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Generalized Hitchin system, spectral curve and $\mathcal{N} = 1$ dynamics

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ABSTRACT: A generalized Hitchin equation was proposed as the BPS equation for a large class of four dimensional $\mathcal{N} = 1$ theories engineered using M5 branes. In this paper, we show how to write down the spectral curve for the moduli space of generalized Hitchin equations, and extract interesting $\mathcal{N} = 1$ dynamics out of it, such as deformed moduli space, chiral ring relation, SUSY breaking, etc. Holomorphy plays a crucial role in our construction.

KEYWORDS: Supersymmetric gauge theory, M-Theory

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1 Introduction

Seiberg and Witten found the exact solution of Coulomb branch of $\mathcal{N} = 2$ theories by using remarkable physical insights [1, 2]. The results can be summarized elegantly using a Seiberg-Witten curve fibered over Coulomb branch. Exact low energy effective action on Coulomb branch is determined by finding a Seiberg-Witten curve. However, the physical methods used in [1, 2] are not easy to use for more complicated theories. There are two

closely related methods which are used very successfully in finding solutions for general $\mathcal{N} = 2$ theories. One is the type IIA brane construction and its M theory lift [3], and the other one is using the connection of the Seiberg-Witten solution and integrable system [4–6], in particular, Hitchin system is playing a crucial role in finding solutions.

It was pointed out by Intriligator and Seiberg [7] that such curves can also be written down for Coulomb branch of $\mathcal{N} = 1$ theories. The curves describe holomorphic gauge couplings of low energy massless U(1) gauge fields.

Later in the context of the M theory lift of type IIA brane setup, curves are obtained for $\mathcal{N} = 1$ Supersymmetric QCD (SQCD) which are obtained by turning on the mass of the adjoint chiral multiplet inside $\mathcal{N} = 2$ vector multiplet [8, 9]. A lot of results were obtained from this approach (see [10] and references therein).

However, those results are more or less relying on results of $\mathcal{N} = 2$ theories which have type IIA brane realization (i.e., linear quiver gauge theories) and then deforming them by mass of adjoint chiral multiplets, and it is difficult to find the curves for pure $\mathcal{N} = 1$ theories, that is, theories in which adjoint masses are infinity or there are no adjoint fields at all. There are many other $\mathcal{N} = 1$ models which have no obvious origin to $\mathcal{N} = 2$ linear quiver theories. There has been little clue how to write down curves of those models.

The purpose of this paper is to propose a general method for finding $\mathcal{N} = 1$ curves for theories [11] engineered using M5 branes compactified on a punctured Riemann surface.¹ A generalized Hitchin equation is proposed in [11] for describing the moduli space of above field theories (see also [24] for another approach to generalized Hitchin systems). It is expected that the moduli space of solutions of this generalized Hitchin equations (the moduli space is denoted as M_{GH}) is describing some kind of ‘‘Coulomb’’² branch of the underlying four dimensional theory compactified on a circle. Given the similarity between the generalized Hitchin equations and ordinary Hitchin equations, we expect that a similar spectral curve for M_{GH} could be written which will then describe the holomorphic aspects of $\mathcal{N} = 1$ gauge theory dynamics.

Indeed, one can write down a spectral curve for M_{GH} , and surprisingly one can extract lots of dynamical information of the low energy dynamics in simply trying to write down the curves. The procedure of determining such curves is surprisingly simple, and the crucial thing is the holomorphy, which agrees with the philosophy taken by Seiberg [25].

Let’s summarize our main results for determining $\mathcal{N} = 1$ curves. There are two Higgs fields Φ_1 and Φ_2 in generalized Hitchin equations. They are sections of $L_1 \otimes \text{ad}(E)$ and $L_2 \otimes \text{ad}(E)$ respectively, where L_1 and L_2 are line bundles such that $L_1 \otimes L_2$ is equal to the canonical bundle K , and $\text{ad}(E)$ is the holomorphic vector bundle in the adjoint representation of gauge group. We take fiber coordinates of L_1 and L_2 as v and w , and the coordinates of the Riemann surface where M5 brane wraps is denoted as z . These three coordinates parameterize a non-compact local Calabi-Yau manifold [17]. Our spectral curve is an N cover of the Riemann surface, and it is described by a set of polynomial equations depending on v, w, z .

¹ $\mathcal{N} = 1$ field theory dynamics of same or similar class are studied in [12–23]. One of crucial ingredients is the quartic superpotential of [19] as we will see.

²We use quotation mark here because there is no real distinction between the Coulomb and Higgs branches of three dimensional $\mathcal{N} = 2$ theory.

One can write down two obvious spectral curves for them using the compactification data (punctures and bundles)

$$\begin{aligned} \det(v - \Phi_1) = 0 &\rightarrow v^N + \sum_{i=2}^N \phi_{1i}(z)v^{N-i} = 0, \\ \det(w - \Phi_2) = 0 &\rightarrow w^N + \sum_{i=2}^N \phi_{2i}(z)w^{N-i} = 0. \end{aligned} \tag{1.1}$$

The coefficients in those two curves are the ‘‘Coulomb’’ branch moduli. If one of the Higgs fields, say Φ_2 , vanishes, the equation for Φ_1 is valid with arbitrary moduli parameters which are consistent with singularities and bundle structures. It actually gives the moduli space of twisted Higgs bundle and it is shown in [26, 27] that a spectral curve can be written down.

When both Higgs fields are nonzero, there is a crucial commuting condition on those two matrices,

$$[\Phi_1, \Phi_2] = 0. \tag{1.2}$$

There is a simple fact about the commuting matrices: given a matrix A with generic eigenvalues, the matrices commuting with A can be written as a degree $N - 1$ polynomial in A : $B = h_1 A^{N-1} + h_2 A^{N-2} + \dots + h_N$, therefore once this link equation is given, the eigenvalues of B are determined by the eigenvalues of A . Applying this theorem to our context, we need a third equation relating v and w as

$$w = h_1(z)v^{N-1} + h_2(z)v^{N-2} + \dots + h_N. \tag{1.3}$$

These three equations are not independent, namely, given the spectral curve of v and this link equation, we should be able to recover the spectral curve of w . The crucial point is that h_i has to be holomorphic (or meromorphic). The holomorphic property of h_i and the above consistent relation put a lot of constraints on various moduli appearing in spectral equations of v and w .

There are many $\mathcal{N} = 1$ dynamical informations which can be extracted in the attempt of solving the link equation, and those dynamics do not appear in $\mathcal{N} = 2$ context. Let’s list some of them:

- **Deformed moduli space:** in some cases, the moduli in spectral curves satisfy the deformed chiral ring relations due to quantum effect as in $N_f = N$ SQCD [28]. We can recover those type of relations by solving the link equation.
- **Chiral ring relation:** in general, the operators in the v and w spectral curves satisfy interesting chiral ring relations (including deformed moduli constraints), which can be determined exactly. For example, we find interesting chiral ring relations for moduli space of Maldacena-Nunez theory [29].
- **Mass deformation:** in $\mathcal{N} = 2$ theories, masses of hypermultiplets only change the metric of Coulomb branch, but masses in $\mathcal{N} = 1$ theory dramatically change the IR behavior, such as elimination of moduli spaces, SUSY breaking, SUSY restoration, etc.

- **Phase structure:** in $\mathcal{N} = 2$ case, the curve can be used to probe non-abelian (conformal) and abelian Coulomb phases, and there is always a continuous moduli space. In $\mathcal{N} = 1$ case, the curve can also be used to probe non-abelian and abelian Coulomb phases, and we also find Higgs/Confining phases. In some cases, there are only isolated vacua as in pure $\mathcal{N} = 1$ super-Yang-Mills.
- **Supersymmetry breaking:** in some cases, one can not find any solution to the link equation, and one can not write a spectral curve. Then we conclude that SUSY is dynamically broken [30].³

This paper is organized as follows: in section 2, we discuss how to find the spectral curve of generalized Hitchin system. In section 3, we solve theories engineered using six dimensional A_1 theory. In section 4, we solve theories engineered using 6d A_{N-1} theory. Finally, we give a conclusion in section 5.

2 Generalized Hitchin's equations and spectral curve

2.1 Generalized Hitchin equation and four dimensional $\mathcal{N} = 1$ theory

Four dimensional $\mathcal{N} = 1$ theories can be derived by compactifying six dimensional $(2, 0)$ theory on a punctured Riemann surface. The data defining the theory are

- A punctured Riemann surface $M_{g,n}$ and a choice of ADE group G .
- Two line bundles L_1 and L_2 such that $L_1 \otimes L_2 = K$ with K the canonical bundle [17]. The two Higgs fields Φ_1, Φ_2 are holomorphic sections of $L_1 \otimes \text{ad}(E)$ and $L_2 \otimes \text{ad}(E)$ respectively, where $\text{ad}(E)$ is the bundle in the adjoint representation of the gauge group.
- The local puncture types: a commuting nilpotent pair of G [11].

In this paper, we only consider locally $\mathcal{N} = 2$ punctures, namely only one of the Higgs fields is singular at a puncture, with the same types of singularities as in [6, 31]

It is proposed in [11] that the following generalized Hitchin equations are the BPS equations for these $\mathcal{N} = 1$ compactifications:

$$\begin{aligned}
 D_{\bar{z}}\Phi_1 &= D_{\bar{z}}\Phi_2 = 0, \\
 [\Phi_1, \Phi_2] &= 0, \\
 F_{z\bar{z}} + [\Phi_1, \Phi_1^*]h_1 + [\Phi_2, \Phi_2^*]h_2 &= 0,
 \end{aligned}
 \tag{2.1}$$

Here h_1 and h_2 are fixed Hermitian metrics for two line bundles $L_1 \otimes K^{-1}$ and $L_2 \otimes K^{-1}$.

The moduli space of this generalized Hitchin equations is expected to be the target of three dimensional theory derived by compactifying our four dimensional theory on a circle, similar to the $\mathcal{N} = 2$ case [6]. More details about the moduli space will be discussed elsewhere [32]. The purpose of this paper is to try to use spectral curve to understand this moduli space and therefore learn interesting IR dynamics of field theory.

³However, we cannot immediately say whether there exists a stable SUSY breaking vacuum or the potential is of runaway type. We need more detailed field theory analysis in this case.

2.1.1 Field theory description and quartic superpotential

Here let's review the field theory description about those theories constructed from M5 branes. The weakly coupled field theory description is described by taking degeneration limit of the Riemann surface. There are two kinds of matter systems; one is represented by a sphere with three regular punctures, and the other is described by a sphere with one irregular punctures and a regular puncture. The latter part is useful for describing non-conformal theories.

In the degeneration limit, there are two types of matter systems which are called NS and NS' matter in [11]. When two matter systems of the same type are glued together, we get a $\mathcal{N} = 2$ gauge group and a familiar cubic superpotential term involving the moment map and the adjoint chiral field in $\mathcal{N} = 2$ vector multiplet. When an NS matter and an NS' matter are connected, we get a $\mathcal{N} = 1$ gauge group and a quartic superpotential [19]

$$W = c \operatorname{tr}(\mu_1 \mu_2), \quad (2.2)$$

here μ_1 and μ_2 are the moment maps for two glued punctures. Let's give a simple example showing the explicit form of the above superpotential term. Our main example in this paper is SQCD which is described by a sphere with one irregular puncture of Φ_1 and one irregular puncture of Φ_2 . Irregular punctures are defined to have singularities which are more singular than a simple pole $1/z$ (for more details, see [33]). We only use punctures which can be read off [6, 31] from the solutions of IIA brane configurations uplifted to M theory [3]. They are given as

$$\begin{aligned} \Phi_1 &\rightarrow \frac{\zeta}{z^{1+1/(N-k_1)}} \operatorname{diag}(0, \dots, 0, 1, \omega_{N-k_1}, \dots, \omega_{N-k_1}^{N-k_1-1}), & (z \rightarrow 0), \\ \Phi_2 &\rightarrow \zeta z^{1/(N-k_2)} \operatorname{diag}(0, \dots, 0, 1, \omega_{N-k_2}, \dots, \omega_{N-k_2}^{N-k_2-1}), & (z \rightarrow \infty), \end{aligned} \quad (2.3)$$

which describes k_1 flavors and k_2 flavors separately, where $\omega_k = \exp(2\pi i/k)$. The irregular punctures are actually the ones used for $\mathcal{N} = 2$ SQCD, and we just rotate one of the irregular singularity, see figure. 1. The bundle structures are $L_1 = L_2 = \mathcal{O}(-1)$ as we will explain at the beginning of section 3, and we have taken into account $L_2 = \mathcal{O}(-1)$ in the above behavior at $z \rightarrow \infty$.

The quarks are divided into two sets with k_1 and k_2 flavors which are represented by a sphere with an irregular singularity and a regular singularity. The momental maps for $SU(N)$ gauge groups for these two sets of quarks are

$$\begin{aligned} (\mu_1)_\alpha^\beta &= \sum_{i=1}^{k_1} \left(q_\alpha^i \tilde{q}_i^\beta - \frac{1}{N} \operatorname{tr}(q_\gamma^i \tilde{q}_i^\gamma) \delta_\alpha^\beta \right), \\ (\mu_2)_\alpha^\beta &= \sum_{i=1}^{k_2} \left(p_\alpha^i \tilde{p}_i^\beta - \frac{1}{N} \operatorname{tr}(p_\gamma^i \tilde{p}_i^\gamma) \delta_\alpha^\beta \right), \end{aligned} \quad (2.4)$$

here α, β are gauge indices, and i, j are flavor indices. Let's decompose the meson as follows:

$$M = \begin{pmatrix} M_1 & L \\ \tilde{L} & M_2 \end{pmatrix}, \quad (2.5)$$

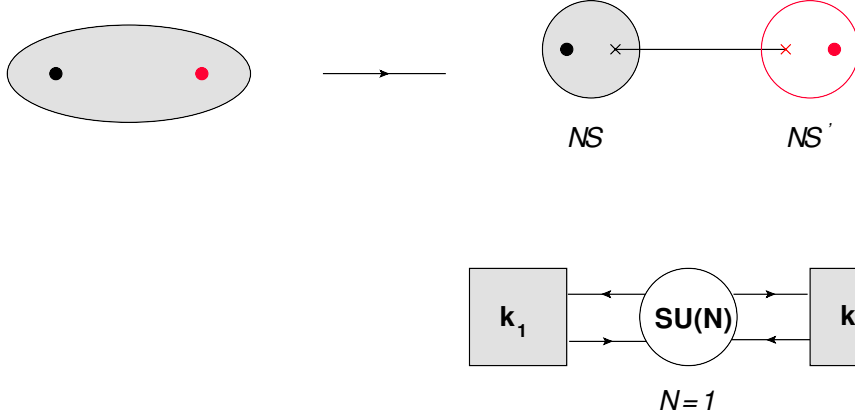


Figure 1. A Riemann sphere with two singularities (left), its degeneration limit (upper-right) and the corresponding field theory quiver (lower-right).

i.e. M_1 (M_2) represents the meson built from k_1 (k_2) flavors, and L, \tilde{L} are the mixed mesons constructed by using two sets of quarks. Then the quartic superpotential is simply

$$W = c \operatorname{tr}(L\tilde{L}) - \frac{c}{N} \operatorname{tr}(M_1) \operatorname{tr}(M_2). \quad (2.6)$$

2.2 Hitchin fibration for moduli space of twisted Higgs bundle

The Hitchin fibration of the ordinary Hitchin system plays an important role in studying the dynamics of four dimensional $\mathcal{N} = 2$ theory. In particular, the spectral curve is identified with the Seiberg-Witten curve.

It is expected that we can also write a spectral curve for the moduli space of generalized Hitchin equations. Here we assume $G = SU(N)$ and consider the moduli space of generalized Hitchin equations on a Riemann surface without punctures (some of the conclusion is also valid when we have punctures). There are two special sub-manifolds inside the full moduli space; A: M_1 which describes the solutions with $\Phi_1 = 0$ & $\Phi_2 \neq 0$; and B: M_2 which describes the solutions with $\Phi_2 = 0$ & $\Phi_1 \neq 0$. These spaces are the so-called moduli space of twisted Higgs bundle [26, 27], namely the single Higgs field is not a section of canonical bundle but the section of a general bundle L . One can define the Hitchin map on L_1 and L_2

$$\begin{aligned} \det(v - \Phi_1) = 0 &\rightarrow v^N + \sum_{i=2}^N \phi_{1i}(z)v^{N-i} = 0, \\ \det(w - \Phi_2) = 0 &\rightarrow w^N + \sum_{i=2}^N \phi_{2i}(z)w^{N-i} = 0, \end{aligned} \quad (2.7)$$

where $\phi_{1i} \in H^0(M_{g,n}, L_1^i)$ and $\phi_{2i} \in H^0(M_{g,n}, L_2^i)$. The Riemann-Roch theorem states

$$\dim H^0(L) - \dim H^0(L^* \otimes K) = \deg(L) - g + 1. \quad (2.8)$$

Here $\dim H^0(L)$ is the dimension of holomorphic sections of the line bundle L .

For example, let's consider $L_1 = L_2$, so $\deg(L_1) = \deg(L_2) = g - 1$. This is the Maldacena-Nunez theory [29]. Then we have $\dim H^0(L^i) = i(g - 1) - g + 1, i > 2$ and $\dim H^0(L^2) = g$, and the dimension of the Coulomb branch from one spectral curve is

$$d_b = \left[\sum_{i=2}^N (i(g - 1) - g + 1) + 1 \right] = \frac{1}{2}(g - 1)(N^2 - N) + 1. \quad (2.9)$$

The dimension of the fibre is computed as the genus of the spectral curve if the gauge group is $U(N)$. We need to subtract the genus of the base Riemann surface in the $SU(N)$ case. It is given as [26, 27]

$$d_f = \frac{1}{2}(N^2 - N)(g - 1) + (g - 1)(N - 1). \quad (2.10)$$

Thus the dimension of the base and the dimension of the fibre are different which is a reflection of $\mathcal{N} = 1$ supersymmetry.

2.3 Commuting matrices

When both Higgs fields are nonzero, the above two spectral equations are still valid, but there are constraints coming from the commuting condition $[\Phi_1, \Phi_2]$, which will relate the spectra of Φ_1 and Φ_2 . Therefore we need a third equation relating v and w . The answer is given by the following simple fact: given two commuting matrices A and B , if they both have distinct eigenvalues, then the matrices A, B could be written as a degree $N - 1$ polynomial of each other

$$\begin{aligned} A &= f_{N-1}B^{N-1} + f_{N-2}B^{N-2} + \dots, \\ B &= g_{N-1}A^{N-1} + g_{N-2}A^{N-2} + \dots \end{aligned} \quad (2.11)$$

In particular, the eigenvalues of A are polynomials of eigenvalues of B , and vice versa. In our case, there are families of commuting matrices defined on points of the Riemann surface, and the above formula should be still valid for a generic point of the Riemann surface. The (v, w) in this general case is defined as N pairs of eigenvalues of (Φ_1, Φ_2) , and hence the above equations are equivalent to

$$\begin{aligned} v &= f_{N-1}(z)w^{N-1} + f_{N-2}(z)w^{N-2} + \dots, \\ w &= g_{N-1}(z)v^{N-1} + g_{N-2}(z)v^{N-2} + \dots \end{aligned} \quad (2.12)$$

Here f_i and g_i are both meromorphic sections of various line bundles whose degree can be easily found, and we call them link equations.

Therefore we have three equations: two of them are the spectral curves for v and w , and the third one is the above link equations relating v and w . Given the spectral curve of v , we should be able to write down the spectral curve of w using the above link equation, and vice versa. Therefore only two equations are independent, and the link equation is the most important one. This put a lot of constraints on coefficients of three equations as we will see later explicitly.

In summary, we have the following three polynomial equations which describes the spectral curve of generalized Hitchin system:

$$\begin{aligned}
 v^N + \sum_{i=2}^N \phi_{1i}(z)v^{N-i} &= 0, \\
 w^N + \sum_{i=2}^N \phi_{2i}(z)w^{N-i} &= 0, \\
 w &= h_1(z)v^{N-1} + h_2(z)v^{N-2} + \dots + h_{N-1}(z).
 \end{aligned}
 \tag{2.13}$$

The form of $h_i(z)$ is found by imposing the condition that the combination of the second and third equations imply the first equation.

2.3.1 SU(2) case

In the SU(2) case, the full set of equations can be reduced to the following simple formula

$$\begin{aligned}
 v^2 &= f(z), \\
 w^2 &= g(z), \\
 vw &= h(z),
 \end{aligned}
 \tag{2.14}$$

where $f(z), g(z)$ and $h(z)$ are sections of L_1^2, L_2^2 and K respectively. In these equations, we have used the fact that Φ_1 and Φ_2 are traceless. From them, we get $h(z)^2 = f(z)g(z)$. This equation puts strong constraint on the coefficients of f and g as the square root of their product should be a holomorphic section (or meromorphic at punctures). This constraint was also discussed in [24]

2.3.2 General case: the use of holomorphy

In the higher rank case, we could derive the coefficients h_i by the requirement that combining the link equation and the spectral equation for v , we should get the spectral equation for w . We would get N nonlinear equations relating N coefficients h_k and f_k, g_k , and h_k can be uniquely fixed by these equations. It would be a formidable task to really solve the above nonlinear equations though. However, using the holomorphic (or meromorphic) property of $h(z)$ and the spectral curves of v and w , we can solve h_k and find the constrains among the moduli in spectral equations. Moreover, the link equation can be significantly simplified in some cases. In this part, we study the constraints on link equation from the holomorphy of h_i .

The most important information about a meromorphic section is its poles and zeros, and we would like to first determine the poles of h_i . First of all, h_i can not have pole at the pole position of v , as otherwise due to the link equation, w would have a pole at the pole position of v , which is against our assumption that either v or w has a singularity at a point. Secondly, h_i has to have a pole at the pole position of w .

Thirdly, let us consider an arbitrary point P on Riemann surface where v and w are non-singular here, and choose a local coordinate such that P has coordinate $z = 0$. Then the spectral curve for v can be expanded as

$$v^N + \sum_{k=2}^{k=N} (\phi_{1,k}(0) + \phi'_{1,k}(0)z + \dots)v^{N-k} = 0$$

$$\rightarrow \prod_{i=1}^N (v - c_i) + z \sum_{k=2}^N a_k v^{N-k} + \dots = 0, \quad (2.15)$$

and the eigenvalues for v near $z = 0$ can be expanded as

$$v_i = c_i + z^{\alpha_i} + \dots \quad (2.16)$$

Here we have defined $\alpha_i = 1/n_i$, where n_i is the degeneracy of c_i among (c_1, \dots, c_N) and we have $\sum n_i = N$. We assumed that $\sum_{k=2}^N a_k v^{N-k}$ is nonzero at $v = c_i$. Now let's assume h_i does have a pole at $z = 0$. Then near $z = 0$, the link equation should have the expansion like, e.g.,

$$w = \frac{\prod_{i=1}^{N-1} (v - c_i)}{z} + \dots, \quad (2.17)$$

so as to cancel the pole as far as possible since w is assumed not to have a singularity. Let's assume that the multiplicities of $(v - c_i)$ in above formula is a_i , and we have $\sum a_i = N - 1$. Then to cancel the pole completely, we have to impose the following condition

$$\frac{a_i}{n_i} \geq 1. \quad (2.18)$$

However, this is impossible as it implies $\sum a_i \geq \sum n_i$. Therefore h_i is non-singular at the above generic point.

Finally, we consider a point where w has a singularity but v is regular, and we would like to determine the order of pole of h_i . Here we do not give a general analysis, and just study a particular important example. Let us assume that the N values of w behave as

$$w \sim \left(\frac{1}{z^{1+1/(N-N_f)}}, \frac{\omega_{N-N_f}}{z^{1+1/(N-N_f)}}, \dots, \frac{\omega_{N-N_f}^{N-N_f-1}}{z^{1+1/(N-N_f)}}, \frac{m_1}{z}, \dots, \frac{m_{N_f}}{z} \right), \quad (2.19)$$

where $\omega_{N-N_f} = \exp(2\pi i/(N - N_f))$. This is a standard behavior for singularities which have type IIA construction. To reproduce this singular behavior, we need to have $h_i \sim 1/z^2$ because w is more singular than $1/z$ and v is not singular. Furthermore, the curves should be given as

$$(v - c_0)^{N-N_f} \sum_{i=1}^{N_f} (v - c_i) + z \sum_{k=2}^N a_k v^{N-k} + \dots = 0, \quad (2.20)$$

$$w = \frac{(v - c_0)^{N-N_f-1} \sum_{i=1}^{N_f} (v - c_i)}{z^2} + \dots \quad (2.21)$$

In this form, the correct behavior of w is reproduced, assuming $\sum_{k=2}^N a_k v^{N-k} \neq 0$. We can use the equation for v to get

$$z(v - c_0)w = \sum_{k=2}^N h'_k v^{N-k}, \quad (2.22)$$

where h'_k is now non-singular.

Let's first apply the above result to the theory defined on a Riemann surface without any puncture. Without loss of generality, we assume $0 \leq \deg(L_2) \leq \deg(L_1)$, since if $\deg L_2 < 0$, we can only have $w = 0$. Now, under the assumption of genericity $\sum_{k=2}^N a_k v^{N-k} \neq 0$ discussed above, h_i is a holomorphic section of a line bundle whose degree is $\deg(L_2) - (N - i) \deg(L_1)$, so only h_N and possibly h_{N-1} have non-negative degree and therefore nonzero. Taking into account the traceless condition, the link equation is simplified as

$$w = h_{N-1}v, \tag{2.23}$$

with h_{N-1} constant for $\deg L_1 = \deg L_2$ and $h_{N-1} = 0$ for $\deg L_2 < \deg L_1$. We will also discuss nongeneric case in section 3.

Next let us consider the case of a Riemann sphere with two punctures at $z = 0, \infty$. In this paper, we are mainly interested in SQCD which is defined on sphere with an irregular singularity for v and w separately, and the bundle structure is $L_1 = L_2 = \mathcal{O}(-1)$. We take the above geometry as our example, and the analysis for more general geometry may be similar. We assume that w is singular at $z = 0$, so we have (2.22). We also assume that there is one singularity of v whose singular behavior near $z = \infty$ is $v' = \frac{1}{z'^{1+\gamma}}$ ($z' = 1/z, v' = zv$). It is obvious that the h'_k in (2.22) should be constant since terms with positive powers of z would imply that w is singular at $z = \infty$, which is against our assumption. Furthermore, by the same reason, only the terms $h'_{N-1}v + h'_N$ are allowed. Therefore, the link equation is given as

$$w = \frac{1}{z} \frac{av + b}{v - c}, \tag{2.24}$$

where $a = h'_{N-1}$, $b = h'_N$ and $c = c_0$ are constants.

3 SU(2) theory

The basic formula is the following

$$\begin{aligned} v^2 &= f(z), \\ w^2 &= g(z), \\ vw &= h(z), \end{aligned} \tag{3.1}$$

with $h^2 = fg$. Typically, we first write down the first two equations using the bundle and puncture structures, and then simply require that their product is a square of holomorphic section (or meromorphic at punctures). This will link the parameters in f and g .

3.1 SU(2) SQCD

We are going to study SU(2) SQCD in detail. There are three types of punctures we are going to use

$$\Phi_A = \frac{1}{z^{1+1/2}} \text{diag}(\zeta, -\zeta) + \dots, \tag{3.2}$$

$$\Phi_B = \frac{1}{z^2} \text{diag}(\zeta, -\zeta) + \frac{1}{z} \text{diag}(m, -m) \dots, \tag{3.3}$$

$$\Phi_C = \frac{1}{z} \text{diag}(m, -m), \tag{3.4}$$

where these expressions are written in the case that the position of the puncture is at $z = 0$. Here the puncture of type A describes zero fundamental, B describes one fundamental, and C is a regular full puncture. The line bundles L_1 and L_2 of the two scalars Φ_1 and Φ_2 are both the bundle $\mathcal{O}(-1)$ of the sphere, i.e., $\text{deg } L_1 = \text{deg } L_2 = -1$.

The fact $\text{deg } L_1 = \text{deg } L_2 = -1$ can be seen as follows. Let us consider a brane setup of the SQCD as in [8, 9]. We prepare two NS5 branes, which we denote as NS5 and NS5', and we suspend N D4 branes between them. The Higgs field Φ_1 has a pole at NS5 and Φ_2 has a pole at NS5'. Originally, v and w are flat coordinates in the brane set up, but as was done by Gaiotto [31], introducing the poles at the two ends of the D4 branes change the bundle structures. In the $\mathcal{N} = 2$ case of [31], introducing poles of v at both ends of the D4 branes makes v a coordinate of the canonical bundle $K = \mathcal{O}(-2)$ of the sphere, since the coordinate⁴ v in the patch near $z = 0$ and v' in the patch near $z = \infty$ is related as $v' = z^2 v$ in the coordinate change $z \rightarrow z' = 1/z$. The w remains to be a coordinate of the trivial bundle. In our $\mathcal{N} = 1$ case, introducing a pole of v at one end of the D4 branes and a pole of w at the other end makes both v and w be coordinates of the $\mathcal{O}(-1)$ bundle. Actually, by requiring that v has the appropriate pole at $z = \infty$ and is smooth at $z = 0$, we get $\text{deg } L_1 \geq -1$. In the same way, we get $\text{deg } L_2 \geq -1$. Taking into account $\text{deg } L_1 + \text{deg } L_2 = \text{deg } K = -2$, we obtain $\text{deg } L_1 = \text{deg } L_2 = -1$. Roughly speaking, a puncture of Φ_1 (or Φ_2) adds -1 to the degree of the line bundle L_1 (or L_2).

3.1.1 Pure SU(2)

There are one A type puncture of Φ_1 at $z = 0$ and one A type puncture of Φ_2 at $z = \infty$. The two spectral curves without imposing any conditions are⁵

$$\begin{aligned} v^2 &= \frac{\zeta_1^2}{z^3} + \frac{u_1}{z^2}, \\ w^2 &= \zeta_2^2 z + u_2, \\ vw &= h(z), \end{aligned} \tag{3.5}$$

where $h(z)^2 = (\frac{\zeta_1^2}{z^3} + \frac{u_1}{z^2})(\zeta_2^2 z + u_2)$. It is easy to see that in order for $h(z)$ to be meromorphic, we need to impose $u_1 = u_2 = 0$, and we get a curve

$$\begin{aligned} v^2 &= \frac{\zeta_1^2}{z^3} \\ w^2 &= \zeta_2^2 z \\ vw &= \zeta_1 \zeta_2 / z \end{aligned} \tag{3.6}$$

⁴In [31], the notation x is used instead of v for the coordinate of the canonical bundle. Here we continue to use v for nontrivial bundles.

⁵Notice that we can use scale invariance to make sure $\zeta_1 = \zeta_2$, so there is only one independent dimensional parameter in SQCD case.

which is in agreement with the solution found by Witten [9] (One need to redefine the coordinates v and w).

3.1.2 SU(2) with one flavor

There are one B type puncture of Φ_1 and one A type puncture of Φ_2 . The curves for v and w are

$$\begin{aligned} v^2 &= \frac{\zeta_1^2}{z^4} + \frac{\zeta_1 m}{z^3} + \frac{u_1}{z^2}, \\ w^2 &= \zeta_2^2 z + u_2, \\ vw &= h(z). \end{aligned} \tag{3.7}$$

Again, $h(z)^2 = (\frac{\zeta_1^2}{z^4} + \frac{\zeta_1 m}{z^3} + \frac{u_1}{z^2})(\zeta_2^2 z + u_2)$. The holomorphy of $h(z)$ will ensure that $u_1 = 0, u_2 = \zeta_1 \zeta_2^2 / m$, and our final curve would be

$$\begin{aligned} v^2 &= \frac{\zeta_1^2}{z^4} + \frac{\zeta_1 m}{z^3}, \\ w^2 &= \zeta_2^2 z + \zeta_1 \zeta_2^2 / m, \\ vw &= \frac{\zeta_2 (\zeta_1 m)^{1/2}}{z^2} \left(z + \frac{\zeta_1}{m} \right). \end{aligned} \tag{3.8}$$

In the massless limit, u_2 is infinity, which means that there is no way to write a meromorphic spectral curve, and we conclude that SUSY is broken. This matches perfectly with the field theory fact that the Affleck-Dine-Seiberg superpotential [34] is generated and there is no supersymmetric vacuum.

3.1.3 SU(2) with two flavors

There are two ways of realizing two flavor theories. We can divide the flavor number as $N_f = N'_f + N''_f$. Then we have the choice $N'_f = N''_f = 1$ or $N'_f = 2, N''_f = 0$. They are different theories because of the existence of quartic superpotential [11, 19],

$$W = c \operatorname{tr} \left[\sum_{i=1}^{N'_f} \left(q_i \tilde{q}_i - \frac{1}{N} \operatorname{tr}(q_i \tilde{q}_i) \right) \sum_{a=1}^{N''_f} \left(p_a \tilde{p}_a - \frac{1}{N} \operatorname{tr}(p_a \tilde{p}_a) \right) \right]. \tag{3.9}$$

First realization. The puncture types of the first arrangement $N'_f = N''_f = 1$ are: there are one B type puncture of Φ_1 and one B type puncture of Φ_2 . The curves are

$$\begin{aligned} v^2 &= \frac{\zeta_1^2}{z^4} + \frac{\zeta_1 m_1}{z^3} + \frac{u_1}{z^2}, \\ w^2 &= \zeta_2^2 z^2 + \zeta_2 m_2 z + u_2. \end{aligned} \tag{3.10}$$

The third equation requires that

$$h(z)^2 = \frac{1}{z^4} (u_1 z^2 + \zeta_1 m_1 z + \zeta_1^2) (\zeta_2^2 z^2 + m_2 \zeta_2 z + u_2). \tag{3.11}$$

There are two ways of satisfying this equation. The first way is to impose that

$$(\zeta_2^2 z^2 + m_2 \zeta_2 z + u_2) \propto (u_1 z^2 + \zeta_1 m_1 z + \zeta_1^2). \quad (3.12)$$

This condition gives

$$u_1 u_2 = \zeta_1^2 \zeta_2^2, \quad m_2 u_1 = \zeta_1 \zeta_2 m_1. \quad (3.13)$$

The second way is to impose

$$(\zeta_2^2 z^2 + m_2 \zeta_2 z + u_2) = (\alpha z + \beta)^2, \quad (3.14)$$

$$(u_1 z^2 + \zeta_1 m_1 z + \zeta_1^2) = (\gamma z + \delta)^2. \quad (3.15)$$

This condition gives

$$u_2 = \frac{m_2^2}{4}, \quad u_1 = \frac{m_1^2}{4}. \quad (3.16)$$

In either case, there are discrete vacua as expected.

In the massless limit, the condition (3.13) becomes

$$u_1 u_2 = \zeta_1^2 \zeta_2^2. \quad (3.17)$$

On the other hand, the condition (3.16) is

$$u_1 = u_2 = 0. \quad (3.18)$$

We may interpret these equations as follows. In the field theory, the low energy superpotential is given by

$$W = X(M_{11}M_{22} - M_{12}M_{21} - B\tilde{B} - \Lambda^4) + c\left(M_{12}M_{21} - \frac{1}{2}M_{11}M_{22}\right), \quad (3.19)$$

where M_{ij} are mesons, B and \tilde{B} are baryons, and X is a Lagrange multiplier. The first term is the deformed moduli constraint of SQCD, and the second term comes from the quartic superpotential (3.9). One can see that there are three branches;

$$(1) : \quad M_{11}M_{22} = \Lambda^4, \quad M_{12} = M_{21} = B = \tilde{B} = 0, \quad X = \frac{c}{2}, \quad (3.20)$$

$$(2) : \quad M_{12}M_{21} = -\Lambda^4, \quad M_{11} = M_{22} = B = \tilde{B} = 0, \quad X = -c, \quad (3.21)$$

$$(3) : \quad B\tilde{B} = -\Lambda^4, \quad M_{11} = M_{22} = M_{12} = M_{21} = 0, \quad X = 0. \quad (3.22)$$

The first branch may correspond to (3.17) by identifying $u_1 \sim (M_{11})^2$ and $u_2 \sim (M_{22})^2$ while the other two branches may correspond to (3.18). See [35] for details about these points.

Second realization. In the realization $(N'_f, N''_f) = (2, 0)$ of the $N_f = 2$ theory, there are three punctures: two regular punctures C of Φ_1 at $z = 0, \infty$ and a type A puncture of Φ_2 at $z = 1$. The two curves read

$$\begin{aligned} v^2 &= \frac{m_1^2}{z^2} + \frac{u_1}{z} + m_2^2, \\ w^2 &= \frac{\zeta^2}{(z-1)^3} + \frac{u_2}{(z-1)^2}. \end{aligned} \tag{3.23}$$

The third equation requires

$$h(z)^2 = \frac{1}{z^2(z-1)^4} (m_2^2 z^2 + u_1 z + m_1^2) (u_2(z-1) + \zeta^2)(z-1). \tag{3.24}$$

This constraint is satisfied if and only if

$$u_1 = -(m_1^2 + m_2^2), \quad (m_2^2 - m_1^2)u_2 = \zeta^2 m_2^2. \tag{3.25}$$

In the massless limit, u_1 has to be zero, and we only get the second curve

$$w^2 = \frac{\zeta^2}{(z-1)^3} + \frac{u_2}{(z-1)^2}, \tag{3.26}$$

with u_2 a free modulus. In the field theory side, the flavor symmetry contains a subgroup $SU(2)_1 \times SU(2)_2 \subset SU(4)$, in the spirit of [31]. We have chiral operators (usually called mesons) μ_1 and μ_2 which are in the adjoint representations of $SU(2)_1$ and $SU(2)_2$, respectively. The deformed moduli constraint is given as $\text{tr } \mu_1^2 - \text{tr } \mu_2^2 = \Lambda^4$. This is just a rewriting of the usual SQCD deformed moduli constraint [28]. We may identify the u_2 in the curve as the flavor invariant operator of the field theory, $u_2 \sim \text{tr } \mu_1^2 + c \sim \mu_2^2 + c'$, where c and c' are constants. After some coordinate changes, the curve is the same as the ones found in [7, 14] by taking some of the dynamical scales of [7, 14] to be zero. There u_2 was really identified as $u_2 \sim \text{tr } \mu_1^2 + c \sim \mu_2^2 + c'$.

3.1.4 $SU(2)$ with three flavors

There are three punctures: two regular punctures C and a type B puncture.

$$\begin{aligned} v^2 &= \frac{m_1^2}{z^2} + \frac{u_1}{z} + m_2^2, \\ w^2 &= \frac{\zeta^2}{(z-1)^4} + \frac{\zeta m_3}{(z-1)^3} + \frac{u_2}{(z-1)^2}. \end{aligned} \tag{3.27}$$

The third equation is

$$h(z)^2 = \frac{1}{z^2(z-1)^4} (m_2^2 z^2 + u_1 z + m_1^2) (u_2(z-1)^2 + m(z-1) + \zeta^2). \tag{3.28}$$

There are two ways to satisfy this equation. The first way is

$$(m_2^2 z^2 + u_1 z + m_1^2) \propto (u_2(z-1)^2 + m_3(z-1) + \zeta^2). \tag{3.29}$$

This condition gives

$$u_2(u_1 + 2m_2^2) = m_2^2 m_3, \quad \zeta^2(u_1 + 2m_2^2) = (u_1 + m_1^2 + m_2^2)m_3. \quad (3.30)$$

The second way is

$$(m_2^2 z^2 + u_1 z + m_1^2) = (\alpha z + \beta)^2, \quad (3.31)$$

$$(u_2(z-1)^2 + m_3(z-1) + \zeta^2) = (\gamma z + \delta)^2. \quad (3.32)$$

This condition gives

$$u_1^2 = \frac{m_1^2 m_2^2}{4}, \quad u_2 = \frac{4m_3^2}{\zeta^2}. \quad (3.33)$$

In the massless limit, we must have $u_1 = 0$. Then the curves are given by $v = 0$ and

$$w^2 = \frac{\zeta^2}{(z-1)^4} + \frac{u_2}{(z-1)^2}. \quad (3.34)$$

In the field theory side, the low energy superpotential is given as follows. We take quarks as $q_{i_1 i_2 \alpha}$ ($i_1, i_2, \alpha = 1, 2$) and $p_\alpha, \tilde{p}_\alpha$. The q quarks are in the trifundamental representation of $SU(2)_1 \times SU(2)_2 \times SU(2)_g$, where $SU(2)_1 \times SU(2)_2$ are flavor groups and $SU(2)_g$ is the gauge group. We define gauge invariant operators as

$$\begin{aligned} (\mu_1)_{i_1}^{j_1} &= \frac{1}{2} q_{i_1 i_2 \alpha} q^{j_1 i_2 \alpha}, & (\mu_2)_{i_2}^{j_2} &= \frac{1}{2} q_{i_1 i_2 \alpha} q^{i_1 j_2 \alpha}, \\ N &= p_\alpha \tilde{p}^\alpha, & B_{i_1 i_2} &= q_{i_1 i_2 \alpha} p^\alpha, & \tilde{B}_{i_1 i_2} &= q_{i_1 i_2 \alpha} \tilde{p}^\alpha, \end{aligned} \quad (3.35)$$

where indices are raised and lowered by the totally antisymmetric tensor. The quartic superpotential is given as

$$W = c(q_{j_1 i_2 \alpha} q^{j_1 j_2 \beta}) \left(p_\beta \tilde{p}^\alpha - \frac{1}{2} \delta_\beta^\alpha (\tilde{p}^\gamma p_\gamma) \right) = c B_{i_1 i_2} \tilde{B}^{i_1 i_2}. \quad (3.36)$$

Including the dynamical superpotential, we obtain

$$W = \frac{1}{\Lambda^3} \left[N(\text{tr } \mu_1^2 - \text{tr } \mu_2^2) + B_{i_1 i_2} \tilde{B}^{j_1 j_2} \left((\mu_1)_{j_1}^{i_1} \delta_{j_2}^{i_2} - \delta_{j_1}^{i_1} (\mu_2)_{j_2}^{i_2} \right) \right] + c B_{i_1 i_2} \tilde{B}^{i_1 i_2}. \quad (3.37)$$

There is a branch such that $B_{i_1 i_2} = \tilde{B}_{i_1 i_2} = N = 0$ and $\text{tr } \mu_1^2 = \text{tr } \mu_2^2$. We may identify $u_2 \sim \text{tr } \mu_1^2 = \text{tr } \mu_2^2$.

3.1.5 $SU(2)$ with four flavors

The spectral curves are

$$\begin{aligned} v^2 &= \frac{m_1^2}{z^2} + \frac{u_1}{z} + m_2^2, \\ w^2 &= \frac{m_3^2}{(z-1)^2} + \frac{u_2}{(z-1)(z-\tau)} + \frac{m_4^2}{(z-\tau)^2}. \end{aligned} \quad (3.38)$$

Again, one can fix u_1 and u_2 using the factorization condition $v^2 w^2 = h(z)^2$. In the massless limit, $u_1 = 0$ or $u_2 = 0$. The curve is at the conformal point if $u_1 = u_2 = 0$. One can assign $U(1)_R$ charges as $v : 1/2$, $w : 1/2$, $u_1 : 1$, $u_2 : 1$. This symmetry is the $U(1)_R$ symmetry of the superconformal algebra.

3.2 Three irregular singularities: three gauge groups

In this subsection, we study theories which are realized as a Riemann sphere with three punctures. Suppose that there are n_1 punctures of Φ_1 and n_2 punctures of Φ_2 with $n_1 + n_2 = 3$. Then, the degrees of the line bundles L_1 and L_2 are given by $(\deg L_1, \deg L_2) = (-n_1, 1 - n_2)$ (or $(\deg L_1, \deg L_2) = (1 - n_1, -n_2)$).

Let's suppose $(\deg L_1, \deg L_2) = (-n_1, 1 - n_2)$. This formula may be interpreted as a result of the fact that the degree of the line bundle L_i is decreased -1 as we introduce a puncture of Φ_i as discussed at the beginning of section 3.1. The term 1 in $\deg L_2 = 1 - n_2$ is interpreted as the Euler number of a sphere with three holes around the punctures. Note that if $(n_1, n_2) = (0, 3)$, we get $(\deg L_1, \deg L_2) = (0, -2)$ as it should be for $\mathcal{N} = 2$ theories.

The field theory interpretation of punctures are as follows. The punctures of Φ_2 are the same as in $\mathcal{N} = 2$ theories. We have a trifundamental chiral field $Q_{i_1 i_2 i_3}$ of $SU(2)_1 \times SU(2)_2 \times SU(2)_3$ (or more generally a copy of the T_N theory with $SU(N)_1 \times SU(N)_2 \times SU(N)_3$ flavor symmetry), and if a puncture is the irregular singularities of type A, (3.2), the corresponding $SU(2)$ is gauged by $\mathcal{N} = 2$ vector multiplet. If it is type B, (3.3), there is an additional fundamental flavor of quarks coupled to the vector multiplet.

The punctures of Φ_1 are kind of “dual” of the above $\mathcal{N} = 2$ punctures in the following sense [11, 19]. If the puncture is a regular puncture of type C, (3.4), then we have a meson $(M_1)_{j_1}^{i_1}$ coupled to the trifundamental field as

$$W = \text{tr}(M_1 \mu_1), \quad (\mu_1)_{i_1}^{j_1} = Q_{i_1 i_2 i_3} Q^{j_1 i_2 i_3}, \tag{3.39}$$

where we have assumed that the puncture corresponds to $SU(2)_1$. This is similar to a superpotential in Seiberg dual; see [19] for details. If the puncture is the irregular singularity of type A, the corresponding $SU(2)_1$ is coupled to $\mathcal{N} = 2$ vector multiplet with the superpotential $\text{tr}(M_1 \phi_1)$, where ϕ_1 is the adjoint chiral multiplet of the $\mathcal{N} = 2$ vector multiplet. By integrating out these massive M_1 and ϕ_1 , the end result is that the $SU(2)_1$ is coupled to $\mathcal{N} = 1$ vector multiplet. The puncture of type B can be treated in a similar way. It is straightforward to extend the above discussion to the T_N theory.

3.2.1 Field theory analysis

We study $SU(2)$ theory with three gauge groups, which is defined by a sphere with three irregular singularities of type A. Before going into the details of the curve, let us sketch what happens in field theory. We only consider the case $n_1 \geq 1$, and assume that $SU(2)_3$ is gauged by $\mathcal{N} = 1$ vector multiplet. Suppose the dynamical scale of the gauge group $SU(2)_3$ is very large. Then, at low energies, we get a deformed moduli space

$$W = X(\text{tr} \mu_1^2 - \text{tr} \mu_2^2 - \Lambda_1^4), \tag{3.40}$$

where X is a Lagrange multiplier and μ_1, μ_2 are defined as in (3.39). We consider the following three cases.

A: $(n_1, n_2) = (3, 0)$, $L_1 = \mathcal{O}(-3)$, $L_2 = \mathcal{O}(1)$. In this case, we couple $\mathcal{N} = 1$ vector multiplets to $SU(2)_1 \times SU(2)_2$ in addition to the $\mathcal{N} = 1$ gauge group $SU(2)_3$. The fields μ_1

and μ_2 are in the adjoint representations of the corresponding gauge groups. Their vevs break $SU(2)_1 \times SU(2)_2$ to $U(1)^2$. There is one modulus field $u \equiv \text{tr } \mu_1^2 = \text{tr } \mu_2^2 + \Lambda_1^4$. Therefore, the low energy theory consists of two massless $U(1)$ fields and one massless modulus u . This is precisely the theory studied in [14]. See [21] for a generalization to the T_N theory.

B: $(n_1, n_2) = (2, 1)$, $L_1 = \mathcal{O}(-2)$, $L_2 = \mathcal{O}(0)$. In this case, we couple $\mathcal{N} = 1$ vector multiplet to $SU(2)_1$ and $\mathcal{N} = 2$ vector multiplet to $SU(2)_2$. The effective superpotential is

$$W = X(\text{tr } \mu_1^2 - \text{tr } \mu_2^2 - \Lambda_1^4) + \text{tr}(\mu_2 \phi_2). \quad (3.41)$$

Equations of motion of ϕ_2 set $\mu_2 = 0$. Then the deformed moduli constraint requires that $\mu_1 = \text{diag}(\Lambda_1^2, -\Lambda_1^2)$ up to gauge transformations. This vev breaks $SU(2)_1$ to $U(1)$. The term $\text{tr}(\mu_2 \phi_2)$ can be regarded as a mass term for μ_2 and ϕ_2 . After integrating out them, the gauge group $SU(2)_2$ becomes a pure $\mathcal{N} = 1$ Yang-Mills and confines at low energies. There are no moduli fields. Therefore, the result is that there are two discrete vacua generated by the gaugino condensation of $SU(2)_2$ and each vacua has a massless $U(1)$ vector multiplet.

C: $(n_1, n_2) = (1, 2)$, $L_1 = \mathcal{O}(-1)$, $L_2 = \mathcal{O}(-1)$. In this case, we couple $\mathcal{N} = 2$ vector multiplets to $SU(2)_1 \times SU(2)_2$. The superpotential is given as

$$W = X(\text{tr } \mu_1^2 - \text{tr } \mu_2^2 - \Lambda_1^4) + \text{tr}(\mu_1 \phi_1) + \text{tr}(\mu_2 \phi_2). \quad (3.42)$$

Equations of motion of ϕ_1 and ϕ_2 set $\mu_1 = \mu_2 = 0$, which is inconsistent with the deformed moduli constraint. Thus we conclude that the supersymmetry is broken. This case is almost the same as the dynamical supersymmetry breaking model of [36, 37]. See [23] for a generalization to the T_N theory.

3.2.2 Curves

Now we would like to discuss the spectral curves of the above theories.

A: $(n_1, n_2) = (3, 0)$, $L_1 = \mathcal{O}(-3)$, $L_2 = \mathcal{O}(1)$. The curves are

$$\begin{aligned} v^2 &= \frac{\zeta_1^2}{z^3(z-1)^2} + \frac{\zeta_2^2}{z^2(z-1)^3} + \frac{\zeta_3^2}{z(z-1)^2} + \frac{u}{z^2(z-1)^2} \\ w^2 &= u_1 z^2 + u_2 z + u_3 \end{aligned} \quad (3.43)$$

By looking at the behaviors at $z = 0, 1, \infty$, one can find that $u_1 = u_2 = u_3 = 0$ to satisfy the constraint $v^2 w^2 = h(z)^2$. Thus $w = 0$ and we get a single curve with one moduli u . This is in perfect agreement with the Seiberg-Witten curve found in [14] after a redefinition $v' = z(z-1)v$.

B: $(n_1, n_2) = (2, 1)$, $L_1 = \mathcal{O}(-2)$, $L_2 = \mathcal{O}(0)$. The curves are

$$\begin{aligned} v^2 &= \frac{\zeta_1^2}{z^3} + \frac{u_1}{z^2} + \frac{\zeta_3^2}{z} \\ w^2 &= \frac{\zeta_2^2}{(z-1)^3} + \frac{u_2}{(z-1)^2} + \frac{u_3}{(z-1)} + u_4 \end{aligned} \quad (3.44)$$

Using the link equation $v^2w^2 = h(z)^2$, one can fix all the parameters:

$$u_1 = -\zeta_1^2 - \zeta_3^2, \quad u_2 = \zeta_2^2 + \frac{\zeta_2^2\zeta_3^2}{\zeta_3^2 - \zeta_1^2}, \quad u_3 = \frac{\zeta_2^2\zeta_3^2}{\zeta_3^2 - \zeta_1^2}, \quad u_4 = 0, \quad (3.45)$$

and the curves are

$$\begin{aligned} v^2 &= \frac{(\zeta_3^2 z - \zeta_1^2)(z - 1)}{z^3}, \\ w^2 &= \frac{\zeta_2^2}{\zeta_3^2 - \zeta_1^2} \frac{(\zeta_3^2 z - \zeta_1^2)z}{(z - 1)^3}, \\ vw &= \left(\frac{\zeta_2^2}{\zeta_3^2 - \zeta_1^2} \right)^{1/2} \frac{(\zeta_3^2 z - \zeta_1^2)}{z(z - 1)} \end{aligned} \quad (3.46)$$

There are two isolated vacua in this theory corresponding to the square root. This is the same as the prediction from gauge theory. These curves should give the Seiberg-Witten curve of the massless U(1) field discussed above. Actually, one can check that the genus of the curve is 1, consistent with the fact that there is one massless U(1) field at low energies.

C: $(n_1, n_2) = (1, 2)$, $L_1 = \mathcal{O}(-1)$, $L_2 = \mathcal{O}(-1)$. The curves are

$$\begin{aligned} v^2 &= z\zeta_3^2 + u_1, \\ w^2 &= \frac{\zeta_1^2}{z^3(z - 1)^2} + \frac{\zeta_2^2}{z^2(z - 1)^3} + \frac{u_2 z^2 + u_3 z + u_4}{z^2(z - 1)^2}. \end{aligned} \quad (3.47)$$

By looking at the behavior of $v^2w^2 = h(z)^2$ at $z \rightarrow 0$ and $z \rightarrow 1$, one can check that there is no solution at all. This is consistent with the supersymmetry breaking in the field theory.

3.3 Chiral ring relation for Maldacena-Nunez theory

It is not hard to do calculation for general theory defined on a sphere, and one can also do the calculation on higher genus case by using the explicit information about holomorphic sections of various bundles. The general procedure is exactly like what we have done earlier. Here we will consider A_1 Maldacena-Nunez theory [29] to show the interesting chiral ring relation one can find by writing down the curve .

Maldacena-Nunez (MN) theory is defined by taking a genus g Riemann surface without any puncture, and the bundle is $L_1 = L_2 = K^{\frac{1}{2}}$, where K is the canonical bundle. We assume that the complex structure of the Riemann surface is such that it is a hyperelliptic surface for genus g Riemann surface, and the algebraic curve defining it is

$$y^2 = \prod_{k=1}^{2g+2} (z - p_k), \quad p_k \neq p_j. \quad (3.48)$$

The basis for degree one holomorphic differential (holomorphic sections of the canonical bundle) is

$$e_j = \frac{z^j dz}{y}, \quad j = 0, \dots, g - 1, \quad (3.49)$$

and the basis for holomorphic quadratic differential is

$$\begin{aligned} s_j &= \frac{z^j dz^2}{y^2}, \quad j = 0, \dots, 2g - 2, \\ t_j &= \frac{z^j dz^2}{y}, \quad j = 0, \dots, g - 3, \end{aligned} \tag{3.50}$$

Notice that we have the following simple relation for the product of the degree one differential

$$e_j e_k = e_{j'} e_{k'} = s_{j+k}, \quad j + k = j' + k'. \tag{3.51}$$

Let's first study MN theory on genus two Riemann surface. The three curves are

$$\begin{aligned} v^2 &= v_0 e_0 + v_1 e_1, \\ w^2 &= u_0 e_0 + u_1 e_1, \\ vw &= h_0 e_0 + h_1 e_1. \end{aligned} \tag{3.52}$$

The consistency condition gives us the following equations

$$h_0^2 = v_0 u_0, \quad h_1^2 = v_1 u_1, \quad 2h_0 h_1 = v_0 u_1 + v_1 u_0. \tag{3.53}$$

The consistency of above equations implies that

$$v_0 u_1 = v_1 u_0. \tag{3.54}$$

This is the interesting chiral-ring relation we found for A_1 MN theory.

Let's now generalize the above consideration to A_1 MN theory on a genus g Riemann surface, and the curves are

$$\begin{aligned} v^2 &= \sum_{i=0}^{g-1} v_i e_i, \\ w^2 &= \sum_{i=0}^{g-1} u_i e_i, \\ vw &= \sum_{i=0}^{g-1} h_i e_i. \end{aligned} \tag{3.55}$$

The consistency equations are

$$\sum_{i+j=p} h_i h_j = \sum_{i+j=p} v_i u_j, \quad p = 0, 1, \dots, 2g - 2. \tag{3.56}$$

There are only g constant h_i and there are $2g - 1$ equations, so generically we need to impose $(g - 1)$ constraints on v_i and u_i , which would give us the chiral ring relation. The dimension of the moduli space is $g + 1$ generically. One generic branch is given by (2.23), $v \propto w$, which leads to

$$v_i u_j = v_j u_i. \tag{3.57}$$

This is the chiral-ring relation relating the parameters in spectral curve of v and w .

There are several other branches found by solving equation (3.56). We can impose chiral ring relation for the parameters inside the spectral curve of v and w . For example, in the genus three case, we have two branches, one of them is given by

$$\begin{aligned} v^2 &= v_0 e_0 + v_1 e_1 + v_2 e_2, \\ v^2 &= u_0 e_0 + u_1 e_1 + u_2 e_2, \\ v_0 u_1 &= v_1 u_0, \quad v_1 u_2 = v_2 u_1, \quad v_2 u_0 = v_0 u_2. \end{aligned} \tag{3.58}$$

and the other one is given by

$$\begin{aligned} v^2 &= v_0^2 e_0 + 2v_0 v_1 e_1 + v_1^2 e_2, \\ v^2 &= u_0^2 e_0 + 2u_0 u_1 e_1 + u_1^2 e_2. \end{aligned} \tag{3.59}$$

Here v_0, v_1, u_0, u_1 are arbitrary.

One can understand those other branches as follows. First take an arbitrary line bundle L . Next, take two sections of L , $s, t \in H^0(L)$, and a section of $K \otimes L^{-2}$, $u \in H^0(K \otimes L^{-2})$. We can find spectral curves as

$$v^2 = s^2 u, \quad w^2 = t^2 u, \quad vw = stu. \tag{3.60}$$

Therefore there are many branches in the Maldacena-Nunez theory corresponding to the choice of L . The moduli space of this branch is given by

$$(H^0(L) \oplus H^0(L) \oplus H^0(K \otimes L^{-2})) / \mathbb{C}^* \tag{3.61}$$

where division by \mathbb{C}^* means to divide by the equivalence relation $(s, t, u) \cong (\alpha s, \alpha t, \alpha^{-2} u)$ for $\alpha \in \mathbb{C}^*$.

4 SU(N) theory

In this section, we are going to study several interesting examples for $SU(N)$ theories to show how our general procedure of writing down the curve can be implemented in practice.

4.1 SQCD

In this section, we study a few examples of $SU(N)$ theories. The irregular punctures we will use are

$$N_f = 0 : \quad \Phi_A = \frac{\zeta}{z^{1+1/N}} \text{diag}(1, \omega_N, \omega_N^2, \dots, \omega_N^{N-1}) + \dots, \tag{4.1}$$

$$\begin{aligned} N_f = k : \quad \Phi_B &= \frac{\zeta}{z^{1+1/(N-k)}} \text{diag}(0, \dots, 1, \omega_{N-k}, \dots, \omega_{N-k}^{N-k-1}) \\ &+ \frac{1}{z} \text{diag}(m_1, m_2, \dots, m_k, m, \dots, m) \dots, \end{aligned} \tag{4.2}$$

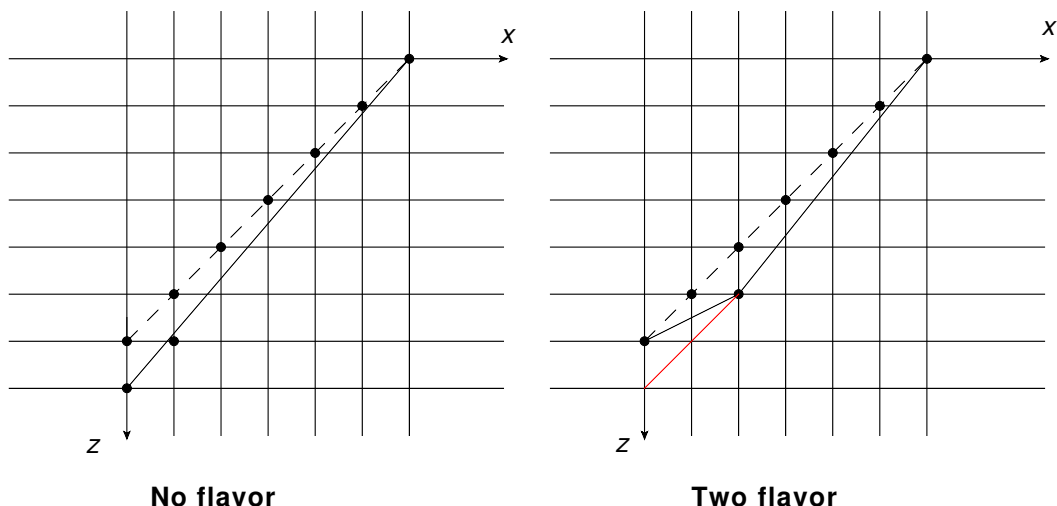


Figure 2. The newton polygon for irregular singularity. The red line represents the mass-deformed theory, and the black one represents the massless theory.

$$N_f = N - 1 : \quad \Phi_C = \frac{\zeta}{z^2} \text{diag}(1, 1, 1, \dots, -(N - 1)) + \frac{1}{z} \text{diag}(m_1, m_2, \dots, m_{N-1}, -(m_1 + \dots + m_{N-1})), \quad (4.3)$$

where $\omega_k = \exp(2\pi i/k)$. The curves for these irregular singularities can be easily read from a Newton polygon, see figure. 2. The slop of the boundary of Newton polygon encodes the pole orders of the irregular singularity, and the integer lattice points inside the Newton polygon represents the deformation in Seiberg-Witten curve. For more details, see [38].

4.1.1 $N_f < N_c$ massive SQCD

Let's consider masssive theory. Let's put one singularity representing N_f flavor at $z = \infty$, and the other singularity representing zero flavor at $z = 0$. The curves look like

$$v^N + \sum_{i=2}^N p_i v^{N-i} + \zeta_1^{N-N_f} z f(v, m) = 0, \\ w^N + \sum_{i=2}^N \frac{q_i}{z^i} w^{N-i} + \frac{\zeta_2^N}{z^{N+1}} = 0, \quad (4.4)$$

where $f(v, m) = \prod_{i=1}^{N_f} (v - m_i)$.

We use the result of section 2.3.2. The condition (2.20) is satisfied at $z = 0$ only if we set all $p_i = 0$. By imposing (2.20) also at $z = \infty$, we have the simple curves

$$v^N + \zeta_1^{N-N_f} z f(v, m_i) = 0, \\ (w - u/z)^{N-N_f} g(z, u_i) + \frac{\zeta_2^N}{z^{N+1}} = 0, \quad (4.5)$$

where $g(z, u_i) = \prod_{i=1}^{N_f} (w - u_i/z)$ and $(N - N_f)u + \sum_{i=1}^{N_f} u_i = 0$.

The link equation (2.24) can be simplified by simple observation that the two equations look very similar: here we take the ansatz that the relation is given by $(w - u/z)v = az^{-1}$. Then after substituting this equation into the second one, we have

$$0 = \left(\frac{a}{z}\right)^{N-N_f} \prod_{i=1}^{N_f} (a/z - v(u_i - u)/z) + \frac{\zeta_2^N}{z^{N+1}} v^N$$

$$\rightarrow 0 = v^N + z \left[\zeta_2^{-N} a^{N-N_f} \prod_{i=1}^{N_f} (u - u_i) \right] \prod_{i=1}^{N_f} \left(v - \frac{a}{u_i - u} \right). \quad (4.6)$$

Then u_i and a can be found as

$$\frac{a}{u_i - u} = m_i, \quad \left[\zeta_2^{-N} a^{N-N_f} \prod_{i=1}^{N_f} (u - u_i) \right] = \zeta_1^{N-N_f}. \quad (4.7)$$

The solution is

$$a^N = (-1)^{N_f} \zeta_1^{N-N_f} \zeta_2^N \prod_{i=1}^{N_f} m_i, \quad u = -\frac{a}{N} \sum_{k=1}^{N_f} \frac{1}{m_k}, \quad u_i - u = \frac{a}{m_i}. \quad (4.8)$$

One can easily check that this solution reduces to the solutions of $N_f = 0$ [9] in the limit $m \rightarrow \infty$, $\zeta_1 \rightarrow 0$ with a fixed. One can also check that this curve is the same as the one in [8] which was derived for $m_1 = \dots = m_{N_f}$.

4.1.2 General massless case

As we argued in section 2, the two curves in the massless case before imposing any constraints are

$$(v - v_0)^{N-N_2} \prod_{i=1}^{N_2} (v - v_i) + \zeta_1^{N-N_1} z v^{N_1} = 0,$$

$$(w - w_0/z)^{N-N_1} \prod_{i=1}^{N_1} (w - w_i/z) + \frac{\zeta_2^{N-N_2}}{z^{N-N_2+1}} w^{N_2} = 0, \quad (4.9)$$

The link equation takes the form

$$w = \frac{1}{z} \left(\frac{av + b}{v - v_0} \right), \quad (4.10)$$

a, b are fixed by familiar condition that combining link equation and the equation of v would give the equation of w . A rather singular limit is when $a = b = v_0 = 0$, then we have to present our link equation in following form:

$$wv = 0. \quad (4.11)$$

We will see that both the general and special equations are useful in finding curves for SQCD.

Generic link equation. Let's first consider the generic link equation and simply substitute the link equation into the equation of w . We get

$$((a-w_0)v+b+v_0w_0)^{N-N_1} \prod_{i=1}^{N_1} ((a-w_i)v+b+w_iv_0) + \frac{\zeta_2^{N-N_2}}{z} (av+b)^{N_2} (v-v_0)^{N-N_2}. \quad (4.12)$$

Comparing this equation with the equation of v to make the above equation vanish, we must have the following conditions:

$$\begin{aligned} v_i &= -b/a, \\ w_i &= r, \quad b + rv_0 = 0, \quad a = w_0. \end{aligned} \quad (4.13)$$

Using the above condition except the $b + rv_0 = 0$, the curve now looks like

$$\begin{aligned} (v-v_0)^{N-N_2} (v+b/w_0)^{N_2} + \zeta^{N-N_1} z v^{N_1} &= 0, \\ (w-w_0/z)^{N-N_1} (w-r/z)^{N_1} + \frac{\zeta^{N-N_2}}{z^{N-N_2+1}} w^{N_2} &= 0, \end{aligned} \quad (4.14)$$

with the following condition from the traceless condition of the eigenvalues of v, w :

$$N_2 b/w_0 = (N - N_2) v_0, \quad N_1 r = -(N - N_1) w_0. \quad (4.15)$$

However, this is consistent with $b + rv_0 = 0$ only if $N_1 = N_2$, assuming v_0, w_0 are nonzero.

Now the parameters $a, b, v_i, w_i = r$ are expressed in terms of v_0 and w_0 . We get a relation between v_0 and w_0 by matching equation (4.12) to equation of v , and we have

$$(v_0 w_0)^{N-N_1} = \left(\frac{N_1}{N} \right)^N \zeta_1^{N-N_1} \zeta_2^{N-N_2}. \quad (4.16)$$

The result is consistent with the following field theory branch. As in section 2, mesons are given as

$$M = \begin{pmatrix} M_1 & L \\ \tilde{L} & M_2 \end{pmatrix}, \quad (4.17)$$

where $M_{1,2}$ is the $N_{1,2} \times N_{1,2}$ mesons and L, \tilde{L} are $N_{1,2} \times N_{2,1}$ mesons. Let us consider the branch where $L = \tilde{L} = 0$ and $M_{1,2}$ have generic vevs. Then, the effective superpotential for $N \neq N_1 + N_2$ is

$$W = (N - N_1 - N_2) \left(\frac{\Lambda^{3N-N_1-N_2}}{\det M_1 \det M_2} \right)^{\frac{1}{(N-N_1-N_2)}} - \frac{c}{N} \text{tr } M_1 \text{tr } M_2. \quad (4.18)$$

The first term is the Affleck-Dine-Seiberg superpotential [34] for $N_1 + N_2 < N$. In the case $N_1 + N_2 > N$, we consider the Seiberg dual [39] of the theory. Then dual quarks have masses M_1 and M_2 and can be integrated out if the rank of the mesons are such that $\text{rank } M = N_1 + N_2$. After integrating out quarks, gaugino condensation generate the above first term. The second term $\text{tr } M_1 \text{tr } M_2$ is the quartic superpotential which is present at the tree level.

The equations of motion requires

$$\left(\frac{\Lambda^{3N-N_1-N_2}}{\det M_1 \det M_2} \right)^{\frac{1}{(N-N_1-N_2)}} M_1^{-1} = \frac{c}{N} \mathbf{1}_{N_1} \text{tr} M_2, \quad (4.19)$$

$$\left(\frac{\Lambda^{3N-N_1-N_2}}{\det M_1 \det M_2} \right)^{\frac{1}{(N-N_1-N_2)}} M_2^{-1} = \frac{c}{N} \mathbf{1}_{N_2} \text{tr} M_1. \quad (4.20)$$

By multiplying these equations by M_1 or M_2 and taking traces, one can see that these equations are consistent only if $N_1 = N_2$. This result is consistent with the above result from the spectral curve. Then, we get

$$M_1 = m_1 \mathbf{1}_{N_1}, \quad M_2 = m_2 \mathbf{1}_{N_2}, \quad (4.21)$$

with a constraint

$$(m_1 m_2)^{N-N_1} = \left(\frac{N}{c N_1 N_2} \right)^{N-2N_1} \Lambda^{3N-2N_1}. \quad (4.22)$$

This precisely matches with (4.16) by identifying $m_1 \propto v_0$ and $m_2 \propto w_0$. Although we have assumed $N_1 + N_2 \neq N$, a similar analysis gives us the same result for $N_1 + N_2 = N$.

Special link equation. Next let's consider the special link equation $wv = 0$. It is easy to see that this is only possible for $N_1 + N_2 \geq N$, because w (v) has at least $N - N_1$ ($N - N_2$) nonzero eigenvalues at the punctures, and the sum of the numbers of the non-zero eigenvalues should not exceed N . The curves consistent with the link equations are

$$\begin{aligned} v^{N-k_1} \left(\prod_{i=1}^{k_1} (v - v_i) + \zeta_1^{N-N_1} z v^{N_1+k_1-N} \right) &= 0, \\ w^{N-k_2} \left(\prod_{i=1}^{k_2} (w - w_i/z) + \frac{\zeta_2^{N-N_2}}{z^{N-N_2+1}} v^{N_2+k_2-N} \right) &= 0, \\ wv &= 0, \end{aligned} \quad (4.23)$$

where $k_1 + k_2 = N$, and $\sum_i v_i = \sum_i w_i = 0$.

This result is consistent with the field theory. Let us neglect the quartic superpotential for the moment. Then, for the $N_1 + N_2 \geq N$ case, there is a branch of the moduli space in the field theory such that $\text{rank} M \leq N$ [39]. This is a classical constraint for $M_j^i = \tilde{q}^i q_j$ since the quarks have at most rank N , and this branch survives quantum mechanically for SQCD with $N_1 + N_2 \geq N$. The maximal possible rank is achieved when $\text{rank} M_1 = k_1$ and $\text{rank} M_2 = k_2$ with $k_1 + k_2 = N$. The inclusion of the quartic superpotential $\text{tr} M_1 \text{tr} M_2$ makes the trace part of these mesons massive and they can be integrated out. Then we get $\text{tr} M_1 = \text{tr} M_2 = 0$. This condition should correspond to $\sum_i v_i = \sum_i w_i = 0$ in the curve. Therefore, v_i and w_i are interpreted as eigenvalues of M_1 and M_2 , respectively.

4.2 A sphere with three irregular singularities

As discussed in section 2, it is easy to write down the curve if one of the Higgs fields is zero. Then the generalized Hitchin system becomes a twisted Higgs bundle.

Let us consider a simple example: Φ_1 has three irregular singularities of type A at $z = 0, 1, \infty$ and $L_1 = \mathcal{O}(-3), L_2 = \mathcal{O}(1)$ as in one of the examples of section 3.2. We set $\Phi_2 = 0$. Then there is no constraint on Φ_1 , and the curve is completely determined by

$$v^N + \sum_{i=2}^N \frac{u_i v^{N-i}}{z^i (z-1)^i} + \frac{\zeta_1^N}{z^{N+1} (z-1)^N} + \frac{\zeta_2^N}{z^N (z-1)^{N+1}} + \frac{\zeta_3^N}{z^{N-1} (z-1)^N} = 0. \quad (4.24)$$

This curve is precisely the same as the Seiberg-Witten curve in [21] after a redefinition $v' = z(z-1)v$.

4.3 Maldacena-Nunez theory

Let's now consider higher rank Maldacena-Nunez theory, and we will find interesting chiral-ring relation. The bundles are $L_1 = L_2 = K^{1/2}$ and there are no punctures. We have shown in section 2 that the link equation describing one branch can be written down as (2.23),

$$w = hv, \quad (4.25)$$

here h is just a constant since it is a section of trivial bundle. The spectral curves are written as

$$\begin{aligned} v^N + \sum f_k v^{N-k} &= 0, \\ w^N + \sum g_k w^{N-k} &= 0, \\ w &= hv. \end{aligned} \quad (4.26)$$

Here f_k and g_k can be expanded using a basis $e_i^{(k)}$ of holomorphic sections of $K^{k/2}$ whose dimension is $d_k = (k-1)(g-1) + \delta_{k,2}$:

$$\begin{aligned} f_k &= \sum_{i=1}^{d_k} v_i^{(k)} e_i^{(k)}, \\ g_k &= \sum_{i=1}^{d_k} u_i^{(k)} e_i^{(k)}. \end{aligned} \quad (4.27)$$

Using the consistency condition of the three equations, we can easily find the chiral ring relation

$$v_i^{(k)} u_j^{(k)} = v_j^{(k)} u_i^{(k)}. \quad (4.28)$$

Another branch may be found by taking the ansatz, e.g.,

$$\begin{aligned} v^N &= f(z), \quad w^N = g(z), \quad wv = h(z), \\ fg &= h^N, \end{aligned} \quad (4.29)$$

so we need to choose the parameters inside f and g to make h holomorphic. This will give us lots of interesting chiral-ring relation.

Finally, we can have the link equation

$$wv = 0. \tag{4.30}$$

In this case, the spectral curves of v and w factorizes as

$$v^{k_1}(v^{N-k_1} + \dots) = 0, \quad w^{k_2}(w^{N-k_2} + \dots) = 0, \tag{4.31}$$

where $k_1 + k_2 = N$ and the moduli fields inside the bracket is arbitrary other than the traceless condition.

5 Conclusion

We developed a general method for finding $\mathcal{N} = 1$ curve for various kinds of theories engineered using M5 branes. We find many dynamical properties of the gauge theories in attempting to write down the curves. It is quite remarkable that we simply start with a mathematically defined spectral curve without using any physical input, and we can recover the highly non-trivial quantum dynamics, such as deformed moduli space, Seiberg-Witten curve, etc. The perfect agreement with the field theory results clearly shows that our method is correct.

Our basic construction follows from the generalized integrability property of the moduli space of twisted Higgs bundle as shown in [26, 27]. It is interesting that there are also connections to integrable system even for $\mathcal{N} = 1$ theory. This rather surprising connection to integrable system definitely deserves further study.

It is interesting that we can probe all kinds of phase structures using the spectral curve, and let's summarize the characteristic feature of the curves in different phases:

- **Non-abelian Coulomb phase:** this phase typically happens at the origin, i.e. when all the moduli are set to zero. If we set all the moduli to be zero, and all the eigenvalues of v and w to be zero, then we can define a $U(1)$ symmetry which does not change the fibre and it is identified as the $U(1)_R$ symmetry. Typical examples are $SU(N)$ SQCD with $N_f = 2N$ theory and Maldacena-Nunez theory.
- **Abelian Coulomb phase:** such phases happens when the genus of the spectral curve is non-zero. Typically there are other moduli, but it is possible that the vacua are discrete. In the field theory, there are massless abelian gauge fields. Typical examples are the theory defined using a sphere with two irregular punctures and one regular puncture, as first studied by Intriligator-Seiberg [7].
- **Higgs phase:** this phase happens when there is a continuous moduli and the genus of the spectral curve is zero. This happens for many SQCD examples. The quartic superpotential is crucially important to match the spectral curve and the deformed moduli space of the field theory.

- **Mass gapped phase:** if there are no moduli in the curve and the genus of the curve is zero, there is no massless fields at low energies. There are discrete vacua in most cases. A typical example is the $\mathcal{N} = 1$ pure super-Yang-Mills.

We found these phases using the spectral curve. It would be interesting to understand these phases using traditional order parameter, such as Wilson loop, 't Hooft loop, surface operators [40] etc. Those extended objects may also have a geometric interpretation in terms of the generalized Hitchin equation and M5 brane construction.

In this paper, we only discuss the $\mathcal{N} = 1$ dynamics relating to the attempt of writing down the curves. It is definitely interesting to extract more physical information out of the curves, such as the position of the singularity where monopoles become massless, etc.

We have only studied some particular examples in this paper to show how our construction works nicely. It is interesting to work out more examples such as new theories defined using three punctured sphere, and more general theory including $\mathcal{N} = 1$ Argyres-Douglas theories. The spectral curve construction presented in this paper is a very important tool, and we believe that our methods provide an extremely powerful way of studying gauge dynamics of four dimensional $\mathcal{N} = 1$ theory.

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