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Generalized Nonlinear Variational Inclusions with Noncompact Valued Mappings

NAN-JING HUANG
Department of Mathematics, Sichuan University
Chengdu, Sichuan 610064, P.R. China

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Abstract—In this paper, we introduce and study a new class of set-valued nonlinear generalized variational inclusion with noncompact valued mappings and construct a new iterative algorithm. We prove the existence of solutions for this class of variational inclusion and the convergence of iterative sequences generated by this algorithm.

Keywords—Variational inclusion, Set-valued mapping, Algorithm, Existence, Convergence.

1. INTRODUCTION

Variational inequalities, introduced by Hartman and Stampacchia [1] in the early sixties, are a very powerful tool of the current mathematical technology. These have been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium and engineering sciences, etc. Quasivariational inequalities are a generalized form of variational inequalities in which the constraint set depends on the solution. These were introduced and studied by Bensoussan, Goursat and Lions [2]. For further details we refer to [3–7].

In 1991, Chang and Huang [8,9] introduced and studied some new class of complementarity problems and variational inequalities for set-valued mappings with compact values in Hilbert spaces. In the recent paper [10], Hassouni and Moudafi have studied a new class of variational inclusions, which included many variational and quasivariational inequalities considered by Noor [11–13], Isac [14], and Siddiqi and Ansari [15,16] as special cases.

The main purpose of this work is to extend their ideas to more general problems. Especially, let H be a real Hilbert space endowed with a norm $\|\cdot\|$, and inner product $\langle\cdot,\cdot\rangle$. Given set-valued mappings $T, A : H \to 2^H$ (where 2^H denotes the family of all nonempty subsets of H) and single-valued mappings $f, p, g : H \to H$ with $\operatorname{Im} g \bigcap \operatorname{dom}(\partial\varphi) \neq \phi$, we consider the following problem.

Find $u \in H$, $w \in Tu$, $y \in Au$, such that $g(u) \bigcap \operatorname{dom}(\partial\varphi) \neq \phi$, and

$$\langle f(w) - p(y), v - g(u)\rangle \geq \varphi(g(u)) - \varphi(v), \qquad \forall v \in H, \tag{1.1}$$

where $\partial\varphi$ denotes the subdifferential of a proper, convex and lower semicontinuous function $\varphi : H \to R \cup \{+\infty\}$. This problem is called a set-valued nonlinear generalized variational inclusion.

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It is clear that the set-valued nonlinear generalized variational inclusion (1.1) includes many kinds of variational inequalities and quasivariational inequalities of [6–19] as special cases.

2. ITERATIVE ALGORITHM

LEMMA 2.1. *u, w and y are solutions of problem (1.1) if and only if there exists $w \in Tu$, $y \in Au$ such that*

$$g(u) = J_\alpha^\varphi \left(g(u) - \alpha \left(f(w) - p(y)\right)\right), \tag{2.1}$$

where $\alpha > 0$ is a constant and $J_\alpha^\varphi = (I + \alpha\partial\varphi)^{-1}$ is the so-called proximal mapping on H.

PROOF. From the definition of J_α^φ one has

$$g(u) - \alpha \left(f(w) - p(y)\right) \in g(u) + \alpha\partial\varphi \left(g(u)\right),$$

and hence

$$p(y) - f(w) \in \partial\varphi \left(g(u)\right).$$

From the definition of $\partial\varphi$ we have

$$\varphi(v) \geq \varphi\left(g(u)\right) + \langle p(y) - f(w), v - g(u)\rangle, \qquad \forall v \in H.$$

Thus u, w and y are solutions of (1.1). This completes the proof.

To obtain an approximate solution of (1.1), we can apply a successive approximation method to the problem of solving

$$u \in F(u) \tag{2.2}$$

where

$$F(u) = u - g(u) + J_\alpha^\varphi \left(g(u) - \alpha \left(f(Tu) - p(Au)\right)\right).$$

Based on (2.1) and (2.2), we proceed with our algorithm.

Let $T, A : H \to CB(H)$ (where $CB(H)$ denotes the family of all nonempty closed bounded subsets of H). For given $u_0 \in H$, let $w_0 \in Tu_0$, $y_0 \in Au_0$ and

$$u_1 = u_0 - g(u_0) + J_\alpha^\varphi \left(g(u_0) - \alpha \left(f(w_0) - p(y_0)\right)\right).$$

By [20], there exists $w_1 \in Tu_1$ and $y_1 \in Au_1$ such that

$$\|w_1 - w_0\| \leq (1+1)\widehat{\mathbf{H}}(Tu_1, Tu_0), \qquad \|y_1 - y_0\| \leq (1+1)\widehat{\mathbf{H}}(Au_1, Au_0),$$

where $\widehat{\mathbf{H}}$ is the Hausdorff metric on H. By induction, we can obtain our algorithm as follows.

ALGORITHM 2.1. *Let $T, A : H \to CB(H)$, and $f, p : H \to H$. For given $u_0 \in H$, we can get an algorithm for (1.1) as follows:*

$$\begin{aligned} & u_{n+1} = u_n - g(u_n) + J_\alpha^\varphi \left(g(u_n) - \alpha \left(f(w_n) - p(y_n)\right)\right), \\ w_n \in Tu_n, \quad & \|w_{n+1} - w_n\| \leq \left(1 + (1+n)^{-1}\right) \widehat{\mathbf{H}} \left(Tu_{n+1}, Tu_n\right), \\ y_n \in Au_n, \quad & \|y_{n+1} - y_n\| \leq \left(1 + (1+n)^{-1}\right) \widehat{\mathbf{H}} \left(Au_{n+1}, Au_n\right), \\ & n = 0, 1, 2, \dots. \end{aligned} \tag{2.3}$$

REMARK 2.1. *Algorithm 2.1 includes several known algorithms of [6,8–13,15–19] as special cases.*

3. EXISTENCE AND CONVERGENCE

DEFINITION 3.1. *A mapping $g : H \to H$ is said to be*

(i) *strongly monotone if there exists some $\delta > 0$ such that*

$$\langle g(u_1) - g(u_2), u_1 - u_2 \rangle \geq \delta \|u_1 - u_2\|^2, \qquad \forall u_i \in H, \quad i = 1, 2,$$

(ii) *Lipschitz continuous if there exists some $\sigma > 0$ such that*

$$\|g(u_1) - g(u_2)\| \leq \sigma \|u_1 - u_2\|, \qquad \forall u_i \in H, \quad i = 1, 2.$$

DEFINITION 3.2. *A set-valued mapping $T : H \to 2^H$ is said to be*

(i) *strongly monotone with respect to a mapping $f : H \to H$ if there exists some $\beta > 0$ such that*

$$\langle f(w_1) - f(w_2), u_1 - u_2 \rangle \geq \beta \|u_1 - u_2\|^2, \qquad \forall u_i \in H, \quad w_i \in Tu_i, \quad i = 1, 2,$$

(ii) *$\widehat{\mathbf{H}}$-Lipschitz continuous if there exists some $\gamma > 0$ such that*

$$\widehat{\mathbf{H}}(Tu_1, Tu_2) \leq \gamma \|u_1 - u_2\|, \qquad \forall u_i \in H, \quad i = 1, 2.$$

THEOREM 3.1. *Let $g : H \to H$ be strongly monotone and Lipschitz continuous, f, $p : H \to H$ be Lipschitz continuous, $T, A : H \to CB(H)$ be $\widehat{\mathbf{H}}$-Lipschitz continuous and T be strongly monotone with respect to f. If the following conditions hold:*

$$\left|\alpha - \frac{\beta + \epsilon\mu(k-1)}{\eta^2\gamma^2 - \epsilon^2\mu^2}\right| < \frac{\sqrt{(\beta + (k-1)\epsilon\mu)^2 - (\eta^2\gamma^2 - \epsilon^2\mu^2)k(2-k)}}{\eta^2\gamma^2 - \epsilon^2\mu^2}, \tag{3.1}$$

$$\beta > (1-k)\epsilon\mu + \sqrt{(\eta^2\gamma^2 - \epsilon^2\mu^2)k(2-k)}, \qquad \eta\gamma > \epsilon\mu, \tag{3.2}$$

$$\alpha\mu\epsilon < 1 - k, \qquad k = 2\sqrt{1 - 2\delta + \sigma^2}, \qquad k < 1, \tag{3.3}$$

where β and δ are strongly monotone constants of T and g, respectively, γ and μ are $\widehat{\mathbf{H}}$-Lipschitz constants of T and A, respectively, and σ, η and ϵ are the Lipschitz constants of g, f and p, respectively, then there exist $u \in H$, $w \in Tu$, $y \in Au$, such that $g(u) \bigcap \operatorname{dom}(\partial\varphi) \neq \phi$ and (1.1) is satisfied. Moreover, $u_n \to u$, $w_n \to w$, $y_n \to y$, $n \to \infty$, where $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ are defined in Algorithm 2.1.

PROOF. From (2.3) we have

$$\|u_{n+1} - u_n\| = \|u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) + J_\alpha^\varphi(h(u_n)) - J_\alpha^\varphi(h(u_{n-1}))\|,$$

where $h(u_n) = g(u_n) - \alpha(f(w_n) - p(y_n))$. Also we have

$$\begin{aligned}\|J_\alpha^\varphi(h(u_n)) - J_\alpha^\varphi(h(u_{n-1}))\| \leq \|h(u_n) - h(u_{n-1})\| &\leq \|u_n - u_{n-1} - \alpha(f(w_n) - f(w_{n-1}))\| \\ &+ \|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| + \alpha\|p(y_n) - p(y_{n-1})\|.\end{aligned}$$

That is

$$\begin{aligned}\|u_{n+1} - u_n\| \leq{} & 2\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| \\ & + \|u_n - u_{n-1} - \alpha(f(w_n) - f(w_{n-1}))\| + \alpha\|p(y_n) - p(y_{n-1})\|.\end{aligned} \tag{3.4}$$

By Lipschitz continuity and strong monotonicity of g, we obtain

$$\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\|^2 \leq (1 - 2\delta + \sigma^2)\|u_n - u_{n-1}\|^2. \tag{3.5}$$

Also from $\widehat{\mathbf{H}}$-Lipschitz continuity and strong monotonicity of T, and Lipschitz continuity of f, we have

$$\|u_n - u_{n-1} - \alpha\left(f(w_n) - f(w_{n-1})\right)\|^2 \le \left(1 - 2\beta\alpha + \alpha^2\eta^2(1+n^{-1})^2\gamma^2\right)\|u_n - u_{n-1}\|^2. \quad (3.6)$$

By $\widehat{\mathbf{H}}$-Lipschitz continuity of A, Lipschitz continuity of p and (2.3), we know

$$\alpha\|p(y_n) - p(y_{n-1})\| \le \alpha\epsilon(1+n^{-1})\mu\|u_n - u_{n-1}\|. \quad (3.7)$$

So by combining (3.4)–(3.7) and denoting

$$\theta_n := 2\sqrt{1-2\delta+\sigma^2} + \sqrt{1-2\beta\alpha+\alpha^2\eta^2(1+n^{-1})^2\gamma^2} + \alpha\epsilon(1+n^{-1})\mu,$$

we get

$$\|u_{n+1} - u_n\| \le \theta_n\|u_n - u_{n-1}\|.$$

Letting $\theta := 2\sqrt{1-2\delta+\sigma^2} + \sqrt{1-2\beta\alpha+\alpha^2\eta^2\gamma^2} + \alpha\epsilon\mu$, we know that $\theta_n \searrow \theta$. It follows from (3.1)–(3.3) that $\theta < 1$. Hence $\theta_n < 1$, for n sufficiently large. Therefore $\{u_n\}$ is a Cauchy sequence and we can suppose that $u_n \to u \in H$.

Now we prove that $w_n \to w \in Tu$, $y_n \to y \in Au$. In fact, it follows from Algorithm 2.1 that

$$\|w_n - w_{n-1}\| \le (1+n^{-1})\gamma\,\|u_n - u_{n-1}\|,$$
$$\|y_n - y_{n-1}\| \le (1+n^{-1})\mu\,\|u_n - u_{n-1}\|;$$

i.e., $\{w_n\}$ and $\{y_n\}$ are Cauchy sequences. Let $w_n \to w$, $y_n \to y$. Further we have

$$\begin{aligned}\varrho(w,Tu) &= \inf\{\|w-z\| : z \in Tu\} \le \|w - w_n\| + \varrho(w_n, Tu)\\ &\le \|w - w_n\| + \widehat{\mathbf{H}}(Tu_n, Tu) \le \|w - w_n\| + \gamma\|u_n - u\| \to 0.\end{aligned}$$

Hence, $w \in Tu$. Similarly, $y \in Au$. This completes the proof.

REMARK 3.1. *For a suitable choice of the operators g, T, A, f, p and the function φ, we can obtain several known results [6,8–19] as special cases of the main result of this paper.*

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