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On the low-x NLO evolution of 4 point colorless operators

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Abstract: The NLO evolution equations for quadrupole and double dipole operators have been obtained within the high energy operator expansion method. The corresponding quasi-conformal evolution equations for the composite operators were constructed.

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Contents

1 Introduction

This paper develops the Wilson line approach to high energy scattering proposed in [1] to the case of quadrupole and double dipole operators in the next to leading order (NLO). Such operators naturally appear when one studies amplitudes for diffractive processes with the production of 3 or 4 particles in the Regge limit. Moreover, the quadrupole operator enters the definition of the Weizsäcker-Williams gluon distribution [2–4] which gives the Fock space number density of gluons inside dense hadrons in light-cone gauge. One can find the NLO evolution equation for the operator necessary for the Weizsäcker-Williams gluon distribution differentiating the quadrupole equation obtained in this paper. This result is going to be presented in a future work.

In the Wilson line approach to high energy scattering [1] the amplitudes are convolutions of impact factors and a Green function. The impact factors describe the decomposition of the colliding particles into quarks and gluons while the Green function is responsible for the interaction of these quarks and gluons with the quarks and gluons from the other colliding particle. In this framework such fast-moving partons are depicted as Wilson lines with the path going along their trajectories. Hence, the corresponding Green functions are the operators constructed of the Wilson lines. These operators obey the evolution equations with respect to the rapidity divide. This rapidity divide separates the gluon field into the fast quantum one and the slow external field of the other particle, through which the current quark or gluon is propagating.

In the most thoroughly studied case of a virtual photon splitting into quark antiquark pair, the corresponding Wilson line operator is a color dipole. The evolution equation for this operator is known as the Balitsky - Kovchegov (BK) equation [1, 5, 6]. The NLO corrections to this equation were calculated in [7–10]. Another interesting case is application of this formalism to a proton. The proton has baryon color structure and can be described as a 3-quark Wilson loop operator (3QWL). The evolution equation for this operator was calculated in the leading order (LO) in [11] and in the NLO in [12]. The latter calculation was based on the NLO hierarchy of the evolution equations for the Wilson lines with open indices [13] and the connected contribution to the 3QWL kernel [14]. These results were also obtained in the Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner (JIMWLK) formalism [15–21]. The hamiltonian equivalent to the NLO hierarchy was obtained in [22] and the evolution equation for the 3QWL in [23]. The NLO kernel for the evolution Wilson line operators was also constructed in [24].

The quadrupole and the double dipole are 4-particle colorless operators. Their LO linear evolution equations were derived in [4, 10, 25–27]. Here the results of [13] and [14] are used to construct the NLO evolution equations for these operators and the results of [10] and [12] are used to check these equations.

The paper is organized as follows. The next section contains the definitions and necessary results. Sections 3 and 4 present the NLO evolution equations for the quadrupole and the double dipole operators in the standard and quasi-conformal forms. Section 5 discusses different checks of the results. Section 6 concludes the paper.

2 Definitions and building blocks

The light cone vectors n_1 and n_2 are defined as

$$n_1 = (1,0,0,1)\,, \quad n_2 = \frac{1}{2}(1,0,0,-1)\,, \quad n_1^+ = n_2^- = n_1 n_2 = 1 \tag{2.1}$$

and any vector p can be decomposed as

$$p^+ = p_- = pn_2 = \frac{1}{2}\left(p^0 + p^3\right), \qquad p_+ = p^- = pn_1 = p^0 - p^3, \tag{2.2}$$

$$p = p^+ n_1 + p^- n_2 + p_\perp, \qquad p^2 = 2p^+p^- - \vec{p}^{\,2}, \tag{2.3}$$

$$p\,k = p^\mu k_\mu = p^+k^- + p^-k^+ - \vec{p}\vec{k} = p_+k_- + p_-k_+ - \vec{p}\vec{k}. \tag{2.4}$$

For brevity the following notation for traces is used

$$tr(U_i U_j{}^\dagger \ldots U_k U_l{}^\dagger) \equiv \mathbf{U}_{ij^\dagger \ldots kl^\dagger}, \tag{2.5}$$

where

$$U_i = U\left(\vec{r}_i, \eta\right) = Pe^{ig\int_{-\infty}^{+\infty} b_\eta^-(r^+,\vec{r})dr^+}, \tag{2.6}$$

and b_η^- is the external shock wave field built from only slow gluons

$$b_\eta^- = \int \frac{d^4p}{(2\pi)^4} e^{-ipz} b^-(p)\,\theta(e^\eta - p^+). \tag{2.7}$$

The parameter η separates the slow gluons entering the Wilson lines from the fast ones in the impact factors. The field

$$b^\mu(r) = b^-(r^+,\vec{r})n_2^\mu = \delta(r^+)b(\vec{r})\,n_2^\mu. \tag{2.8}$$

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The coordinates $\vec{r}_{1,2,3,4}$ denote the quarks, and $\vec{r}_0, \vec{r}_5$ are the coordinates of the gluons. In intermediate formulas the coordinates $\vec{r}_{6,7}$ will also be used. The SU(N_c) identities

$$U_4^{ba} = 2tr(t^b U_4 t^a U_4^\dagger), \quad (t^a)_i^j (t^a)_k^l = \frac{1}{2}\delta_i^l \delta_k^j - \frac{1}{2N_c}\delta_i^j \delta_k^l \tag{2.9}$$

are necessary to rewrite the SU(N_c) operators only through the Wilson lines in the fundamental representation. For a generic operator O the rapidity evolution equation has the form

$$\frac{\partial}{\partial \eta}\langle O\rangle = \langle K_{\rm LO} \otimes O\rangle + \langle K_{\rm NLO} \otimes O\rangle. \tag{2.10}$$

where $K_{\rm LO} \sim \alpha_s$ and $K_{\rm NLO} \sim \alpha_s^2$. The $\langle \ldots \rangle$ brackets were explicitly written to denote that the calculation was performed in the shockwave background. Hereafter they will be often omitted to avoid overloading the notation. The BK equation in this notation reads [1]

$$\frac{\partial \mathbf{U}_{12^\dagger}}{\partial \eta} = \frac{\alpha_s}{2\pi^2}\int d\vec{r}_0 \frac{\vec{r}_{12}{}^2}{\vec{r}_{10}{}^2 \vec{r}_{20}{}^2}\left(\mathbf{U}_{2^\dagger 0}\mathbf{U}_{0^\dagger 1} - N_c \mathbf{U}_{2^\dagger 1}\right), \tag{2.11}$$

where $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$. The LO quadrupole evolution equation reads [4]

$$\begin{aligned}\frac{\partial \mathbf{U}_{12^\dagger 34^\dagger}}{\partial \eta} = \frac{\alpha_s}{4\pi^2}\int d\vec{r}_0 \Bigg\{ &\frac{\vec{r}_{14}{}^2}{\vec{r}_{10}{}^2 \vec{r}_{40}{}^2}\left(\mathbf{U}_{10^\dagger}\mathbf{U}_{02^\dagger 34^\dagger} + \mathbf{U}_{4^\dagger 0}\mathbf{U}_{12^\dagger 30^\dagger} - (0\to 1 \equiv 0 \to 4)\right) \\ &+ \frac{\vec{r}_{12}{}^2}{\vec{r}_{10}{}^2 \vec{r}_{20}{}^2}\left(\mathbf{U}_{10^\dagger}\mathbf{U}_{02^\dagger 34^\dagger} + \mathbf{U}_{2^\dagger 0}\mathbf{U}_{10^\dagger 34^\dagger} - (0\to 1 \equiv 0 \to 2)\right) \\ &- \frac{\vec{r}_{24}{}^2}{2\vec{r}_{20}{}^2 \vec{r}_{40}{}^2}\left(\mathbf{U}_{10^\dagger}\mathbf{U}_{02^\dagger 34^\dagger} + \mathbf{U}_{30^\dagger}\mathbf{U}_{04^\dagger 12^\dagger} - (0\to 4 \equiv 0 \to 2)\right) \\ &- \frac{\vec{r}_{13}{}^2}{2\vec{r}_{10}{}^2 \vec{r}_{30}{}^2}\left(\mathbf{U}_{4^\dagger 0}\mathbf{U}_{12^\dagger 30^\dagger} + \mathbf{U}_{2^\dagger 0}\mathbf{U}_{34^\dagger 10^\dagger} - (0\to 1 \equiv 0 \to 3)\right) + (1\leftrightarrow 3, 2\leftrightarrow 4)\Bigg\}.\end{aligned} \tag{2.12}$$

Here $(0 \to 1 \equiv 0 \to 4)$ stands for the substitution $U_0 \to U_1$ or $U_0 \to U_4$, which gives the same result. In addition $(1 \leftrightarrow 3, 2 \leftrightarrow 4)$ means that one has to change $\vec{r}_1 \leftrightarrow \vec{r}_3, \vec{r}_2 \leftrightarrow \vec{r}_4$ and $U_1 \leftrightarrow U_3, U_2 \leftrightarrow U_4$. We will also need the LO evolution equations for the double dipole, sextupole and the dipole-quadrupole product. All these equations follow from the LO hierarchy [1] directly.

$$\begin{aligned}\frac{\partial \mathbf{U}_{12^\dagger}\mathbf{U}_{34^\dagger}}{\partial \eta} &= \frac{\alpha_s}{4\pi^2}\int d\vec{r}_0 \left(\frac{\vec{r}_{13}{}^2}{\vec{r}_{10}{}^2 \vec{r}_{30}{}^2} - \frac{\vec{r}_{23}{}^2}{\vec{r}_{20}{}^2 \vec{r}_{30}{}^2} - \frac{\vec{r}_{14}{}^2}{\vec{r}_{10}{}^2 \vec{r}_{40}{}^2} + \frac{\vec{r}_{24}{}^2}{\vec{r}_{20}{}^2 \vec{r}_{40}{}^2}\right) \\ &\times \left(\mathbf{U}_{2^\dagger 14^\dagger 3} + \mathbf{U}_{2^\dagger 34^\dagger 1} - \mathbf{U}_{2^\dagger 10^\dagger 34^\dagger 0} - \mathbf{U}_{2^\dagger 04^\dagger 30^\dagger 1}\right) + \mathbf{U}_{4^\dagger 3}\frac{\partial \mathbf{U}_{12^\dagger}}{\partial \eta} + \mathbf{U}_{2^\dagger 1}\frac{\partial \mathbf{U}_{4^\dagger 3}}{\partial \eta}.\end{aligned} \tag{2.13}$$

$$\begin{aligned}\frac{\partial \mathbf{U}_{12^\dagger 34^\dagger}\mathbf{U}_{76^\dagger}}{\partial \eta} &= \frac{\alpha_s}{4\pi^2}\int d\vec{r}_0 \Bigg\{ \left(\mathbf{U}_{0^\dagger 76^\dagger 02^\dagger 34^\dagger 1} + \mathbf{U}_{0^\dagger 12^\dagger 34^\dagger 06^\dagger 7} - (0 \to 7 \equiv 0 \to 6)\right) \\ &\times \left(\frac{\vec{r}_{16}{}^2}{\vec{r}_{01}{}^2 \vec{r}_{06}{}^2} - \frac{\vec{r}_{17}{}^2}{\vec{r}_{01}{}^2 \vec{r}_{07}{}^2}\right) + \left(\frac{\vec{r}_{27}{}^2}{\vec{r}_{02}{}^2 \vec{r}_{07}{}^2} - \frac{\vec{r}_{26}{}^2}{\vec{r}_{02}{}^2 \vec{r}_{06}{}^2}\right) \\ &\times \left(\mathbf{U}_{0^\dagger 76^\dagger 02^\dagger 34^\dagger 1} + \mathbf{U}_{0^\dagger 34^\dagger 12^\dagger 06^\dagger 7} - (0 \to 7 \equiv 0 \to 6)\right) \\ &+ (1 \leftrightarrow 3, 2 \leftrightarrow 4)\Bigg\} + \mathbf{U}_{76^\dagger}\frac{\partial \mathbf{U}_{12^\dagger 34^\dagger}}{\partial \eta} + \mathbf{U}_{12^\dagger 34^\dagger}\frac{\partial \mathbf{U}_{76^\dagger}}{\partial \eta}.\end{aligned} \tag{2.14}$$

$$\begin{aligned}\frac{\partial \mathbf{U}_{12^\dagger 34^\dagger 56^\dagger}}{\partial\eta} &= \frac{\alpha_s}{4\pi^2}\int d\vec{r}_0\Bigg\{\frac{\vec{r}_{25}{}^2}{\vec{r}_{02}{}^2\vec{r}_{05}{}^2}(\mathbf{U}_{0^\dagger 34^\dagger 5}\mathbf{U}_{2^\dagger 06^\dagger 1}+\mathbf{U}_{0^\dagger 56^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 0}-(0\to 5\equiv 0\to 2))\\ &- \frac{\vec{r}_{15}{}^2}{\vec{r}_{01}{}^2\vec{r}_{05}{}^2}(\mathbf{U}_{0^\dagger 56^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 0}+\mathbf{U}_{6^\dagger 0}\mathbf{U}_{0^\dagger 12^\dagger 34^\dagger 5}-(0\to 5\equiv 0\to 1))\\ &- \frac{\vec{r}_{26}{}^2}{\vec{r}_{02}{}^2\vec{r}_{06}{}^2}(\mathbf{U}_{0^\dagger 34^\dagger 5}\mathbf{U}_{2^\dagger 06^\dagger 1}+\mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 56^\dagger 0}-(0\to 2\equiv 0\to 6))\\ &+ \frac{\vec{r}_{16}{}^2}{\vec{r}_{01}{}^2\vec{r}_{06}{}^2}(\mathbf{U}_{6^\dagger 0}\mathbf{U}_{0^\dagger 12^\dagger 34^\dagger 5}+\mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 56^\dagger 0}-(0\to 1\equiv 0\to 6))\\ &+ \frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}(\mathbf{U}_{2^\dagger 0}\mathbf{U}_{0^\dagger 34^\dagger 56^\dagger 1}+\mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 56^\dagger 0}-(0\to 1\equiv 0\to 2))\\ &+(1\to 3\to 5\to 1, 2\to 4\to 6\to 2)+(1\to 5\to 3\to 1, 2\to 6\to 4\to 2)\Bigg\}. \end{aligned} \tag{2.15}$$

Here $1\to 3\to 5\to 1$ stands for permutation, i.e. one has to change $\vec{r}_1\to\vec{r}_3$, $\vec{r}_3\to\vec{r}_5$, $\vec{r}_5\to\vec{r}_1$ and $U_1\to U_3$, $U_3\to U_5$, $U_5\to U_1$.

For the self and the pairwise NLO interactions one can take the results of [13] while the triple-interaction diagrams were already calculated in [14]. The results of these papers were derived using sharp cutoff on the rapidity variable. Since this paper is devoted to color singlet operators one can drop the kernels which vanish acting on the colorless operators, as was shown in [22]. The rest reads

$$\begin{aligned}\left.\frac{\partial\,(U_1)^j_i}{\partial\eta}\right|_{\rm NLO} &= \frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_5 d\vec{r}_0 J_{11105}\left[if^{ad'e'}(\{t^dt^e\}U_1t^a)^j_i - if^{ade}(t^aU_1\{t^{d'}t^{e'}\})^j_i\right]\\ &\times U_0^{dd'}(U_5^{ee'}-U_0^{ee'})+\frac{\alpha_s^2N_c}{4\pi^3}\int\frac{d\vec{r}_5}{\vec{r}_{15}{}^4}(U_5^{ab}-U_1^{ab})(t^aU_1t^b)^j_i\beta\ln\left(\frac{\vec{r}_{15}^{\,2}}{\tilde{\mu}^2}\right),\end{aligned} \tag{2.16}$$

$$\beta=\left(\frac{11}{3}-\frac{2}{3}\frac{n_f}{N_c}\right),\quad \beta\ln\frac{1}{\tilde{\mu}^2}=\left(\frac{11}{3}-\frac{2}{3}\frac{n_f}{N_c}\right)\ln\left(\frac{\mu^2}{4e^{2\psi(1)}}\right)+\frac{67}{9}-\frac{\pi^2}{3}-\frac{10}{9}\frac{n_f}{N_c}, \tag{2.17}$$

n_f is the number of the quark flavours, μ^2 is the renormalization scale in the $\overline{MS}$-scheme and $J_{ijklm}\equiv J(\vec{r}_i,\vec{r}_j,\vec{r}_k,\vec{r}_l,\vec{r}_m)$

$$\begin{aligned}J_{12305} &= \left(\frac{\vec{r}_{01}\vec{r}_{52}}{\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{25}{}^2}+\frac{2\,(\vec{r}_{01}\vec{r}_{03})\,(\vec{r}_{05}\vec{r}_{25})}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{05}{}^2\vec{r}_{25}{}^2}-\frac{2\,(\vec{r}_{01}\vec{r}_{03})\,(\vec{r}_{25}\vec{r}_{35})}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{25}{}^2\vec{r}_{35}{}^2}+\frac{2\,(\vec{r}_{01}\vec{r}_{05})\,(\vec{r}_{25}\vec{r}_{35})}{\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{25}{}^2\vec{r}_{35}{}^2}\right)\\ &\times\ln\left(\frac{\vec{r}_{03}{}^2}{\vec{r}_{35}{}^2}\right).\end{aligned} \tag{2.18}$$

This function has the properties

$$J_{ijk05}=-J_{jik50},\quad J_{11105}=\frac{(\vec{r}_{51}\vec{r}_{01})}{\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{15}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2}{\vec{r}_{15}{}^2}\right). \tag{2.19}$$

$$\left.\frac{\partial\,(U_1)^j_i\,(U_2)^l_k}{\partial\eta}\right|_{\rm NLO} = \frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_5 d\vec{r}_0\,(\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3)+\frac{\alpha_s^2}{8\pi^3}\int d\vec{r}_5\,(\mathcal{B}_1+N_c\mathcal{B}_2). \tag{2.20}$$

Here

$$\begin{aligned}\mathcal{A}_1 &= \left[(t^aU_1)^j_i(U_2t^b)^l_k+(t^aU_2)^l_k(U_1t^b)^j_i\right]\Big[f^{ade}f^{bd'e'}U_0^{dd'}(U_5^{ee'}-U_0^{ee'})4L_{12}\\ &+4n_fL^q_{12}tr(t^aU_5t^b(U_0^\dagger-U_5^\dagger))\Big],\end{aligned} \tag{2.21}$$

where $L_{ij} \equiv L(\vec{r}_i, \vec{r}_j)$ and $L^q_{ij} \equiv L^q(\vec{r}_i, \vec{r}_j)$ were introduced in this form in [12]

$$\begin{aligned}
L_{12} = &\left[\frac{1}{\vec{r}_{01}{}^2\vec{r}_{25}{}^2 - \vec{r}_{02}{}^2\vec{r}_{15}{}^2}\left(-\frac{\vec{r}_{12}{}^4}{8}\left(\frac{1}{\vec{r}_{01}{}^2\vec{r}_{25}{}^2} + \frac{1}{\vec{r}_{02}{}^2\vec{r}_{15}{}^2}\right) + \frac{\vec{r}_{12}{}^2}{\vec{r}_{05}{}^2} - \frac{\vec{r}_{02}{}^2\vec{r}_{15}{}^2 + \vec{r}_{01}{}^2\vec{r}_{25}{}^2}{4\vec{r}_{05}{}^4}\right)\right.\\
&\left. + \frac{\vec{r}_{12}{}^2}{8\vec{r}_{05}{}^2}\left(\frac{1}{\vec{r}_{02}{}^2\vec{r}_{15}{}^2} - \frac{1}{\vec{r}_{01}{}^2\vec{r}_{25}{}^2}\right)\right]\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{25}{}^2}{\vec{r}_{15}{}^2\vec{r}_{02}{}^2}\right) + \frac{1}{2\vec{r}_{05}{}^4}, \qquad (2.22)\\
L^q_{12} = &\frac{1}{\vec{r}_{05}{}^4}\left\{\frac{\vec{r}_{02}{}^2\vec{r}_{15}{}^2 + \vec{r}_{01}{}^2\vec{r}_{25}{}^2 - \vec{r}_{05}{}^2\vec{r}_{12}{}^2}{2(\vec{r}_{02}{}^2\vec{r}_{15}{}^2 - \vec{r}_{01}{}^2\vec{r}_{25}{}^2)}\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{15}{}^2}{\vec{r}_{01}{}^2\vec{r}_{25}{}^2}\right) - 1\right\}. \qquad (2.23)
\end{aligned}$$

These functions have the unintegrable singularity at $\vec{r}_{05} = 0$, which is canceled by the subtraction in the color structure. They are symmetric conformally invariant functions $L^{(q)}_{ij} = L^{(q)}_{ji} = L^{(q)}_{ij}|_{\vec{r}_0\leftrightarrow\vec{r}_5}$.

$$\begin{aligned}
\mathcal{A}_2 = &\,4(U_0 - U_1)^{dd'}(U_5 - U_2)^{ee'}\left\{i\left[f^{ad'e'}(t^dU_1t^a)^j_i(t^eU_2)^l_k - f^{ade}(t^aU_1t^{d'})^j_i(U_2t^{e'})^l_k\right]J_{12105}\right.\\
&\left. + i\left[f^{ad'e'}(t^dU_1)^j_i(t^eU_2t^a)^l_k - f^{ade}(U_1t^{d'})^j_i(t^aU_2t^{e'})^l_k\right]J_{12205}\right\}, \qquad (2.24)\\
\mathcal{A}_3 = &\,2U_0^{dd'}\left\{i[f^{ad'e'}(U_1t^a)^j_i(t^dt^eU_2)^l_k - f^{ade}(t^aU_1)^j_i(U_2t^{e'}t^{d'})^l_k]\,(U_5 - U_2)^{ee'}\right.\\
&\times(J_{22105} + J_{21205} - J_{12205}) + (J_{12105} + J_{11205} - J_{21105})\\
&\left.\times\, i\left[f^{ad'e'}(t^dt^eU_1)^j_i(U_2t^a)^l_k - f^{ade}(U_1t^{e'}t^{d'})^j_i(t^aU_2)^l_k\right](U_5 - U_1)^{ee'}\right\}. \qquad (2.25)\\
\mathcal{B}_1 = &\,2\ln\left(\frac{\vec{r}^{\,2}_{15}}{\vec{r}^{\,2}_{12}}\right)\ln\left(\frac{\vec{r}^{\,2}_{25}}{\vec{r}^{\,2}_{12}}\right) \qquad (2.26)\\
&\times\left\{(U_5 - U_1)^{ab}i\left[f^{bde}(t^aU_1t^d)^j_i(U_2t^e)^l_k + f^{ade}(t^eU_1t^b)^j_i(t^dU_2)^l_k\right]\left(\frac{(\vec{r}_{15}\vec{r}_{25})}{\vec{r}^{\,2}_{15}\vec{r}^{\,2}_{25}} - \frac{1}{\vec{r}^{\,2}_{15}}\right)\right.\\
&\left. + (U_5 - U_2)^{ab}i\left[f^{bde}(U_1t^e)^j_i(t^aU_2t^d)^l_k + f^{ade}(t^dU_1)^j_i(t^eU_2t^b)^l_k\right]\left(\frac{(\vec{r}_{15}\vec{r}_{25})}{\vec{r}^{\,2}_{15}\vec{r}^{\,2}_{25}} - \frac{1}{\vec{r}^{\,2}_{25}}\right)\right\},\\
\mathcal{B}_2 = &\,\beta\,(2U_5 - U_1 - U_2)^{ab}\left[(t^aU_1)^j_i(U_2t^b)^l_k + (U_1t^b)^j_i(t^aU_2)^l_k\right]\\
&\times\left\{\frac{(\vec{r}_{15}\vec{r}_{25})}{\vec{r}^{\,2}_{15}\vec{r}^{\,2}_{25}}\ln\left(\frac{\vec{r}^{\,2}_{12}}{\tilde{\mu}^2}\right) + \frac{1}{2\vec{r}^{\,2}_{15}}\ln\left(\frac{\vec{r}^{\,2}_{25}}{\vec{r}^{\,2}_{12}}\right) + \frac{1}{2\vec{r}^{\,2}_{25}}\ln\left(\frac{\vec{r}^{\,2}_{15}}{\vec{r}^{\,2}_{12}}\right)\right\}. \qquad (2.27)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial\eta}(U_1)^j_i\,(U_2)^l_k\,(U_3)^n_m\Big|_{\text{NLO}} = &\,\frac{i\alpha_s^2}{2\pi^4}\int d\vec{r}_5 d\vec{r}_0 \qquad (2.28)\\
&\times\left\{f^{cde}\left[(t^aU_1)^j_i\,(t^bU_2)^l_k\,(U_3t^c)^n_m\,(U_0 - U_1)^{ad}(U_5 - U_2)^{be}\right.\right.\\
&\left. - (U_1t^a)^j_i\,(U_2t^b)^l_k\,(t^cU_3)^n_m\,(U_0 - U_1)^{da}(U_5 - U_2)^{eb}\right]J_{12305}\\
& + f^{ade}\left[(U_1t^a)^j_i\,(t^bU_2)^l_k\,(t^cU_3)^n_m\,(U_0 - U_3)^{cd}(U_5 - U_2)^{be}\right.\\
&\left. - (t^aU_1)^j_i\,(U_2t^b)^l_k\,(U_3t^c)^n_m\,(U_0 - U_3)^{dc}(U_5 - U_2)^{eb}\right]J_{32105}\\
& + f^{bde}\left[(t^aU_1)^j_i\,(U_2t^b)^l_k\,(t^cU_3)^n_m\,(U_0 - U_1)^{ad}(U_5 - U_3)^{ce}\right.\\
&\left.\left. - (U_1t^a)^j_i\,(t^bU_2)^l_k\,(U_3t^c)^n_m\,(U_0 - U_1)^{da}(U_5 - U_3)^{ec}\right]J_{13205}\right\}.
\end{aligned}$$

We will also need the following functions. The function $M_i^{jk} \equiv M(\vec{r}_i, \vec{r}_j, \vec{r}_k)$ was introduced in [12]

$$
\begin{aligned}
M_2^{13} &= \frac{1}{2}(J_{12205} + J_{23205} - J_{13205} - J_{22205}) \\
&= \frac{1}{4\vec{r}_{01}{}^2\vec{r}_{35}{}^2}\left(\frac{\vec{r}_{12}{}^2\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{25}{}^2} - \frac{\vec{r}_{15}{}^2\vec{r}_{23}{}^2}{\vec{r}_{05}{}^2\vec{r}_{25}{}^2} - \frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2\vec{r}_{05}{}^2} + \frac{\vec{r}_{13}{}^2}{\vec{r}_{05}{}^2}\right)\ln\left(\frac{\vec{r}_{02}{}^2}{\vec{r}_{25}{}^2}\right).
\end{aligned}
\tag{2.29}
$$

It was also introduced as M_2 in [28]. It has the property

$$
M_k^{ij}|_{5\leftrightarrow 0} = -M_k^{ji}. \tag{2.30}
$$

The functions $\tilde{L}_{ij} \equiv \tilde{L}(\vec{r}_i, \vec{r}_j)$ and $M_{ij} \equiv M(\vec{r}_i, \vec{r}_j)$ were introduced in [12] as well

$$
\tilde{L}_{12} = \frac{1}{2}(M_1^{22} - M_2^{11}) = \frac{\vec{r}_{12}{}^2}{8\vec{r}_{05}{}^2}\left[\frac{\vec{r}_{12}{}^2\vec{r}_{05}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{15}{}^2\vec{r}_{25}{}^2} - \frac{1}{\vec{r}_{01}{}^2\vec{r}_{25}{}^2} - \frac{1}{\vec{r}_{02}{}^2\vec{r}_{15}{}^2}\right]\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{25}{}^2}{\vec{r}_{15}{}^2\vec{r}_{02}{}^2}\right), \tag{2.31}
$$

$$
M_{12} = \frac{1}{4}(M_2^{11} + M_1^{22}) = \frac{\vec{r}_{12}{}^2}{16\vec{r}_{05}{}^2}\left[\frac{\vec{r}_{12}{}^2\vec{r}_{05}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{15}{}^2\vec{r}_{25}{}^2} - \frac{1}{\vec{r}_{01}{}^2\vec{r}_{25}{}^2} - \frac{1}{\vec{r}_{02}{}^2\vec{r}_{15}{}^2}\right]\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}{\vec{r}_{15}{}^2\vec{r}_{25}{}^2}\right). \tag{2.32}
$$

Here $\tilde{L}_{ij}$ is conformally invariant. Moreover, $\tilde{L}_{ij}$ is antisymmetric w.r.t. both $5 \leftrightarrow 0$ and $i \leftrightarrow j$ transformations while M_{ij} is antisymmetric w.r.t. $5 \leftrightarrow 0$. One can also combine all the terms $\sim \beta$ into $M_{ij}^{\beta} \equiv M^{\beta}(\vec{r}_i, \vec{r}_j)$

$$
M_{12}^{\beta} = \frac{N_c\beta}{2}\left\{\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right) + \frac{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}{\vec{r}_{12}^{\,2}}\ln\left(\frac{\vec{r}_{02}^{\,2}}{\vec{r}_{01}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right)\right\}. \tag{2.33}
$$

The NLO BK kernel reads [10]

$$
\begin{aligned}
\langle K_{\mathrm{NLO}} \otimes \mathbf{U}_{12^\dagger}\rangle &= \frac{\alpha_s^2}{4\pi^3}\int d\vec{r}_0 \frac{\vec{r}_{12}{}^2}{\vec{r}_{10}{}^2\vec{r}_{20}{}^2}\left\{M_{12}^{\beta} - N_c\ln\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{10}{}^2}\right)\ln\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{20}{}^2}\right)\right\} \\
&\quad\times(\mathbf{U}_{2^\dagger 0}\mathbf{U}_{0^\dagger 1} - N_c\mathbf{U}_{2^\dagger 1}) + \frac{\alpha_s^2}{4\pi^4}\int d\vec{r}_0 d\vec{r}_5\{\tilde{L}_{12}(\mathbf{U}_{0^\dagger 5}\mathbf{U}_{2^\dagger 0}\mathbf{U}_{5^\dagger 1} - (0 \leftrightarrow 5)) \\
&\quad + L_{12}((\mathbf{U}_{0^\dagger 5 2^\dagger 0 5^\dagger 1} - \mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{5^\dagger 0} - (0 \to 5)) + (0 \leftrightarrow 5)) \\
&\quad - 2n_f L_{12}^q(tr(t^a U_1 t^b U_2^\dagger)tr(t^a U_5 t^b(U_0^\dagger - U_5^\dagger)) + (5 \leftrightarrow 0))\}.
\end{aligned}
\tag{2.34}
$$

3 Construction of the kernel

First, one has to discuss the singularities of the building blocks from the previous section. All the ultraviolet (UV) singularities in (2.16), (2.20), and (2.28) were removed by the renormalization. It means that these expressions converge at $\vec{r}_0 = \vec{r}_{1,2,3,4,5}$ and $\vec{r}_5 = \vec{r}_{0,1,2,3,4}$. In particular, the functions J in $\mathcal{A}_2$ (2.24), $\mathcal{A}_3$ (2.25), and (2.28) are convergent at these points, which ensures UV-safety of these expressions. However, the function J_{11105} in the first line of (2.16), has the UV singularity at $\vec{r}_0 = \vec{r}_5 = \vec{r}_1$. As in (2.22) this singularity is removed by the subtraction in the color structure.

Nevertheless, these expressions have infrared (IR) singularities, which appear as both $\vec{r}_{0,5} \to \infty$. Indeed, changing the variables e.g. as $r_0 = ut$, $r_5 = u\bar{t}, \bar{t} = 1 - t$, one faces a logarithmic singularity integrating w.r.t. u first

$$\int d\vec{r}_5 d\vec{r}_0 J_{12305} = \int d\phi_5 d\phi_0 \int_0^1 dt \int_0^{+\infty} du \left(\frac{2\cos(\phi_{05})}{u(t^2+\bar{t}^2-2t\bar{t}\cos(\phi_{05}))} \ln\left(\frac{t}{\bar{t}}\right) + O\left(\frac{1}{u^2}\right)\right). \tag{3.1}$$

Hence this double integral is ill-defined and requires either regularization or definition in terms of the iterated integrals. To understand how to correctly treat the IR singularities one can either return to the diagrams and keep the regularization, or calculate the known dipole equation and fix the definition from there. The latter way is attempted here. Assembling BK kernel (2.34) from (2.16)–(2.20), one can see that all the β-functional terms go to M_{12}^{β}, $\mathcal{A}_1$ (2.21) reshapes to the terms $\sim L_{12}, L_{12}^{q}$, the Wilson line operators from (2.16), (2.24)–(2.25) depending on both $\vec{r}_5$ and $\vec{r}_0$ give the term $\sim \tilde{L}_{12}$ after the symmetrization

$$\begin{aligned} A(U)F(\vec{r}) &\to [AF]^{\mathrm{sym}} \\ &= \frac{[A + A\,(0 \leftrightarrow 5)]\,[F + F\,(0 \leftrightarrow 5)] + [A - A\,(0 \leftrightarrow 5)]\,[F - F\,(0 \leftrightarrow 5)]}{4}. \end{aligned} \tag{3.2}$$

Next, $\mathcal{B}_1$ (2.26) gives one half of the double logarithm contribution. All the remaining terms are to be equal to the other half of the double logarithm contribution. They read

$$\begin{aligned} \Delta K = \frac{\alpha_s^2 N_c}{16\pi^4} \int d\vec{r}_0 d\vec{r}_5 \{ &(\mathbf{U}_{2\dagger 5}\mathbf{U}_{5\dagger 1} - \mathbf{U}_{0\dagger 1}\mathbf{U}_{2\dagger 0})\,(J_{22105} + J_{11205}) \\ &+ (J_{21105} - J_{12105} + J_{12205} - J_{21205})\,(2N_c\mathbf{U}_{2\dagger 1} - \mathbf{U}_{0\dagger 1}\mathbf{U}_{2\dagger 0} - \mathbf{U}_{2\dagger 5}\mathbf{U}_{5\dagger 1})\}. \end{aligned} \tag{3.3}$$

This term is IR safe. The second line is the product of 2 expressions symmetric w.r.t. $0 \leftrightarrow 5$ permutation. Therefore one can set $U_5 \to U_0$ there. In the first line there is a product of 2 expressions antisymmetric w.r.t. $0 \leftrightarrow 5$ permutation. Hence, one could add and subtract $N_c\mathbf{U}_{2\dagger 1}$ in the first brackets and write

$$\begin{aligned} \Delta K \to \frac{\alpha_s^2 N_c}{8\pi^4} \int d\vec{r}_0 d\vec{r}_5 \{ &(N_c\mathbf{U}_{2\dagger 1} - \mathbf{U}_{2\dagger 0}\mathbf{U}_{0\dagger 1})\,(J_{22105} + J_{11205}) \\ &+ (J_{21105} + J_{12205} - J_{12105} - J_{21205})\,(N_c\mathbf{U}_{2\dagger 1} - \mathbf{U}_{0\dagger 1}\mathbf{U}_{2\dagger 0})\} \end{aligned} \tag{3.4}$$

$$\begin{aligned} = \frac{\alpha_s^2 N_c}{8\pi^4} \int d\vec{r}_0 \int d\vec{r}_5\, &(N_c\mathbf{U}_{2\dagger 1} - \mathbf{U}_{0\dagger 1}\mathbf{U}_{2\dagger 0}) \\ &\times \{J_{22105} - J_{21205} + J_{11205} - J_{12105} + J_{21105} + J_{12205}). \end{aligned} \tag{3.5}$$

One could understand the latter integral as an iterated one. Then, using the integrals

$$\int \frac{d\vec{r}_5}{\pi}(J_{ijk05} - J_{ikj05}) = \left(\frac{\vec{r}_{0i}\vec{r}_{0j}}{\vec{r}_{0i}^{\,2}\vec{r}_{0j}^{\,2}} - \frac{\vec{r}_{0i}\vec{r}_{0k}}{\vec{r}_{0i}^{\,2}\vec{r}_{0k}^{\,2}}\right) \ln\left(\frac{\vec{r}_{jk}^{\,2}}{\vec{r}_{0k}^{\,2}}\right) \ln\left(\frac{\vec{r}_{jk}^{\,2}}{\vec{r}_{0j}^{\,2}}\right), \qquad \int d\vec{r}_5 J_{ijj05} = 0, \tag{3.6}$$

one could get the other half of the double logarithm term in the BK kernel

$$\Delta K \to \frac{\alpha_s^2 N_c}{8\pi^3} \int d\vec{r}_0\, (N_c\mathbf{U}_{2\dagger 1} - \mathbf{U}_{0\dagger 1}\mathbf{U}_{2\dagger 0}) \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{20}^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{10}^{\,2}}\right) \ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{20}^{\,2}}\right). \tag{3.7}$$

Although such treatment gives the correct result, it does not take into account the IR singularity of J. Indeed if one introduces the dimensional regularization into (3.3) then one gets

$$\Delta K \to \frac{\alpha_s^2 N_c}{8\pi^4} \int d^d r_0 d^d r_5 \left(N_c \mathbf{U}_{2^\dagger 1} - \mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 0}\right) \times \left(J_{22105} - J_{21205} + J_{11205} - J_{12105} + J_{21105} + J_{12205}\right). \tag{3.8}$$

However, in the dimensional regularization the integral $\int d^d r_5 J_{ijj05}$ would be $\sim \epsilon$ rather than 0 and the double integral

$$\int d^d r_0 d^d r_5 J_{ijj05} = 2\pi^2 \zeta(3) \tag{3.9}$$

because the second integral w.r.t. r_0 has an IR divergence as $r_0 \to \infty$ and starts from $\frac{1}{\epsilon}$. Therefore if one wants to integrate the coefficient of $\mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 0}$ w.r.t. $d^d r_5$, one has to keep the result in the dimension d without expanding the series. At the same time the coefficient of $N_c \mathbf{U}_{2^\dagger 1}$ gets the doubled contribution since

$$\int d\vec{r}_0 \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{20}^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{10}^{\,2}}\right) \ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{20}^{\,2}}\right) = 4\pi^2 \zeta(3). \tag{3.10}$$

Thus, the result depends on the regularization. Such an ambiguity is the consequence of the fact that the initial expressions do not have the IR regularization. To avoid this ambiguity one needs the evolution equations for Wilson lines (2.24)–(2.25) with the IR regularization. Alternatively, one can write them in the form where the terms which do not depend on both U_5^{ab} and U_0^{ab} are integrated w.r.t. the coordinate of the other gluon.

In this paper the procedure discussed in (3.2)–(3.7) is used. Technically it means that for the terms $\sim U_5^{ab} U_0^{a'b'}$, the gluons are treated equally and the kernel is represented in the form of symmetrized sum (3.2). In the terms depending only on U_5^{ab} or only on U_0^{ab}, the integration order is fixed as $\int d\vec{r}_0 \int d\vec{r}_5$ or $\int d\vec{r}_5 \int d\vec{r}_0$ correspondingly and the integrals are understood as iterated. As a result, one can take the inner integral via (3.6). The terms independent of U_5^{ab} and U_0^{ab} are also symmetrized according to (3.2) and in them the substitution $J_{ijj05} \to J_{jji05}$ is made. This substitution can be understood as follows. First one drops the terms with $\int d\vec{r}_5 J_{ijj05}$. They vanish (3.6) if one treats the integrals as iterated. Next, one adds the totally antisymmetric w.r.t. $(5 \leftrightarrow 0)$ terms J_{jji05}. These terms vanish if they are integrated w.r.t. $\vec{r}_0$ and $\vec{r}_5$ in the double integral. After that the first integral in (3.6) is enough to calculate all the integrals. Again, I stress that although such treatment gives the correct dipole result (as well as the evolution equation for the baryon operator coinciding with [12]) it involves the cavalier treatment of the IR singularities.

Taking the contributions of the self-interaction of one Wilson line (2.16), the connected contributions of 2 Wilson lines (2.20) and the connected contributions of 3 Wilson lines (2.28) with the appropriate charge conjugation, and using the integration procedure described above, one can write the full NLO evolution of the quadrupole operator $tr\left(U_1 U_2{}^\dagger U_3 U_4{}^\dagger\right) \equiv \mathbf{U}_{12^\dagger 34^\dagger}$ as

$$\langle K_{\mathrm{NLO}} \otimes \mathbf{U}_{12^\dagger 34^\dagger}\rangle = \frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_5 \left(\mathbf{G}_s + \mathbf{G}_a\right) + \frac{\alpha_s^2}{8\pi^3} \int d\vec{r}_0 \left(\mathbf{G}_\beta + \mathbf{G}\right), \tag{3.11}$$

Following (3.2) the 2-gluon contribution can be decomposed into the product $\mathbf{G}_s$ of the symmetric coordinate and color structures and the product $\mathbf{G}_a$ of the antisymmetric ones w.r.t. $0 \leftrightarrow 5$ transposition, i.e. the substitution $\vec{r}_0 \leftrightarrow \vec{r}_5$ and $U_0 \leftrightarrow U_5$. After color convolution and integration w.r.t. $\vec{r}_0$ or $\vec{r}_5$ of the contributions which do not depend on the other variable one comes to the 1-gluon part. It contains the contribution proportional to β-function $\mathbf{G}_\beta$ ($\beta = \frac{11}{3} - \frac{2}{3}\frac{n_f}{N_c}$) and the rest $\mathbf{G}$. One can see that all the $\mathbf{G}$'s separately vanish without the shockwave, i.e. if all the $U \to 0$.

Doing the same for the double dipole operator $tr\left(U_1 U_2{}^\dagger\right) tr(U_3 U_4{}^\dagger) \equiv \mathbf{U}_{12^\dagger}\mathbf{U}_{34^\dagger}$, one can write its full NLO evolution equation as

$$\langle K_{\rm NLO} \otimes \mathbf{U}_{12^\dagger}\mathbf{U}_{34^\dagger}\rangle = \mathbf{U}_{12^\dagger}\langle K_{\rm NLO} \otimes \mathbf{U}_{34^\dagger}\rangle + \mathbf{U}_{34^\dagger}\langle K_{\rm NLO} \otimes \mathbf{U}_{12^\dagger}\rangle + \frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_5\; (\tilde{\mathbf{G}}_s + \tilde{\mathbf{G}}_a) + \frac{\alpha_s^2}{8\pi^3}\int d\vec{r}_0\; (\tilde{\mathbf{G}}_\beta + \tilde{\mathbf{G}}). \tag{3.12}$$

Here the NLO dipole kernel is written in our notation in (2.34), $\tilde{\mathbf{G}}_s(\tilde{\mathbf{G}}_a)$ is the product of the coordinate and color structures (anti)symmetric w.r.t. $0 \leftrightarrow 5$ transposition, $\tilde{\mathbf{G}}_\beta$ is proportional to β-function and $\tilde{\mathbf{G}}$ is the remaining contribution with 1 gluon crossing the shockwave.

3.1 Quadrupole

We start from the product of the symmetric structures

$$\mathbf{G}_s = \mathbf{G}_{s1} + n_f \mathbf{G}_q + \mathbf{G}_{s2}. \tag{3.13}$$

$$\begin{aligned}\mathbf{G}_{s1} = &\left(\{\mathbf{U}_{0^\dagger 34^\dagger 15^\dagger 02^\dagger 5} - \mathbf{U}_{5^\dagger 0}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{0^\dagger 34^\dagger 1} - (5\to 0)\} + (5 \leftrightarrow 0)\right)(L_{12} + L_{32} - L_{13})\\ &+ \left(\{\mathbf{U}_{0^\dagger 15^\dagger 02^\dagger 34^\dagger 5} - \mathbf{U}_{0^\dagger 5}\mathbf{U}_{5^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 0} - (5\to 0)\} + (5 \leftrightarrow 0)\right)(L_{12} + L_{14} - L_{42})\\ &+ (1 \leftrightarrow 3, 2 \leftrightarrow 4),\end{aligned} \tag{3.14}$$

where L was introduced in (2.22). It is a conformally invariant contribution.

$$\begin{aligned}\mathbf{G}_q = &\left(\left\{\frac{\mathbf{U}_{0^\dagger 34^\dagger 12^\dagger 5} + \mathbf{U}_{2^\dagger 34^\dagger 15^\dagger 0}}{N_c} - \frac{\mathbf{U}_{0^\dagger 5}\mathbf{U}_{2^\dagger 34^\dagger 1}}{N_c^2} - \mathbf{U}_{2^\dagger 5}\mathbf{U}_{0^\dagger 34^\dagger 1} - (5\to 0)\right\} + (5\leftrightarrow 0)\right)\\ &\times \frac{1}{2}\left(L^q_{12} + L^q_{32} - L^q_{13}\right) + \frac{1}{2}\left(L^q_{12} + L^q_{14} - L^q_{42}\right)\\ &\times \left(\left\{\frac{\mathbf{U}_{0^\dagger 12^\dagger 34^\dagger 5} + \mathbf{U}_{2^\dagger 34^\dagger 15^\dagger 0}}{N_c} - \frac{\mathbf{U}_{0^\dagger 5}\mathbf{U}_{2^\dagger 34^\dagger 1}}{N_c^2} - \mathbf{U}_{5^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 0} - (5\to 0)\right\} + (5\leftrightarrow 0)\right)\\ &+ (1 \leftrightarrow 3, 2 \leftrightarrow 4),\end{aligned} \tag{3.15}$$

Here L^q is defined in (2.23).

$$\begin{aligned}\mathbf{G}_{s2} = &\frac{1}{2}\left(\mathbf{U}_{0^\dagger 34^\dagger 52^\dagger 05^\dagger 1} - \mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{4^\dagger 05^\dagger 3} + (5\leftrightarrow 0)\right)\left(M_1^{34} - M_1^{24} + M_2^{43} - M_2^{13} + (5\leftrightarrow 0)\right)\\ &+ \frac{1}{2}\left(\mathbf{U}_{0^\dagger 35^\dagger 02^\dagger 54^\dagger 1} - \mathbf{U}_{0^\dagger 3}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{4^\dagger 15^\dagger 0} + (5\leftrightarrow 0)\right)\left(M_3^{14} - M_3^{24} + M_2^{41} - M_2^{31} + (5\leftrightarrow 0)\right)\\ &+ \frac{1}{2}\left(\mathbf{U}_{0^\dagger 15^\dagger 02^\dagger 34^\dagger 5} - \mathbf{U}_{0^\dagger 5}\mathbf{U}_{5^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 0} + (5\leftrightarrow 0)\right)\left(M_2^{14} + M_4^{12} + (5\leftrightarrow 0)\right)\\ &+ \frac{1}{2}\left(\mathbf{U}_{0^\dagger 34^\dagger 15^\dagger 02^\dagger 5} - \mathbf{U}_{5^\dagger 0}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{0^\dagger 34^\dagger 1} + (5\leftrightarrow 0)\right)\left(M_1^{23} + M_3^{21} + (5\leftrightarrow 0)\right)\\ &+ (1 \leftrightarrow 3, 2 \leftrightarrow 4),\end{aligned} \tag{3.16}$$

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where M_i^{jk} is defined in (2.29). Using property (2.30) one can show that $\mathbf{G}_{s2}$ vanishes without the shockwave, i.e. when all the $U \to 1$. Indeed, it is clear from the representation

$$\begin{aligned}
2\mathbf{G}_{s2} =& (\mathbf{U}_{2^\dagger 5}\mathbf{U}_{5^\dagger 0}\mathbf{U}_{0^\dagger 34^\dagger 1} - \mathbf{U}_{4^\dagger 0}\mathbf{U}_{5^\dagger 1}\mathbf{U}_{0^\dagger 52^\dagger 3} + \mathbf{U}_{0^\dagger 15^\dagger 04^\dagger 52^\dagger 3} - \mathbf{U}_{0^\dagger 34^\dagger 15^\dagger 02^\dagger 5} + (5\leftrightarrow 0))(M_1^{32} - M_1^{23}) \\
&+ (\mathbf{U}_{4^\dagger 0}\mathbf{U}_{5^\dagger 1}\mathbf{U}_{0^\dagger 52^\dagger 3} - \mathbf{U}_{2^\dagger 0}\mathbf{U}_{5^\dagger 1}\mathbf{U}_{0^\dagger 34^\dagger 5} - \mathbf{U}_{0^\dagger 15^\dagger 04^\dagger 52^\dagger 3} + \mathbf{U}_{0^\dagger 15^\dagger 34^\dagger 02^\dagger 5} + (5 \leftrightarrow 0))(M_1^{42} - M_1^{24}) \\
&+ (\mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{4^\dagger 05^\dagger 3} - \mathbf{U}_{4^\dagger 5}\mathbf{U}_{5^\dagger 0}\mathbf{U}_{0^\dagger 12^\dagger 3} + \mathbf{U}_{0^\dagger 12^\dagger 35^\dagger 04^\dagger 5} - \mathbf{U}_{0^\dagger 15^\dagger 34^\dagger 02^\dagger 5} + (5 \leftrightarrow 0))(M_1^{43} - M_1^{34}) \\
&+ (\mathbf{U}_{0^\dagger 3}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{4^\dagger 15^\dagger 0} - \mathbf{U}_{2^\dagger 0}\mathbf{U}_{5^\dagger 1}\mathbf{U}_{0^\dagger 34^\dagger 5} + \mathbf{U}_{0^\dagger 15^\dagger 34^\dagger 02^\dagger 5} - \mathbf{U}_{0^\dagger 52^\dagger 04^\dagger 15^\dagger 3} + (5 \leftrightarrow 0))(M_2^{31} - M_2^{13}) \\
&+ (\mathbf{U}_{0^\dagger 5}\mathbf{U}_{5^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 0} - \mathbf{U}_{0^\dagger 3}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{4^\dagger 15^\dagger 0} - \mathbf{U}_{0^\dagger 15^\dagger 02^\dagger 34^\dagger 5} + \mathbf{U}_{0^\dagger 35^\dagger 02^\dagger 54^\dagger 1} + (5 \leftrightarrow 0))(M_2^{41} - M_2^{14}) \\
&+ (\mathbf{U}_{0^\dagger 5}\mathbf{U}_{5^\dagger 3}\mathbf{U}_{2^\dagger 04^\dagger 1} - \mathbf{U}_{2^\dagger 0}\mathbf{U}_{5^\dagger 1}\mathbf{U}_{0^\dagger 34^\dagger 5} + \mathbf{U}_{0^\dagger 15^\dagger 34^\dagger 02^\dagger 5} - \mathbf{U}_{0^\dagger 54^\dagger 12^\dagger 05^\dagger 3} + (5 \leftrightarrow 0))(M_2^{43} - M_2^{34}) \\
&+ (1 \leftrightarrow 3, 2 \leftrightarrow 4).
\end{aligned} \tag{3.17}$$

The contribution which is the product of the antisymmetric w.r.t. $5 \leftrightarrow 0$ parts reads

$$\mathbf{G}_a = \mathbf{G}_{a1} + \mathbf{G}_{a2} + \mathbf{G}_{a3}. \tag{3.18}$$

$$\begin{aligned}
\mathbf{G}_{a1} =& \frac{1}{2}(\mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{4^\dagger 05^\dagger 3} + \mathbf{U}_{0^\dagger 34^\dagger 52^\dagger 05^\dagger 1} - (5 \leftrightarrow 0))(M_2^{31} - M_2^{34} - M_1^{42} + M_1^{43} - (5 \leftrightarrow 0)) \\
&+ \frac{1}{2}(\mathbf{U}_{0^\dagger 3}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{4^\dagger 15^\dagger 0} + \mathbf{U}_{0^\dagger 35^\dagger 02^\dagger 54^\dagger 1} - (5 \leftrightarrow 0))(M_2^{13} - M_2^{14} - M_3^{42} + M_3^{41} - (5 \leftrightarrow 0)) \\
&+ (1 \leftrightarrow 3, 2 \leftrightarrow 4).
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
\mathbf{G}_{a2} =& \frac{1}{2}(\mathbf{U}_{0^\dagger 34^\dagger 15^\dagger 02^\dagger 5} - (5 \leftrightarrow 0))(\tilde{L}_{13} + 2M_{21} - 2M_{23} - M_1^{23} + M_3^{21} - (5 \leftrightarrow 0)) \\
&+ \frac{1}{2}(\mathbf{U}_{0^\dagger 15^\dagger 02^\dagger 34^\dagger 5} - (5 \leftrightarrow 0))(\tilde{L}_{42} - 2M_{12} + 2M_{14} + M_2^{14} - M_4^{12} - (5 \leftrightarrow 0)) \\
&+ (1 \leftrightarrow 3, 2 \leftrightarrow 4).
\end{aligned} \tag{3.20}$$

Here the functions $\tilde{L}$ and M_{ij} are defined in (2.31) and (2.32).

$$\begin{aligned}
\mathbf{G}_{a3} =& \frac{1}{2}(\mathbf{U}_{0^\dagger 5}\mathbf{U}_{5^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 0} - (5 \leftrightarrow 0))(\tilde{L}_{12} + \tilde{L}_{14} - 2M_{24} + M_2^{14} + M_4^{12} - (5 \leftrightarrow 0)) \\
&+ \frac{1}{2}(\mathbf{U}_{5^\dagger 0}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{0^\dagger 34^\dagger 1} - (5 \leftrightarrow 0))(\tilde{L}_{21} + \tilde{L}_{23} - 2M_{13} + M_1^{23} + M_3^{21} - (5 \leftrightarrow 0)) \\
&+ (1 \leftrightarrow 3, 2 \leftrightarrow 4).
\end{aligned} \tag{3.21}$$

From (2.31) and (2.32) one can see that it is possible to express $\mathbf{G}_a$ in terms of only one function M_i^{jk} (2.29).

The β-functional part of 1-gluon contribution $\mathbf{G}_\beta$ (3.11) has the same structure as LO kernel (2.12)

$$\begin{aligned}
\mathbf{G}_\beta =& \frac{\vec{r}_{14}{}^2}{\vec{r}_{10}{}^2\vec{r}_{40}{}^2}M_{14}^\beta(\mathbf{U}_{10^\dagger}\mathbf{U}_{02^\dagger 34^\dagger} + \mathbf{U}_{4^\dagger 0}\mathbf{U}_{12^\dagger 30^\dagger} - (0 \to 1 \equiv 0 \to 4)) \\
&+ \frac{\vec{r}_{12}{}^2}{\vec{r}_{10}{}^2\vec{r}_{20}{}^2}M_{12}^\beta(\mathbf{U}_{10^\dagger}\mathbf{U}_{02^\dagger 34^\dagger} + \mathbf{U}_{2^\dagger 0}\mathbf{U}_{10^\dagger 34^\dagger} - (0 \to 1 \equiv 0 \to 2)) \\
&- \frac{\vec{r}_{24}{}^2}{2\vec{r}_{20}{}^2\vec{r}_{40}{}^2}M_{24}^\beta(\mathbf{U}_{10^\dagger}\mathbf{U}_{02^\dagger 34^\dagger} + \mathbf{U}_{30^\dagger}\mathbf{U}_{04^\dagger 12^\dagger} - (0 \to 4 \equiv 0 \to 2)) \\
&- \frac{\vec{r}_{13}{}^2}{2\vec{r}_{10}{}^2\vec{r}_{30}{}^2}M_{13}^\beta(\mathbf{U}_{4^\dagger 0}\mathbf{U}_{12^\dagger 30^\dagger} + \mathbf{U}_{2^\dagger 0}\mathbf{U}_{34^\dagger 10^\dagger} - (0 \to 1 \equiv 0 \to 3)) + (1 \leftrightarrow 3, 2 \leftrightarrow 4).
\end{aligned} \tag{3.22}$$

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Here M^β is defined in (2.33). The 1-gluon term without beta function reads

$$\mathbf{G} = \mathbf{G}_1 + \mathbf{G}_0. \tag{3.23}$$

In $\mathbf{G}$ one can pick the terms independent of $\vec{r}_0$ and integrate them out if they are convergent. We call these terms $\mathbf{G}_0$. In fact the choice of $\mathbf{G}_0$ is not unique. We have

$$\begin{aligned}
\mathbf{G}_0 = \frac{N_c}{4}(\mathbf{U}_{4^\dagger 1}\mathbf{U}_{2^\dagger 3} - \mathbf{U}_{4^\dagger 3}\mathbf{U}_{2^\dagger 1}) \Bigg\{ & \left(\frac{\vec{r}_{14}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{40}^{\,2}} + \frac{\vec{r}_{23}^{\,2}}{\vec{r}_{20}^{\,2}\vec{r}_{30}^{\,2}} - \frac{\vec{r}_{13}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{30}^{\,2}} - \frac{\vec{r}_{24}^{\,2}}{\vec{r}_{20}^{\,2}\vec{r}_{40}^{\,2}} \right) \\
& \times \ln\left(\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{12}^{\,2}}\right) \ln\left(\frac{\vec{r}_{20}^{\,2}}{\vec{r}_{12}^{\,2}}\right) + \left(\frac{\vec{r}_{13}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{30}^{\,2}} + \frac{\vec{r}_{24}^{\,2}}{\vec{r}_{20}^{\,2}\vec{r}_{40}^{\,2}} - \frac{\vec{r}_{34}^{\,2}}{\vec{r}_{30}^{\,2}\vec{r}_{40}^{\,2}} - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{20}^{\,2}\vec{r}_{10}^{\,2}} \right) \\
& \times \ln\left(\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{14}^{\,2}}\right) \ln\left(\frac{\vec{r}_{40}^{\,2}}{\vec{r}_{14}^{\,2}}\right) + \left(\ln\left(\frac{\vec{r}_{20}^{\,2}}{\vec{r}_{24}^{\,2}}\right) \ln\left(\frac{\vec{r}_{40}^{\,2}}{\vec{r}_{24}^{\,2}}\right) + \ln\left(\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{13}^{\,2}}\right) \ln\left(\frac{\vec{r}_{30}^{\,2}}{\vec{r}_{13}^{\,2}}\right) \right) \\
& \times \left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{20}^{\,2}} - \frac{\vec{r}_{14}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{40}^{\,2}} \right) \Bigg\} + (1 \leftrightarrow 3, 2 \leftrightarrow 4).
\end{aligned} \tag{3.24}$$

It has zero dipole limit.

$$\begin{aligned}
\mathbf{G} = {} & \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{20}^{\,2}} \ln\left(\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{12}^{\,2}}\right) \ln\left(\frac{\vec{r}_{20}^{\,2}}{\vec{r}_{12}^{\,2}}\right) \Bigg\{ \frac{N_c}{2} (2N_c \mathbf{U}_{2^\dagger 34^\dagger 1} - \mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 0} - \mathbf{U}_{2^\dagger 0}\mathbf{U}_{4^\dagger 10^\dagger 3}) \\
& + (\mathbf{U}_{2^\dagger 10^\dagger 34^\dagger 0} - \mathbf{U}_{2^\dagger 0}\mathbf{U}_{4^\dagger 3}\mathbf{U}_{0^\dagger 1} - (0 \to 1)) \Bigg\} \\
& + \frac{\vec{r}_{14}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{40}^{\,2}} \ln\left(\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{14}^{\,2}}\right) \ln\left(\frac{\vec{r}_{40}^{\,2}}{\vec{r}_{14}^{\,2}}\right) \Bigg\{ \frac{N_c}{2} (2N_c \mathbf{U}_{2^\dagger 34^\dagger 1} - \mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 0} - \mathbf{U}_{4^\dagger 0}\mathbf{U}_{2^\dagger 30^\dagger 1}) \\
& + (\mathbf{U}_{2^\dagger 30^\dagger 14^\dagger 0} - \mathbf{U}_{4^\dagger 0}\mathbf{U}_{2^\dagger 3}\mathbf{U}_{0^\dagger 1} - (0 \to 1)) \Bigg\} \\
& + \frac{1}{2} \left(\frac{\vec{r}_{13}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{30}^{\,2}} \ln\left(\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{13}^{\,2}}\right) \ln\left(\frac{\vec{r}_{30}^{\,2}}{\vec{r}_{13}^{\,2}}\right) + \frac{\vec{r}_{24}^{\,2}}{\vec{r}_{20}^{\,2}\vec{r}_{40}^{\,2}} \ln\left(\frac{\vec{r}_{20}^{\,2}}{\vec{r}_{24}^{\,2}}\right) \ln\left(\frac{\vec{r}_{40}^{\,2}}{\vec{r}_{24}^{\,2}}\right) \right) \\
& \times \{ (\mathbf{U}_{4^\dagger 0}\mathbf{U}_{2^\dagger 1} + \mathbf{U}_{4^\dagger 1}\mathbf{U}_{2^\dagger 0})\mathbf{U}_{0^\dagger 3} - \mathbf{U}_{2^\dagger 04^\dagger 10^\dagger 3} - \mathbf{U}_{2^\dagger 04^\dagger 30^\dagger 1} - (0 \to 3) \} \\
& + \{ \mathbf{U}_{2^\dagger 0}\mathbf{U}_{4^\dagger 1}\mathbf{U}_{0^\dagger 3} - \mathbf{U}_{2^\dagger 0}\mathbf{U}_{0^\dagger 1}\mathbf{U}_{34^\dagger} + \mathbf{U}_{2^\dagger 10^\dagger 34^\dagger 0} - \mathbf{U}_{0^\dagger 32^\dagger 04^\dagger 1} \} \\
& \times \frac{1}{2\vec{r}_{20}^{\,2}} \left(\frac{\vec{r}_{23}^{\,2}}{\vec{r}_{30}^{\,2}} - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{10}^{\,2}} \right) \ln\left(\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{13}^{\,2}}\right) \ln\left(\frac{\vec{r}_{30}^{\,2}}{\vec{r}_{13}^{\,2}}\right) + \frac{1}{2\vec{r}_{10}^{\,2}} \left(\frac{\vec{r}_{14}^{\,2}}{\vec{r}_{40}^{\,2}} - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{20}^{\,2}} \right) \ln\left(\frac{\vec{r}_{20}^{\,2}}{\vec{r}_{24}^{\,2}}\right) \ln\left(\frac{\vec{r}_{40}^{\,2}}{\vec{r}_{24}^{\,2}}\right) \\
& \times \{ \mathbf{U}_{2^\dagger 3}\mathbf{U}_{4^\dagger 0}\mathbf{U}_{0^\dagger 1} - \mathbf{U}_{2^\dagger 0}\mathbf{U}_{0^\dagger 1}\mathbf{U}_{34^\dagger} + \mathbf{U}_{2^\dagger 10^\dagger 34^\dagger 0} - \mathbf{U}_{0^\dagger 14^\dagger 02^\dagger 3} \} \\
& + \{ \mathbf{U}_{4^\dagger 0}\mathbf{U}_{2^\dagger 1}\mathbf{U}_{0^\dagger 3} - N_c \mathbf{U}_{2^\dagger 0}\mathbf{U}_{4^\dagger 10^\dagger 3} + \mathbf{U}_{2^\dagger 34^\dagger 1} - \mathbf{U}_{2^\dagger 04^\dagger 30^\dagger 1} \} \\
& \times \frac{1}{2\vec{r}_{30}^{\,2}} \left(\frac{\vec{r}_{23}^{\,2}}{\vec{r}_{20}^{\,2}} - \frac{\vec{r}_{13}^{\,2}}{\vec{r}_{10}^{\,2}} \right) \ln\left(\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{12}^{\,2}}\right) \ln\left(\frac{\vec{r}_{20}^{\,2}}{\vec{r}_{12}^{\,2}}\right) + \frac{1}{2\vec{r}_{40}^{\,2}} \left(\frac{\vec{r}_{14}^{\,2}}{\vec{r}_{10}^{\,2}} - \frac{\vec{r}_{24}^{\,2}}{\vec{r}_{20}^{\,2}} \right) \ln\left(\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{12}^{\,2}}\right) \ln\left(\frac{\vec{r}_{20}^{\,2}}{\vec{r}_{12}^{\,2}}\right) \\
& \times \{ \mathbf{U}_{4^\dagger 0}\mathbf{U}_{2^\dagger 1}\mathbf{U}_{0^\dagger 3} - N_c \mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 0} + \mathbf{U}_{2^\dagger 34^\dagger 1} - \mathbf{U}_{2^\dagger 04^\dagger 30^\dagger 1} \} \\
& + \{ \mathbf{U}_{2^\dagger 0}\mathbf{U}_{4^\dagger 1}\mathbf{U}_{0^\dagger 3} - N_c \mathbf{U}_{4^\dagger 0}\mathbf{U}_{12^\dagger 30^\dagger} + \mathbf{U}_{2^\dagger 34^\dagger 1} - \mathbf{U}_{2^\dagger 04^\dagger 10^\dagger 3} \} \\
& \times \frac{1}{2\vec{r}_{30}^{\,2}} \left(\frac{\vec{r}_{34}^{\,2}}{\vec{r}_{40}^{\,2}} - \frac{\vec{r}_{13}^{\,2}}{\vec{r}_{10}^{\,2}} \right) \ln\left(\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{14}^{\,2}}\right) \ln\left(\frac{\vec{r}_{40}^{\,2}}{\vec{r}_{14}^{\,2}}\right) + \frac{1}{2\vec{r}_{20}^{\,2}} \left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{10}^{\,2}} - \frac{\vec{r}_{24}^{\,2}}{\vec{r}_{40}^{\,2}} \right) \ln\left(\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{14}^{\,2}}\right) \ln\left(\frac{\vec{r}_{40}^{\,2}}{\vec{r}_{14}^{\,2}}\right) \\
& \times \{ \mathbf{U}_{2^\dagger 0}\mathbf{U}_{4^\dagger 1}\mathbf{U}_{0^\dagger 3} - N_c \mathbf{U}_{0^\dagger 1}\mathbf{U}_{02^\dagger 34^\dagger} + \mathbf{U}_{2^\dagger 34^\dagger 1} - \mathbf{U}_{2^\dagger 04^\dagger 10^\dagger 3} \} + (1 \leftrightarrow 3, 2 \leftrightarrow 4).
\end{aligned} \tag{3.25}$$

All the integrals with the functions $\mathbf{G}_s$, $\mathbf{G}_a$, $\mathbf{G}_\beta$ and $\mathbf{G}$ are convergent. It is clear from the explicit expressions for $\mathbf{G}_\beta$ and $\mathbf{G}$. For $\mathbf{G}_s$ and $\mathbf{G}_a$ one can see it recalling that $L_{ij}^{(q)}$ has

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unintegrable singularity at $\vec{r}_0 = \vec{r}_5$ and M_k^{ij} has unintegrable singularity at $\vec{r}_0 = \vec{r}_5 = \vec{r}_k$. In all expressions in this section these singularities cancel.

3.2 Double dipole

The symmetric contribution reads

$$\tilde{\mathbf{G}}_s = \tilde{\mathbf{G}}_{s1} + n_f \tilde{\mathbf{G}}_q + \tilde{\mathbf{G}}_{s2}, \tag{3.26}$$

$$\begin{aligned}\tilde{\mathbf{G}}_{s1} = &\left(\{\mathbf{U}_{0^\dagger 1 2^\dagger 5}\mathbf{U}_{4^\dagger 3 5^\dagger 0} - \mathbf{U}_{0^\dagger 5}\mathbf{U}_{2^\dagger 1 5^\dagger 3 4^\dagger 0} - (5 \to 0)\} + (5 \leftrightarrow 0)\right)(L_{14} - L_{13} + L_{23} - L_{24}) \\ &+ (1 \leftrightarrow 3, 2 \leftrightarrow 4),\end{aligned} \tag{3.27}$$

$$\begin{aligned}\tilde{\mathbf{G}}_q = &\frac{1}{2}\left(\left\{\frac{\mathbf{U}_{4^\dagger 3}}{N_c}\left(\mathbf{U}_{0^\dagger 1 2^\dagger 5} + \mathbf{U}_{0^\dagger 5 2^\dagger 1} - \frac{\mathbf{U}_{2^\dagger 1}\mathbf{U}_{0^\dagger 5}}{N_c}\right) - \mathbf{U}_{0^\dagger 1 2^\dagger 5 4^\dagger 3} - (5 \to 0)\right\} + (5 \leftrightarrow 0)\right) \\ &\times (L_{14}^q - L_{13}^q + L_{23}^q - L_{24}^q) + (1 \leftrightarrow 3, 2 \leftrightarrow 4),\end{aligned} \tag{3.28}$$

$$\begin{aligned}\tilde{\mathbf{G}}_{s2} = &\frac{1}{2}(\mathbf{U}_{0^\dagger 5 4^\dagger 3}\mathbf{U}_{2^\dagger 0 5^\dagger 1} - \mathbf{U}_{0^\dagger 5}\mathbf{U}_{2^\dagger 1 5^\dagger 3 4^\dagger 0} + (5 \leftrightarrow 0))(M_4^{12} + M_3^{21} - M_1^{34} - M_2^{43} + (5 \leftrightarrow 0)) \\ &+ \frac{1}{2}(\mathbf{U}_{4^\dagger 0}\mathbf{U}_{0^\dagger 3 5^\dagger 1 2^\dagger 5} + \mathbf{U}_{4^\dagger 0}\mathbf{U}_{0^\dagger 5 2^\dagger 1 5^\dagger 3} - \mathbf{U}_{0^\dagger 3}\mathbf{U}_{2^\dagger 1 5^\dagger 0 4^\dagger 5} - \mathbf{U}_{0^\dagger 3}\mathbf{U}_{2^\dagger 5 4^\dagger 0 5^\dagger 1} + (5 \leftrightarrow 0)) \\ &\times \left(M_4^{13} + M_3^{14} - M_4^{23} - M_3^{24} + (5 \leftrightarrow 0)\right) + (1 \leftrightarrow 3, 2 \leftrightarrow 4).\end{aligned} \tag{3.29}$$

Here L, L^q, and M_i^{jk} are introduced in (2.22), (2.23), and (2.29). The antisymmetric contribution reads

$$\begin{aligned}\tilde{\mathbf{G}}_a = &\frac{1}{2}(\mathbf{U}_{0^\dagger 5 4^\dagger 3}\mathbf{U}_{2^\dagger 0 5^\dagger 1} - \mathbf{U}_{0^\dagger 5 2^\dagger 1}\mathbf{U}_{4^\dagger 0 5^\dagger 3} + \mathbf{U}_{5^\dagger 0}\mathbf{U}_{0^\dagger 1 2^\dagger 5 4^\dagger 3} - \mathbf{U}_{0^\dagger 5}\mathbf{U}_{2^\dagger 1 5^\dagger 3 4^\dagger 0} - (5 \leftrightarrow 0)) \\ &\times (M_4^{11} - M_3^{11} + M_3^{12} - M_4^{12} + M_3^{21} - M_4^{21} - M_3^{22} + M_4^{22}) \\ &+ \frac{1}{2}(\mathbf{U}_{5^\dagger 3}\mathbf{U}_{0^\dagger 1 2^\dagger 0 4^\dagger 5} + \mathbf{U}_{5^\dagger 3}\mathbf{U}_{0^\dagger 5 4^\dagger 0 2^\dagger 1} - \mathbf{U}_{4^\dagger 5}\mathbf{U}_{0^\dagger 1 2^\dagger 0 5^\dagger 3} - \mathbf{U}_{4^\dagger 5}\mathbf{U}_{0^\dagger 3 5^\dagger 0 2^\dagger 1} - (5 \leftrightarrow 0)) \\ &\times (M_4^{23} + M_3^{24} - M_4^{13} - M_3^{14} - M_4^{31} + M_4^{32} - M_3^{41} + M_3^{42}) + (1 \leftrightarrow 3, 2 \leftrightarrow 4).\end{aligned} \tag{3.30}$$

The β-functional contribution has the form

$$\begin{aligned}\tilde{\mathbf{G}}_\beta = &\left(\frac{\vec{r}_{13}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{30}^{\,2}}M_{13}^\beta - \frac{\vec{r}_{23}^{\,2}}{\vec{r}_{20}^{\,2}\vec{r}_{30}^{\,2}}M_{23}^\beta - \frac{\vec{r}_{14}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{40}^{\,2}}M_{14}^\beta + \frac{\vec{r}_{24}^{\,2}}{\vec{r}_{20}^{\,2}\vec{r}_{40}^{\,2}}M_{24}^\beta\right) \\ &\times (\mathbf{U}_{2^\dagger 1 4^\dagger 3} + \mathbf{U}_{2^\dagger 3 4^\dagger 1} - \mathbf{U}_{2^\dagger 1 0^\dagger 3 4^\dagger 0} - \mathbf{U}_{2^\dagger 0 4^\dagger 3 0^\dagger 1}),\end{aligned} \tag{3.31}$$

where M^β is introduced in (2.33). The remaining contribution reads

$$\tilde{\mathbf{G}} = \tilde{\mathbf{G}}_1 + \tilde{\mathbf{G}}_0. \tag{3.32}$$

$$\begin{aligned}\tilde{\mathbf{G}}_0 = &\frac{1}{4}(2\mathbf{U}_{2^\dagger 1}\mathbf{U}_{4^\dagger 3} - N_c\mathbf{U}_{2^\dagger 1 4^\dagger 3} - N_c\mathbf{U}_{2^\dagger 3 4^\dagger 1})\left[\left(\frac{2\vec{r}_{13}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{03}^{\,2}} - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} + \frac{\vec{r}_{14}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{04}^{\,2}}\right)\right. \\ &\times \ln\left(\frac{\vec{r}_{13}^{\,2}}{\vec{r}_{01}^{\,2}}\right)\ln\left(\frac{\vec{r}_{13}^{\,2}}{\vec{r}_{03}^{\,2}}\right) + \left(\frac{2\vec{r}_{24}^{\,2}}{\vec{r}_{02}^{\,2}\vec{r}_{04}^{\,2}} - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} + \frac{\vec{r}_{14}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{04}^{\,2}}\right)\ln\left(\frac{\vec{r}_{24}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\ln\left(\frac{\vec{r}_{24}^{\,2}}{\vec{r}_{04}^{\,2}}\right) \\ &+ \left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} - \frac{\vec{r}_{13}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{03}^{\,2}} - \frac{4\vec{r}_{14}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{04}^{\,2}} - \frac{\vec{r}_{24}^{\,2}}{\vec{r}_{02}^{\,2}\vec{r}_{04}^{\,2}} + \frac{\vec{r}_{34}^{\,2}}{\vec{r}_{03}^{\,2}\vec{r}_{04}^{\,2}}\right)\ln\left(\frac{\vec{r}_{14}^{\,2}}{\vec{r}_{01}^{\,2}}\right)\ln\left(\frac{\vec{r}_{14}^{\,2}}{\vec{r}_{04}^{\,2}}\right) \\ &\left.+ \left(\frac{\vec{r}_{13}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{03}^{\,2}} - \frac{\vec{r}_{14}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{04}^{\,2}} - \frac{\vec{r}_{23}^{\,2}}{\vec{r}_{02}^{\,2}\vec{r}_{03}^{\,2}} + \frac{\vec{r}_{24}^{\,2}}{\vec{r}_{02}^{\,2}\vec{r}_{04}^{\,2}}\right)\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}}\right)\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\right] + (1 \leftrightarrow 3, 2 \leftrightarrow 4).\end{aligned} \tag{3.33}$$

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$$
\begin{aligned}
\tilde{\mathbf{G}}_1 &= \frac{1}{2}(\mathbf{U}_{2^\dagger 0}\mathbf{U}_{0^\dagger 14^\dagger 3} + \mathbf{U}_{2^\dagger 0}\mathbf{U}_{0^\dagger 34^\dagger 1} - \mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 04^\dagger 3} - \mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 0}) \left[\ln\left(\frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2}\right)\ln\left(\frac{\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2}\right)\right. \\
&\times \left(\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2} - \frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2} - \frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right) + \left(\frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2} - \frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2} - \frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2}\right) \\
&\times \ln\left(\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2}\right)\ln\left(\frac{\vec{r}_{13}{}^2}{\vec{r}_{03}{}^2}\right) + \ln\left(\frac{\vec{r}_{14}{}^2}{\vec{r}_{01}{}^2}\right)\ln\left(\frac{\vec{r}_{14}{}^2}{\vec{r}_{04}{}^2}\right)\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2} + \frac{\vec{r}_{14}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2} - \frac{\vec{r}_{24}{}^2}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2}\right) \\
&+ \left.\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2} - \frac{\vec{r}_{14}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2} + \frac{\vec{r}_{24}{}^2}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2}\right)\ln\left(\frac{\vec{r}_{24}{}^2}{\vec{r}_{02}{}^2}\right)\ln\left(\frac{\vec{r}_{24}{}^2}{\vec{r}_{04}{}^2}\right)\right] \\
&+ \frac{1}{2}(2\mathbf{U}_{2^\dagger 1}\mathbf{U}_{4^\dagger 3} - N_c\mathbf{U}_{0^\dagger 12^\dagger 04^\dagger 3} - N_c\mathbf{U}_{0^\dagger 34^\dagger 02^\dagger 1})\ln\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2}\right)\ln\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2}\right) \\
&\times \left(\frac{\vec{r}_{14}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2} - \frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2} + \frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2} - \frac{\vec{r}_{24}{}^2}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2}\right) + (1 \leftrightarrow 3, 2 \leftrightarrow 4). \qquad (3.34)
\end{aligned}
$$

As for the quadrupole, it is straightforward to check that none of the functions $\tilde{\mathbf{G}}_s$, $\tilde{\mathbf{G}}_a$, $\tilde{\mathbf{G}}_\beta$, $\tilde{\mathbf{G}}$ has unintegrable singularities.

4 Quasi-conformal evolution equation for composite operators

To construct composite conformal operators we use the prescription [10] (see also [29])

$$
O^{\rm conf} = O + \frac{1}{2}\frac{\partial O}{\partial\eta}\bigg|_{\frac{\vec{r}_{mn}^{\,2}}{\vec{r}_{im}^{\,2}\vec{r}_{\rm in}^{\,2}} \to \frac{\vec{r}_{mn}^{\,2}}{\vec{r}_{im}^{\,2}\vec{r}_{\rm in}^{\,2}}\ln\left(\frac{\vec{r}_{mn}^{\,2}a}{\vec{r}_{im}^{\,2}\vec{r}_{\rm in}^{\,2}}\right)}, \qquad (4.1)
$$

where a is an arbitrary constant. The conformal dipole reads [10]

$$
\mathbf{U}^{\rm conf}_{12^\dagger} = \mathbf{U}_{2^\dagger 1} + \frac{\alpha_s}{4\pi^2}\int d\vec{r}_0 \frac{\vec{r}_{12}{}^2}{\vec{r}_{10}{}^2\vec{r}_{20}{}^2}\ln\left(\frac{a\vec{r}_{12}{}^2}{\vec{r}_{10}{}^2\vec{r}_{20}{}^2}\right)(\mathbf{U}_{2^\dagger 0}\mathbf{U}_{0^\dagger 1} - N_c\mathbf{U}_{2^\dagger 1}). \qquad (4.2)
$$

The evolution equation for this operator [10] is quasi-conformal

$$
\begin{aligned}
\frac{\partial \mathbf{U}^{\rm conf}_{12^\dagger}}{\partial\eta} &= \frac{\alpha_s}{2\pi^2}\int d\vec{r}_0 \frac{\vec{r}_{12}{}^2}{\vec{r}_{10}{}^2\vec{r}_{20}{}^2}\left(1 + \frac{\alpha_s}{2\pi}M^\beta_{12}\right)(\mathbf{U}_{2^\dagger 0}\mathbf{U}_{0^\dagger 1} - N_c\mathbf{U}_{2^\dagger 1})^{\rm conf} \\
&+ \frac{\alpha_s^2}{4\pi^4}\int d\vec{r}_0 d\vec{r}_5 \Big\{ L^C_{12}((\mathbf{U}_{0^\dagger 52^\dagger 05^\dagger 1} - \mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{5^\dagger 0} - (0 \to 5)) + (0 \leftrightarrow 5)) \\
&+ \tilde{L}^C_{12}(\mathbf{U}_{0^\dagger 5}\mathbf{U}_{2^\dagger 0}\mathbf{U}_{5^\dagger 1} - (0 \leftrightarrow 5)) \\
&- 2n_f L^q_{12}\left(tr(t^a U_1 t^b U_2^\dagger) tr\left(t^a U_5 t^b (U_0^\dagger - U_5^\dagger)\right) + (5 \leftrightarrow 0)\right)\Big\}, \qquad (4.3)
\end{aligned}
$$

where M^β_{12} is defined in (2.33); $L^C_{ij} \equiv L^C(\vec{r}_i, \vec{r}_j)$ and $\tilde{L}^C_{ij} \equiv \tilde{L}^C(\vec{r}_i, \vec{r}_j)$ were introduced in this form in [12]

$$
L^C_{12} = L_{12} + \frac{\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{25}{}^2}\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{15}{}^2}{\vec{r}_{05}{}^2\vec{r}_{12}{}^2}\right) + \frac{\vec{r}_{12}{}^2}{4\vec{r}_{02}{}^2\vec{r}_{05}{}^2\vec{r}_{15}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{25}{}^2}{\vec{r}_{05}{}^2\vec{r}_{12}{}^2}\right), \qquad (4.4)
$$

$$
\tilde{L}^C_{12} = \tilde{L}_{12} + \frac{\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{25}{}^2}\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{15}{}^2}{\vec{r}_{05}{}^2\vec{r}_{12}{}^2}\right) - \frac{\vec{r}_{12}{}^2}{4\vec{r}_{02}{}^2\vec{r}_{05}{}^2\vec{r}_{15}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{25}{}^2}{\vec{r}_{05}{}^2\vec{r}_{12}{}^2}\right). \qquad (4.5)
$$

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For the conformal quadrupole operator using (2.12) we have

$$\begin{aligned}
\mathbf{U}^{\text{conf}}_{12^\dagger 34^\dagger} &= \mathbf{U}_{12^\dagger 34^\dagger} + \frac{\alpha_s}{8\pi^2}\int d\vec{r}_0 \\
&\times \Big\{ \frac{\vec{r}_{14}{}^2}{\vec{r}_{10}{}^2\vec{r}_{40}{}^2} \ln\left(\frac{a\vec{r}_{14}{}^2}{\vec{r}_{10}{}^2\vec{r}_{40}{}^2}\right)(\mathbf{U}_{10^\dagger}\mathbf{U}_{02^\dagger 34^\dagger} + \mathbf{U}_{4^\dagger 0}\mathbf{U}_{12^\dagger 30^\dagger} - (0\to 1)) \\
&+ \frac{\vec{r}_{12}{}^2}{\vec{r}_{10}{}^2\vec{r}_{20}{}^2} \ln\left(\frac{a\vec{r}_{12}{}^2}{\vec{r}_{10}{}^2\vec{r}_{20}{}^2}\right)(\mathbf{U}_{10^\dagger}\mathbf{U}_{02^\dagger 34^\dagger} + \mathbf{U}_{2^\dagger 0}\mathbf{U}_{10^\dagger 34^\dagger} - (0\to 1)) \\
&- \frac{\vec{r}_{24}{}^2}{2\vec{r}_{20}{}^2\vec{r}_{40}{}^2} \ln\left(\frac{a\vec{r}_{24}{}^2}{\vec{r}_{20}{}^2\vec{r}_{40}{}^2}\right)(\mathbf{U}_{10^\dagger}\mathbf{U}_{02^\dagger 34^\dagger} + \mathbf{U}_{30^\dagger}\mathbf{U}_{04^\dagger 12^\dagger} - (0\to 4)) \\
&- \frac{\vec{r}_{13}{}^2}{2\vec{r}_{10}{}^2\vec{r}_{30}{}^2} \ln\left(\frac{a\vec{r}_{13}{}^2}{\vec{r}_{10}{}^2\vec{r}_{30}{}^2}\right)(\mathbf{U}_{4^\dagger 0}\mathbf{U}_{12^\dagger 30^\dagger} + \mathbf{U}_{2^\dagger 0}\mathbf{U}_{34^\dagger 10^\dagger} - (0\to 1)) \\
&+ (1\leftrightarrow 3, 2\leftrightarrow 4)\Big\}. \qquad (4.6)
\end{aligned}$$

The conformal double dipole operator reads

$$\begin{aligned}
(\mathbf{U}_{12^\dagger}\mathbf{U}_{34^\dagger})^{\text{conf}} &= \mathbf{U}_{12^\dagger}\mathbf{U}_{34^\dagger} + \frac{\alpha_s}{8\pi^2}\int d\vec{r}_0 (\mathbf{U}_{2^\dagger 14^\dagger 3} + \mathbf{U}_{2^\dagger 34^\dagger 1} - \mathbf{U}_{2^\dagger 10^\dagger 34^\dagger 0} - \mathbf{U}_{2^\dagger 04^\dagger 30^\dagger 1}) \\
&\times \left(\frac{\vec{r}_{13}{}^2}{\vec{r}_{10}{}^2\vec{r}_{30}{}^2}\ln\left(\frac{a\vec{r}_{13}{}^2}{\vec{r}_{10}{}^2\vec{r}_{30}{}^2}\right) - \frac{\vec{r}_{23}{}^2}{\vec{r}_{20}{}^2\vec{r}_{30}{}^2}\ln\left(\frac{a\vec{r}_{23}{}^2}{\vec{r}_{20}{}^2\vec{r}_{30}{}^2}\right) \right. \\
&\left. - \frac{\vec{r}_{14}{}^2}{\vec{r}_{10}{}^2\vec{r}_{40}{}^2}\ln\left(\frac{a\vec{r}_{14}{}^2}{\vec{r}_{10}{}^2\vec{r}_{40}{}^2}\right) + \frac{\vec{r}_{24}{}^2}{\vec{r}_{20}{}^2\vec{r}_{40}{}^2}\ln\left(\frac{a\vec{r}_{24}{}^2}{\vec{r}_{20}{}^2\vec{r}_{40}{}^2}\right)\right) \\
&+ \mathbf{U}_{4^\dagger 3}(\mathbf{U}^{\text{conf}}_{2^\dagger 1} - \mathbf{U}_{2^\dagger 1}) + \mathbf{U}_{2^\dagger 1}(\mathbf{U}^{\text{conf}}_{4^\dagger 3} - \mathbf{U}_{4^\dagger 3}). \qquad (4.7)
\end{aligned}$$

The evolution equations for the conformal quadrupole and double dipole operators in the conformal basis have the general form

$$\begin{aligned}
\frac{\partial \mathbf{U}^{\text{conf}}_{12^\dagger 34^\dagger}}{\partial\eta} &= \frac{\alpha_s}{4\pi^2}\int d\vec{r}_0 \Big\{ \frac{\vec{r}_{14}{}^2}{\vec{r}_{10}{}^2\vec{r}_{40}{}^2}(\mathbf{U}_{10^\dagger}\mathbf{U}_{02^\dagger 34^\dagger} + \mathbf{U}_{4^\dagger 0}\mathbf{U}_{12^\dagger 30^\dagger} - (0\to 1))^{\text{conf}} \\
&+ \frac{\vec{r}_{12}{}^2}{\vec{r}_{10}{}^2\vec{r}_{20}{}^2}(\mathbf{U}_{10^\dagger}\mathbf{U}_{02^\dagger 34^\dagger} + \mathbf{U}_{2^\dagger 0}\mathbf{U}_{10^\dagger 34^\dagger} - (0\to 1))^{\text{conf}} \\
&- \frac{\vec{r}_{24}{}^2}{2\vec{r}_{20}{}^2\vec{r}_{40}{}^2}(\mathbf{U}_{10^\dagger}\mathbf{U}_{02^\dagger 34^\dagger} + \mathbf{U}_{30^\dagger}\mathbf{U}_{04^\dagger 12^\dagger} - (0\to 4))^{\text{conf}} \\
&- \frac{\vec{r}_{13}{}^2}{2\vec{r}_{10}{}^2\vec{r}_{30}{}^2}(\mathbf{U}_{4^\dagger 0}\mathbf{U}_{12^\dagger 30^\dagger} + \mathbf{U}_{2^\dagger 0}\mathbf{U}_{34^\dagger 10^\dagger} - (0\to 1))^{\text{conf}} + (1\leftrightarrow 3, 2\leftrightarrow 4)\Big\} \\
&+ \frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_5\, (\mathbf{G}^{\text{conf}}_s + \mathbf{G}^{\text{conf}}_a) + \frac{\alpha_s^2}{8\pi^3}\int d\vec{r}_0\, (\mathbf{G}_\beta + \mathbf{G}^{\text{conf}}), \qquad (4.8)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(\mathbf{U}_{12^\dagger}\mathbf{U}_{34^\dagger})^{\text{conf}}}{\partial\eta} &= \Big\{ \frac{\alpha_s}{8\pi^2}\int d\vec{r}_0 \Big[\frac{4\vec{r}_{12}{}^2}{\vec{r}_{10}{}^2\vec{r}_{20}{}^2}(\mathbf{U}_{4^\dagger 3}\mathbf{U}_{2^\dagger 0}\mathbf{U}_{0^\dagger 1} - N_c\mathbf{U}_{4^\dagger 3}\mathbf{U}_{2^\dagger 1})^{\text{conf}} \\
&+ \left(\frac{\vec{r}_{13}{}^2}{\vec{r}_{10}{}^2\vec{r}_{30}{}^2} - \frac{\vec{r}_{23}{}^2}{\vec{r}_{20}{}^2\vec{r}_{30}{}^2} - \frac{\vec{r}_{14}{}^2}{\vec{r}_{10}{}^2\vec{r}_{40}{}^2} + \frac{\vec{r}_{24}{}^2}{\vec{r}_{20}{}^2\vec{r}_{40}{}^2}\right) \\
&\times (\mathbf{U}_{2^\dagger 14^\dagger 3} + \mathbf{U}_{2^\dagger 34^\dagger 1} - \mathbf{U}_{2^\dagger 10^\dagger 34^\dagger 0} - \mathbf{U}_{2^\dagger 04^\dagger 30^\dagger 1})^{\text{conf}}\Big] \\
&+ \mathbf{U}_{34^\dagger}\langle K_{\text{NLO}} \otimes \mathbf{U}^{\text{conf}}_{12^\dagger}\rangle + (1\leftrightarrow 3, 2\leftrightarrow 4)\Big\} \\
&+ \frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_5\, (\tilde{\mathbf{G}}^{\text{conf}}_s + \tilde{\mathbf{G}}^{\text{conf}}_a) + \frac{\alpha_s^2}{8\pi^3}\int d\vec{r}_0\, (\tilde{\mathbf{G}}_\beta + \tilde{\mathbf{G}}^{\text{conf}}). \qquad (4.9)
\end{aligned}$$

As in the previous section, the individual NLO evolution of the dipoles here is taken out of the functions $\tilde{\mathbf{G}}$

$$\langle K_{\rm NLO} \otimes \mathbf{U}_{12^\dagger}^{\rm conf} \rangle = \frac{\partial \mathbf{U}_{12^\dagger}^{\rm conf}}{\partial \eta} - \frac{\alpha_s}{2\pi^2} \int d\vec{r}_0 \frac{\vec{r}_{12}{}^2}{\vec{r}_{10}{}^2 \vec{r}_{20}{}^2} \left(\mathbf{U}_{2^\dagger 0} \mathbf{U}_{0^\dagger 1} - N_c \mathbf{U}_{2^\dagger 1}\right)^{\rm conf}. \tag{4.10}$$

Therefore one can rewrite (4.9)

$$\begin{aligned} \frac{\partial (\mathbf{U}_{12^\dagger}\mathbf{U}_{34^\dagger})^{\rm conf}}{\partial \eta} = & \Bigg\{ \mathbf{U}_{34^\dagger} \frac{\partial \mathbf{U}_{12^\dagger}^{\rm conf}}{\partial \eta} + \frac{\alpha_s}{8\pi^2} \int d\vec{r}_0 \Bigg[\frac{4\vec{r}_{12}{}^2}{\vec{r}_{10}{}^2 \vec{r}_{20}{}^2} \\ & \times \{ (\mathbf{U}_{4^\dagger 3} \mathbf{U}_{2^\dagger 0} \mathbf{U}_{0^\dagger 1} - N_c \mathbf{U}_{4^\dagger 3} \mathbf{U}_{2^\dagger 1})^{\rm conf} - \mathbf{U}_{4^\dagger 3} \left(\mathbf{U}_{2^\dagger 0} \mathbf{U}_{0^\dagger 1} - N_c \mathbf{U}_{2^\dagger 1}\right)^{\rm conf} \} \\ & + \left(\frac{\vec{r}_{13}{}^2}{\vec{r}_{10}{}^2 \vec{r}_{30}{}^2} - \frac{\vec{r}_{23}{}^2}{\vec{r}_{20}{}^2 \vec{r}_{30}{}^2} - \frac{\vec{r}_{14}{}^2}{\vec{r}_{10}{}^2 \vec{r}_{40}{}^2} + \frac{\vec{r}_{24}{}^2}{\vec{r}_{20}{}^2 \vec{r}_{40}{}^2} \right) \\ & \times \left(\mathbf{U}_{2^\dagger 1 4^\dagger 3} + \mathbf{U}_{2^\dagger 3 4^\dagger 1} - \mathbf{U}_{2^\dagger 1 0^\dagger 3 4^\dagger 0} - \mathbf{U}_{2^\dagger 0 4^\dagger 3 0^\dagger 1} \right)^{\rm conf} \Bigg] + (1 \leftrightarrow 3, 2 \leftrightarrow 4) \Bigg\} \\ & + \frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_5 \, (\tilde{\mathbf{G}}_s^{\rm conf} + \tilde{\mathbf{G}}_a^{\rm conf}) + \frac{\alpha_s^2}{8\pi^3} \int d\vec{r}_0 \, (\tilde{\mathbf{G}}_\beta + \tilde{\mathbf{G}}^{\rm conf}). \end{aligned} \tag{4.11}$$

Plainly, $\mathbf{G}_\beta$ and $\tilde{\mathbf{G}}_\beta$ are the same as in (3.11) and (3.12). The other functions $\mathbf{G}^{\rm conf}$ will be given below.

To obtain these functions one has to calculate the evolution equations for conformal operators (4.6), (4.7) using (2.11)–(2.15) and express the results in terms of conformal operators via (4.1). Technically, it means that one has to add to the kernels of the evolution equations from the previous section the corrections in the form of double integrals w.r.t. $\vec{r}_0$ and $\vec{r}_5$ [10]. To get the conformally invariant results one has to symmetrize these corrections according to (3.2). Then, the terms with color operators independent of $\vec{r}_0$ (or $\vec{r}_5$) can be integrated w.r.t. $\vec{r}_0$ (or $\vec{r}_5$) via the integrals from appendix A of [30]. Finally, the terms with color operators independent of both $\vec{r}_0$ and $\vec{r}_5$ can be integrated with respect to both $\vec{r}_0$ and $\vec{r}_5$. In addition to the integrals from appendix A of [30], one needs the following integral

$$\int d\vec{r}_0 \left(\frac{\vec{r}_{01}\vec{r}_{02}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} - \frac{\vec{r}_{01}\vec{r}_{03}}{\vec{r}_{01}^{\,2}\vec{r}_{03}^{\,2}} \right) \ln^2 \left(\frac{\vec{r}_{02}^{\,2}\vec{r}_{13}^{\,2}}{\vec{r}_{03}^{\,2}\vec{r}_{12}^{\,2}} \right) = \frac{\pi}{3} \ln^3 \left(\frac{\vec{r}_{13}^{\,2}}{\vec{r}_{12}^{\,2}} \right). \tag{4.12}$$

4.1 Quadrupole

For the symmetric contribution $\mathbf{G}_s^{\rm conf}$ we have

$$\mathbf{G}_s^{\rm conf} = \mathbf{G}_{s1}^{\rm conf} + n_f \mathbf{G}_q + \mathbf{G}_{s2}^{\rm conf}, \tag{4.13}$$

where $\mathbf{G}_q$ did not change. It is defined in (3.15) .

$$\begin{aligned} \mathbf{G}_{s1}^{\rm conf} = & \left(\{\mathbf{U}_{0^\dagger 3 4^\dagger 1 5^\dagger 0 2^\dagger 5} - \mathbf{U}_{5^\dagger 0} \mathbf{U}_{2^\dagger 5} \mathbf{U}_{0^\dagger 3 4^\dagger 1} - (5 \to 0)\} + (5 \leftrightarrow 0)\right) \left(L_{12}^C + L_{32}^C - L_{13}^C\right) \\ & + \left(\{\mathbf{U}_{0^\dagger 1 5^\dagger 0 2^\dagger 3 4^\dagger 5} - \mathbf{U}_{0^\dagger 5} \mathbf{U}_{5^\dagger 1} \mathbf{U}_{2^\dagger 3 4^\dagger 0} - (5 \to 0)\} + (5 \leftrightarrow 0)\right) \left(L_{12}^C + L_{14}^C - L_{42}^C\right) \\ & + (1 \leftrightarrow 3, 2 \leftrightarrow 4), \end{aligned} \tag{4.14}$$

where L^C is defined in (4.4).

$$
\begin{aligned}
\mathbf{G}_{s2}^{\mathrm{conf}} &= \frac{1}{2}\left(\mathbf{U}_{0^\dagger 34^\dagger 52^\dagger 05^\dagger 1} - \mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{4^\dagger 05^\dagger 3} + (5 \leftrightarrow 0)\right) \\
&\times \left(M_1^{C34} - M_1^{C43} + M_1^{C42} - M_1^{C24} + M_2^{C43} - M_2^{C34} + M_2^{C31} - M_2^{C13}\right) \\
&+ \frac{1}{2}\left(\mathbf{U}_{0^\dagger 35^\dagger 02^\dagger 54^\dagger 1} - \mathbf{U}_{0^\dagger 3}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{4^\dagger 15^\dagger 0} + (5 \leftrightarrow 0)\right) \\
&\times \left(M_3^{C14} - M_3^{C41} + M_3^{C42} - M_3^{C24} + M_2^{C41} - M_2^{C14} + M_2^{C13} - M_2^{C31}\right) \\
&+ \frac{1}{2}\left(\mathbf{U}_{0^\dagger 15^\dagger 02^\dagger 34^\dagger 5} - \mathbf{U}_{0^\dagger 5}\mathbf{U}_{5^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 0} + (5 \leftrightarrow 0)\right)\left(M_2^{C14} - M_2^{C41} + M_4^{C12} - M_4^{C21}\right) \\
&+ \frac{1}{2}\left(\mathbf{U}_{0^\dagger 34^\dagger 15^\dagger 02^\dagger 5} - \mathbf{U}_{5^\dagger 0}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{0^\dagger 34^\dagger 1} + (5 \leftrightarrow 0)\right)\left(M_1^{C23} - M_1^{C32} + M_3^{C21} - M_3^{C12}\right) \\
&+ (1 \leftrightarrow 3, 2 \leftrightarrow 4).
\end{aligned} \tag{4.15}
$$

Here $M_i^{Cjk} \equiv M^C(\vec{r}_i, \vec{r}_j, \vec{r}_k)$ reads

$$
\begin{aligned}
M_2^{C13} &= M_2^{13} + \frac{\vec{r}_{15}{}^2\vec{r}_{23}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{25}{}^2\vec{r}_{35}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{23}{}^2}{\vec{r}_{15}{}^2\vec{r}_{25}{}^2\vec{r}_{35}{}^2}\right) - \frac{\vec{r}_{12}{}^2}{4\vec{r}_{02}{}^2\vec{r}_{05}{}^2\vec{r}_{15}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{25}{}^2}{\vec{r}_{05}{}^2\vec{r}_{12}{}^2}\right) \\
&+ \frac{\vec{r}_{13}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{35}{}^2}\ln\left(\frac{\vec{r}_{05}{}^2\vec{r}_{13}{}^2\vec{r}_{35}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^4}\right) - \frac{\vec{r}_{12}{}^2\vec{r}_{23}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{25}{}^2\vec{r}_{35}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{23}{}^2}{\vec{r}_{12}{}^2\vec{r}_{25}{}^2\vec{r}_{35}{}^2}\right) \\
&- \frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{05}{}^2\vec{r}_{35}{}^2}\ln\left(\frac{\vec{r}_{05}{}^2\vec{r}_{12}{}^2\vec{r}_{35}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right) \\
&= \frac{\vec{r}_{15}{}^2\vec{r}_{23}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{25}{}^2\vec{r}_{35}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{23}{}^2\vec{r}_{25}{}^2}{\vec{r}_{02}{}^4\vec{r}_{15}{}^2\vec{r}_{35}{}^2}\right) - \frac{\vec{r}_{12}{}^2}{4\vec{r}_{02}{}^2\vec{r}_{05}{}^2\vec{r}_{15}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{25}{}^2}{\vec{r}_{05}{}^2\vec{r}_{12}{}^2}\right) \\
&- \frac{\vec{r}_{13}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{35}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{03}{}^4\vec{r}_{25}{}^4}{\vec{r}_{02}{}^4\vec{r}_{05}{}^2\vec{r}_{13}{}^2\vec{r}_{35}{}^2}\right) + \frac{\vec{r}_{23}{}^2\vec{r}_{12}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{25}{}^2\vec{r}_{35}{}^2}\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{12}{}^2\vec{r}_{35}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2\vec{r}_{25}{}^2}\right) \\
&+ \frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{05}{}^2\vec{r}_{35}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{25}{}^4}{\vec{r}_{02}{}^2\vec{r}_{05}{}^2\vec{r}_{12}{}^2\vec{r}_{35}{}^2}\right).
\end{aligned} \tag{4.16}
$$

The function M_i^{Cjk} is conformally invariant. It does not have property (2.30). Nevertheless like $\mathbf{G}_{s2}$, $\mathbf{G}_{s2}^{\mathrm{conf}}$ can be rearranged to form (3.17) where instead of M_i^{jk} will be M_i^{Cjk}. As a result $\mathbf{G}_{s2}^{\mathrm{conf}}$ vanishes without the shockwave. Finally, one can see that $\mathbf{G}_{s}^{\mathrm{conf}}$ can be formally obtained from $\mathbf{G}_s$ via the substitution $M \to M^C, L \to L^C$.

The contribution which is the product of the antisymmetric w.r.t. $5 \leftrightarrow 0$ parts reads

$$
\mathbf{G}_{a}^{\mathrm{conf}} = \mathbf{G}_{a1}^{\mathrm{conf}} + \mathbf{G}_{a2}^{\mathrm{conf}} + \mathbf{G}_{a3}^{\mathrm{conf}}. \tag{4.17}
$$

$$
\begin{aligned}
\mathbf{G}_{a1}^{\mathrm{conf}} &= \frac{1}{2}\left(\mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{4^\dagger 05^\dagger 3} + \mathbf{U}_{0^\dagger 34^\dagger 52^\dagger 05^\dagger 1} - (5 \leftrightarrow 0)\right) \\
&\times \left(M_2^{C31} + M_2^{C13} - M_2^{C34} - M_2^{C43} - M_1^{C42} - M_1^{C24} + M_1^{C43} + M_1^{C34} - R_{2134}\right) \\
&+ \frac{1}{2}\left(\mathbf{U}_{0^\dagger 3}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{4^\dagger 15^\dagger 0} + \mathbf{U}_{0^\dagger 35^\dagger 02^\dagger 54^\dagger 1} - (5 \leftrightarrow 0)\right) \\
&\times \left(M_2^{C13} + M_2^{C31} - M_2^{C14} - M_2^{C41} - M_3^{C42} - M_3^{C24} + M_3^{C41} + M_3^{C14} + R_{3241}\right) \\
&+ (1 \leftrightarrow 3, 2 \leftrightarrow 4).
\end{aligned} \tag{4.18}
$$

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Here $R_{ijkl} \equiv R(\vec{r}_i, \vec{r}_j, \vec{r}_k, \vec{r}_l)$ is a conformally invariant function. It reads

$$\begin{aligned} R_{2134} &= \frac{\vec{r}_{12}{}^2}{2\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{25}{}^2} \ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{15}{}^2}{\vec{r}_{05}{}^2\vec{r}_{12}{}^2}\right) - \frac{\vec{r}_{12}{}^2}{2\vec{r}_{02}{}^2\vec{r}_{05}{}^2\vec{r}_{15}{}^2} \ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{25}{}^2}{\vec{r}_{05}{}^2\vec{r}_{12}{}^2}\right) \\ &+ \frac{\vec{r}_{24}{}^2}{2\vec{r}_{02}{}^2\vec{r}_{05}{}^2\vec{r}_{45}{}^2} \ln\left(\frac{\vec{r}_{04}{}^2\vec{r}_{25}{}^2}{\vec{r}_{05}{}^2\vec{r}_{24}{}^2}\right) - \frac{\vec{r}_{13}{}^2}{2\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{35}{}^2} \ln\left(\frac{\vec{r}_{03}{}^2\vec{r}_{15}{}^2}{\vec{r}_{05}{}^2\vec{r}_{13}{}^2}\right). \end{aligned} \tag{4.19}$$

$$\begin{aligned} \mathbf{G}^{\text{conf}}_{a2} &= \frac{1}{2}\left(\mathbf{U}_{0^\dagger 34^\dagger 15^\dagger 02^\dagger 5} - (5 \leftrightarrow 0)\right)(R_{231} - R_{123} \\ &+ 2\tilde{L}^C_{13} - M_1^{C23} - M_1^{C32} + M_3^{C21} + M_3^{C12} + M_2^{C11} + M_1^{C22} - M_3^{C22} - M_2^{C33}) \\ &+ \frac{1}{2}\left(\mathbf{U}_{0^\dagger 15^\dagger 02^\dagger 34^\dagger 5} - (5 \leftrightarrow 0)\right)(R_{124} - R_{142} \\ &+ 2\tilde{L}^C_{42} - M_1^{C22} - M_2^{C11} + M_1^{C44} + M_4^{C11} + M_2^{C14} + M_2^{C41} - M_4^{C21} - M_4^{C12}) \\ &+ (1 \leftrightarrow 3, 2 \leftrightarrow 4). \end{aligned} \tag{4.20}$$

Here the function $\tilde{L}^C$ is defined in (4.5) and $R_{ijk} \equiv R(\vec{r}_i, \vec{r}_j, \vec{r}_k)$ is another conformally invariant function. It reads

$$R_{123} = \frac{\vec{r}_{13}{}^2}{2\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{35}{}^2} \ln\left(\frac{\vec{r}_{03}{}^2\vec{r}_{15}{}^2}{\vec{r}_{05}{}^2\vec{r}_{13}{}^2}\right) + \frac{\vec{r}_{23}{}^2}{2\vec{r}_{02}{}^2\vec{r}_{05}{}^2\vec{r}_{35}{}^2} \ln\left(\frac{\vec{r}_{03}{}^2\vec{r}_{25}{}^2}{\vec{r}_{05}{}^2\vec{r}_{23}{}^2}\right). \tag{4.21}$$

In fact, there is freedom in the definition of the functions M_i^{Cjk}, R_{ijk} and R_{ijkl} since one can redistribute terms between them. For example, one can try to redefine M_i^{Cjk} so that to make the functions R zero.

The remaining antisymmetric contribution reads

$$\begin{aligned} \mathbf{G}^{\text{conf}}_{a3} &= \frac{1}{2}\left(\mathbf{U}_{0^\dagger 5}\mathbf{U}_{5^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 0} - (5 \leftrightarrow 0)\right) \\ &\times (2\tilde{L}^C_{12} + 2\tilde{L}^C_{14} - M_2^{C44} - M_4^{C22} + M_2^{C14} + M_2^{C41} + M_4^{C21} + M_4^{C12} + R_{241}) \\ &+ \frac{1}{2}\left(\mathbf{U}_{5^\dagger 0}\mathbf{U}_{2^\dagger 5}\mathbf{U}_{0^\dagger 34^\dagger 1} - (5 \leftrightarrow 0)\right) \\ &\times (2\tilde{L}^C_{21} + 2\tilde{L}^C_{23} - M_1^{C33} - M_3^{C11} + M_1^{C32} + M_1^{C23} + M_3^{C21} + M_3^{C12} + R_{132}) \\ &+ (1 \leftrightarrow 3, 2 \leftrightarrow 4). \end{aligned} \tag{4.22}$$

The integrated w.r.t. $\vec{r}_5$ part of the kernel has the form

$$\mathbf{G}^{\text{conf}} = \mathbf{G}^{\text{conf}}_1 + \mathbf{G}^{\text{conf}}_2. \tag{4.23}$$

Here

$$\begin{aligned} \mathbf{G}^{\text{conf}}_1 &= (\mathbf{U}_{0^\dagger 3}\mathbf{U}_{2^\dagger 1}\mathbf{U}_{4^\dagger 0} - \mathbf{U}_{0^\dagger 12^\dagger 04^\dagger 3} - \mathbf{U}_{2^\dagger 34^\dagger 1}(N_c^2 - 1)) \\ &\times \frac{1}{4}\left[\frac{\vec{r}_{14}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2}\left(\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2}\right) - \ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{14}{}^2}{\vec{r}_{04}{}^2\vec{r}_{13}{}^2}\right)\right) - \frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2}\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}\right)\right. \\ &+ \frac{\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{02}{}^2}\left(\ln^2\left(\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}\right) - \ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{24}{}^2}{\vec{r}_{04}{}^2\vec{r}_{23}{}^2}\right)\right) - \frac{\vec{r}_{24}{}^2}{\vec{r}_{04}{}^2\vec{r}_{02}{}^2}\ln^2\left(\frac{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2}\right) \\ &\left.+ \frac{\vec{r}_{34}{}^2}{\vec{r}_{04}{}^2\vec{r}_{03}{}^2}\left(\ln^2\left(\frac{\vec{r}_{01}{}^2\vec{r}_{34}{}^2}{\vec{r}_{04}{}^2\vec{r}_{13}{}^2}\right) + \ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{34}{}^2}{\vec{r}_{03}{}^2\vec{r}_{24}{}^2}\right)\right)\right] \end{aligned}$$

$$
\begin{aligned}
&+ (\mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 3}\mathbf{U}_{4^\dagger 0} - \mathbf{U}_{0^\dagger 14^\dagger 02^\dagger 3} - \mathbf{U}_{2^\dagger 34^\dagger 1}(N_c^2-1)) \\
&\times \frac{1}{4}\left[\frac{\vec{r}_{34}{}^2}{\vec{r}_{03}{}^2\vec{r}_{04}{}^2}\left(\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{34}{}^2}{\vec{r}_{04}{}^2\vec{r}_{23}{}^2}\right) - \ln^2\left(\frac{\vec{r}_{01}{}^2\vec{r}_{34}{}^2}{\vec{r}_{04}{}^2\vec{r}_{13}{}^2}\right)\right) - \frac{\vec{r}_{24}{}^2}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2}\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{24}{}^2}{\vec{r}_{04}{}^2\vec{r}_{23}{}^2}\right)\right. \\
&+ \frac{\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2\vec{r}_{01}{}^2}\left(\ln^2\left(\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}\right) - \ln^2\left(\frac{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2}\right)\right) - \frac{\vec{r}_{13}{}^2}{\vec{r}_{03}{}^2\vec{r}_{01}{}^2}\ln^2\left(\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right) \\
&\left.+ \frac{\vec{r}_{14}{}^2}{\vec{r}_{04}{}^2\vec{r}_{01}{}^2}\left(\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{14}{}^2}{\vec{r}_{04}{}^2\vec{r}_{13}{}^2}\right) + \ln^2\left(\frac{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}\right)\right)\right] + (1\leftrightarrow 3, 2\leftrightarrow 4). \qquad (4.24)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{G}_2^{\rm conf} &= \frac{N_c}{4}(\mathbf{U}_{2^\dagger 0}\mathbf{U}_{0^\dagger 34^\dagger 1} - N_c\mathbf{U}_{2^\dagger 34^\dagger 1})\left[\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2}\left(\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}\right) + \ln^2\left(\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right)\right)\right. \\
&\left.- \frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\ln^2\left(\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}\right) - \frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}\ln^2\left(\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}\right)\right] \\
&+ \frac{N_c}{4}(\mathbf{U}_{0^\dagger 3}\mathbf{U}_{2^\dagger 04^\dagger 1} - N_c\mathbf{U}_{2^\dagger 34^\dagger 1})\left[\frac{\vec{r}_{24}{}^2}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2}\left(\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{24}{}^2}{\vec{r}_{04}{}^2\vec{r}_{23}{}^2}\right) + \ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{34}{}^2}{\vec{r}_{03}{}^2\vec{r}_{24}{}^2}\right)\right)\right. \\
&\left.- \frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{34}{}^2}{\vec{r}_{04}{}^2\vec{r}_{23}{}^2}\right) - \frac{\vec{r}_{34}{}^2}{\vec{r}_{03}{}^2\vec{r}_{04}{}^2}\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{34}{}^2}{\vec{r}_{04}{}^2\vec{r}_{23}{}^2}\right)\right] + (1\leftrightarrow 3, 2\leftrightarrow 4). \qquad (4.25)
\end{aligned}
$$

It was straightforwardly checked that all the integrals of $\mathbf{G}_s^{\rm conf}$, $\mathbf{G}_a^{\rm conf}$, and $\mathbf{G}^{\rm conf}$ do not have unintegrable singularities.

4.2 Double dipole

For symmetric contribution $\tilde{\mathbf{G}}_s^{\rm conf}$ we have

$$
\tilde{\mathbf{G}}_s^{\rm conf} = \tilde{\mathbf{G}}_{s1}^{\rm conf} + n_f\tilde{\mathbf{G}}_q + \tilde{\mathbf{G}}_{s2}^{\rm conf}, \qquad (4.26)
$$

where $\tilde{\mathbf{G}}_q$ (3.15) did not change.

$$
\begin{aligned}
\tilde{\mathbf{G}}_{s1}^{\rm conf} &= (\{\mathbf{U}_{0^\dagger 12^\dagger 5}\mathbf{U}_{4^\dagger 35^\dagger 0} - \mathbf{U}_{0^\dagger 5}\mathbf{U}_{2^\dagger 15^\dagger 34^\dagger 0} - (5\to 0)\} + (5\leftrightarrow 0))\left(L_{14}^C - L_{13}^C + L_{23}^C - L_{24}^C\right) \\
&+ (1\leftrightarrow 3, 2\leftrightarrow 4), \qquad (4.27) \\
\tilde{\mathbf{G}}_{s2}^{\rm conf} &= \left(M_4^{C13} - M_4^{C31} + M_3^{C14} - M_3^{C41} - M_4^{C23} + M_4^{C32} - M_3^{C24} + M_3^{C42}\right) \\
&\times \frac{1}{2}(\mathbf{U}_{4^\dagger 0}\mathbf{U}_{0^\dagger 35^\dagger 12^\dagger 5} + \mathbf{U}_{4^\dagger 0}\mathbf{U}_{0^\dagger 52^\dagger 15^\dagger 3} - \mathbf{U}_{0^\dagger 3}\mathbf{U}_{2^\dagger 15^\dagger 04^\dagger 5} - \mathbf{U}_{0^\dagger 3}\mathbf{U}_{2^\dagger 54^\dagger 05^\dagger 1} + (5\leftrightarrow 0)) \\
&+ (M_4^{C12} - M_4^{C21} + M_3^{C21} - M_3^{C12} - M_1^{C34} + M_1^{C43} - M_2^{C43} + M_2^{C34}) \\
&\times \frac{1}{2}(\mathbf{U}_{0^\dagger 54^\dagger 3}\mathbf{U}_{2^\dagger 05^\dagger 1} - \mathbf{U}_{0^\dagger 5}\mathbf{U}_{2^\dagger 15^\dagger 34^\dagger 0} + (5\leftrightarrow 0)) + (1\leftrightarrow 3, 2\leftrightarrow 4). \qquad (4.28)
\end{aligned}
$$

The antisymmetric contribution reads

$$
\begin{aligned}
\tilde{\mathbf{G}}_a =& \frac{1}{2}(\mathbf{U}_{5^\dagger 3}\mathbf{U}_{0^\dagger 12^\dagger 04^\dagger 5} + \mathbf{U}_{5^\dagger 3}\mathbf{U}_{0^\dagger 54^\dagger 02^\dagger 1} - \mathbf{U}_{4^\dagger 5}\mathbf{U}_{0^\dagger 12^\dagger 05^\dagger 3} - \mathbf{U}_{4^\dagger 5}\mathbf{U}_{0^\dagger 35^\dagger 02^\dagger 1} - (5\leftrightarrow 0)) \\
&\times (M_4^{23} + M_3^{24} - M_4^{13} - M_3^{14} - M_4^{31} + M_4^{32} - M_3^{41} + M_3^{42} - R_{341} + R_{342}) \\
&+ \frac{1}{2}(\mathbf{U}_{0^\dagger 54^\dagger 3}\mathbf{U}_{2^\dagger 05^\dagger 1} - \mathbf{U}_{0^\dagger 52^\dagger 1}\mathbf{U}_{4^\dagger 05^\dagger 3} + \mathbf{U}_{5^\dagger 0}\mathbf{U}_{0^\dagger 12^\dagger 54^\dagger 3} - \mathbf{U}_{0^\dagger 5}\mathbf{U}_{2^\dagger 15^\dagger 34^\dagger 0} - (5\leftrightarrow 0)) \\
&\times (M_4^{C11} - M_3^{C11} + M_3^{C12} - M_4^{C12} + M_3^{C21} - M_4^{C21} - M_3^{C22} + M_4^{C22}) + (1\leftrightarrow 3, 2\leftrightarrow 4). \qquad (4.29)
\end{aligned}
$$

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The contribution with 1 gluon crossing the shockwave has the form

$$\begin{aligned}
\tilde{\mathbf{G}}^{\rm conf} &= \frac{1}{4}(\mathbf{U}_{2^\dagger 0}\mathbf{U}_{0^\dagger 14^\dagger 3} + \mathbf{U}_{2^\dagger 0}\mathbf{U}_{0^\dagger 34^\dagger 1} - \mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 04^\dagger 3} - \mathbf{U}_{0^\dagger 1}\mathbf{U}_{2^\dagger 34^\dagger 0}) \\
&\times \left[\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}\left(\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2}\right) - \ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}\right) - \ln^2\left(\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}\right) + \ln^2\left(\frac{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2}\right)\right)\right. \\
&+ \left.\left(\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2} + \frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right) - \left(\frac{\vec{r}_{14}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2} + \frac{\vec{r}_{24}{}^2}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2}\right)\ln^2\left(\frac{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}\right)\right] \\
&+ \frac{1}{4}\left(N_c\mathbf{U}_{0^\dagger 12^\dagger 04^\dagger 3} + N_c\mathbf{U}_{0^\dagger 34^\dagger 02^\dagger 1} - 2\mathbf{U}_{2^\dagger 1}\mathbf{U}_{34^\dagger}\right)\left[\frac{\vec{r}_{24}{}^2}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2}\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{34}{}^2}{\vec{r}_{03}{}^2\vec{r}_{24}{}^2}\right)\right. \\
&+ \left.\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2}\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}\right) - \frac{\vec{r}_{14}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2}\left(\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2}\right) + \ln^2\left(\frac{\vec{r}_{01}{}^2\vec{r}_{34}{}^2}{\vec{r}_{03}{}^2\vec{r}_{14}{}^2}\right)\right)\right] \\
&+ (1 \leftrightarrow 3, 2 \leftrightarrow 4). \qquad (4.30)
\end{aligned}$$

As for the quadrupole, it was straightforwardly checked that all the integrals of $\tilde{\mathbf{G}}^{\rm conf}_s$, $\tilde{\mathbf{G}}^{\rm conf}_a$, and $\tilde{\mathbf{G}}^{\rm conf}$ do not have unintegrable singularities.

5 Checks

There are two checks which can be done for the results of this paper. The evolution equations for the quadrupole and double dipole operators can be obtained from the NLO JIMWLK hamiltonian [23] and the general evolution equations from [24].

In this paper the following two checks were done. First, quadrupole kernels (3.11) and (4.8) go into BK ones (2.34) and (4.3) in the dipole limits $1 \to 2$, $2 \to 3$, $3 \to 4$, and $4 \to 1$. Double dipole kernels (3.12) and (4.9) also have the correct dipole limits $1 \to 2$ and $3 \to 4$. In these limits they also go into the BK ones (2.34) and (4.3) times N_c. This statement can be checked straightforwardly going to the dipole limits in explicit expressions (3.11), (4.8), (3.12), and (4.9). Our kernels match the Balitsky-Fadin-Kuraev-Lipatov NLO kernel [31] in these limits.

The second check is that in SU(3) our kernels respect the identity

$$B_{123} \equiv \mathbf{U}_{12^\dagger}\mathbf{U}_{32^\dagger} - \mathbf{U}_{12^\dagger 32^\dagger}, \qquad (5.1)$$

where B_{123} is the 3-quark Wilson loop (baryon) operator defined as

$$B_{123} \equiv U_1 \cdot U_2 \cdot U_3 \equiv \varepsilon^{i'j'h'}\varepsilon_{ijh}U^i_{1i'}U^j_{2j'}U^h_{3h'}. \qquad (5.2)$$

The evolution equations for the l.h.s. of (5.1) in the standard and quasi-conformal forms are given in [12]. They read

$$\begin{aligned}
\frac{\partial B_{123}}{\partial \eta} &= \frac{\alpha_s(\mu^2)}{8\pi^2}\int d\vec{r}_0\left[(B_{100}B_{320} + B_{200}B_{310} - B_{300}B_{210} - 6B_{123})\right. \\
&\times \left\{\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{01}\vec{r}^{\,2}_{02}} - \frac{3\alpha_s}{4\pi}\beta\left[\ln\left(\frac{\vec{r}^{\,2}_{01}}{\vec{r}^{\,2}_{02}}\right)\left(\frac{1}{\vec{r}^{\,2}_{02}} - \frac{1}{\vec{r}^{\,2}_{01}}\right) - \frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{01}\vec{r}^{\,2}_{02}}\ln\left(\frac{\vec{r}^{\,2}_{12}}{\mu^2}\right)\right]\right\} \\
&- \frac{\alpha_s}{\pi}\ln\frac{\vec{r}^{\,2}_{20}}{\vec{r}^{\,2}_{21}}\ln\frac{\vec{r}^{\,2}_{10}}{\vec{r}^{\,2}_{21}}\left\{\frac{1}{2}\left[\frac{\vec{r}^{\,2}_{13}}{\vec{r}^{\,2}_{10}\vec{r}^{\,2}_{30}} - \frac{\vec{r}^{\,2}_{32}}{\vec{r}^{\,2}_{30}\vec{r}^{\,2}_{20}}\right](B_{100}B_{320} - B_{200}B_{310})\right.
\end{aligned}$$

$$- \frac{\vec r_{12}^2}{\vec r_{10}^2 \vec r_{20}^{\,2}} \left(9B_{123} - \frac{1}{2}\left[2\left(B_{100}B_{320} + B_{200}B_{130}\right) - B_{300}B_{120}\right]\right)\Big\} + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\Big]$$

$$- \frac{\alpha_s^2 n_f}{16\pi^4} \int d\vec r_0 d\vec r_5 \left[\left\{\left(\frac{1}{3}(U_1 U_0{}^\dagger U_5 + U_5 U_0{}^\dagger U_1)\cdot U_2 \cdot U_3 - \frac{1}{9} B_{123} tr(U_0{}^\dagger U_5)\right.\right.\right.$$

$$+ (U_1 U_0{}^\dagger U_2)\cdot U_3 \cdot U_5 + \frac{1}{6}B_{123} - \frac{1}{4}(B_{013}B_{002} + B_{001}B_{023} - B_{012}B_{003})$$

$$\left.\left.\left. + (1 \leftrightarrow 2)\right) + (0 \leftrightarrow 5)\right\} L_{12}^q + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\right]$$

$$- \frac{\alpha_s^2}{8\pi^4} \int d\vec r_0 d\vec r_5 \ \left[\left\{ \tilde L_{12} \left(U_0 U_5{}^\dagger U_2\right)\cdot \left(U_1 U_0{}^\dagger U_5\right)\cdot U_3\right.\right.$$

$$+ L_{12}\left[\left(U_0 U_5{}^\dagger U_2\right)\cdot\left(U_1 U_0{}^\dagger U_5\right)\cdot U_3 + tr\left(U_0 U_5{}^\dagger\right)\left(U_1 U_0{}^\dagger U_2\right)\cdot U_3 \cdot U_5\right.$$

$$\left. - \frac{3}{4}[B_{155}B_{235} + B_{255}B_{135} - B_{355}B_{125}] + \frac{1}{2}B_{123}\right]$$

$$+ (M_{13} - M_{12} - M_{23} + M_2^{13})\left[(U_0 U_5{}^\dagger U_3)\cdot(U_2 U_0{}^\dagger U_1)\cdot U_5 + (U_1 U_0{}^\dagger U_2)\cdot(U_3 U_5{}^\dagger U_0)\cdot U_5\right]$$

$$\left.\left. + \text{ (all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3)\right\} + (0 \leftrightarrow 5)\right], \tag{5.3}$$

where L, L^q, M_i^{jk} and M_{ij} are introduced in (2.22), (2.23), (2.29), and (2.32). (5.3) and (5.4).

$$\frac{\partial B_{123}^{\rm conf}}{\partial \eta} = \frac{\alpha_s\left(\mu^2\right)}{8\pi^2} \int d\vec r_0 \left[\left((B_{100}B_{320} + B_{200}B_{310} - B_{300}B_{210}) - 6B_{123}\right)^{\rm conf}\right.$$

$$\times\left(\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}} - \frac{3\alpha_s}{4\pi}\beta\left[\ln\left(\frac{\vec r_{01}^{\,2}}{\vec r_{02}^{\,2}}\right)\left(\frac{1}{\vec r_{02}^{\,2}} - \frac{1}{\vec r_{01}^{\,2}}\right) - \frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}}{\tilde\mu^2}\right)\right]\right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\Big]$$

$$- \frac{\alpha_s^2}{32\pi^3} \int d\vec r_0 \left(B_{003}B_{012}\left[\frac{\vec r_{32}{}^2}{\vec r_{03}{}^2\vec r_{02}{}^2}\ln^2\left(\frac{\vec r_{32}{}^2\vec r_{10}{}^2}{\vec r_{13}{}^2\vec r_{20}{}^2}\right) - \frac{\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{02}{}^2}\ln^2\left(\frac{\vec r_{12}{}^2\vec r_{30}{}^2}{\vec r_{13}{}^2\vec r_{20}{}^2}\right)\right]\right.$$

$$\left. + \text{ (all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3)\right)$$

$$- \frac{\alpha_s^2 n_f}{16\pi^4} \int d\vec r_0 d\vec r_5 \left[\left\{\left(\frac{1}{3}(U_1 U_0{}^\dagger U_5 + U_5 U_0{}^\dagger U_1)\cdot U_2 \cdot U_3 - \frac{1}{9} B_{123} tr(U_0{}^\dagger U_5)\right.\right.\right.$$

$$+ (U_1 U_0{}^\dagger U_2)\cdot U_3 \cdot U_5 + \frac{1}{6}B_{123} - \frac{1}{4}(B_{013}B_{002} + B_{001}B_{023} - B_{012}B_{003})$$

$$\left.\left.\left. + \ (1 \leftrightarrow 2)\right) + (0 \leftrightarrow 5)\right\} L_{12}^q + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\right]$$

$$- \frac{\alpha_s^2}{8\pi^4} \int d\vec r_0 d\vec r_5 \ \left(\left\{ \tilde L_{12}^C \left(U_0 U_5{}^\dagger U_2\right)\cdot \left(U_1 U_0{}^\dagger U_5\right)\cdot U_3\right.\right.$$

$$+ L_{12}^C\left[\left(U_0 U_5{}^\dagger U_2\right)\cdot\left(U_1 U_0{}^\dagger U_5\right)\cdot U_3 + tr\left(U_0 U_5{}^\dagger\right)\left(U_1 U_0{}^\dagger U_2\right)\cdot U_3 \cdot U_5\right.$$

$$\left. - \frac{3}{4}[B_{155}B_{235} + B_{255}B_{135} - B_{355}B_{125}] + \frac{1}{2}B_{123}\right]$$

$$+ \tilde M_{12}^C\left[\left(U_0 U_5{}^\dagger U_3\right)\cdot\left(U_2 U_0{}^\dagger U_1\right)\cdot U_5 + \left(U_1 U_0{}^\dagger U_2\right)\cdot\left(U_3 U_5{}^\dagger U_0\right)\cdot U_5\right]$$

$$\left.\left. + \ \text{(all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3)\right\} + (0 \leftrightarrow 5)\right), \tag{5.4}$$

where L^C is defined in (4.4), $\tilde{L}^C$ — in (4.5), and

$$
\begin{aligned}
M_{12}^C = & \frac{\vec{r}_{12}{}^2}{16\vec{r}_{02}{}^2\vec{r}_{05}{}^2\vec{r}_{15}{}^2} \ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{35}{}^4}{\vec{r}_{03}{}^4\vec{r}_{15}{}^2\vec{r}_{25}{}^2}\right) + \frac{\vec{r}_{12}{}^2}{16\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{25}{}^2} \ln\left(\frac{\vec{r}_{03}{}^4\vec{r}_{05}{}^4\vec{r}_{12}{}^4\vec{r}_{25}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^6\vec{r}_{15}{}^2\vec{r}_{35}{}^4}\right) \\
& + \frac{\vec{r}_{23}{}^2}{16\vec{r}_{02}{}^2\vec{r}_{05}{}^2\vec{r}_{35}{}^2} \ln\left(\frac{\vec{r}_{01}{}^4\vec{r}_{03}{}^2\vec{r}_{25}{}^6\vec{r}_{35}{}^2}{\vec{r}_{02}{}^2\vec{r}_{05}{}^4\vec{r}_{15}{}^4\vec{r}_{23}{}^4}\right) + \frac{\vec{r}_{23}{}^2}{16\vec{r}_{03}{}^2\vec{r}_{05}{}^2\vec{r}_{25}{}^2} \ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{03}{}^2\vec{r}_{15}{}^4}{\vec{r}_{01}{}^4\vec{r}_{25}{}^2\vec{r}_{35}{}^2}\right) \\
& + \frac{\vec{r}_{13}{}^2}{16\vec{r}_{03}{}^2\vec{r}_{05}{}^2\vec{r}_{15}{}^2} \ln\left(\frac{\vec{r}_{02}{}^4\vec{r}_{15}{}^2\vec{r}_{35}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{25}{}^4}\right) + \frac{\vec{r}_{13}{}^2}{16\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{35}{}^2} \ln\left(\frac{\vec{r}_{02}{}^4\vec{r}_{15}{}^2\vec{r}_{35}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{25}{}^4}\right) \\
& + \frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{05}{}^2\vec{r}_{35}{}^2} \ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{25}{}^4}{\vec{r}_{02}{}^2\vec{r}_{05}{}^2\vec{r}_{12}{}^2\vec{r}_{35}{}^2}\right) + \frac{\vec{r}_{23}{}^2\vec{r}_{12}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{25}{}^2\vec{r}_{35}{}^2} \ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{12}{}^2\vec{r}_{35}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2\vec{r}_{25}{}^2}\right) \\
& + \frac{\vec{r}_{15}{}^2\vec{r}_{23}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{25}{}^2\vec{r}_{35}{}^2} \ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{05}{}^2\vec{r}_{23}{}^2\vec{r}_{25}{}^2}{\vec{r}_{02}{}^4\vec{r}_{15}{}^2\vec{r}_{35}{}^2}\right). \qquad (5.5)
\end{aligned}
$$

In order to check identity (5.1) one has to rewrite the evolution of its l.h.s. in the same operator basis as the r.h.s. To this end one can use the SU(3) identities

$$
\begin{aligned}
0 = & \Big[U_0 \cdot U_1 \cdot U_2 tr\left(U_0{}^\dagger U_5\right) tr\left(U_5{}^\dagger U_3\right) \\
& - tr\left(U_5 U_0{}^\dagger\right)\left(U_1 U_5{}^\dagger U_3 + U_3 U_5{}^\dagger U_1\right) \cdot U_0 \cdot U_2 + \left(U_0 U_5{}^\dagger U_1\right) \cdot \left(U_3 U_0{}^\dagger U_5\right) \cdot U_2 \\
& + \left(U_1 U_5{}^\dagger U_0\right) \cdot \left(U_5 U_0{}^\dagger U_3\right) \cdot U_2 + (1 \leftrightarrow 2)\Big] - (5 \leftrightarrow 0), \\
0 = & \, 2tr\left(U_5 U_0{}^\dagger\right)\left(U_2 U_5{}^\dagger U_3 + U_3 U_5{}^\dagger U_2\right) \cdot U_0 \cdot U_1 \\
& + \left(U_0 U_5{}^\dagger U_1 + U_1 U_5{}^\dagger U_0\right) \cdot \left(U_2 U_0{}^\dagger U_3 + U_3 U_0{}^\dagger U_2\right) \cdot U_5 \\
& + \left(U_0 U_5{}^\dagger U_2 - U_2 U_5{}^\dagger U_0\right) \cdot \left(U_3 U_0{}^\dagger U_1 - U_1 U_0{}^\dagger U_3\right) \cdot U_5 \\
& + \left(U_0 U_5{}^\dagger U_3 - U_3 U_5{}^\dagger U_0\right) \cdot \left(U_2 U_0{}^\dagger U_1 - U_1 U_0{}^\dagger U_2\right) \cdot U_5 - (5 \leftrightarrow 0)
\end{aligned}
$$

in the antisymmetric color structures and then

$$
U_i \cdot U_j \cdot U_k = (U_i U_l^\dagger) \cdot (U_j U_l^\dagger) \cdot (U_k U_l^\dagger) = (U_l^\dagger U_i) \cdot (U_l^\dagger U_j) \cdot (U_l^\dagger U_k) \qquad (5.6)
$$

with $l = 2$ to express U_2 in terms of $U_2^\dagger$. After that one can expand the products of Levi-Civita symbols as

$$
\varepsilon_{ijh}\varepsilon^{i'j'h'} = \begin{vmatrix} \delta_i^{i'} & \delta_i^{j'} & \delta_i^{h'} \\ \delta_j^{i'} & \delta_j^{j'} & \delta_j^{h'} \\ \delta_h^{i'} & \delta_h^{j'} & \delta_h^{h'} \end{vmatrix} \qquad (5.7)
$$

and see that (5.1) is satisfied.

6 Discussion and conclusion

This paper presents the evolution equations for the double dipole and quadrupole operators in the standard (3.11), (3.12) and quasi-conformal forms (4.8), (4.9). They have correct dipole limits and in SU(3) obey group identity (5.1) with the corresponding evolution equations for the 3QWL operator obtained in [12]. This fact ensures the correctness of all the 3 results. To construct the composite operators, prescription (4.1) was used. It was proposed in [10] for the dipole and proved in [29]. Here it gave the quasi-conformal kernels for both double dipole and quadrupole operators, thus being checked by the specific calculation of the evolution of the 4-point operators.

Unlike the dipole and the 3QWL operators, the evolution of the quadrupole and the double dipole ones generates several operators in the virtual part. Indeed, the virtual gluons do not change the color structure of a dipole or a baryon. New color structures appear in the evolution of these operators only when the gluons cross the shockwave. Therefore, one can write the virtual part of the evolution equations for them without calculation. The double dipole and the quadrupole, on the contrary, mix in the virtual part with each other and with the double dipoles and quadrupoles with the other order of the Wilson lines. Therefore they had to be calculated explicitly. Using the evolution equations for Wilson lines from [13] in this calculation, one encounters ill-defined integrals which were treated here so as to obtain the known result for the dipole and the 3QWL operators. Although such treatment gave the equations satisfying all the checks, it is important to have the initial expressions with the regularization of the IR singularities and to check the results of this paper. Such checks may be performed starting from the evolution equations found in [23] and [24].

The equations obtained in this paper may be used to derive the NLO evolution equation for Weizsäcker-Williams gluon distribution. This work is in progress. They can also be important in the analysis of higher than dipole Fock components of the virtual photon in the diffractive processes.

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