$, where $\Gamma_R \subset \mathbb{P}^n \times \mathbb{P}^n$ is the graph of σ_R.

It is important to view Γ_R as the full scheme rather than only its closed points. (Theorem 4.3 will consider a scheme consisting of a quadric and an embedded line.)

For algebras that have been of interest so far in noncommutative algebraic geometry, the Artin–Schelter regular [2] algebras of global dimensions two, three, and four, it makes sense to ask for R to have associated geometric data or to be determined by such data: all Artin–Schelter regular algebras of global dimension two have for $\mathscr{V}(W)$ the graph of an automorphism σ of a projective line $\mathbb{P}^1$ and are determined by this $(\mathbb{P}^1, \sigma)$; cf. Section 3. In the case in which R is Artin–Schelter regular of global dimension three, generated in degree one, it is shown in [3, Thm. 1] that $\mathscr{V}(W)$ is the graph of an automorphism of a cubic divisor in $\mathbb{P}^2$ and by [3, Thm. 6.7] such an R is determined by these geometric data.

The foregoing no longer holds for algebras of global dimension four. It is true that under some conditions, $\mathscr{V}(W)$ is a graph; in [21] the authors show that this is the case if R is noetherian, finitely generated in degree one with Hilbert series $(1-t)^{-4}$, Auslander regular of global dimension four, satisfying the Cohen Macaulay property, and such that both projections $\Pi_i \mathscr{V}(W)$ contain two distinct points. However, not all four-dimensional Artin–Schelter regular algebras are determined by a scheme and an automorphism of it. As mentioned in the Introduction, Theorem 4.3 provides the first known example of such an algebra.

DEFINITION 2.8 [3]. Let R be a graded algebra which is generated by R_1. A (left) point (respectively line, respectively plane) module over R is a graded cyclic (left) R-module M with Hilbert series $H_M(t) = (1-t)^{-1}$ (respectively $(1-t)^{-2}$, respectively $(1-t)^{-3}$).

For A, B and associated $(P_A, \sigma_A), (P_B, \sigma_B)$, we write the image of $P_A \times P_B$ under the Segre embedding as $P_A \circ P_B$ and points of $P_A \circ P_B$ as $p \circ q$ with $p \in P_A$ and $q \in P_B$. The two automorphisms σ_A and σ_B then induce an automorphism $\sigma_S = \sigma_{A \circ B} = \sigma_A \circ \sigma_B$ of $P_A \circ P_B$ by $\sigma_S(p \circ q) = \sigma_A(p) \circ \sigma_B(q)$ and let $\Gamma_S = \{(v, \sigma_S(v)) | v \in P_A \circ P_B\}$ be the graph of σ_S on $P_A \circ P_B$.

For algebras having a graph of an automorphism as the zero locus of the defining relations there is a one-to-one correspondence [3, Cor. 3.13] between point modules and points on (the underlying variety of) the scheme P_R. Therefore P_R is called the point scheme of R. The next theorem shows that the point scheme of the Segre product of A and B is the (Segre) product of the point schemes of A and B.

THEOREM 2.9. *If A and B have associated geometric data (P_A, σ_A) and (P_B, σ_B), then $S = A \circ B$ has associated geometric data $(P_A \circ P_B, \sigma_A \circ \sigma_B)$.*

Proof. We must show that if $\mathscr{V}(\mathscr{R}_A) = \Gamma_A$ and $\mathscr{V}(\mathscr{R}_B) = \Gamma_B$ then $\mathscr{V}(\mathscr{R}_S) = \Gamma_S$.

Put $i = \dim_k A_1$, $j = \dim_k B_1$ and let $x_1, \ldots, x_i$ and $u_1, \ldots, u_j$ be generators of A_1 and B_1. For (v, w) in Γ_S there exists a p in P_A and an r in P_B such that

$$(v, w) = (p \circ r, \sigma_S(p \circ r)) = \big((p_k r_m)_{(k,m)}, ((\sigma_A p)_l (\sigma_B r)_n)_{(l,n)}\big),$$

where, for the rest of this proof, k and l belong to $\{1, \ldots, i\}$ and m and n to $\{1, \ldots, j\}$, and $p = (x_1(p), \ldots, x_i(p))$ is written as $(p_1, \ldots, p_i)$ (similarly for r).

If h belongs to $\mathscr{R}_S$ then h is a linear combination of terms of the form $\mathscr{S}_{(23)}(f \otimes b + a \otimes g)$ with $f \in \mathscr{R}_A$, $g \in \mathscr{R}_B$, $a \in A_1 \otimes A_1$, and $b \in B_1 \otimes B_1$. In order to show that $h(v, w) = 0$ for all h in $\mathscr{R}_S$, it suffices to take

$h = \mathscr{S}_{(23)}(f \otimes u_m \otimes u_n)$, with $m, n \leq j$ and $f = \sum_{k,l \leq i} \alpha_{kl} x_k x_l$. Then, because $f \in \mathscr{R}_A$,

$$\begin{aligned} h(v,w) &= \sum_{k,l} \alpha_{kl}(x_k \otimes u_m) \otimes (x_l \otimes u_n)(v,w) \\ &= \sum_{k,l} \alpha_{k,l} p_k r_m (\sigma_A p)_l (\sigma_B r)_n \\ &= [f(p, \sigma_A p)] r_m (\sigma_B r)_n \\ &= 0 \end{aligned}$$

and so (v,w) belong to $\mathscr{V}(\mathscr{R}_S)$.

On the other hand, suppose that $h(v,w) = 0$ for all h in $\mathscr{R}_S$. We must show that $(v,w) = (p \circ r, \sigma_S(p \circ r))$ with $(p,r) \in P_A \times P_B$. Suppose that $\mathscr{R}_A$ is generated by $(f^\gamma)_\gamma$, $f^\gamma = \sum_{k,l} \alpha^\gamma_{kl} x_k \otimes x_l \in A_1 \otimes A_1$ $(\gamma \in \{1, \dots, a_1^2 - a_2\})$ and $\mathscr{R}_B$ is generated by $(g^\delta)_\delta$, $g^\delta = \sum_{m,n} \beta^\delta_{mn} u_m \otimes u_n \in B_1 \otimes B_1$ $(\delta \in \{1, \dots, b_1^2 - b_2\})$ and put $(v,w) = ((v_{km})_{(k,m)}, (w_{ln})_{(l,n)})$. From $(f^\gamma \otimes u_m \otimes u_n)(v,w) = 0$ and $(x_k \otimes x_l \otimes g^\delta)(v,w) = 0$, it follows that

$$\forall m, n \leq j: \qquad \sum_{k,l} \alpha^\gamma_{kl} v_{km} w_{ln} = 0 \tag{1}$$

$$\forall k, l \leq i: \qquad \sum_{m,n} \beta^\delta_{mn} v_{km} w_{ln} = 0. \tag{2}$$

For any point $v = (v_{km})_{(k,m)}$ in $\mathbb{P}^{ij-1}$ there exist $(p_k)_k$ $(k \in \{1, \dots, i\})$ and $(r_m)_m$ $(m \in \{1, \dots, j\})$ such that $v_{km} = p_k r_m$ if $k = i$ or $m = j$. Equations (1) imply that also for $k \neq i$ and $m \neq j$, $v_{km} = p_k r_m$, hence $v = p \circ r$. Repeat this for w and Eqs. (2) to get $w = q \circ s$. Then rewrite Eqs. (1) as $r_m s_n f^\gamma(p,q) = 0$ $(\forall \gamma, \forall m, n \leq j)$ such that $q = \sigma_A(p)$ with $p \in P_A$ and similarly $s = \sigma_B(r)$ with $r \in P_B$. ∎

If ${}_AM$ and ${}_BN$ are point modules over A and B, respectively, then $M \otimes_K N$ is a point module over $A \otimes B$ (for the double gradation this means $\dim_k(M \otimes_k N)_{(\alpha,\beta)} = 1$ for all α and β) and by Theorem 2.4 this determines a point module over $A \circ B$, namely $M \circ N = (M \otimes_k N)_0$. The converse is also true and follows from Theorem 2.9 and [3, Cor. 3.1.3].

COROLLARY 2.10. *Point modules over the Segre product $A \circ B$ are in one-to-one correspondence with products of point modules over A and B.*

More concretely, any point module over A, corresponding to $p \in P_A$, is of the form $M(p) = A/(A\alpha_1 + \cdots + A\alpha_{i-1})$ with $\alpha_m \in A_1$ and $\mathscr{V}(\alpha_1, \dots, \alpha_{i-1}) = p$. If similarly for B, $N(q) = B/(B\beta_1 + \cdots + B\beta_{j-1})$

with $\beta_m \in B_1$ and $\mathscr{V}(\beta_1, \dots, \beta_{j-1}) = q \in P_B$, then $M(p) \circ N(q) = S/(S\gamma_1 + \cdots + S\gamma_{ij-1})$ with $\gamma_m \in S_1$ and $\mathscr{V}(\gamma_1, \dots, \gamma_{ij-1}) = p \circ q \in P_S$ is the corresponding point module over S and all point modules over S are of this form.

THEOREM 2.11. *If A and B are determined by (P_A, σ_A) and (P_B, σ_B), then $S = A \circ B$ is determined by $(P_S = P_A \circ P_B,\ \sigma_S = \sigma_A \circ \sigma_B)$.*

Proof. It suffices to show that any h in $S_1 \otimes S_1$, vanishing on Γ_S, belongs to the defining relations $\mathscr{R}_S$ of S; that is, h must be a linear combination of terms of the form $\mathscr{S}_{(23)}(f \otimes b \otimes b' + a \otimes a' \otimes g)$ for certain a, a' in A_1, f in $\mathscr{R}_A, b, b'$ in B_1, and g in $\mathscr{R}_B$. Write h as $\mathscr{S}_{(23)}\sum_{i=1}^n a_i \otimes a_i' \otimes b_i \otimes b_i'$ for some a_i, a_i' in A_1 and b_i, b_i' in B_1. We give a proof by induction on the length n of this sum.

If $n = 1$ then $h = \mathscr{S}_{(23)}(a \otimes a' \otimes b \otimes b')$ and for all p in P_A and q in P_B we have $a(p)a'(\sigma_A p)b(q)b'(\sigma_B q) = 0$. If $a \otimes a'$ does not belong to $\mathscr{R}_A$, then there is a point p on P_A such that $a(p)a'(\sigma_A p) \neq 0$ and therefore $b \otimes b'$ vanishes on Γ_B and belongs to $\mathscr{R}_B$.

Suppose the statement holds for length $n = k - 1$ and take h of length $n = k$. If $h = \mathscr{S}_{(23)}\sum_{i=1}^k a_i \otimes a_i' \otimes b_i \otimes b_i'$ then $\sum_{i=1}^k a_i(p)a_i'(\sigma_A p) \times b_i(q)b_i'(\sigma_B q) = 0$ for all p in P_A and q in P_B. If for all $i \in \{1, \dots, k\}$, $a_i(p)a_i'(\sigma_A p) = 0$ for all p in P_A, then we are done. So suppose there is a j in $\{1, \dots, k\}$ such that there is a p in P_A with $a_j(p)a_j'(\sigma_A p) \neq 0$. Then $b_j \otimes b_j' + (1/a_j(p)a_j'(\sigma_A p))\sum_{i \neq j}^k a_i(p)a_i'(\sigma_A p)b_i \otimes b_i'$ vanishes on Γ_B and $b_j \otimes b_j' = (-1/a_j(p)a_j'(\sigma_A p))\sum_{i \neq j}^k a_i(p)a_i'(\sigma_A p)b_i \otimes b_i'$ in B and the length of h can be reduced by one. ∎

For the sequel it is worthwhile to pay attention to the construction of the twisted algebra (cf. [4, Sect. 8]). If τ is a graded algebra automorphism of a $\mathbb{Z}$-graded algebra R, then the twist of R by τ is a graded algebra R_τ and is defined as follows. As a graded abelian group, R_τ is isomorphic to R, but the multiplication has been changed. If $r_\tau \in R_\tau$ corresponds to $r \in R$, then the product of two homogeneous elements r_τ, s_τ is defined to be $r_\tau \cdot s_\tau = (r \cdot s^{\tau^d})_\tau$, where $d = \deg(r)$. So if $r_1, \dots, r_n$ belong to R_1 and if $f = \sum \alpha_{i_1 \dots i_n} r_{i_1} \dots r_{i_n}$ is a relation of R, then $f_\tau = \sum \alpha_{i_1 \dots i_n}(r_{i_1})_\tau (r_{i_2}^{\tau^{-1}})_\tau \dots (r_{i_n}^{\tau^{-n+1}})_\tau$ holds in R_τ.

The functor from R-Gr to R_τ-Gr which associates to $M \in R$-Gr the module M_τ in R_τ-Gr by $r_\tau m_\tau = (r^{\tau^d} m)_\tau$, where $d = \deg(r)$, defines a category equivalence R-Gr $\simeq R_\tau$-Gr [3, Cor. 8.5], hence Proj $R \simeq$ Proj R_τ. For the sequel it is important that properties like Artin−Schelter regularity, global dimension, and so on, are twisting invariant; cf. [3, Sect. 8; 24, Sect. 5].

It is clear that the Segre product behaves well with respect to twisting:

LEMMA 2.12. *Let A and B be connected. If σ and τ are algebra automorphisms of A and B, respectively, then*

$$A_\sigma \circ B_\tau \cong (A \circ B)_{\sigma \circ \tau}.$$

3. THE SEGRE PRODUCT OF ARTIN–SCHELTER REGULAR ALGEBRAS OF GLOBAL DIMENSION TWO

The Artin–Schelter regular algebras of global dimension two are easily described. As recalled in the introduction of [2], there are only two types:

- for every $q \in k^*$, a quantum plane $C(q) = k\langle x, y\rangle/\langle yx - qxy\rangle$, and
- the Jordan plane $J = k\langle x, y\rangle/\langle yx - xy - x^2\rangle$.

These algebras are connected, finitely generated in degree one with Hilbert series $\sum_{n \geq 0}(n+1)t^n = 1/(1-t)^2$, and noetherian (because they are iterated Ore extensions), and finally, they determine and are determined by some geometric data in the sense of Definition 2.7. The algebra $C(q)$ is determined by $(\mathbb{P}^1, \sigma_q(x, y) = (qx, y))$ and J by $(\mathbb{P}^1, \sigma_J(x, y) = (x, y - x))$. Since $C(q)$ and J are twists of the commutative $C(1) = k[x, y]$ by the algebra automorphisms σ_q and σ_J, we call the Proj of such an algebra a quantum $\mathbb{P}^1$.

Henceforth $S = A \circ B$ will be the Segre product of two-dimensional Artin–Schelter regular algebras A and B determined by $(\mathbb{P}^1, \sigma_A)$ and (P^1, σ_B). If (x, y) are the degree one generators of A and (u, v) those of B, then the generators of S are $(U, X, Y, Z) = (x \otimes u,\ y \otimes u,\ x \otimes v,\ y \otimes v)$. Because of Proposition 2.6, $H_S(t) = \sum_{n \geq 0}(n+1)^2 t^n = (1+t)/(1-t)^3$. Theorem 2.11 and the choice of the coordinates (U, X, Y, Z) imply that S is determined by the smooth quadric $Q = \mathscr{V}(XY - UZ)$ and the automorphism $\sigma_S = \sigma_A \circ \sigma_B$.

Notation. There are three kinds of Segre products to consider,

- $S_{a \circ b} := C(a) \circ C(b)$ (Segre product of two quantum planes) and $\sigma_{a \circ b} := \sigma_a \circ \sigma_b$,
- $S_{J \circ J} := J \circ J$ (Segre product of two Jordan planes) and $\sigma_{J \circ J} := \sigma_J \circ \sigma_J$,
- $S_{J \circ q} := J \circ C(q)$ (Segre product of a Jordan plane and a quantum plane) and $\sigma_{J \circ q} := \sigma_J \circ \sigma_q$.

Because $C(a)$ and J are twists of $C(1)$, Lemma 2.12 implies that $S_{a \circ b} = (S_{1 \circ 1})_{\sigma_a \circ \sigma_b}$, $S_{J \circ J} = (S_{1 \circ 1})_{\sigma_J \circ \sigma_J}$, and $S_{J \circ q} = (S_{1 \circ 1})_{\sigma_J \circ \sigma_q}$. All Segre products of the regular algebras of global dimension two are therefore twists of $S_{1 \circ 1} = C(1) \circ C(1) = k[u, x, y, z]/\langle XY - UZ \rangle$, the commutative homogeneous coordinate ring of Q.

We continue to write $S = A \circ B$ when it is not specified what type A and B are.

PROPOSITION 3.1.

1. *S is a noetherian domain and Auslander Gorenstein.*
2. *S is a twisted homogeneous coordinate ring of the smooth quadric $Q = \mathscr{V}(XY - UZ)$ in the sense of* [5].

Proof. As A and B are both iterated Ore extensions, so is $A \otimes B$ and therefore $A \otimes B$ is a noetherian domain and $S \subset A \otimes B$ a domain. As the degree zero part of a $\mathbb{Z}$-graded noetherian ring, S is noetherian [16, Lemma II.3.2]. On the other hand, S is a twist of the commutative homogeneous coordinate ring and by [24, Prop. 5.1, 5.2], being Auslander Gorenstein is (among other properties) twisting invariant.

For the second statement we refer to [3, 5] for the definition of a twisted homogeneous coordinate ring or to [20, Sect. 3] for the specific case of a quadric. ∎

PROPOSITION 3.2.

1. *The point modules over S are in bijective correspondence with the points on Q.*
2. *The line modules over S are in bijective correspondence with the lines on Q.*

Proof. The first statement follows from Corollary 2.10. The second statement holds for the commutative homogeneous coordinate ring and because of [4, Cor. 8.5], an algebra and its twisted algebras have equivalent categories of graded modules, so the same bijective correspondence holds. ∎

Recall from [10, Exercise I.2.15] that the lines on $Q = \mathscr{V}(XY - UZ)$ belong to two families (rulings) of lines, either of the form $\mathscr{V}(\alpha U - \beta X, \alpha Y - \beta Z)$ or of the form $\mathscr{V}(\gamma X - \delta Z, \gamma U - \delta Y)$ $(\alpha, \beta, \gamma, \delta \in k)$, and any two lines on Q intersect if and only if they belong to different rulings.

4. EMBEDDING SEGRE PRODUCTS OF QUANTUM PLANES IN QUANTUM $\mathbb{P}^3$'s

If $S = S_{1\circ 1} = k[x,y]\circ k[u,v] = k[U,X,Y,Z]/\langle XY - UZ\rangle$, the (commutative) homogeneous coordinate ring of the quadric $\mathbb{P}^1\circ\mathbb{P}^1 = \mathscr{V}(XY - UZ) \subset \mathbb{P}^3$, then it is the surjection $R = k[U,X,Y,Z] \twoheadrightarrow S$ that leads to the (commutative) projective embedding of $\operatorname{Proj} S$ into $\operatorname{Proj} R = \mathbb{P}^3$. The sequel of this paper studies a "quantization" of this embedding.

If the Proj of the coordinate ring of a quantum plane is viewed as a quantum $\mathbb{P}^1$, then (the Proj of) the Segre product S of two such algebras is a twisted homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$, and so may be viewed as a quantum quadric. For a and b in k^*, we write

$$S = S_{a\circ b} = C(a)\circ C(b),$$

where $C(a) = k\langle x,y\rangle/\langle yx - axy\rangle$ and $C(b) = k\langle u,v\rangle/\langle vu - buv\rangle$ are the algebras determined by $(\mathbb{P}^1, \sigma_a(x,y) = (ax,y))$ and $(\mathbb{P}^1, \sigma_b(u,v) = (bu,v))$. By Theorem 2.11, S is determined by the quadric $Q = \mathscr{V}(XY - UZ) \subset \mathbb{P}^3$ and the automorphism $\sigma_Q = \sigma_{a\circ b}$, given by $\sigma_{a\circ b}(U,X,Y,Z) = \sigma_a\circ\sigma_b(U,X,Y,Z) = (abU, bX, aY, Z)$. More concretely, $S_{a\circ b} = k\langle U,X,Y,Z\rangle/\langle f_1,f_2,f_3,f_4,f_5,f_6,f_7\rangle$ with

$$\begin{aligned} f_1 &= ZY - aYZ,\\ f_2 &= ZX - bXZ,\\ f_3 &= ZU - abUZ,\\ f_4 &= YX - bUZ,\\ f_5 &= YU - bUY,\\ f_6 &= XY - aUZ,\\ f_7 &= XU - aUX. \end{aligned}$$

A quantum $\mathbb{P}^3$ will be the Proj of an Artin–Schelter [2] regular graded algebra R of global dimension four with Hilbert series $H_R(t) = (1-t)^{-4}$ (properties shared with the polynomial ring on four variables).

In order to view $\operatorname{Proj} R$ as a quantum $\mathbb{P}^3$ containing $\operatorname{Proj} S$ (or to have a map $\operatorname{Proj} S \hookrightarrow \operatorname{Proj} R$ in the sense of [6, Sect. 2]), it suffices to have $R_1 \simeq S_1$ as k-vectorspaces and a graded degree zero onto homomorphism $R \twoheadrightarrow S$.

The goal is to classify all such algebras R. Our methods rely heavily on Schelter's computer program Affine. The claim that Theorem 4.3 lists *all* such algebras is based upon these calculations, which were omitted from the paper. This problem is avoided in [21]. The current strategy will be explained, but the proof of Theorem 4.3 can only show that the algebras listed there satisfy the above properties.

Strategy. The defining relations of R, given by a six-dimensional subspace of the seven-dimensional space $\langle f_1, \ldots, f_7 \rangle$, can be written as

$$\begin{cases} f_1 + A_1 f_7 = 0 \\ f_2 + A_2 f_7 = 0 \\ f_3 + A_3 f_7 = 0 \\ f_4 + A_4 f_7 = 0 \\ f_5 + A_5 f_7 = 0 \\ f_6 + A_6 f_7 = 0 \end{cases} \quad \text{or} \quad \begin{cases} f_1 + A_1 f_6 = 0 \\ f_2 + A_2 f_6 = 0 \\ f_3 + A_3 f_6 = 0 \\ f_4 + A_4 f_6 = 0 \\ f_5 + A_5 f_6 = 0 \\ f_7 = 0 \end{cases}$$

$$\text{or} \ldots \text{or} \quad \begin{cases} f_1 + A_1 f_2 = 0 \\ f_3 = 0 \\ f_4 = 0 \\ f_5 = 0 \\ f_6 = 0 \\ f_7 = 0 \end{cases} \quad \text{or} \quad \begin{cases} f_2 = 0 \\ f_3 = 0 \\ f_4 = 0 \\ f_5 = 0 \\ f_6 = 0 \\ f_7 = 0 \end{cases}$$

with parameters $(A_i)_i \in k$.

A first selection, which turned out be the most important, is to take only those six-dimensional subspaces yielding already a correct Hilbert series up to a certain degree. This was done with the use of Affine; cf. [2]. Affine uses the Diamond lemma (cf. [8]) to calculate $\dim_k R_n$ for fixed degree n. Therefore one takes all monomials of degree n and reduces the number of independent ones using the given relations of lower degree. At the same time some overlaps can occur (for example, in degree three, one has $ZXY = (ZX)Y = Z(XY)$). If such an overlap is not associative, this adds a new relation and lowers the degree n part of the Hilbert series. When one checks the associativity of an overlap, the parameters a, b, A_i appear. Taking the right values for a, b, A_i is then the only way to change the associativity and to control the Hilbert series. With the use of Affine, we checked Hilbert series up to degree five, which imposed certain relations on the a, b, A_i. Once such algebras were found, general techniques were used (cf. the proof of Theorem 4.3) to check regularity and the Hilbert series in higher degree. All calculations were done over $\mathbb{C}$ but Theorem 4.3 holds for any algebraically closed field. The following claim lists all solutions found in this way, but Theorem 4.3 classifies them in a more constructive (geometric) manner.

Remark 4.1. All algebras are classified up to isomorphism. For example, the exchange $a \leftrightarrow b$ (which corresponds to $S = C(b) \circ C(a)$ instead of $C(a) \circ C(b)$) is obtained by interchanging $X \leftrightarrow Y$.

CLAIM 4.2. *The following six-dimensional spaces of relations are, up to isomorphisms* (*the only ones*), *determining a four-dimensional Artin–Schelter regular algebra with Hilbert series* $(1-t)^{-4}$ *surjecting on* $S_{a \circ b}$:

1. *for a and b arbitrary*: $\langle f_1, f_2, f_3, f_4 + A_4 f_6, f_5, f_7 \rangle$ *with* $A_4 \neq 0$,
2. *for* $a = 1$, $b = 1$: $\langle f_1 + (-A_3 A_6 - A_2 A_5 + A_3 A_4) f_7, f_2 + A_2 f_7, f_3 + A_3 f_7, f_4 + A_4 f_7, f_5 + A_5 f_7, f_6 + A_6 f_7 \rangle$ *with* $A_2 A_5 + A_4(A_6 - A_3) \neq 0$,
3. *for a arbitrary*, $b = 1$: $\langle f_1 + (-A_3 A_6 + aA_3) f_7, f_2, f_3 + A_3 f_7, f_4 + f_7, f_5, f_6 + A_6 f_7 \rangle$ *with* $A_6 - A_3 \neq 0$,
4. *for* $a = -1$, $b = -1$: $\langle f_1 - A_5 f_7, f_2 + f_7, f_3, f_4, f_5 + A_5 f_7, f_6 \rangle$ *with* $A_5 \neq 0$,
5. *for* $a = 1$, $b = -1$: $\langle f_1 + (-A_2 A_5 + A_4^2) f_7, f_2 + A_2 f_7, f_3 + A_4 f_7, f_4 + A_4 f_7, f_5 + A_5 f_7, f_6 \rangle$ *with* $A_4^2 \neq A_2 A_5$,
6. *for a arbitrary*, $b = -1$: $\langle f_1 + a^2 f_7, f_2, f_3 + af_7, f_4 + f_7, f_5, f_6 \rangle$.

Observation. A detailed study of the previous list reveals the following facts that lead to the classification of Theorem 4.3.

- If $R = T(R_1)/\langle W \rangle$ with $W \subset R_1 \otimes R_1$, then $\mathscr{V}(W) = \Gamma_\sigma$, the graph of an automorphism σ of a scheme $P \subset \mathbb{P}^3$, and therefore P is the (point) scheme parametrizing the point modules over R. Further $Q \subset P$ and $\sigma|_Q = \sigma_Q = \sigma_{a \circ b}$. Recall that this scheme P is the zero locus of the 4×4 minors of the matrix M determining R; that is, if $W = (g_1, \ldots, g_6)$ then M is such that $(g_i)^t = M(U\,X\,Y\,Z)^t$.

- If $P \neq Q$, which happens in cases 1 to 4 of Claim 4.2, and if $P \neq \mathbb{P}^3$ (so R is not a twist of a polynomial ring), then $P = Q \cup L$ for some line $L \subset \mathbb{P}^3$. If P consists of Q and an embedded line $L \subset Q$, e.g., $P = \mathscr{V}(Y(XY - UZ), Z(XY - UZ))$, we still denote $P = Q \cup L$ (although this "union" does not satisfy $Q \cup L = Q$ in the case $L \subset Q$). In cases 1 to 4, L is respectively $\mathscr{V}(U, Z)$, $\mathscr{V}(Z + A_3 X - A_2 U, Y + A_5 X + (A_6 - A_4)U)$, $\mathscr{V}(Z + A_3 X, Y + (A_6 - 1)U)$, and $\mathscr{V}(A_5 X + Y, U + Z)$. However, a good choice of generators for R_1 enables the two linear terms defining L, as well as the defining relations of R, to be expressed more simply. By [20, Lemma 1.1] any line L intersecting $Q = \mathscr{V}(XY - UZ)$ in two distinct points can be written in the form $\mathscr{V}(U, Z)$. In the same way we may assume that any tangent line is of the form $\mathscr{V}(\alpha X + Y, Z)$ for some scalar $\alpha \neq 0$, and any embedded line $L \subset Q$ of the form $L = \mathscr{V}(Y, Z)$ (e.g., the line $\mathscr{V}(Y - U, Z - X)$ belongs to $Q = \mathscr{V}(XY - UZ)$

$= \mathscr{V}((X+Z)(Y-U)-(U+Y)(Z-X)))$. Such R are determined by the geometric data $(Q \cup L, \sigma)$ in the sense of Definition 2.7 and $\sigma_Q = \sigma|_Q$ defines an algebra automorphism on R (recall that we identified $\mathbb{P}(R_1^*) \simeq \mathbb{P}^3$). Hence all these algebras twist to an algebra mapping onto $S_{1\circ 1}$, the commutative homogeneous coordinate ring of Q. It is shown in Theorem 4.3 that all these algebras are (twists of) central extensions of three-dimensional Artin–Schelter regular algebras [11].

- If $P = Q$, which is the case in 5 and 6 of Claim 4.2, the σ_Q is not defined on R and R is not determined by the geometric data $(P, \sigma) = (Q, \sigma_Q)$. Furthermore, all algebras of case 6 twist to an algebra of case 5, which maps onto $S_{1\circ -1}$.

THEOREM 4.3. (i) *The following algebras R are Artin–Schelter regular algebras of global dimension four with Hilbert series* $1/(1-t)^4$:

Type 1. The four-dimensional polynomial ring, determined by the geometric data $(P = \mathbb{P}^3, \sigma = \mathrm{id})$.

Type 2. Algebras R mapping onto the commutative homogeneous coordinate ring $S_{1\circ 1}$ of Q and determined by the geometric data (P, σ). The point scheme $P = Q \cup L$ is the union of the quadric Q and a line $L \subset \mathbb{P}^3$ and σ is an automorphism of P with $\sigma|_Q = \mathrm{id}_Q$ and $\sigma|_L$ an automorphism of L. These $(L, \sigma|_L)$ are either

1. $L = \mathscr{V}(U, Z)$, intersecting Q in two distinct points, and $\sigma|_L(0, X, Y, 0) = (0, \alpha X, Y, 0)$ for some scalar $\alpha \in k \setminus \{0,1\}$; then $R = k\langle U, X, Y, Z\rangle$ with defining relations

$$\begin{aligned} &ZY = YZ, && ZX = XZ, && ZU = UZ,\\ &YX - \alpha XY = -(\alpha-1)UZ, && YU = UY, && XU = UX \end{aligned}$$

or

2. $L = \mathscr{V}(\alpha X + Y, Z)$, tangent to Q, and $\sigma|_L(U, X, -\alpha X, 0) = (U + \beta X, X, -\alpha X, 0)$ for some scalars α and β with $\alpha\beta \neq 0$; then $R = k\langle U, X, Y, Z\rangle$ with defining relations

$$\begin{aligned} &ZY = YZ, && ZX = XZ, && ZU = UZ,\\ &YX = XY, && YU - UY = -\alpha(XU - UX), && XY - UZ = -\frac{\alpha}{\beta}(XU - UX) \end{aligned}$$

or

3. $L = \mathscr{V}(Y, Z) \subset Q$ and σ is uniquely defined on P as follows:
on $\mathscr{V}(U-1)$: $\sigma(1, X, Y, Z) = (1, X + (1/\alpha)XY -$

$(1/\alpha)Z, Y, Z)$,

on $\mathscr{V}(X-1)$: $\sigma(U,1,Y,Z) = (U - (1/\alpha)Y + (1/\alpha)UZ, 1, Y, Z)$,
on $\mathscr{V}(Y-1)$: $\sigma(U,X,1,Z) = (U,X,1,Z)$ and
on $\mathscr{V}(Z-1)$: $\sigma(U,X,Y,1) = (U,X,Y,1)$

with $\alpha \neq 0$; then $R = k\langle U,X,Y,Z\rangle$ with defining relations

$$ZY = YZ, \qquad ZX = XZ, \qquad ZU = UZ,$$
$$YX = XY, \qquad YU = UY, \qquad XY - UZ = -\alpha(XU - UX).$$

Type 3. Algebras R mapping onto $S_{1\circ -1}$ with point scheme Q, not determined by the geometric data $(P,\sigma) = (Q, \sigma_Q)$; $R = k\langle U,X,Y,Z\rangle$ with defining relations of the form

$$ZY - YZ = -\left(A_4^2 - A_2 A_5\right)(XU - UX), \qquad YX + UZ = -A_4(XU - UX),$$
$$ZX + XZ = -A_2(XU - UX), \qquad YU + UY = -A_5(XU - UX),$$
$$ZU + UZ = -A_4(XU - UX), \qquad XY = UZ,$$

with $A_2, A_4, A_5 \in k$ and $A_2 A_5 - A_4^2 \neq 0$.

Type 4. For all a and b in k^*: twists by $\sigma_{a\circ b}$ of the Type 1 algebra R; determined by $(\mathbb{P}^3, \sigma_{a\circ b})$.

For those a and b in k^* for which $\sigma_{a\circ b}$ is defined on a Type 2 algebra R: twists by $\sigma_{a\circ b}$ of such Type 2 algebras R; determined by $(Q \cup L, \sigma_{a\circ b} \circ \sigma)$.

For those a and b in k^* for which $\sigma_{a\circ b}(\sigma_{1\circ -1})^{-1}$ is defined on a Type 3 algebra: twists by $\sigma_{a\circ b}(\sigma_{1\circ -1})^{-1}$ of such Type 3 algebras R; determined by $(Q, \sigma_{a\circ b} \circ (\sigma_{1\circ -1})^{-1} \circ \sigma)$.

(ii) Every Artin–Schelter regular algebra R of global dimension four and with Hilbert series $1/(1-t)^4$ which maps onto the Segre product $S_{a\circ b}$ $(a, b \in k^*)$ is isomorphic to an algebra of the above types.

Proof. As stated above, the proof of (ii) relies on the calculations with Affine. We prove (i) here.

Type 1. The algebra of Type 1, the four-dimensional polynomial ring, determined by the geometric data $(\mathbb{P}^3, \mathrm{id})$, clearly satisfies the hypotheses.

Type 2. These algebras are determined by $Q \cup L$ and an automorphism σ of $Q \cup L$ with $\sigma|_Q = \sigma_{1\circ 1} = \mathrm{id}_Q$ and $\sigma|_L$ an automorphism of L (therefore fixing the intersection points of $Q \cap L$). We first describe the geometric data that can occur and give the defining relations of the corresponding R. As mentioned before, L is assumed to be either $\mathscr{V}(U,Z), \mathscr{V}(\alpha X + Y, Z)$, or $\mathscr{V}(Y,Z)$.

1. If $L = \mathscr{V}(U, Z)$ then any automorphism of L fixing the intersection points with Q is of the form $\sigma|_L(0, X, Y, 0) = (0, \alpha X, Y, 0)$ for some scalar α in k^*. To exclude twists of a polynomial ring (classified in Type 1), take $\alpha \neq 1$ (otherwise σ extends to an automorphism of $\mathbb{P}^3$). By [20, Lemma 1.3(a)], this geometry determines the defining relations of R as given above.

2. If $L = \mathscr{V}(\alpha X + Y, Z)$ with $\alpha \neq 0$, tangent to Q, the situation is slightly more complicated. Any automorphism of L fixing the point of intersection is of the form $\sigma|_L(U, X, -\alpha X, 0) = (U + \beta X, \gamma X, -\alpha\gamma X, 0)$ for some β, γ in k, but only $\gamma = 1$ is a candidate for an algebra with a correct Hilbert series. In the case $\gamma \neq 1$, this geometry determines only five defining relations instead of six. A better argument will be given later on in the proof. If $\gamma = 1$, exclude $\beta = 0$ (twists of the polynomial ring) and in a way similar to that above the defining relations of R can be determined.

3. If L is an embedded line on Q, then L is of the form $\mathscr{V}(Y, Z)$. If the matrix M is such that the defining relations of R can be written in the form $M \cdot (U, X, Y, Z)^t = 0$, then $M(p)\sigma(p) = 0$ for p on P implies that the vector $\sigma(p)$ is orthogonal to the six row vectors of M. The matrix M has rank three at every point of $\mathbb{P}^3$; so if m^l_{ijk} denotes the minor of the 3×3 matrix obtained by deleting rows i, j, k and column l from M, then there always exist i, j, k such that $\sigma(U, X, Y, Z) = (-m^1_{ijk}, m^2_{ijk}, -m^3_{ijk}, m^4_{ijk})$ defines a point in $\mathbb{P}^3$ and in particular $M(p)\sigma(p) = 0$ for p on P. Then σ can be described on the affine opens $\mathscr{V}(U - 1), \mathscr{V}(X - 1), \mathscr{V}(Y - 1), \mathscr{V}(Z - 1)$ if one takes respectively $(i, j, k) = (1, 2, 4), (1, 3, 5), (2, 3, 6)$, and $(4, 5, 6)$. Thus σ is defined on P and this determines the given relations of R if only those f in $R_1 \otimes R_1$ are taken that vanish on the full *scheme* Γ_Q, where $Q \supset L$.

It remains to show that all Type 2 algebras are Artin–Schelter regular with Hilbert series $(1 - t)^{-4}$. In [20, Cor. 1.9, Prop. 2.3], this has been done in the case $L = \mathscr{V}(U, Z)$ and the same technique can be applied to tangent and embedded lines. We prefer the methods of [11] because of their use later. All algebras whose defining relations are given above are shown to be regular as central extensions of three-dimensional Artin–Schelter regular algebras; i.e., R has a central, regular element c in R_1 such that $C = R/Rc$ is a three-dimensional Artin–Schelter regular algebra.

It is clear that for all algebras considered, Z is central in R_1. If $C = R/RZ$ and if R is Artin–Schelter regular, then C must be a three-dimensional Artin–Schelter regular algebra. For such a regular C with generators U, X, Y we can choose a basis of the three-dimensional space of quadratic relations (h_1, h_2, h_3) such that there is a 3×3 matrix M with entries in C and a matrix N of $GL_3(k)$ such that the relations of C can be written as $(h_1, h_2, h_3)^t = M(U, X, Y)^t$ and $(U, X, Y)M = (N(h_1, h_2, h_3)^t)^t$. (This is exactly the property of being a standard algebra in [3, Thm. 1].) Take representatives for U, X, Y in R_1, still denoted by U, X, Y, and consider their span in R_1 as a copy of C_1. Therefore the equations of R may be written in the form

$$g_j = h_j + Zl_j + \alpha_j Z^2 = 0, \qquad ZU - UZ = 0, \qquad ZX - XZ = 0,$$
$$ZY - YZ = 0 \tag{3}$$

with $j = 1, \dots, 3$, $l_1, l_2, l_3 \in C_1$, and $\alpha_1, \alpha_2, \alpha_3 \in k$. Let $(U^*, X^*, Y^*)^t \in C_1^3$ be defined by $(U^*, X^*, Y^*)^t = N^t(U, X, Y)^t$. Theorem 3.1.3 of [11] describes when regularity can be lifted from C to R:

Equations (3) define a four-dimensional regular algebra if and only if there exist $(\gamma_j)_j \in k$, $j = 1, \dots, 3$, that form a solution to the following system of linear equations in $C_1^{\otimes 2}$, C_1, and k,

$$\sum_j \gamma_j h_j = \sum_j (x_j l_j - l_j x_j^*),$$
$$\sum_j \gamma_j l_j = \sum_j \alpha_j (x_j - x_j^*),$$
$$\sum_j \gamma_j \alpha_j = 0,$$

where one takes $(x_1, x_2, x_3) = (U, X, Y)$. If such $(\gamma_j)_j$ exist, then they are uniquely determined by $(l_j)_j$.

This theorem may be applied to all three algebras of Type 2. For instance, in the case of the tangent line $\mathscr{V}(\alpha X + Y, Z)$,

$$(h_1, h_2, h_3) = \left(YX - XY, YX - YU + UY, \frac{\beta}{\alpha} XY + XU - UX\right),$$

$$(l_1, l_2, l_3) = \left(0, -\beta U, -\frac{\beta}{\alpha} U\right), \qquad (\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0),$$

$$M = \begin{pmatrix} 0 & Y & -X \\ -Y & \beta Y & U \\ X & -U & (\beta/\alpha) X \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & 0 & 0 \\ \beta & 1 & 0 \\ \beta/\alpha & 0 & 1 \end{pmatrix}.$$

Then $C = k\langle U, X, Y\rangle/(h_1, h_2, h_3)$ determines a three-dimensional Artin−Schelter regular algebra: from the above, C is standard and because the nine conics defined by the 2×2 minors of M have no common zero, C is nondegenerate; cf. [3, Thm. 1]. Its point variety, the locus of zeros of $\det M$, is the intersection $(Q \cup L) \cap \mathscr{V}(z)$ consisting of the three lines $\mathscr{V}(X), \mathscr{V}(Y), \mathscr{V}(\alpha X + Y)$ in $\mathbb{P}^2 \simeq \mathscr{V}(z) \subset \mathbb{P}^3$. (For the other positions of L, the point variety will be a conic union a line or a line union a double line.)

An easy calculation shows that $(\gamma_1, \gamma_2, \gamma_3) = (-\beta^2/\alpha, \beta/\alpha, -\beta)$; hence, by [11, Cor. 2.3, 2.7, Thm. 3.1.3], R is an Artin−Schelter regular noetherian domain with Hilbert series $(1-t)^{-4}$, Auslander regular, and Koszul and satisfies the Cohen−Macaulay property. More details on central extensions can be found in [11].

Remark. The above imposes the restriction $\gamma = 1$ on the automorphism $\sigma|_L(U, X, -\alpha X, 0) = (U + \beta X, \gamma X, -\alpha\gamma X, 0)$ of the tangent line, because otherwise the quotient R/RZ would not be three-dimensional Artin−Schelter regular: the geometric data $((Q \cup L) \cap \mathscr{V}(Z)$, the identity on $Q \cap \mathscr{V}(Z)$, and $\sigma|_L$ on $L \cap \mathscr{V}(Z))$ can only determine a regular algebra if $\gamma = 1$; cf. condition (1.2) in [3, Thms. 2, 3].

Type 3. The algebras of Type 3, whose relations are stated above, all have the same associated data $(Q, \sigma_{1\circ -1})$ and are therefore not determined by these geometric data (which determine $S_{1\circ -1}$). Because any normal degree one element of R must be normal in $S_{1\circ -1}$ (a twist of the commutative $S_{1\circ 1}$ by $\sigma_{1\circ -1}$), such a normal element must commute by means of a scalar multiple of $\sigma_{1\circ -1}$. However, $\sigma_{1\circ -1}$ does not define an algebra automorphism of R so no normal elements can exist in R_1 and other techniques must be used. Furthermore, this means that R can never twist to an algebra mapping onto the (commutative) homogeneous coordinate ring of Q.

First, $S_{1\circ -1}$ is a noetherian domain and Auslander Gorenstein by Proposition 3.1. Furthermore $S_{1\circ -1}$ is a Koszul algebra (cf. [11, Sect. 2; 18, Sect. 4] because by [20], $S_{1\circ -1}$ is a quotient of a Koszul algebra by a normal degree two element, hence a Koszul algebra itself (cf. [9]).

The sequel of this proof depends on some calculations, easily checked with Affine: if $A_2 A_5 - A_4^2 \neq 0$ then

- $\Omega = XU - UX$ is normal in R_2 and

- $R^! = T(V^*)/W^\perp$, the dual of $R = T(V)/W$, is finite dimensional with Hilbert series $H_{R^!}(t) = (1+t)^4$ (V^* is the vector space dual of V and $W^\perp$ the orthogonal of W in $V^* \otimes V^*$; cf. [17, Sect. 4.2]).

More precisely, if $\{u, x, y, z\}$ is the basis of V^* dual to $\{U, X, Y, Z\}$ then $R^! = k\langle u, x, y, z\rangle$ with defining relations

$$
\begin{aligned}
&u^2 = x^2 = y^2 = z^2 = 0\\
&\left(A_2A_5 - A_4^2\right)zy - (A_2xz + A_4xy + A_4uz + A_5uy + ux) = 0\\
&zx - xz = 0\\
&zu + yx - xy - uz = 0\\
&\left(A_2A_5 - A_4^2\right)yz + (A_2xz + A_4xy + A_4uz + A_5uy + ux) = 0\\
&yu - uy = 0\\
&xu + ux = 0
\end{aligned}
$$

and the elements (u, x, y, z) form a basis for $R^!_1$, (xy, xz, yx, ux, uy, uz) for $R^!_2$, (xyx, uxy, uxz, uyx) for $R^!_3$, and $uxyx$ for $R^!_4$.

Since S is a Koszul algebra, $H_{S^!}(t) = (H_S(-t))^{-1} = (1+t)^3(1-t)^{-1}$ and so $H_{R^!}(t) = (1-t^2)H_{S^!}(t)$. As $S = R/\Omega R$ with Ω normal in R_2, it is clear that $R^! = S^!/S^!\omega S^!$ for some ω in $S^!_2$. One may check (with Affine) that $\omega = (A_2A_5 - A_4^2)zy - (A_2xz + A_4xy + A_4uz + A_5uy + ux)$ is 1-regular in $S^!$, i.e., $\dim S^!_1\omega = \dim S^!_1 = \dim \omega S^!_1$. Then ω is normal because

$$
\begin{aligned}
4 = \dim R^!_3 &= \dim S^!_3 - \dim\left(S^!_1\omega + \omega S^!_1\right)\\
&\le \dim S^!_3 - \dim S^!_1\omega = \dim S^!_3 - 4
\end{aligned}
$$

and $\dim S^!_3 = 8$ implies that $S^!_1\omega = S^!_1\omega + \omega S^!_1 = \omega S^!_1$. The exact sequence

$$
0 \to \operatorname{Ann}(\omega) \to S^! \xrightarrow{\cdot\omega} S^! \to S^!/S^!\omega = R^! \to 0
$$

combined with $H_{R^!}(t) = (1-t^2)H_{S^!}(t)$ then implies that ω is regular.

Since S is a Koszul algebra we get that $S^!$ is a Koszul algebra, and so are R and $R^!$ (a quotient of a Koszul algebra by the degree two normal regular element ω; cf. [9]). It follows that R has finite global dimension since R is a Koszul algebra and $R^!$ is finite dimensional.

Comparison of $H_R(t) = (H_{R^!}(-t))^{-1} = (1-t)^{-4}$ with $H_S(t)$ shows that Ω is regular. So by [12, Prop. 3.5, 3.6] we can lift the fact that S is noetherian and Auslander Gorenstein to R, which is then Gorenstein and Artin–Schelter regular but also Auslander regular and Koszul and satisfies the Cohen–Macaulay property.

Type 4. It is clear that, if the Type 1 algebra or a Type 2 algebra $R \twoheadrightarrow S_{1\circ 1}$ (respectively Type 3 algebra $R \twoheadrightarrow S_{1\circ -1}$) and a and b are such that $\sigma_{a\circ b}$ (resp. $\sigma_{a\circ b}(\sigma_{1\circ -1})^{-1}$) are defined on R, then the twisted algebra $R_{\sigma_{a\circ b}} \twoheadrightarrow S_{a\circ b}$ (resp. $R_{\sigma_{a\circ b}(\sigma_{1\circ -1})^{-1}} \twoheadrightarrow S_{a\circ b}$) satisfies the desired properties. To end this proof we briefly indicate which $\sigma_{a\circ b}$ can occur.

If R is the four-dimensional polynomial ring, then $\sigma_{a \circ b}$ is defined on R for all a and b in k^*.

In order to twist algebras of Type 2, $\sigma_{a \circ b}$ is defined on R

- for all a and b if L intersects Q in two distinct points, or

- for $a = b = 1$ if L is tangent (so algebras with an associated tangent line always map onto $S_{1 \circ 1}$), or

- for arbitrary a and $b = 1$ if L is a line of the following ruling of lines on Q: $\mathscr{V}(\gamma X - \delta Z, \gamma U - \delta Y)$ (by symmetry the lines of the other ruling will occur for arbitrary b but $a = 1$).

As algebras R determined by $(Q \cup L, \sigma)$, mapping onto the homogeneous coordinate ring of Q, have two central elements v, w in R_1 (where $L = \mathscr{V}(v, w)$), their twists will have at least one normal element in degree one, the eigenvector for $\sigma_{a \circ b}$ in $kv + kw$. To twist the algebras of Type 3, $\sigma_{a \circ b}(\sigma_{1 \circ -1})^{-1}$ is defined on R

- for $a = -1, b = 1, A_4 = 0$ (by symmetry of the case $a = 1, b = -1$, cf. Remark 4.1), or

- for a arbitrary, $b = -1$ and $A_2 = A_5 = 0$, which is the solution 6 of Claim 4.2. ∎

From the proof of the previous theorem it follows that

PROPOSITION 4.4. *The algebras listed in Theorem* 4.3 *are Koszul, noetherian, and Auslander regular and satisfy the Cohen–Macaulay property.*

Let R be one of the regular algebras described in Theorem 4.3. The following theorems describe the point, line, and plane modules over such an R. It is well known that such modules are respectively of the form $R/Ru + Rv + Rw$, $R/Ru + Rv$, and R/Ru for u, v, w in R_1 (cf. [11, Thms. 4.1.1, 5.1.1], hence determine a point, line, and plane in $\mathbb{P}^3$. We now classify these points, lines, and planes.

THEOREM 4.5. *Let R be as above; then the point modules over R are in bijective correspondence with the points on*

1. $\mathbb{P}^3$ *if R is a twist of the polynomial ring,*
2. $Q \cup L$ *if R is a twist of the algebras of Type* 2 *in Theorem* 4.3,
3. Q *if R is a twist of the algebras of Type* 3 *in Theorem* 4.3.

Proof. This is an easy consequence of [3, Cor. 3.13] and Theorem 4.3. ∎

Since R is a domain with Hilbert series $(1-t)^{-4}$ it is clear that

THEOREM 4.6. *Let R be as above; then the plane modules over R are in bijective correspondence with the planes in $\mathbb{P}^3$.*

THEOREM 4.7.

1. *The line modules over (twists of) the four-dimensional polynomial ring are in bijective correspondence with the lines in $\mathbb{P}^3$.*

2. *The line modules over (twists of) the algebras of Type 2 in Theorem 4.3, determined by $(Q \cup L, \sigma)$, are in bijective correspondence with the lines in $\mathbb{P}^3$ that either lie on Q or meet L.*

3. *The line modules over (twists of) the algebras of Type 3 in Theorem 4.3 with associated (Q, σ_Q) are in bijective correspondence with the lines on Q or lines that can be described as follows: for every point $p \in \mathbb{P}^3$ there is a plane T_p with the property that all lines in T_p containing this point p are line modules. For the Type 3 algebras with parameters A_2, A_4, A_5 as in Theorem 4.3, T_p is given by*

$$\begin{aligned} T_p = \mathscr{V}\big(&((-A_2A_5 + A_4^2)p_1 - A_2p_2 + A_4p_3\big)U \\ &-\big((-A_2A_5 + A_4^2)p_0 - A_4p_2 + A_5p_3\big)X \\ &+(A_2p_0 - A_4p_1 + p_3)Y - (A_4p_0 - A_5p_1 + p_2)Z\big), \end{aligned}$$

where $p = (p_0, p_1, p_2, p_3)$.

Proof. 1. This correspondence holds for the commutative polynomial ring, and therefore for its twists.

2. A classification of the line modules in the case $L = \mathscr{V}(U, Z)$ can be found in [20, Sect. 2.2]. Instead of generalizing this to arbitrary L, we use [11, Sect. 5], which describes the line modules over central extensions of three-dimensional Artin–Schelter regular algebras.

As in the proof of Theorem 4.3, Z is a central element of R_1 such that $C = R/RZ$ is three-dimensional Artin–Schelter regular. By [11, Sect. 5] we find that all line modules over C (which are in bijective correspondence with the lines in $\mathbb{P}^3$ lying in $\mathscr{V}(Z)$), are line modules over R. If a line module $R/Ru + Rv$ corresponds to a line $l = \mathscr{V}(u, v)$ not lying in the plane $\mathscr{V}(Z)$, then $R/Ru + Rv + RZ$ is a point module over C. So $l \cap \mathscr{V}(Z)$ belongs to P_C, the point variety of C, which is the intersection of the point variety of R with $\mathscr{V}(Z)$. By [11, Thm. 5.1.6], a line through $p \in P_C \subset \mathscr{V}(Z)$ corresponds to a line module if and only if it is contained in either $\mathscr{V}(Z)$ or Q_p. This Q_p is a scheme in $\mathbb{P}^3$, defined as follows: with notations as in the proof of Theorem 4.3, we can write the relations of R as $\langle g_j = h_j + Zl_j + \alpha_j Z^2 = 0,\ ZU - UZ = 0,\ ZX - XZ = 0,\ ZY - YZ =$

$0\rangle$. If we take $\zeta = (\zeta_1, \zeta_2, \zeta_3)^t$ to be the coordinates of $\sigma|_{P_C}(p)$, $g = (g_1, g_2, g_3)^t$ and the matrix N as before, then $Q_p = \mathscr{V}(\zeta^t N g)$.

Let us briefly indicate what this means for the tangent $L = \mathscr{V}(\alpha X + Y, Z)$. In this case P_C is the union of the three lines $\mathscr{V}(X), \mathscr{V}(Y)$, $\mathscr{V}(\alpha X + Y)$ in $\mathscr{V}(Z) \subset \mathbb{P}^3$ and σ is the identity on the first two lines and $\sigma(U, X, -\alpha X) = (U + \beta X, X, -\alpha X)$ on the third one. It is easy to see that for $p \in \mathscr{V}(X), Q_p$ is either $\mathbb{P}^3$ or Q; for $p \in \mathscr{V}(Y), Q_p$ is Q and for $p \in \mathscr{V}(\alpha X + Y), Q_p$ is $\mathbb{P}^3$. The lines through those $p \in P_C$ lying on Q_p and the lines in $\mathscr{V}(Z)$ are the lines on Q and the lines meeting L.

3. By [13, Prop. 2.8; 11, Thm. 5.1.1], there is a bijective correspondence between the (left) line modules M and the space of rank two tensors $u \otimes a - v \otimes b \in \mathscr{R}_R$ (the space of relations of R) such that $M \cong R/Ra + Rb$. Because a plane intersects Q in either a nondegenerate conic or two distinct lines, there are only a few possibilities for $\text{div}_Q(a)$ and $\text{div}_Q(b)$.

- If $\text{div}_Q(a) = \text{div}_Q(b)$ then $u \otimes a - v \otimes b$ is of rank 1.

- Let $C \neq C'$ be two conics, $l_1 \neq l_3$ (resp. $l_2 \neq l_4$) be two lines on Q belonging to the first (resp. second) family of lines.

—If $(\text{div}_Q(a), \text{div}_Q(b)) = (C, C'), (C, l_1 + l_2)$, or $(l_1 + l_2, l_3 + l_4)$ then $u = \sigma(b)$ and $v = \sigma(a)$ is the only way to make $u \otimes a - v \otimes b$ vanish on the graph of σ on Q. Note that we have put $\sigma(b)(p) = b(\sigma(p))$ for p in Q.

—If $\text{div}_Q(a) = l_1 + l_2$ and $\text{div}_Q(b) = l_1 + l_4$ then $l = \mathscr{V}(a, b) = l_1$ so l lies on Q and these lines already correspond to line modules over R because of Proposition 3.2.

So we must classify the lines $l = \mathscr{V}(a, b)$ such that $\sigma(b)a - \sigma(a)b$ belongs to $\mathscr{R}_R$.

Put $a = pU + qX + rY + sZ$ and $b = p'U + q'X + r'Y + s'Z$ in R_1. In order to have $\sigma(b) \otimes a - \sigma(a) \otimes b$ vanish on the graph of σ on Q, $\sigma(b)a - \sigma(a)b$ must belong to $\mathscr{R}_S$ (the space of relations of S). As before, $\mathscr{R}_S = \langle f_1, \ldots, f_7 \rangle$ and the relations of R are $\langle f_1 + (A_4^2 - A_2 A_5) f_7, f_2 + A_2 f_7, \ldots, f_6 + A_6 f_7 \rangle$ (in Thm. 4.3, f_7 was called Ω). Because $\mathscr{R}_S = \mathscr{R}_R + k f_7$, there is a surjective homomorphism ϕ: $\mathscr{R}_S \to k$ with kernel $\mathscr{R}_R$, sending $(f_1, f_2, f_3, f_4, f_5, f_6, f_7)$ to $(-(-A_2 A_5 + A_4^2), -A_2, -A_4, -A_4, -A_5, 0, 1)$. A calculation shows that $\sigma(b)a - \sigma(a)b = pq' f_7 + pr'(-f_5) + ps'(-f_3) + qp'(-f_7) + qr'(-f_4 - f_6) + qs'(-f_2) + rp' f_5 + rq'(f_4 + f_6) + rs'(-f_1) + sp' f_3 + sq' f_2 + sr' f_1$. The condition for $l = \mathscr{V}(a, b)$ to be a line module is then

$$\begin{aligned} 0 &= \phi(\sigma(b)a - \sigma(a)b) \\ &= 1(pq' - qp') + A_5(pr' - rp') + A_4(ps' - sp') \\ &\quad + A_4(qr' - rq') + A_2(qs' - sq') + \left(-A_2A_5 + A_4^2\right)(rs' - sr') \end{aligned}$$

or

$$0 = N_{23} - A_5N_{13} + A_4N_{03} + A_4N_{12} - A_2N_{02} + \left(-A_2A_5 + A_4^2\right)N_{01}, \quad (4)$$

where N_{ij} is the corresponding minor of N, determined by the line $\mathscr{V}(a, b)$:

$$N = \begin{pmatrix} s & -r & q & -p \\ s' & -r' & q' & -p' \end{pmatrix}.$$

Equation (4) can also be written as $a \wedge b \wedge T = 0$ (cf. [7]), where a, b, $a \wedge b$ and T are given by the tuples

$$a = (s \;\; -r \; q \;\; -p), \qquad b = (s' \;\; -r' \; q' \;\; -p'),$$
$$a \wedge b = (N_{01} \; N_{02} \; N_{03} \; N_{12} \; N_{13} \; N_{23}), \qquad T = (T_{01} \; T_{02} \; T_{03} \; T_{12} \; T_{13} \; T_{23})$$

and

$$a \wedge b \wedge T = N_{01}T_{23} - N_{02}T_{13} + N_{03}T_{12} + N_{12}T_{03} - N_{13}T_{02} + N_{23}T_{01}.$$

Of course $a = (s, -r, q, -p)$ corresponds to the plane $a = \mathscr{V}(pU + qX + rY + sZ)$ and $a \wedge b$ to the line $\mathscr{V}(a, b)$.

Consider $p = (p_0, p_1, p_2, p_3)$ a point in $\mathbb{P}^3$ and define T_p to be the plane $p \vee T = \delta U - \gamma X + \beta Y - \alpha Z$ containing p and given by the 4-tuple

$$\begin{aligned} (\alpha, \beta, \gamma, \delta) = (&p_0T_{12} - p_1T_{02} + p_2T_{01}, p_0T_{13} - p_1T_{03} + p_3T_{01}, p_0T_{23} \\ &-p_2T_{03} + p_3T_{02}, p_1T_{23} - p_2T_{13} + p_3T_{12}). \end{aligned}$$

The condition $a \wedge b \wedge T = 0$ is then equivalent to $p \in a \wedge b$ and $a \wedge b \subset T_p$ (or $p \wedge a \wedge b = 0$ and $a \wedge b \wedge T_p = 0$).

Then T corresponds to a line, i.e., $T = c \wedge d$ if and only if $T_{01}T_{23} - T_{02}T_{13} + T_{03}T_{12} = 0$ (this is the so-called Plücker relation), and otherwise T corresponds to some $c \wedge d + e \wedge f$; cf. [7, Prop. 5.5]. (Note that we could so far repeat this proof for the Type 2 algebras, determined by $(Q \cup L, \sigma)$. In that case T is the line L and T_p the plane through p and L.) Here T is given by $T = (1 \;\; A_5 \;\; A_4 \;\; A_4 \;\; A_2 \;\; (-A_2A_5 + A_4^2))$ and the Plücker relation is $-A_2A_5 + A_4^2 = 0$, which was excluded from Theorem

4.3. Therefore T is not a line and the plane T_p is given by

$$
\begin{aligned}
T_p\colon & \big(\big(-A_2A_5 + A_4^2\big)p_1 - A_2p_2 + A_4p_3\big)U \\
& -\big(\big(-A_2A_5 + A_4^2\big)p_0 - A_4p_2 + A_5p_3\big)X \\
& +(A_2p_0 - A_4p_1 + p_3)Y - (A_4p_0 - A_5p_1 + p_2)Z = 0
\end{aligned}
$$

or $T_p = \{q \in \mathbb{P}^3 | p\mathscr{M}q = 0\}$, where $\mathscr{M}$ is an anti-symmetric matrix $(p\mathscr{M}p = 0$ because $p \in T_p)$

$$
\mathscr{M} = \begin{pmatrix} 0 & -\big(-A_2A_5 + A_4^2\big) & A_2 & -A_4 \\ -A_2A_5 + A_4^2 & 0 & -A_4 & A_5 \\ -A_2 & A_4 & 0 & -1 \\ A_4 & -A_5 & 1 & 0 \end{pmatrix}.
$$

Observe that $\mathscr{M}$ is a regular matrix with determinant $4(-A_2A_5 + A_4^2)^2$. If the line modules intersect a fixed line (as for the algebras with point scheme $Q \cup L$) this matrix is singular. ∎

Comparison of the Hilbert series of R and S yields that $R/\langle\Omega\rangle = S$ for some Ω in R_2. One can check that for all algebras of Theorem 4.3, Ω is normal. Such an Ω commutes by means of an algebra automorphism θ such that $\Omega r = \theta(r)\Omega$ for all r in R. For the Type 3 algebras with point scheme Q, the automorphism θ is given by

$$
\theta(U, X, Y, Z) = \left(\frac{A_4Y - A_5Z}{A_2A_5 - A_4^2}, \frac{A_2Y - A_4Z}{A_2A_5 - A_4^2}, A_4U - A_5X, A_2U - A_4X\right)
$$

and $\theta^{-1} = -\theta$. If s is the automorphism of $\mathbb{P}^3$ given by

$$
s\begin{pmatrix} U \\ X \\ Y \\ Z \end{pmatrix}^t = \begin{pmatrix} A_4U - A_5X + Y \\ A_2U - A_4X + Z \\ \big(A_2A_5 - A_4^2\big)U + A_4Y - A_5Z \\ \big(A_2A_5 - A_4^2\big)X + A_2Y - A_4Z \end{pmatrix}^t
$$

then $s^2 = \theta$ and s is not an automorphism of Q. However, in [23] we show the existence of a quadric K such that s is an automorphism of $K \cap Q$. The quadric K consists of those points p of Q for which the line through p and θp lies on Q. For such points p, the plane T_p intersects Q in a degenerate conic. In general, the plane T_p is the tangent plane to K in $\theta s(p)$. We remark that this gives us three points on T_p which (in most cases) determine T_p.

COROLLARY 4.8. *The plane T_p of Theorem 4.7 is the plane through the three points p, θp, and θsp. If $p = \theta p$ then T_p is the tangent plane to Q in p.*

Because any line $l \subset \mathbb{P}^3$ intersects Q it suffices to look at the planes T_p for p in Q. An explicit description of the $\mathbb{P}^1$ of lines through p can be given (by describing all points q_p for which the line through p and q_p is a line module); cf. [23]. Furthermore one can check that there are only four points on Q for which $\theta p = p$ and then $p = sp$.

To conclude, we remark on some material for further study in [22]. The algebras R of Type 3 with point scheme Q have no normal elements in degree one but three centrals in degree two, respectively of the form $v^2, w^2, vw + wv$ for some v and w in R_1. Apart from the normal degree two element Ω with (finite order) automorphism θ, R has only two other normal degree two elements. They commute with even degree and anti-commute with odd degree elements. The former implies that R has no central elements in odd degrees and we claim there will even be no normals in odd degrees. This and a proof that these algebras are finite over their center appears in [22]. Of course these remarks fail for twists of such algebras.

Remark 4.9. Some of the algebras determined by $(Q \cup L, \sigma)$ with $L = \mathscr{V}(U, Z)$ can also be found in the following way. In [14] one considers a matrix $\mathscr{R}$ a solution of the quantum Yang–Baxter equation and constructs an algebra of quantum vectors $V(\mathscr{R})$, quantum covectors $V^*(\mathscr{R})$ (cf. [14, Examples 2.4, 2.5]), and quantum matrices $A(\mathscr{R})$. If these algebras have generators $(v^i), (x_i), (t_j^i)$, $i, j = 1 \ldots n$, then they satisfy the relations $v^i v^k = \mathscr{R}_{jl}^{ik} v^l v^j$, $x_i x_k = x_n x_m \mathscr{R}_{ik}^{mn}$, and $\mathscr{R}_{mn}^{ik} t_j^m t_l^n = \mathscr{R}_{jl}^{mn} t_n^k t_m^i$. If $\mathscr{R}_{21}$ is the transpose of $\mathscr{R}$, i.e., $(\mathscr{R}_{12})_{kl}^{ij} = \mathscr{R}_{lk}^{ji}$, then the image of the algebra homomorphism $A(\mathscr{R}) \to V(\mathscr{R}_{21}) \otimes V^*(\mathscr{R})$ (cf. [14, Prop. 2.7]) is what we called the Segre product of $V(\mathscr{R}_{21})$ and $V^*(\mathscr{R})$.

For the multiparameter solution $\mathscr{R}_{kl}^{ij} = \delta_k^i \delta_l^j (\delta_j^i + \theta^{ji}(q_{ij})^{-1} + \theta^{ij} q_{ji}/r^2) + \delta_l^i \delta_k^j \theta^{ij}(1 - r^{-2})$ ($\theta_{ij} = 1$ if $i > j$ and zero otherwise), vectors in $V(\mathscr{R}_{21})$ and covectors in $V^*(\mathscr{R})$ satisfy the relations $v^j v^i = (q_{ij})^{-1} v^i v^j$, $x_i x_j = r^2 (q_{ij})^{-1} x_j x_i$. The algebra $A(\mathscr{R})$ is then surjective on their Segre product. For this $\mathscr{R}$, $A(\mathscr{R})$ is an iterated Ore extension and therefore Artin–Schelter regular. This provides some examples of embeddings in higher dimensions. For the two-parameter $\mathscr{R}$, $A(\mathscr{R})$ is the algebra determined by $(Q \cup \mathscr{V}(U, Z), \sigma_{a \circ b}$ on Q and the identity on $L)$, a twist of a Type 2 algebra of Theorem 4.3.

The discussion in Section 4 focused on embeddings of the Segre product of two quantum planes; instead one may consider the Segre product of two Jordan planes or a Jordan plane with a quantum plane. We believe these cases introduce no new ideas and so we leave them for the general treatment in [21]. Indeed, some calculations with Affine established that in the case of two Jordan planes J no algebras with point scheme Q occur. For algebras with point scheme $Q \cup L$ we now do have, in contrast to Theorem 4.3, an algebra with an associated tangent line that maps onto an algebra different from $S_{1 \circ 1}$. On the other hand, no lines can occur that intersect Q in two distinct points because of the structure of $\sigma_J \circ \sigma_J$ (an automorphism of $Q \cup L$ must fix $Q \cap L$ but no such two intersection points on Q exist for $\sigma_J \circ \sigma_J$).

ACKNOWLEDGMENTS

The author thanks M. Artin, L. Le Bruyn, M. Vancliff, M. Van den Bergh, F. Van Oystaeyen, and the referee for their advice on this subject.

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