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Letter to the Editor

On the Filon and Levin methods for highly oscillatory integral $\int_a^b f(x)e^{i\omega g(x)}\,dx$ ☆

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Abstract

This paper shows that for any suitably smooth function $f(x)$ and arbitrarily selected interpolation nodes $c_1, c_2, \ldots, c_v$ in $[a, b]$, the Filon method and the Levin method for $\int_a^b f(x)e^{i\omega g(x)}\,dx$ with the polynomial interpolation approach are identical when $g(x)$ is a linear function. Based on this result, a new efficient Levin quadrature for $\int_a^b f(x)e^{i\omega g(x)}\,dx$ is presented.

MSC: 65D32; 65D30

Keywords: Oscillatory integral; Filon method; Levin method

1. Introduction

The quadrature of highly oscillating integrals is important in many areas of applied mathematics. Highly oscillatory integrals are allegedly difficult to calculate by the standard classic integration formulas when the frequency is significantly larger than the number of quadrature points. Many methods have been developed since Filon [2], such as in [12,14,11,8,9,3,4,6,5], etc.

The Filon quadrature of the form $\int_a^b f(x)e^{i\omega g(x)}\,dx$ is achieved by approximating $f(x)$ by a polynomial $p(x)$ with degree $\leqslant v-1$ at interpolation nodes $c_1, c_2, \ldots, c_v$ and calculating $\int_a^b p(x)e^{i\omega g(x)}\,dx$ instead of $\int_a^b f(x)e^{i\omega g(x)}\,dx$. Iserles [3,4] analyzed the convergent behavior in a range of frequency regimes and showed that the accuracy increases when oscillation becomes faster. Recently Iserles and Nørsett [5,6] extended the approach of Iserles [3,4] and defined the generalized Filon method for $\int_a^b f(x)e^{i\omega g(x)}\,dx$ and showed that the rate of decay of the error, once frequency grows, can be increased. Both the Filon method and the generalized Filon method, an approach of $f(x)$ by splines, are efficient for suitably smooth functions under the condition that the first few moments $\int_a^b x^k e^{i\omega g(x)}\,dx$ can be explicitly calculated (cf. [6]).

The Levin method (cf. [8]) is also efficient and applicable to a wider class of $\int_a^b f(x)e^{i\omega g(x)}\,dx$ without explicit computation of the moments. The integration problem is transformed into a certain O.D.E. problem, and this is solved by a collocation technique at nodes $c_1, c_2, \ldots, c_v$. Levin [10] showed that for ω large enough the accuracy increases

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when oscillation becomes faster. In this paper, we show that for any suitably smooth function $f(x)$ and arbitrarily selected interpolation nodes $c_1, c_2, \ldots, c_v$ in $[a, b]$, the Filon method and the Levin method for $\int_a^b f(x)e^{i\omega g(x)}\,dx$ with the polynomial interpolation approach are identical in case that $g(x)$ is a linear function. Based on this result, we present a special efficient Levin quadrature for $\int_a^b f(x)e^{i\omega g(x)}\,dx$.

2. The identity of the two methods with the polynomial interpolation approach in case that $g(x)$ is a linear function

Let $I(f)$ denote the following integral:

$$I(f) = \int_a^b f(x)e^{i\omega g(x)}\,dx, \tag{2.1}$$

where f and g are suitably smooth functions.

The Filon quadrature is achieved by approximating $f(x)$ by a polynomial $p(x)$ with degree $\leqslant v-1$ at interpolation nodes $c_1, c_2, \ldots, c_v$ and calculating

$$Q_F(f) = \int_a^b p(x)e^{i\omega g(x)}\,dx. \tag{2.2}$$

The spirit of the Levin method (cf. [8]) is based upon the fact that if f were of the form

$$f(x) = \phi'(x) + i\omega g'(x)\phi(x) \equiv L^{(1)}\phi(x), \quad a \leqslant x \leqslant b,$$

then the integral could be evaluated as

$$I(f) = \int_a^b (\phi'(x) + i\omega g'(x)\phi(x))e^{i\omega g(x)}\,dx = \phi(b)e^{i\omega g(b)} - \phi(a)e^{i\omega g(a)}.$$

Select a polynomial $\phi(x)$ with degree $\leqslant v-1$ such that

$$\phi'(c_j) + i\omega g'(c_j)\phi(c_j) = f(c_j), \quad j = 1, 2, \ldots, v, \tag{2.3}$$

and calculate

$$Q_L(f) = \int_a^b (\phi'(x) + i\omega g'(x)\phi(x))e^{i\omega g(x)}\,dx = \phi(b)e^{i\omega g(b)} - \phi(a)e^{i\omega g(a)}. \tag{2.4}$$

Theorem 2.1. *The Levin method and the Filon method are identical for* $\int_a^b f(x)e^{i\omega Ax}\,dx$ $(A \neq 0)$ *for arbitrarily selected interpolation nodes* $c_1, c_2, \ldots, c_v$ *in* $[a, b]$ *and any suitably smooth function* $f(x)$.

Proof. Let $p(x)$ be the interpolation polynomial for the Filon method and $\phi(x)$ be the polynomial for the Levin collocation method. Then

$$\phi'(c_j) + i\omega A\phi(c_j) = f(c_j) = p(c_j), \quad j = 1, \ldots, v. \tag{2.5}$$

Note that $\phi'(x) + i\omega A\phi(x)$ is also a polynomial with degree $\leqslant v-1$ and by the Fundamental Theorem of Algebra (see [1, pp. 101–103]),

$$\phi'(x) + i\omega A\phi(x) \equiv p(x).$$

Hence

$$\begin{aligned} Q_L(f) &= \phi(b)e^{i\omega Ab} - \phi(a)e^{i\omega Aa} \\ &= \int_a^b (\phi'(x) + i\omega A\phi(x))e^{i\omega Ax}\,dx \\ &= \int_a^b p(x)e^{i\omega Ax}\,dx \\ &= Q_F(f). \quad \square \end{aligned}$$

Remark 2.1. (i) The Filon method and Levin method are identical when the interpolation degree is greater than or equal to 3 and Gauss–Legendre nodes are used, a negative answer to a statement in [3].

(ii) If we consider Hermite interpolation approximation at $c_1 = a$ and $c_2 = b$ for the Filon method and Levin method in Theorem 2.1, Theorem 2.1 is also true since in this case $\phi'(x) + \mathrm{i}\omega A\phi(x)$ is also the Hermite interpolation of $f(x)$ at $c_1 = a$ and $c_2 = b$ and $\phi'(x) + \mathrm{i}\omega A\phi(x) \equiv p(x)$.

(iii) Theorem 2.1 is true only in the case that $g(x)$ is linear. If $g(x)$ is nonlinear with $g'(x) \neq 0, \forall x \in [a, b]$, let $f(x)=p(x) \equiv 1, v=1$ and $c_v=x_0 \in [a, b]$, then $Q_F(f)=I(f)=\int_a^b \mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x$ and $Q_L(f)=\mathrm{e}^{\mathrm{i}\omega g(b)}-\mathrm{e}^{\mathrm{i}\omega g(a)}/\mathrm{i}\omega g'(x_0)$. We can select an $x_0 \in [a, b]$ such that $Q_F(f) \not\equiv Q_L(f)$ since $g(x)$ is nonlinear and $g'(x)$ cannot be a constant in $[a, b]$. Then the two quadratures are not identical. For nonlinear function $g(x)$, in most cases, the moment $\int_a^b x^k \mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x$ cannot be computed explicitly.

In the following we present a new efficient Levin quadrature for $I(f) = \int_a^b f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x$ and extend the Filon method to $I(f) = \int_a^b f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x$ without computing the moments where $g'(x) \neq 0, \forall x \in [a, b]$.

By Darboux's Intermediate Value Theorem (cf. [15, p. 84]), $g'(x)$ has the same sign in $[a, b]$ and $g(x)$ is monotonic in $[a, b]$. Let $y = g(x)$, then x can be written as $x = g^{-1}(y)$ and

$$I(f) = \int_a^b f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x = \int_{g(a)}^{g(b)} \frac{f(x)}{g'(x)}\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y = \int_{g(a)}^{g(b)} \frac{f(g^{-1}(y))}{g'(g^{-1}(y))}\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y. \tag{2.6}$$

Let $\phi(y)$ be the Levin collocation polynomial at nodes $d_j = g(c_j)$, $j = 1, 2, \ldots, v$. Then $\phi(y)$ satisfies

$$\phi'(d_j) + \mathrm{i}\omega\phi(d_j) = \frac{f(g^{-1}(d_j))}{g'(g^{-1}(d_j))}, \quad d_j = g(c_j), \quad j = 1, 2, \ldots, v.$$

That is,

$$\phi'(g(c_j))g'(c_j) + \mathrm{i}\omega\phi(g(c_j))g'(c_j) = f(c_j), \quad j = 1, 2, \ldots, v.$$

By Theorem 2.1 in the case of Fourier transform (2.6), we get that:

Corollary 2.1. *For* $\int_a^b f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x$, *let* $\phi(g(x)) = \sum_{k=0}^{v-1} a_k(g(x))^k$ *satisfy*

$$\{[\phi(g(x))]' + \mathrm{i}\omega g'(x)\phi(g(x))\}|_{c_j} = f(c_j), \quad j = 1, 2, \ldots, v,$$

then

$$Q_F(f, y) = Q_L(f, y) = Q_L(f) = \phi(g(b))\mathrm{e}^{\mathrm{i}\omega g(b)} - \phi(g(a))\mathrm{e}^{\mathrm{i}\omega g(a)}, \tag{2.7}$$

where $Q_F(f, y)$ *and* $Q_L(f, y)$ *are the Filon and Levin Quadratures at nodes* $d_j = g(c_j)$, $j = 1, 2, \ldots, v$ *for* $\int_{g(a)}^{g(b)} f(g^{-1}(y))/g'(g^{-1}(y))\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y$.

Based on the van der Corput lemma (cf. Stein [13, p. 332]) and Corollary 2.1, we give a numerical analysis for (2.7). For convenience, here we only consider the case $\omega > 0$.

Lemma 2.1 (*van der Corput*). *Suppose* $g(x)$ *is real-valued and smooth in* (a, b), *and that* $|g^{(k)}(x)| \geqslant 1$ *for all* $x \in (a, b)$ *for a fixed value of* k. *Then*

$$\left|\int_a^b \mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x\right| \leqslant c(k)\omega^{-1/k}$$

holds when (i) $k \geqslant 2$, *or* (ii) $k = 1$ *and* $g'(x)$ *is monotonic. The bound* $c(k)$ *is independent of* g *and* ω, $c(k) = 5 \cdot 2^{k-1} - 2$.

Lemma 2.2 (*Stein [13, p. 334]*). *Under the assumptions on* $g(x)$ *in Lemma* 2.1, *we can conclude that*

$$\left|\int_a^b \mathrm{e}^{i\omega g(x)}\varphi(x)\,\mathrm{d}x\right| \leqslant c(k)\omega^{-1/k}\left[|\varphi(b)| + \int_a^b |\varphi'(x)|\mathrm{d}x\right].$$

Theorem 2.2. *Suppose that $f(x)$ and $g(x)$ are suitably smooth and $g'(x) \neq 0$ for all x in $[a, b]$. Then the special quadrature* (2.7) *at arbitrary interpolation nodes $c_1 < \cdots < c_v$ satisfies*

$$E(f) = |I(f) - Q_L(f)| \leqslant \frac{3(1+v)\|G^{(v)}(y)\|_\infty |g(b) - g(a)|^v}{\omega v!},$$

where $G(y) = f(g^{-1}(y))/g'(g^{-1}(y))$ and $y = g(x)$.

Proof. Let $p(y)$ be the interpolation polynomial of $f(g^{-1}(y))/g'(g^{-1}(y))$ at nodes $d_j = g(c_j)$, $j = 1, 2, \ldots, v$. By Corollary 2.1 and (2.7)

$$E(f) = |I(f) - Q_{\mathrm{L}}(f)| = |I(f) - Q_{\mathrm{F}}(f, y)| = \left| \int_{g(a)}^{g(b)} \Phi(y) \mathrm{e}^{\mathrm{i}\omega y} \, \mathrm{d}y \right|, \tag{2.8}$$

where $\Phi(y) := f(g^{-1}(y))/g'(g^{-1}(y)) - p(y)$ with $\Phi(d_j) = 0$, $j = 1, 2, \ldots, v$. Then for $\Phi(y)$ there exists $y_j \in (d_j, d_{j+1})$ such that

$$\Phi'(y_j) = 0, \quad j = 1, 2, \ldots, v-1.$$

By Theorem 8.1 in [7, p. 157], $\Phi(y)$ and $\Phi'(y)$ can be represented by

$$\Phi(y) = \frac{\Phi^{(v)}(\xi_1)}{v!} \prod_{j=1}^{v} (y - d_j), \quad \Phi'(y) = \frac{\Phi^{(v)}(\xi_2)}{(v-1)!} \prod_{j=1}^{v-1} (y - y_j), \tag{2.9}$$

for some $\xi_1, \xi_2 \in [g(a), g(b)]$ depending on y. Then by Lemma 2.2 for $k = 1$ and $c(1) = 3$, (2.9) and noticing that $p^{(v)}(y) \equiv 0$, we have

$$\begin{aligned} E(f) = |I(f) - Q_{\mathrm{L}}(f)| = |I(f) - Q_{\mathrm{F}}(f)| &= \left| \int_{g(a)}^{g(b)} \Phi(y) \mathrm{e}^{\mathrm{i}\omega y} \, \mathrm{d}y \right| \\ &\leqslant c(k)\omega^{-1} \left(|\Phi(g(b))| + \int_{g(a)}^{g(b)} |\Phi'(y)| dy \right) \\ &\leqslant \frac{3(1+v)\|G^{(v)}(y)\|_\infty |g(b) - g(a)|^v}{\omega v!}. \qquad \square \end{aligned}$$

Remark 2.2. (i) In Theorem 2.2, if $c_1 = a$ and $c_v = b$, integrating by parts, we can get

$$E(f) = \frac{1}{\omega} \left| \int_{g(a)}^{g(b)} \Phi'(y) \mathrm{e}^{\mathrm{i}\omega y} \, \mathrm{d}y \right|.$$

Similar to the proof in Theorem 2.2, the above error can be represented by

$$E(f) \leqslant \frac{3v\|G^{(v)}(y)\|_\infty |g(b) - g(a)|^{v-1}}{\omega^2 (v-1)!}. \tag{2.10}$$

(ii) The error of the composite quadrature of (2.7) with v interpolation nodes including the endpoints of each subinterval is

$$E_{h,L}(f) \leqslant \frac{3v\|G^{(v)}(y)\|_\infty (b-a)\|g'(x)\|_\infty^{v-1} h^{v-2}}{\omega^2 (v-1)!}, \tag{2.11}$$

where h is the length of each subinterval.

Example 2.1. Let us consider the numerical quadrature for $\int_0^1 \cos(\sin x) \cos x \mathrm{e}^{\mathrm{i}\omega \sin x} \, \mathrm{d}x$ by (2.7) and the error bound (2.10) (Table 1; Figs. 1, 2)

Table 1
Error bounds of Example 2.1 by (2.10) at $c_1 = 0, c_2 = 1 (v = 2)$ or $c_1 = 0, c_2 = 0.5, c_3 = 1 (v = 3)$: $G(y) = \cos y$

ω	10	10^2	10^3	10^4
E_L(f)	0.0022	2.3599e-5	7.2509e-7	5.4012e-9
bound ($v = 2$)	0.0600	6.0000e-4	6.0000e-6	6.0000e-8
E_L(f)	5.2170e-4	4.5711e-5	1.1629e-8	3.2875e-10
bound ($v = 3$)	0.0450	4.5000e-4	4.5000e-6	4.5000e-8

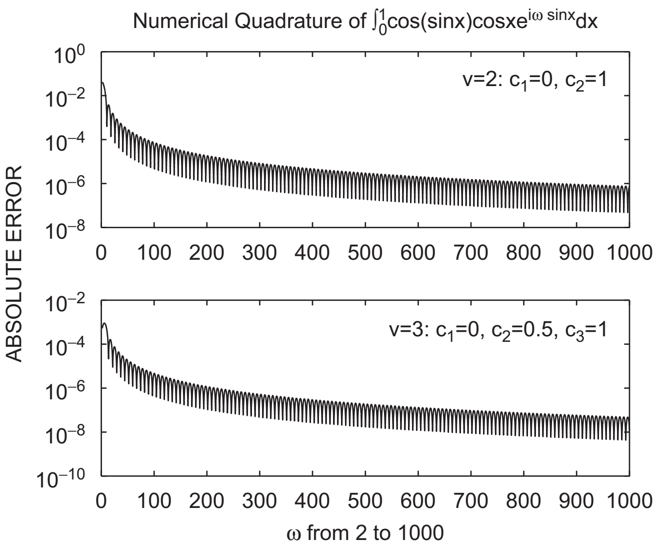

Fig. 1. The error of numerical quadrature for $\int_0^1 \cos(\sin(x)) \cos(x) e^{i\omega \sin(x)}\, dx$ by (2.7).

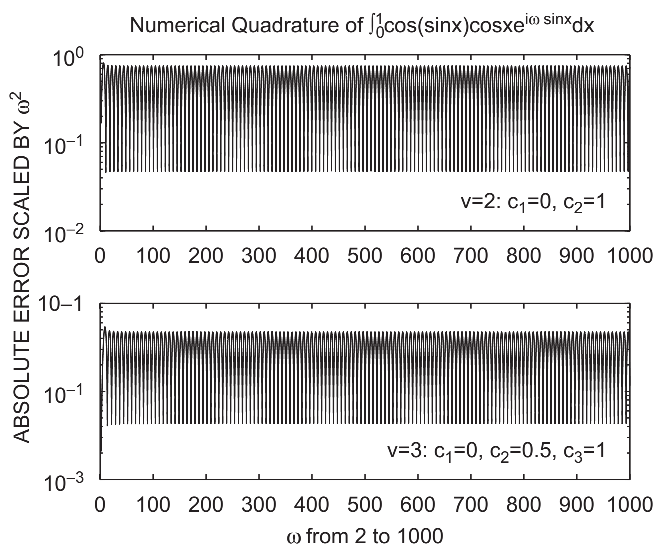

Fig. 2. The error for $\int_0^1 \cos(\sin(x)) \cos(x) e^{i\omega \sin(x)}\, dx$ by (2.7), scaled by ω^2.

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