

On exact symmetries and massless vectors in holographic flows and other flux vacua

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ABSTRACT: We analyze the isometries of Type IIB flux vacua based on the Papadopoulos-Tseytlin ansatz and identify the related massless bulk vector fields. To this end we devise a general ansatz, valid in any flux compactification, for the fluctuations of the metric and p -forms that diagonalizes the coupled equations. We then illustrate the procedure in the simple case of holographic flows driven by the RR 3-form flux only. Specifically we study the fate of the isometries of the Maldacena-Núñez solution associated to wrapped D5-branes.

KEYWORDS: Flux compactifications, Gauge-gravity correspondence, AdS-CFT Correspondence, Superstring Vacua

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1 Introduction

Flux vacua [1] find many interesting applications in String Theory, ranging from holographic flows dual to non (super)conformal boundary theories [2–8] to moduli stabilization in phenomenologically viable models with open and unoriented strings and otherwise [9]. More recently unexpected implications of the AdS/CFT correspondence [10–13] in condensed matter physics [14, 15], astrophysics [16, 17] and gravity at a Lifschitz point [18] have attracted a lot of attention.

Due to the presence of fluxes, fields belonging to different sectors tend to mix with one another, compatibly with the residual (super)symmetry. Resolving the mixing and finding the spectrum of excitations is extremely laborious [19–21] as witnessed by the enormous effort needed to accomplish the task for the metric and active¹ scalar modes in holographic flows described by the Papadopoulos-Tseytlin (PT) ansatz [22, 23, 26, 27]. Our aim is to

¹By *active* we indicate those scalars with a non trivial profile in the background. All the other (pseudo)scalars are said to be *inert*.

extend this kind of analysis to the vector sector [19–21, 28, 29]. To this end, we will start by studying the fate of bulk symmetries of the Type IIB supergravity solutions. Although we will mostly adopt a 10-d perspective, we will also present the 5-d viewpoint, that has a more direct applicability in Holographic Renormalization [20, 21, 30–36] and Holographic QCD [37–41].

Symmetries can be divided into three classes

- Exact Symmetries: not only the metric admits Killing vectors but also fluxes are invariant [19–21, 28, 29].
- Partially Broken Symmetries: Metric invariant, some fluxes are not
- Broken Symmetries: Metric and fluxes only asymptotically invariant [22, 23, 42, 43]

The PT ansatz [5] enjoys $SU(2) \times \widetilde{SU(2)}$ isometry for arbitrary choices of the ‘radial’ functions. On the contrary, the $U(1)_R$, associated to shifts of the coordinate ψ , is broken except for very special cases. The breaking is spontaneous from the bulk viewpoint, i.e. the would-be massless vector field becomes massive after ‘eating’ a Goldstone boson. The Stückelberg formalism for the gauging of axionic shift symmetries is particularly convenient in this respect [19–23, 28, 29, 42, 43]. Except for some very general remarks, we will neither have much further to say about broken symmetries and massive vectors nor discuss at all massless vectors related to harmonic forms [7] and to probe branes [37]. The latter give rise to chiral ‘flavor’ symmetries (breaking) and mesons. The former to baryonic symmetries.

The plan of the paper is as follows. In section 2 we describe the 5-d Lagrangian governing the dynamics of gauge fields and their mixing with would-be axions. Next, in section 3, we check that the PT ansatz indeed admits full $SU(2) \times \widetilde{SU(2)}$ symmetry, in that not only the metric but also the other background field (strengths) are invariant. In section 4, we identify the bulk vector fields that remain massless by means of an ansatz for the fluctuations of the metric and p -forms, that diagonalizes the coupled equations. Finally, in section 5 we illustrate the procedure in the simple case of holographic flows and other flux vacua with F_3 only, which are invariant under Ω . Specifically we study the MN solution [4] associated to wrapped D5-branes, i.e. ‘fractional’ D3-branes. Our conclusions and summary are contained in section 6.

2 Vector fields in holographic renormalization

The 5-d Lagrangian describing vector fields and their possible mixing with inert (pseudo)scalars e.g. axions reads [19–21, 28, 29]

$$\mathcal{L}_{v-a} = -\frac{1}{4}\mathcal{K}_{ij}F_{\mu\nu}^i F^{\mu\nu j} + \frac{1}{2}h_{AB}(\partial_\mu\beta^A - M_i^A A_\mu^i)(\partial^\mu\beta^B - M_j^B A^{\mu j}) \quad (2.1)$$

The gauge kinetic function \mathcal{K}_{ij} and the axion metric h_{AB} may depend on the (active) scalars, while, in the present parametrization M_i^A are constant mass parameters.² The

²In general, after gauging some isometry of the scalar metric $\mathcal{G}_{ab}(\phi)$, covariant derivatives are given by $D\phi^a = \partial\phi^a + \mathcal{K}_i^a(\phi)A^i$, where \mathcal{K}_i^a are Killing vectors i.e. $\nabla_{\mathcal{G}}^a\mathcal{K}_i^b(\phi) + \nabla_{\mathcal{G}}^b\mathcal{K}_i^a(\phi) = 0, \forall i$. Here we focus on the gauging of axionic shifts $\delta\beta^A = M_i^A\alpha_0^i$.

Lagrangian is invariant under gauge transformations of the form

$$\delta A^i = \partial \alpha^i \quad , \quad \delta \beta^A = M_i^A \alpha^i. \quad (2.2)$$

The square mass matrix

$$\mathcal{M}_{ij}^2(\phi) = h_{AB} M_i^A M_j^B \quad (2.3)$$

is semi-positive definite. Zero eigenvalues correspond to exactly massless vectors, associated to isometries or to harmonic forms present in the background solution. Non-zero eigenvalues correspond to massive vectors and broken symmetries. Introducing gauge invariant combinations for the latter

$$\mathcal{B}_\mu^{\hat{i}} = A_\mu^{\hat{i}} - (\mathcal{M}_{\pm 0}^{-2})^{\hat{i}\hat{j}} M_j^A h_{AB} \partial_\mu \beta^B \quad (2.4)$$

and denoting by $A_\mu^{i_0}$ the former yields

$$\mathcal{L}_v = -\frac{1}{4} \mathcal{K}_{i_0 j_0}(\phi) F_{\mu\nu}^{i_0} F^{\mu\nu j_0} - \frac{1}{4} \mathcal{K}_{\hat{i}\hat{j}}(\phi) \mathcal{F}_{\mu\nu}^{\hat{i}} \mathcal{F}^{\mu\nu \hat{j}} + \frac{1}{2} \mathcal{M}_{\hat{i}\hat{j}}^2(\phi) \mathcal{B}_\mu^{\hat{i}} \mathcal{B}^{\mu \hat{j}} \quad (2.5)$$

After diagonalization, one finds a collection of decoupled vector bosons³ each described by

$$\mathcal{L}_v = -\frac{1}{4} \mathcal{K}(\phi) \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} + \frac{1}{2} \mathcal{M}^2(\phi) \mathcal{A}_\mu \mathcal{A}^\mu \quad (2.6)$$

where $\mathcal{K}(\phi)$ is the resulting gauge kinetic function and $\mathcal{M}(\phi)$ is the possibly vanishing mass.

Putting the kinetic term in canonical form one has [19]

$$\mathcal{M}_{\text{eff}}^2 = \frac{1}{2} \frac{\mathcal{K}''}{\mathcal{K}} + \frac{A' \mathcal{K}'}{\mathcal{K}} - \frac{1}{4} \left(\frac{\mathcal{K}'}{\mathcal{K}} \right)^2 + \frac{\mathcal{M}^2}{\mathcal{K}} \quad (2.7)$$

where primes denote derivatives wrt the holographic radial variable. Clearly $\mathcal{M}^2 = 0$ for bulk vector fields associated to unbroken boundary currents (exact global symmetries), while $\mathcal{M}^2 \neq 0$ for broken symmetries. In most if not all known cases [19]

$$\mathcal{M}_{\text{eff}}^2 = -2A'' \quad (2.8)$$

Quite remarkably but without a clear explanation, the above relation has been verified for (transverse) vectors fields in all known solutions: Coulomb branch flow with $\text{SO}(6) \rightarrow \text{SO}(4) \times \text{SO}(2)$ [19–21, 28, 29], GPPZ flow $\text{SO}(6) \rightarrow \text{SO}(3) \times \text{U}(1)_R$ [19–21], KT (and partially KS) solution with (broken) $\text{U}(1)_R$ R-symmetry [22–25].

3 Field equations and PT ansatz

To set the stage for our analysis, let us now briefly recall Papadopoulos and Tseytlin (PT) ansatz for flux vacua in Type IIB supergravity and its symmetries. The main motivation behind PT ansatz is to identify a subset of fields that form a consistent truncation of Type IIB supergravity and allow to study flux vacua with reduced or no supersymmetry at all.

³At the quadratic level this is true for non-abelian symmetries, too.

The reader familiar with Type IIB supergravity and the PT ansatz can skip the following part and go directly to section 3.2.

In the Einstein frame, Type IIB supergravity equations read

$$R_{MN} = \frac{1}{2}\partial_M\phi\partial_N\phi + \frac{1}{2}e^{2\phi}\partial_M\chi\partial_N\chi + \frac{1}{96}\hat{G}_{MPQKL}\hat{G}_N{}^{PQKL} + \frac{1}{4}e^\phi\hat{F}_{MPQ}\hat{F}_N{}^{PQ} + \frac{1}{4}e^{-\phi}H_{MPQ}H_N{}^{PQ} - \frac{1}{48}g_{MN}[e^\phi\hat{F}_{LPQ}\hat{F}^{LPQ} + e^{-\phi}H_{LPQ}H^{LPQ}] \quad (3.1)$$

$$\nabla^2\phi = e^{2\phi}\partial_M\chi\partial^M\chi + \frac{1}{12}e^\phi\hat{F}_{LMN}\hat{F}^{LMN} - \frac{1}{12}e^{-\phi}H_{LMN}H^{LMN} \quad (3.2)$$

$$\nabla^M(e^{2\phi}\partial_M\chi) = -\frac{1}{6}e^\phi H_{LMN}F^{LMN} \quad (3.3)$$

$$\nabla^M(e^\phi\hat{F}_{MNP}) = \frac{1}{6}G_{NPQRS}H^{QRS} \quad (3.4)$$

$$\nabla^M(e^{-\phi}H_{MNP} - e^\phi\chi\hat{F}_{MNP}) = -\frac{1}{6}G_{NPQRS}F^{QRS} \quad (3.5)$$

$$\hat{G}_{M_1\dots M_5} = \frac{1}{120}\varepsilon_{M_1\dots M_5M_6\dots M_{10}}\hat{G}^{M_6\dots M_{10}} \quad (3.6)$$

where

$$\hat{F}_3 = F_3 - \chi H_3 \quad \hat{F}_5 = F_5 + B_2 F_3 \quad (3.7)$$

with

$$F_1 = d\chi \quad F_3 = dC_2 \quad F_5 = dA_4 \quad H_3 = dB_2 \quad (3.8)$$

3.1 PT ansatz

The consistent truncation of 10-d Type IIB supergravity found by Papadopoulos and Tseytlin is based on the following ansatz.

- Metric

$$ds_{10}^2 = e^{2p(u)-x(u)} \left(e^{2A(u)} dx \cdot dx + N_5 du^2 \right) + N_5 \left[e^{x(u)+g(u)} (e_1^2 + e_2^2) + \frac{1}{4}e^{x(u)-g(u)} (\tilde{\omega}_1^2 + \tilde{\omega}_2^2) + \frac{1}{4}e^{-6p(u)-x(u)}\tilde{\omega}_3^2 \right] \quad (3.9)$$

where u denotes the holographic radial variable, the functions A, p, x, g depend on u , and the ‘invariant’ one-forms read

$$\begin{aligned} e_1 &= d\theta, & e_2 &= -\sin\theta d\varphi \\ \tilde{\omega}_1 &= \omega_1 - a(u)e_1, & \tilde{\omega}_2 &= \omega_2 - a(u)e_2, & \tilde{\omega}_3 &= \omega_3 - \cot\theta e_2 \\ \omega_1 &= \sin\psi \sin\tilde{\theta}d\tilde{\varphi} + \cos\psi d\tilde{\theta}, & \omega_2 &= -\sin\psi d\tilde{\theta} + \cos\psi \sin\tilde{\theta}d\tilde{\varphi} \\ \omega_3 &= d\psi + \cos\tilde{\theta}d\tilde{\varphi} \end{aligned} \quad (3.10)$$

- NS-NS dilaton and R-R axion

$$\phi = \phi(u) \quad , \quad \chi = 0 \quad (3.11)$$

- NS-NS 3-form

$$H_3 = h_2(u)\tilde{\omega}_3 \wedge (\omega_1 \wedge e_1 + \omega_2 \wedge e_2) + du \wedge [h'_1(u)(\omega_1 \wedge \omega_2 + e_1 \wedge e_2) + h'_2(u)(\omega_1 \wedge e_2 - \omega_2 \wedge e_1) + h'_3(u)(-\omega_1 \wedge \omega_2 + e_1 \wedge e_2)] \quad (3.12)$$

where ' denotes derivative wrt u . Since $dH_3 = 0$, one has $H_3 = dB_2$ with

$$B_2 = h_1(u)(e_1 \wedge e_2 + \omega_1 \wedge \omega_2) + h_2(u)(\omega_1 \wedge e_2 - \omega_2 \wedge e_1) + h_3(u)(-\omega_1 \wedge \omega_2 + e_1 \wedge e_2) \quad (3.13)$$

- R-R 3-form

$$F_3 = \frac{N_5}{4} \{ \tilde{\omega}_3 \wedge [(\omega_1 \wedge \omega_2 + e_1 \wedge e_2) - b(u)(\omega_1 \wedge e_2 - \omega_2 \wedge e_1)] + b'(u)du \wedge (\omega_1 \wedge e_1 + \omega_2 \wedge e_2) \} \quad (3.14)$$

since $dF_3 = 0$, one has $F_3 = dC_2$ with

$$C_2 = \frac{N_5}{4} [\psi(e_1 \wedge e_2 + \omega_1 \wedge \omega_2) + b(u)(\omega_1 \wedge e_1 + \omega_2 \wedge e_2) + \cos \theta \cos \tilde{\theta} d\varphi d\tilde{\varphi}] \quad (3.15)$$

- R-R self-dual 5-form

$$\hat{G}_5 = \mathcal{G}_5 + *\mathcal{G}_5 \quad \text{with} \quad \mathcal{G}_5 = K(u)e_1 \wedge e_2 \wedge \omega_1 \wedge \omega_2 \wedge \omega_3 \quad (3.16)$$

where $\hat{G}_5 = G_5 + B_2 \wedge F_3$ with $G_5 = dC_4$.

Integrating $\hat{G}_5 = dC_4 + B_2 \wedge F_3$ over a closed 'internal' 5-d section at fixed u yields

$$K(u) = N_3 + 2N_5[h_1(u) + b(u)h_2(u)] \quad (3.17)$$

that allows to eliminate K in terms of b, h_1, h_2 and the integers N_3 and N_5 (i.e. number of D3- and D5-branes in the UV). The Bianchi identity for H_3 yields

$$dh_3 = \frac{(e^{2g} + 2a^2 + e^{-2g}a^4 - e^{-2g})dh_1 + 2a(1 - e^{-2g} + a^2e^{-2g})dh_2}{e^{2g} + (1 - a^2)^2e^{-2g} + 2a^2} \quad (3.18)$$

that allows one to eliminate h_3 , too.

The remaining scalar fields $\{p, x, g, a, b, \phi, h_1, h_2\}$ are governed by a 5-d effective Lagrangian with (almost) diagonal metric \mathcal{G}_{ab} (only h_1 and h_2 mix with each other) and a complicated potential that play no role in our analysis.

3.2 Killing vectors

For arbitrary choices of the functions $x, g, p, a, \phi, b, h_1, h_2, (h_3, K)$ of the radial coordinate u , the metric and p -forms are invariant under $SU(2) \times \widehat{SU(2)}$ isometry generated by the six

Killing vectors ξ_a

$$\begin{aligned}
 \xi_+ &\equiv \xi_1 = e^{i\varphi} (0, 0, 0, 0, 0, -1, 0, -i \cot \theta, 0, i \csc \theta) \\
 \xi_- &\equiv \xi_2 = e^{-i\varphi} (0, 0, 0, 0, 0, 1, 0, -i \cot \theta, 0, i \csc \theta) \\
 \xi_3 &\equiv \xi_3 = (0, 0, 0, 0, 0, 0, 0, 1, 0, 0) \\
 \tilde{\xi}_+ &\equiv \xi_4 = e^{i\tilde{\varphi}} (0, 0, 0, 0, 0, 0, -1, 0, -i \cot \tilde{\theta}, i \csc \tilde{\theta}) \\
 \tilde{\xi}_- &\equiv \xi_5 = e^{-i\tilde{\varphi}} (0, 0, 0, 0, 0, 0, 1, 0, -i \cot \tilde{\theta}, i \csc \tilde{\theta}) \\
 \tilde{\xi}_3 &\equiv \xi_6 = (0, 0, 0, 0, 0, 0, 0, 1, 0)
 \end{aligned} \tag{3.19}$$

Notice that ξ_a have only components in the internal directions, i.e. $\xi^M = \delta_i^M \xi^i$ with $M = 1, \dots, 10$ and $i = 6, \dots, 10$, and the contra-variant components displayed above only depend on the internal ‘angular’ variables, i.e. $\partial_\mu \xi^M = 0$ with $\mu = 1, \dots, 5$. Although the metric does not mix the angular variables with the non-compact variables, after lowering the indices the components of the Killing vector acquire a u dependence due to warping. Clearly PT preserves Poincaré symmetry in the ‘boundary’ space-time directions, too.

It is easy to check that also the following two-forms

$$e_1 \wedge e_2 = -\sin \theta d\theta \wedge d\varphi = d \cos \theta \wedge d\varphi \tag{3.20}$$

$$\omega_1 \wedge \omega_2 = +\sin \tilde{\theta} d\tilde{\theta} \wedge d\tilde{\varphi} = -d \cos \tilde{\theta} \wedge d\tilde{\varphi} \tag{3.21}$$

$$\omega_1 \wedge e_1 + \omega_2 \wedge e_2 = (\sin \psi \sin \tilde{\theta} d\tilde{\varphi} + \cos \psi d\tilde{\theta}) \wedge d\theta + \sin \theta (\sin \psi d\tilde{\theta} - \cos \psi \sin \tilde{\theta} d\tilde{\varphi}) \wedge d\varphi \tag{3.22}$$

$$\omega_1 \wedge e_2 - \omega_2 \wedge e_1 = -\sin \theta (\sin \psi \sin \tilde{\theta} d\tilde{\varphi} + \cos \psi d\tilde{\theta}) \wedge d\varphi + (\sin \psi d\tilde{\theta} - \cos \psi \sin \tilde{\theta} d\tilde{\varphi}) \wedge d\theta \tag{3.23}$$

as well as the one-form

$$\tilde{\omega}_3 = d\psi + \cos \tilde{\theta} d\tilde{\varphi} + \cos \theta d\varphi \tag{3.24}$$

are $SU(2) \times \widetilde{SU(2)}$ invariant, in the sense that $\mathcal{L}_\xi(\dots) = 0$.⁴

As a consequence, all background field-strengths are invariant i.e.

$$\mathcal{L}_{\xi_a} H_3 = 0, \quad \mathcal{L}_{\xi_a} F_3 = 0, \quad \mathcal{L}_{\xi_a} G_5 = 0 \tag{3.25}$$

Moreover, since

$$\mathcal{L}_{\xi_a} B_2 = 0 \tag{3.26}$$

one also has

$$\mathcal{L}_{\xi_a} \hat{F}_3 = 0, \quad \mathcal{L}_{\xi_a} \hat{G}_5 = 0 \tag{3.27}$$

while, in the chosen gauge,

$$\mathcal{L}_{\xi_a} C_2 \neq 0 \tag{3.28}$$

⁴Lie derivatives act according to

$$\mathcal{L}_v T_{M_1 \dots M_p}^{N_1 \dots N_q} = v^L \partial_L T_{M_1 \dots M_p}^{N_1 \dots N_q} + \sum_i T_{M_1 \dots L \dots M_p}^{N_1 \dots N_q} \partial_{M_i} v^L - \sum_j T_{M_1 \dots M_p}^{N_1 \dots L \dots N_q} \partial_L v^{N_j}.$$

By a change of gauge $\delta C_2 = d\lambda_1^C$ we expect

$$\mathcal{L}_{\xi_a} C'_2 = 0 \tag{3.29}$$

Finally, though rather obviously, $\mathcal{L}_{\xi_a} \phi = 0$, $\mathcal{L}_{\xi_a} \chi = 0$.

While admitting $SU(2) \times \widetilde{SU(2)}$ isometry, the PT ansatz generically ‘breaks’ the abelian isometry associated to the vector field $\hat{\xi}^M \partial_M = \partial/\partial\psi$. The latter may be identified with the ‘anomalous’ $U(1)$ R-symmetry of the dual $\mathcal{N} = 1$ SYM theory on the boundary. In the bulk it is broken to Z_{2N} by the background 3-form and 5-form and then broken to Z_2 by non-perturbative effects (string or D-brane instantons, depending on the choice of wrapped branes). As discussed in section 2, the bulk counterpart of the anomalous divergence of the R-symmetry current is a Higgs or rather Stückelberg mechanism [42], whereby a would-be massless vector field eats an axion and becomes massive. This effect has been studied in some details in [22, 23] in the case of the KT background (a singular ‘relative’ of KS solution), confirming the expected value for the ‘mass’ predicted by [19]. For the case of MN solution, some considerations about the required axion can be found in [4, 43].

3.3 Discrete symmetries and closed subsectors

There are two Z_2 symmetries and their product that allow to truncate Type IIB field equations in $D = 10$ to closed sets of fields mixing only with one another. The first is world-sheet parity Ω . The second is fermion parity in the L-moving sector $(-)^{FL}$, which is S-dual to Ω i.e. $(-)^{FL} = S\Omega S^{-1}$, where S exchanges F_3 and H_3 and sends $\tau = \chi + ie^{-\phi}$ to $-1/\tau$. The Einstein-frame metric and the dilaton are invariant under both Ω and $(-)^{FL}$, while the action on the other bosonic fields is

	Ω	$(-)^{FL}$
χ	–	–
B_2	–	+
C_2	+	–
A_4	–	–

Later on we will focus on the subsector invariant under Ω . For the PT ansatz, this means

$$h_1 = h_2 = 0 \rightarrow h_3 = K = 0 \tag{3.30}$$

MN solution for wrapped D5-branes [4] belongs to this class, i.e. it is invariant under Ω . Its dual wrapped NS5-brane solution belongs to the class invariant under $(-)^{FL}$. Standard $AdS_5 \times S^5$, i.e. near-horizon D3-branes, is invariant under $(-)^{FL}\Omega$. Finally KS and KT solutions (related to the conifold) do not preserve any of the above discrete symmetries and are thus more involved to study [26, 27].

4 Exact symmetries and massless vectors

In this section, we would like to discuss the fate of the $SU(2) \times \widetilde{SU(2)}$ isometry, that should correspond to the global ‘flavor’ symmetry of the boundary theory, possibly acting

trivially on the lowest states relevant in the deep IR. Even if from the vantage point of the holographic duality, the presence of this isometry might be annoying, the analysis is quite general and applies to any isometry in any flux compactification.

We will find that the Killing vectors generating $\widetilde{\text{SU}(2)}$ are associated to truly massless vectors in the bulk that correspond to an exact global ‘flavor’ symmetry of any solution based on PT ansatz. For the first $\text{SU}(2)$ factor the situation is subtler, at least in the case of MN solution [4].

First of all notice that invariance of the metric under isometry generated by a Killing vector ξ^M reads

$$\mathcal{L}_\xi g_{MN} \equiv \nabla_M \xi_N + \nabla_N \xi_M = 0 \tag{4.1}$$

that implies

$$\nabla_M \xi^M = 0 \tag{4.2}$$

as well as

$$\nabla_M \xi_N = \frac{1}{2}(\nabla_M \xi_N - \nabla_N \xi_M) \tag{4.3}$$

and

$$\nabla_L \nabla_M \xi_N = -R_{MNLK} \xi^K \tag{4.4}$$

Invariance under diffeomorphisms suggests the existence of a trivial massless zero-mode for the metric fluctuations

$$\delta_{\text{diff}} g_{MN} = \nabla_M \beta_N + \nabla_N \beta_M \tag{4.5}$$

Taking $\beta_M = \alpha(x) \xi_M$

$$\delta_{\text{diff}} g_{MN} = \xi_N \nabla_M \alpha + \xi_M \nabla_N \alpha \tag{4.6}$$

It suggests an ansatz for the metric fluctuations of the form

$$\delta_{\text{phys}} g_{MN} = -\xi_N A_M - \xi_M A_N \tag{4.7}$$

with $\xi^M A_M = 0$ (i.e. $\delta g^L_L = 0$) and $\mathcal{L}_\xi A_M = 0$, since $A_M = A_M(x)$ is to be independent of the five internal coordinates the Killing vectors act on. Gauge invariance under $\delta A_M^{(0)} = -\nabla_M \alpha$ would then result from general covariance and should imply massless-ness of the vector field A_M . However, due to the presence of fluxes in the background, the story is not so simple. The metric fluctuations mix with p -form fluctuations, which we turn now our attention onto.

Let us then consider the general case of an n -form X_n , whose background $(n+1)$ -form field strength $Y_{n+1} = dX_n$ is invariant under some isometry generated by a Killing vector ξ

$$\mathcal{L}_\xi Y_{n+1} = i_\xi dY_{n+1} + d(i_\xi Y_{n+1}) = 0 \tag{4.8}$$

Thanks to Bianchi identity $dY_{n+1} = 0$ one has (locally)

$$i_\xi Y_{n+1} = dZ_{n-1}^\xi \tag{4.9}$$

where Z_{n-1}^ξ is a $(n-1)$ -form defined up to an exact form $\delta Z_{n-1}^\xi = dW_{n-2}$.

Under a diffeomorphism generated by $v^M = \alpha \xi^M$

$$\delta_{\text{Diff}} X_n = \alpha i_\xi Y_{n+1} + d(\alpha i_\xi X_n) = \alpha dZ_{n-1}^\xi + d(\alpha i_\xi X_n) = d\alpha \wedge Z_{n-1}^\xi + d[\alpha(i_\xi X_n - Z_{n-1}^\xi)] \quad (4.10)$$

The last term can be cancelled by a gauge transformation of the n -form X_n . This suggests that the correct ansatz for the ‘massless’ vector A_1 associated to the coupled fluctuations of the metric and n -form X_n along ξ be of the form

$$\delta X_n = A_1 \wedge Z_{n-1}^\xi \quad (4.11)$$

In this way gauge invariance under $\delta A_1 = d\alpha$ would not only be a consequence of general covariance but also of the n -form gauge invariance. For the fluctuations of the $(n+1)$ -form field-strength Y_{n+1} one then finds

$$\delta Y_{n+1} = dA \wedge Z_{n-1}^\xi - A \wedge dZ_{n-1}^\xi = dA \wedge Z_{n-1}^\xi - A_1 \wedge i_\xi Y_{n+1} \quad (4.12)$$

In principle the procedure applies to any background n -form in PT or even more general flux vacua. The analysis can be performed in quite general terms but it drastically simplifies in backgrounds where $F_5 = 0, H_3 = 0, F_1 = 0$, thanks to invariance under world-sheet parity Ω , or else where $F_5 = 0, F_3 = 0, F_1 = 0$, thanks to invariance under $(-)^{F_L}$. In both cases mixing between C_2 and B_2 are excluded, and one can safely set $A_4 = 0$ and even $\delta\phi = 0$, as we will see.

Henceforth we will focus on the sub-sector invariant under Ω i.e. C_2, g, ϕ and set B_2, A_4, χ to zero both in the background and in the fluctuations.

4.1 Ω invariant massless vectors (i.e. $F_5 = 0$ and $H_3 = 0$)

Taking into account that $\mathcal{L}_\xi F_3 = d(i_\xi F_3) = 0$, for any exact Killing vector, one can locally write

$$i_\xi F_3 = d\mu_1^\xi \quad (4.13)$$

that suggests the following combined ansatz for the physical fluctuations

$$\delta g^{MN} = A^M \xi^N + A^N \xi^M, \quad \delta C_{MN} = A_M \mu_N^\xi - A_N \mu_M^\xi \quad (4.14)$$

or, equivalently, for the latter $\delta C_2 = A_1 \wedge \mu_N^\xi$ so that

$$\delta F_3 = d\delta C_2 = dA \wedge \mu_1^\xi - A \wedge i_\xi F_3 \quad (4.15)$$

Setting $\delta B_2 = 0, \delta\chi = 0, \delta A_4 = 0$ as well as $\delta\phi = 0, \delta g_{\mu\nu} = 0, \delta g_{ij} = 0$ one has $g^{MN} \delta g_{MN} = 0$, i.e. $\delta\sqrt{||g||} = 0$.

Moreover, $\delta g_{MN} = -A_M \xi_N - A_N \xi_M$ so that

$$\delta g_{\mu i} = -A_\mu \xi_i = \delta g_{i\mu} \quad (4.16)$$

while

$$\delta C_{\mu i} = A_\mu \mu_i^\xi = -\delta C_{i\mu} \quad (4.17)$$

and

$$\begin{aligned}
 \delta F_{ijk} &= 0 \\
 \delta F_{\mu\nu\rho} &= 0 \\
 \delta F_{\mu ij} &= -A_\mu \left(\partial_i \mu_j^\xi - \partial_j \mu_i^\xi \right) \\
 \delta F_{\mu\nu i} &= (\partial_\mu A_\nu - \partial_\nu A_\mu) \mu_i^\xi - \left(A_\mu \partial_\nu \mu_i^\xi - A_\nu \partial_\mu \mu_i^\xi \right)
 \end{aligned} \tag{4.18}$$

4.1.1 Dilaton equation (consistency check)

Let us first check that it be consistent to set $\delta\phi = 0$. Using the 3-form ansatz, one finds

$$\begin{aligned}
 \delta F^2 &= \delta F_{LMN} F^{LMN} + 3 F_{LMN} \delta g^{LP} F_P^{MN} \\
 &= F^{ijk} \delta F_{ijk} + 3 F^{ujk} \delta F_{ujk} + 6 F_{ujk} \delta g^{ui} F_i^{jk} \\
 &= -3 A_u F^{ujk} \left(\partial_j \mu_k^\xi - \partial_k \mu_j^\xi \right) + 6 \delta g^{ui} F_{ujk} F_i^{jk}
 \end{aligned} \tag{4.19}$$

For a solution of the PT kind one has $F_{ujk} F_i^{jk} = 0$, $i = 6, \dots, 10$. Also one obtains $F^{ujk} \left(\partial_j \mu_k^\xi - \partial_k \mu_j^\xi \right) = 0$ for each one of the six Killing vectors ξ_a . Therefore $\delta F^2 = 0$, consistently with the ansatz $\delta\phi = 0$.

4.1.2 3-form equation

Let us focus on the 3-form equation

$$\begin{aligned}
 \delta \left(\nabla_M \left(\sqrt{g} e^\phi F^{MNP} \right) \right) &= \frac{1}{\sqrt{g}} \partial_M \left(\sqrt{g} e^\phi \left[\delta g^{ML} F_L^{NP} + \delta g^{NL} F^M{}_L{}^P \right. \right. \\
 &\quad \left. \left. + \delta g^{PL} F^{MN}{}_L + g^{ML} g^{NK} g^{PQ} \delta F_{LKQ} \right] \right)
 \end{aligned} \tag{4.20}$$

Decomposing into space-time ($\nu = 1, \dots, 5$) and internal indices ($j = 6, \dots, 10$) one has

- Equations $N = \nu$, $P = \rho$

$$\begin{aligned}
 \delta \left(\nabla_M \left(\sqrt{g} e^\phi F^{M\nu\rho} \right) \right) &\equiv \frac{1}{\sqrt{g}} \partial_M \left(\sqrt{g} e^\phi \left[\delta g^{ML} F_L{}^{\nu\rho} + \delta g^{\nu L} F^M{}_L{}^\rho \right. \right. \\
 &\quad \left. \left. + \delta g^{\rho L} F^{M\nu}{}_L + g^{ML} g^{\nu K} g^{\rho Q} \delta F_{LKQ} \right] \right) = 0
 \end{aligned} \tag{4.21}$$

Keeping only non-vanishing components yields

$$\frac{1}{\sqrt{g}} \partial_M \left(\sqrt{g} e^\phi \left[\delta g^{\nu l} F^M{}_l{}^\rho + \delta g^{\rho L} F^{M\nu}{}_L + g^{ML} g^{\nu k} g^{\rho\mu} \delta F_{Lk\mu} \right] \right) = 0 \tag{4.22}$$

Plugging the ansatz one eventually finds

$$e^\phi \partial_i \left\{ \sqrt{\hat{g}} \left[A^\nu \xi^l F^i{}_l{}^u \delta_u^\rho + A^\rho \xi^l \delta_u^\nu F^{iu}{}_l + g^{il} g^{\nu\lambda} g^{\rho\sigma} \left(\mu_l^\xi f_{\lambda\sigma} - A_\lambda \partial_\sigma \mu_l^\xi + A_\sigma \partial_\lambda \mu_l^\xi \right) \right] \right\} = 0 \tag{4.23}$$

where $f_{MN} = \partial_M A_N - \partial_N A_M$ and $\sqrt{\hat{g}}$ denote the dependence of \sqrt{g} on the internal coordinates. Finally, using $i_\xi F_3 = d\mu_\xi$ one arrives at the following constraint for μ_ξ

$$0 = \partial_i \left(\sqrt{\hat{g}} \mu^i \right) f^{\nu\rho} \Rightarrow \nabla_M \mu^M = 0 \quad (4.24)$$

One can check that this is satisfied for all μ_ξ in any background of the PT kind. Anyway, we expect that it should always be possible to satisfy the constraint by adding to μ^ξ an exact form $d\eta^\xi$, where η^ξ is an appropriate function.

- Equations for $N = i, P = j$

$$\delta \left(\nabla_M \left(\sqrt{g} e^\phi F^{Mij} \right) \right) = \frac{1}{\sqrt{g}} \partial_M \left(\sqrt{g} e^\phi \left[\delta g^{ML} F_L^{ij} + \delta g^{iL} F^M_{L^j} + \delta g^{jL} F^M_{L^i} + g^{ML} g^{iK} g^{jQ} \delta F_{LKQ} \right] \right) = 0 \quad (4.25)$$

Keeping only non-vanishing components yields

$$\frac{1}{\sqrt{g}} \partial_M \left(\sqrt{g} e^\phi \left[\delta g^{ML} F_L^{ij} + \delta g^{iL} F^M_{L^j} - \delta g^{jL} F^M_{L^i} + g^{ML} g^{iK} g^{jL} \delta F_{Lkl} \right] \right) = 0 \quad (4.26)$$

Plugging the ansatz one eventually finds

$$\begin{aligned} & \frac{e^\phi}{\sqrt{g}} A^u \partial_m \left\{ \sqrt{\hat{g}} \left[\xi^m F_u^{ij} + \xi^i F_u^{mj} - \xi^j F_u^{mi} \right] \right\} \\ & + \frac{1}{\sqrt{g}} \partial_\mu \left\{ e^\phi \sqrt{g} \left[A^\mu \xi^k F_k^{ij} - A^\mu (\nabla^i \mu_\xi^j - \nabla^j \mu_\xi^i) \right] \right\} = 0 \end{aligned} \quad (4.27)$$

that is satisfied after using $\mathcal{L}_\xi F_3 = 0$ i.e. $i_\xi F_3 = d\mu_\xi$, $\nabla_M \xi^M = 0$ and the background 3-form equation $\partial_m (\sqrt{\hat{g}} F_u^{mj}) = 0$.

- Equations for $N = \nu, P = l$

$$\delta \left(\nabla_M \left(\sqrt{g} e^\phi F^{M\nu l} \right) \right) = \frac{1}{\sqrt{g}} \partial_M \left(\sqrt{g} e^\phi \left[\delta g^{ML} F_L^{\nu l} + \delta g^{\nu L} F^M_{L^l} + \delta g^{lL} F^M_{L^\nu} + g^{ML} g^{\nu K} g^{lQ} \delta F_{LKQ} \right] \right) = 0 \quad (4.28)$$

Keeping only non-zero components yields

$$\begin{aligned} 0 &= \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} e^\phi \left[\delta g^{\mu k} F_k^{\nu l} + \delta g^{\nu k} F^{\mu}_{k^l} + g^{\mu\rho} g^{\nu\lambda} g^{lj} \delta F_{\rho\lambda j} \right] \right) \\ &+ e^\phi \partial_i \left(\delta g^{\nu k} F_k^{i l} + g^{ik} g^{\nu\mu} g^{lj} \delta F_{k\mu j} \right). \end{aligned} \quad (4.29)$$

Plugging in the ansatz for the fluctuations yields

$$\begin{aligned} & \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} e^\phi \left[A^\mu \xi^k F_k^{\nu l} + A^\nu \xi^k F^{\mu}_{k^l} + g^{\mu\rho} g^{\nu\lambda} g^{lj} (f_{\rho\lambda\mu j} - A_\rho \xi^n F_{n\lambda j} \right. \right. \\ & \left. \left. + A_\lambda \xi^n F_{n\rho j}) \right] \right) + e^\phi \partial_i \left(A^\nu \xi^k F_k^{i l} + g^{ik} g^{\nu\mu} g^{lj} (-A_\mu \xi^n F_{njk}) \right) = 0 \end{aligned} \quad (4.30)$$

After various cancellations, one finally arrives at the Dynamical Equation for the vector fields

$$\partial_\mu \left(\sqrt{g} e^\phi g^{\mu\rho} g^{\nu\lambda} f_{\rho\lambda\mu\xi} \right) = 0 \quad (4.31)$$

When $\mu_\xi^i = e^{-\phi/2}\xi^i$, this further simplifies into

$$\xi^i \partial_\mu \left(\sqrt{g} e^{\phi/2} g^{\mu\rho} g^{\nu\lambda} f_{\rho\lambda} \right) = 0 \quad (4.32)$$

which neatly displays the correspondence between bulk 5-d massless vector fields and exact Killing vectors.

4.1.3 Einstein equations

It is straightforward but very laborious to show that Einstein equations lead to the same results, i.e. the very same dynamical equation for A_μ . For simplicity we will restrict our attention on the case in which $\mu_M^\xi = e^{-\phi/2}\xi_M$ i.e. $i_\xi F_3 = d(e^{-\frac{\phi}{2}}\lambda^\xi)$ where $\lambda^\xi = g_{MN}\xi^M dx^N$

Let us start with the source term

$$\begin{aligned} \delta S_{MN} &= \frac{1}{4} e^\phi \left(\delta F_{MPQ} F_N^{PQ} + \delta F_{NPQ} F_M^{PQ} + 2\delta g^{PL} F_{MPQ} F_{NL}{}^Q \right) - \frac{1}{48} e^\phi \delta g_{MN} F^2 \\ &= \frac{1}{4} \left[e^{\frac{\phi}{2}} F_N^{PQ} \left(f_{[MP\xi Q]} - e^{\frac{\phi}{2}} A_{[M} F_{PQ]L} \xi^L \right) + e^\phi \left(A^P \xi^L + A^L \xi^P \right) F_{MPQ} F_{NL}{}^Q \right. \\ &\quad \left. + (M \leftrightarrow N) \right] - \frac{1}{48} e^\phi \delta g_{MN} F^2 \end{aligned} \quad (4.33)$$

Defining

$$h_{LM} = \nabla_L \xi_M - \nabla_M \xi_L = \frac{1}{2} \left(\partial_L \phi \xi_M - \partial_M \phi \xi_L \right) + e^\phi \left(e^{-\frac{\phi}{2}} \xi^P F_{PLM} \right) \quad (4.34)$$

the first order fluctuation of the Ricci tensor reads

$$\begin{aligned} \delta R_{MN} &= -\frac{1}{2} \left(\nabla^2 \delta g_{MN} - \nabla^L \nabla_M \delta g_{NL} - \nabla^L \nabla_N \delta g_{ML} + \nabla_M \nabla_N \delta g^L{}_L \right) \\ &= -\frac{1}{2} \nabla^L \left[\xi_N f_{ML} + \xi_M f_{NL} + \xi_L \left(\nabla_M A_N + \nabla_N A_M \right) + A_M h_{NL} + A_N h_{ML} \right] \\ &= \frac{1}{2} e^{\frac{\phi}{2}} \left[f_{LM} \nabla^L \left(e^{-\frac{\phi}{2}} \xi_N \right) + f_{LN} \nabla^L \left(e^{-\frac{\phi}{2}} \xi_M \right) \right] + \frac{1}{2} e^{-\frac{\phi}{2}} \left[\xi_N \nabla^L \left(e^{\frac{\phi}{2}} f_{LM} \right) \right. \\ &\quad \left. + \xi_M \nabla^L \left(e^{\frac{\phi}{2}} f_{LN} \right) \right] - \frac{1}{2} \xi^L \nabla_L \left(\nabla_M A_N + \nabla_N A_M \right) + \frac{1}{2} \nabla^L \left(A_M h_{LN} + A_N h_{LM} \right) \end{aligned} \quad (4.35)$$

Moreover one finds

$$\begin{aligned} \frac{1}{2} \nabla^L \left(A_M h_{LN} + A_N h_{LM} \right) &= \frac{1}{2} \nabla^L \left(A_M e^{\frac{\phi}{2}} \xi^P F_{PLN} + A_N e^{\frac{\phi}{2}} \xi^P F_{PLM} \right) \\ &\quad + \frac{1}{4} \nabla^L \left(A_M \xi_N \partial_L \phi - A_M \xi_L \partial_N \phi + A_N \xi_M \partial_L \phi - A_N \xi_L \partial_M \phi \right) \\ &= \frac{1}{2} e^{\frac{\phi}{2}} \xi^P F_{PLN} \nabla^L A_M - \frac{1}{4} e^\phi \xi^k F_k{}^{LP} F_{LPN} A_M - \frac{1}{48} e^\phi \delta g_{MN} F^2 \\ &\quad + \frac{1}{4} \partial_L \phi \nabla^L \left(A_M \xi_N + A_N \xi_M \right) - \frac{1}{4} \xi^L \nabla_L \left(A_M \partial_N \phi + A_N \partial_M \phi \right) \end{aligned} \quad (4.36)$$

and also

$$-\frac{1}{4} e^\phi \xi^k F_k{}^{LP} F_{LPN} A_M = -\frac{1}{4} e^\phi F_N{}^{PQ} \xi^L A_{[M} F_{LPQ]} + \frac{1}{4} e^\phi \left(A^P \xi^L + A^L \xi^P \right) F_{MPQ} F_{NL}{}^Q \quad (4.37)$$

Simplifying terms in the left and right hand side yields

$$\delta' S_{MN} = \frac{1}{4} e^{\frac{\phi}{2}} F_N{}^{PQ} f_{[MP}\xi_{Q]} + (M \leftrightarrow N) \quad (4.38)$$

and

$$\begin{aligned} \delta' R_{MN} &= \frac{1}{2} e^{\frac{\phi}{2}} f_{LM} \nabla^L \left(e^{-\frac{\phi}{2}} \xi_N \right) + \frac{1}{2} e^{-\frac{\phi}{2}} \xi_N \nabla^L \left(e^{\frac{\phi}{2}} f_{LM} \right) - \frac{1}{2} \xi^L \nabla_L \nabla_M A_N \\ &+ \frac{1}{2} e^{\frac{\phi}{2}} \xi^P F_{PLN} \nabla^L A_M + \frac{1}{4} \partial_L \phi \nabla^L (A_M \xi_N) - \frac{1}{4} \xi^L \nabla_L (A_M \partial_N \phi) + (M \leftrightarrow N) \end{aligned} \quad (4.39)$$

In particular for $M = \mu$ and $N = i$, the remaining terms in the source fluctuations are

$$\delta S_{\mu i} = \frac{1}{2} e^{-\frac{\phi}{2}} \left(\frac{1}{2} \phi' \xi_i - \xi'_i \right) (\nabla_\mu A_u - \nabla_u A_\mu) \quad (4.40)$$

while the remaining terms in the Ricci fluctuations read

$$\frac{1}{2} e^{\frac{\phi}{2}} f_{LM} \nabla^L \left(e^{-\frac{\phi}{2}} \xi_N \right) + (M \leftrightarrow N) = \frac{1}{4} e^{-\frac{\phi}{2}} (\nabla_u A_\mu - \nabla_\mu A_u) \left(\xi'_i - \phi' \xi_i \right), \quad (4.41)$$

$$-\frac{1}{2} \xi^L \nabla_L \nabla_M A_N + (M \leftrightarrow N) = -\frac{1}{4} e^{-\frac{\phi}{2}} \xi'_i (\nabla_\mu A_u + \nabla_u A_\mu) + \frac{1}{4} e^{-\frac{\phi}{2}} \delta_\mu^u \xi'_j g^{jk} A_u \partial_u g_{ki}, \quad (4.42)$$

$$\begin{aligned} \frac{1}{2} e^{\frac{\phi}{2}} \xi^P F_{PLN} \nabla^L A_M + (M \leftrightarrow N) = \\ e^{-\frac{\phi}{2}} \left\{ -\frac{1}{4} \phi' \xi_i \nabla_u A_\mu + \frac{1}{2} \xi'_i \nabla_u A_\mu - \delta_\mu^u \left(-\frac{1}{8} \phi' A_u \xi'_i + \frac{1}{4} \xi'_j g^{jk} A_u \partial_u g_{ki} \right) \right\}, \end{aligned} \quad (4.43)$$

$$\frac{1}{4} \partial_L \phi \nabla^L (A_M \xi_N) + (M \leftrightarrow N) = e^{-\frac{\phi}{2}} \left(\frac{1}{4} \phi' \xi_i \nabla_u A_\mu + \frac{1}{8} \xi'_i \phi' A_\mu \right) \quad (4.44)$$

and finally

$$-\frac{1}{4} \xi^L \nabla_L (A_M \partial_N \phi) + (M \leftrightarrow N) = e^{-\frac{\phi}{2}} \left(-\frac{1}{8} \xi'_i A_\mu \phi' - \frac{1}{8} \xi'_i A_u \phi' \delta_\mu^u \right) \quad (4.45)$$

After a number of cancellation one eventually finds

$$\delta S_{\mu i} = \delta R_{\mu i} \quad (4.46)$$

when the 3-form equations are satisfied, showing that the fluctuations ansatz then satisfies Einstein equations, too.

It is easy to check that the other components (i, j) and (μ, ν) are satisfied as well. For brevity, we refrain to present the details here.

4.2 One-forms associated to Killing vectors

For completeness and later use, let us display in the following the expressions of the one-forms associated to the Killing vectors in a generic PT background. Setting⁵ $\lambda_\xi =$

⁵The constant factor N_5^{-1} is inserted for later notational convenience.

$N_5^{-1} g_{ij} \xi^i dx^j$, one has

$$\begin{aligned}
\lambda_1 = & -\frac{1}{4} e^{i\varphi-g+x} (a^2 + 4e^{2g}) d\theta + \frac{1}{4} e^{i\varphi-g+x} a (\cos \psi + i \cos \theta \sin \psi) d\tilde{\theta} \\
& -\frac{1}{4} i e^{i\varphi-g-6p-x} \left(e^{6p+2x} a^2 - e^g + 4e^{2(g+3p+x)} \right) \cos \theta \sin \theta d\varphi \\
& +\frac{1}{4} e^{i\varphi-g-6p-x} \left[i e^g \cos \tilde{\theta} \sin \theta + e^{6p+2x} a (\sin \psi - i \cos \psi \cos \theta) \sin \tilde{\theta} \right] d\tilde{\varphi} \\
& +\frac{1}{4} i e^{i\varphi-6p-x} \sin \theta d\psi
\end{aligned} \tag{4.47}$$

$$\begin{aligned}
\lambda_2 = & \frac{1}{4} e^{-i\varphi-g+x} (a^2 + 4e^{2g}) d\theta - \frac{1}{4} e^{-i\varphi-g+x} a (\cos \psi - i \cos \theta \sin \psi) d\tilde{\theta} \\
& -\frac{1}{4} i e^{-i\varphi-g-6p-x} \left[e^{6p+2x} a^2 - e^g + 4e^{2(g+3p+x)} \right] \cos \theta \sin \theta d\varphi \\
& +\frac{1}{4} i e^{-i\varphi-g-6p-x} \left[e^g \cos \tilde{\theta} \sin \theta - e^{6p+2x} a (\cos \psi \cos \theta - i \sin \psi) \sin \tilde{\theta} \right] d\tilde{\varphi} \\
& +\frac{1}{4} i e^{-i\varphi-6p-x} \sin \theta d\psi
\end{aligned} \tag{4.48}$$

$$\begin{aligned}
\lambda_3 = & -\frac{1}{4} e^{x-g} a \sin \psi \sin \theta d\tilde{\theta} + \frac{1}{4} \left[e^{-6p-x} \cos^2 \theta + e^{x-g} (a^2 + 4e^{2g}) \sin^2 \theta \right] d\varphi \\
& +\frac{1}{4} \left(e^{-6p-x} \cos \theta \cos \tilde{\theta} + e^{x-g} a \cos \psi \sin \theta \sin \tilde{\theta} \right) d\tilde{\varphi} + \frac{1}{4} e^{-6p-x} \cos \theta d\psi
\end{aligned} \tag{4.49}$$

$$\begin{aligned}
\lambda_4 = & \frac{1}{4} e^{i\tilde{\varphi}-g+x} a (\cos \psi + i \cos \tilde{\theta} \sin \psi) d\theta - \frac{1}{4} e^{i\tilde{\varphi}-g+x} d\tilde{\theta} \\
& +\frac{1}{4} e^{i\tilde{\varphi}-g-6p-x} \left[e^{6p+2x} a (\sin \psi - i \cos \psi \cos \tilde{\theta}) \sin \theta + i e^g \cos \theta \sin \tilde{\theta} \right] d\varphi \\
& -\frac{1}{8} i e^{i\tilde{\varphi}-g-6p-x} (-e^g + e^{6p+2x}) \sin(2\tilde{\varphi}) d\tilde{\varphi} + \frac{1}{4} i e^{i\tilde{\varphi}-6p-x} \sin \tilde{\theta} d\psi
\end{aligned} \tag{4.50}$$

$$\begin{aligned}
\lambda_5 = & -\frac{1}{4} e^{-i\tilde{\varphi}-g+x} a (\cos \psi - i \cos \tilde{\theta} \sin \psi) d\theta + \frac{1}{4} e^{-i\tilde{\varphi}-g+x} d\tilde{\theta} \\
& -\frac{1}{4} i e^{-i\tilde{\varphi}-g-6p-x} \left[(e^{6p+2x} a (\cos \psi \cos \tilde{\theta} - i \sin \psi) \sin \theta - e^g \cos \theta \sin \tilde{\theta}) \right] d\varphi \\
& +\frac{1}{4} i e^{-i\tilde{\varphi}-g-6p-x} (e^g - e^{6p+2x}) \cos \tilde{\theta} \sin \tilde{\theta} d\tilde{\varphi} + \frac{1}{4} i e^{-i\tilde{\varphi}-6p-x} \sin \tilde{\theta} d\psi
\end{aligned} \tag{4.51}$$

$$\begin{aligned}
\lambda_6 = & -\frac{1}{4} e^{x-g} a \sin \psi \sin \tilde{\theta} d\theta + \frac{1}{4} \left[e^{-6p-x} \cos \theta \cos \tilde{\theta} + e^{x-g} a \cos \psi \sin \theta \sin \tilde{\theta} \right] d\varphi \\
& +\frac{1}{4} \left[e^{-6p-x} \cos^2 \tilde{\theta} + e^{x-g} \sin^2 \tilde{\theta} \right] d\tilde{\varphi} + \frac{1}{4} e^{-6p-x} \cos \tilde{\theta} d\psi
\end{aligned} \tag{4.52}$$

Then using $N_5^{-1} i_\xi F_3 = d\mu^\xi$ one finds

$$\begin{aligned}
\mu_1 = & \frac{1}{4} \left[e^{i\varphi} f - e^{i\varphi} (f-1) \right] d\theta - \frac{1}{4} e^{i\varphi} b (\cos \psi + i \cos \theta \sin \psi) d\tilde{\theta} \\
& +\frac{1}{8} \left[i e^{i\varphi} (f-1) \sin(2\theta) - 2i e^{i\varphi} \cos \theta (f-1) \sin \theta \right] d\varphi \\
& -\frac{1}{4} e^{i\varphi} \left[i \cos \tilde{\theta} \sin \theta + b (\sin \psi - i \cos \psi \cos \theta) \sin \tilde{\theta} \right] d\tilde{\varphi} \\
& -\frac{1}{4} i e^{i\varphi} \sin \theta d\psi
\end{aligned} \tag{4.53}$$

$$\begin{aligned}
 \mu_2 = & \frac{1}{4} [e^{-i\varphi}(f-1) - e^{-i\varphi}f] d\theta + \frac{1}{4} e^{-i\varphi} b(\cos\psi - i\cos\theta\sin\psi) d\tilde{\theta} \\
 & + \frac{1}{8} [ie^{-i\varphi}(f-1)\sin(2\theta) - 2ie^{-i\varphi}\cos\theta(f-1)\sin\theta] d\varphi \\
 & - \frac{1}{4} ie^{-i\varphi} [\cos\tilde{\theta}\sin\theta + b(i\sin\psi - \cos\psi\cos\theta)\sin\tilde{\theta}] d\tilde{\varphi} \\
 & - \frac{1}{4} ie^{-i\varphi} \sin\theta d\psi
 \end{aligned} \tag{4.54}$$

$$\begin{aligned}
 \mu_3 = & \frac{1}{4} b\sin\psi\sin\theta d\tilde{\theta} + \frac{1}{4} [(f-1)\sin^2\theta + (-\cos^2\theta - f\sin^2\theta)] d\varphi \\
 & + \frac{1}{4} (-\cos\theta\cos\tilde{\theta} - b\cos\psi\sin\theta\sin\tilde{\theta}) d\tilde{\varphi} - \frac{\cos\theta}{4} d\psi
 \end{aligned} \tag{4.55}$$

$$\begin{aligned}
 \mu_4 = & \frac{1}{4} e^{i\tilde{\varphi}} b(\cos\psi + i\cos\tilde{\theta}\sin\psi) d\theta - \frac{e^{i\tilde{\varphi}}}{4} d\tilde{\theta} \\
 & \frac{1}{4} e^{i\tilde{\varphi}} [b(\sin\psi - i\cos\psi\cos\tilde{\theta})\sin\theta + i\cos\theta\sin\tilde{\theta}] d\varphi + \frac{1}{4} ie^{i\tilde{\varphi}} \sin\tilde{\theta} d\psi
 \end{aligned} \tag{4.56}$$

$$\begin{aligned}
 \mu_5 = & -\frac{1}{4} e^{-i\tilde{\varphi}} b(\cos\psi - i\cos\tilde{\theta}\sin\psi) d\theta + \frac{e^{-i\tilde{\varphi}}}{4} d\tilde{\theta} \\
 & - \frac{1}{4} ie^{-i\tilde{\varphi}} [b(\cos\psi\cos\tilde{\theta} - i\sin\psi)\sin\theta - \cos\theta\sin\tilde{\theta}] d\varphi + \frac{1}{4} ie^{-i\tilde{\varphi}} \sin\tilde{\theta} d\psi
 \end{aligned} \tag{4.57}$$

$$\begin{aligned}
 \mu_6 = & -\frac{1}{4} b\sin\psi\sin\tilde{\theta} d\theta + \frac{1}{4} (\cos\theta\cos\tilde{\theta} + b\cos\psi\sin\theta\sin\tilde{\theta}) d\varphi \\
 & + \frac{1}{4} d\tilde{\varphi} + \frac{1}{4} \cos\tilde{\theta} d\psi
 \end{aligned} \tag{4.58}$$

which are valid in any PT background. We also defined $f = 4e^{2g} + a^2$.

For MN background [4] a compact form for μ^ξ in terms of the (rescaled) KV obtains

$$\begin{aligned}
 \mu_{(1)i} &= -e^{-\frac{\phi}{2}} \xi_{(1)i} - \frac{1}{4} (f-1) e^{i\varphi} (\delta_i^6 + i\cos\theta\sin\theta\delta_i^8) \\
 \mu_{(2)i} &= -e^{-\frac{\phi}{2}} \xi_{(2)i} + \frac{1}{4} (f-1) e^{-i\varphi} (\delta_i^6 - i\cos\theta\sin\theta\delta_i^8) \\
 \mu_{(3)i} &= -e^{-\frac{\phi}{2}} \xi_{(3)i} + \frac{1}{4} (f-1) \sin^2\theta\delta_i^8 \\
 \mu_{(4)i} &= e^{-\frac{\phi}{2}} \xi_{(4)i} \\
 \mu_{(5)i} &= e^{-\frac{\phi}{2}} \xi_{(5)i} \\
 \mu_{(6)i} &= e^{-\frac{\phi}{2}} \xi_{(6)i}
 \end{aligned} \tag{4.59}$$

4.3 Scalar products and gauge kinetic functions

The scalar products of the Killing Vectors are diagonal as a result of the $SU(2) \times \widetilde{SU(2)}$ symmetry. For $SU(2)$ one finds

$$\int \xi_a^{*i} g_{ij} \xi_b^j d^5\Omega = \frac{2}{3} N_5^{\frac{7}{2}} \pi^3 \delta_{ab} \kappa_a e^{-g-9p+\frac{\pi}{2}} [2e^{6p+2x}(f-1) + (e^g + 2e^{6p+2x})] \tag{4.60}$$

with $\kappa_1 = \kappa_2 = 2$ and $\kappa_3 = 1$, while for $\widetilde{SU(2)}$ one finds

$$\int \xi_a^{*i} g_{ij} \xi_b^j d^5\Omega = \frac{2}{3} N_5^{\frac{7}{2}} \pi^3 \delta_{ab} \kappa_a e^{-g-9p+\frac{\pi}{2}} (e^g + 2e^{6p+2x}) \tag{4.61}$$

with $\kappa_4 = \kappa_5 = 2$ and $\kappa_6 = 1$.

Similarly for the one-forms μ_a^ξ , the scalar products are diagonal. For $SU(2)$ one gets

$$\int \mu_a^{*i} g_{ij} \mu_b^j d^5\Omega = \frac{1}{3} e^{-g-3p+\frac{x}{2}} \delta_{ab} \kappa_a N_5^{\frac{7}{2}} \pi^3 [(f-1)(a^2-1) + 2e^g(2e^g + e^{6p+2x})] \quad (4.62)$$

with $\kappa_1 = \kappa_2 = 2$ and $\kappa_3 = 1$, while for $\widetilde{SU(2)}$ one gets

$$\int \mu_a^{*i} g_{ij} \mu_b^j d^5\Omega = \frac{2}{3} N_5^{\frac{7}{2}} \pi^3 e^{\frac{x}{2}-3p} \delta_{ab} \kappa_a (2e^g + e^{6p+2x}) \quad (4.63)$$

with $\kappa_4 = \kappa_5 = 2$ and $\kappa_6 = 1$, exposing the $SU(2) \times \widetilde{SU(2)}$ symmetry.

From the above scalar products one can read the gauge kinetic functions for the massless vectors of $SU(2) \times \widetilde{SU(2)}$ in 5-d. For $SU(2)$ one finds

$$\begin{aligned} \mathcal{K}_{SU(2)} = \frac{1}{2} \mathcal{V}^{-\frac{1}{3}} (||\xi||^2 + e^\phi ||\mu||^2) = \frac{N_5^{\frac{8}{3}} \pi^2 \delta_{ab} \kappa_a e^{-2p-g}}{12} \left\{ (f-1) [4e^{2x} + e^\phi (a^2-1)] \right. \\ \left. + [2e^{g-6p} + 4e^{2x} + 2e^{g+\phi} (2e^g + e^{6p+2x})] \right\} \quad (4.64) \end{aligned}$$

where the first term is a contribution from the Einstein-Hilbert term, while the second comes from the F_3^2 in the Type IIB action.⁶ For $\widetilde{SU(2)}$, $\mu = e^{-\phi/2} \xi$ and one simply finds

$$\mathcal{K}_{\widetilde{SU(2)}} = \mathcal{V}^{-\frac{1}{3}} ||\xi||^2 = \frac{N_5^{\frac{8}{3}} \pi^2}{3} \delta_{ab} \kappa_a e^{-g-8p} (e^g + 2e^{6p+2x}) \quad (4.65)$$

The internal volume factor $\mathcal{V}^{-\frac{1}{3}}$ arises from the Weyl scaling of the 10-d metric with pure space-time components so as to have canonical E-H term in 5-d, i.e. $g_{\mu\nu}^{(10)} = \mathcal{V}^{-\frac{2}{3}}(u) g_{\mu\nu}^{(5)}$.

5 Massless vectors in MN background

We will now explicitly apply the above analysis to the case of MN solution for wrapped D5-branes [4]. For simplicity we will focus on the $\widetilde{SU(2)}$ factor, for which $\mu^\xi = e^{-\phi/2} \xi$.

5.1 MN Solution

In MN solution for wrapped D5-branes [4] one has $h_1 = h_2 = 0$ (no D3-branes $F_5 = 0$, no NS5-branes $H_3 = 0$, $\chi = 0$) and $b = a$. Denoting the radial variable by u , the metric reads

$$ds^2 = e^{\frac{\phi}{2}} \left[dx^2 + N_5 \left\{ du^2 + e^{2g} (e_1^2 + e_2^2) + \frac{1}{4} (\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2) \right\} \right] \quad (5.1)$$

where

$$\begin{aligned} e^{-2\phi} &= \frac{2e^g}{\sinh 2u}, & e^{2g} &= u \coth 2u - \frac{1}{4}(1+a^2) \\ a &= \frac{2u}{\sinh 2u}, & f &= 4e^{2g} + a^2 \end{aligned} \quad (5.2)$$

⁶Which is well defined for $A_4 = 0$.

The RR 3-form flux is given by

$$F_3 = \frac{N_5}{4} \left\{ \tilde{\omega}_3 \wedge [(\omega_1 \wedge \omega_2 + e_1 \wedge e_2) - a(u)(\omega_1 \wedge e_2 - \omega_2 \wedge e_1)] + a'(u)du \wedge (\omega_1 \wedge e_1 + \omega_2 \wedge e_2) \right\} \quad (5.3)$$

The asymptotic behavior of the radial functions in the UV ($u \rightarrow 0$) and IR ($u \rightarrow \infty$) are found to be

$$a(u \rightarrow 0) \rightarrow 1 - \frac{2}{3}u^2, \quad a(u \rightarrow \infty) \rightarrow 0 \quad (5.4)$$

$$e^{2g}(u \rightarrow 0) \rightarrow u^2, \quad e^{2g}(u \rightarrow \infty) \rightarrow u \quad (5.5)$$

$$f(u \rightarrow 0) \rightarrow 1 + \frac{8}{3}u^2 - \frac{32}{45}u^4, \quad f(u \rightarrow \infty) \rightarrow 4u \quad (5.6)$$

For later use, notice that

$$g_{\mu\nu} = e^{\frac{\phi}{2}} \delta_{\mu\nu}, \quad g_{ij} = e^{\frac{\phi}{2}} \hat{g}_{ij}, \quad \det g_{MN} = \frac{N_5^6}{64} e^{5\phi} e^{4g} \sin^2 \theta \sin^2 \tilde{\theta}$$

5.2 Spectrum of massless vector harmonics

As we have seen, in order to find the spectrum of bound-states that are holographically dual to the massless bulk vectors associated to the three Killing vectors of $\widehat{\text{SU}}(2)$, one should solve

$$\frac{1}{\sqrt{g}} \partial_\mu [\sqrt{g} e^{\phi/2} f^{\mu\nu}] = 0 \quad (5.7)$$

There are two cases to consider $\nu = \hat{\nu}$ and $\nu = u$.

For $\nu = u$ one simply gets

$$\partial^{\hat{\mu}} f_{\hat{\mu}u} = 0 \quad \rightarrow \quad \partial^{\hat{\mu}} \partial_{\hat{\mu}} A_u - \partial_u \partial^{\hat{\mu}} A_{\hat{\mu}} = 0 \quad (5.8)$$

that allows to express A_u in terms of the longitudinal component of $A_{\hat{\mu}}$.

For $\nu = \hat{\nu}$ one gets

$$\partial_{\hat{\mu}} f^{\hat{\mu}\hat{\nu}} + \frac{1}{\sqrt{g(u)}} \partial_u [\sqrt{g(u)} e^{\phi/2} g^{uu} g^{\hat{\nu}\hat{\lambda}} (\partial_u A_{\hat{\lambda}} - \partial_{\hat{\lambda}} A_u)] = 0 \quad (5.9)$$

Setting $A_{\hat{\mu}} = a_{\hat{\mu}}(u) e^{ip \cdot \hat{x}}$ and $A_u = b(u) e^{ip \cdot \hat{x}}$ one can solve for $a_{\hat{\mu}}(u)$ and $b(u)$. Decomposing $a_{\hat{\mu}}(u)$ into longitudinal and transverse components according to

$$a_{\hat{\mu}}(u) = a_{\hat{\mu}}^T(u) + ip_{\hat{\mu}} a_L(u) \quad (5.10)$$

one finds

$$b(u) = a'_L(u) \quad (5.11)$$

that can be set to zero by gauge transformations. The surviving transverse components then satisfy an equation

$$\frac{1}{\sqrt{g(u)}} \partial_u (\sqrt{g(u)} e^{-\phi/2} \partial_u a_{\hat{\mu}}^T) - e^{-\phi/2} p^2 a_{\hat{\mu}}^T = 0 \quad (5.12)$$

that is identical to the equation for a canonical massless scalar Φ . After setting

$$\Phi = e^{-\phi-g\mathcal{Y}} \tag{5.13}$$

the equation is put in canonical form with an effective potential given by

$$V_{\text{eff}} = \phi'' + g'' \approx -\frac{1}{u^2} \quad \text{in UV} \tag{5.14}$$

Unfortunately due to the ‘pathological’ UV behavior there is no spectrum of discrete states associate to the bulk massless vectors. For $SU(2)$ the story is similar. This behavior is analogous to the one found for the fluctuations of the metric in MN solution [4]. Indeed, the transverse traceless components of the metric fluctuations $h_{ij}^{TT}(u, x) = e_{ij}(p)f_p(u)e^{ipx}$ decouple from the rest and satisfy a free massless scalar equation of the form⁷

$$[\partial_u^2 + 4A'\partial_u - e^{2A}p^2]f_p(u) = 0 \tag{5.15}$$

For MN solution [4], the relevant equation has been studied in [26, 27] and shown to have a continuous spectrum without a mass gap. Longitudinal and radial components of the metric mix with the active scalars and behave better [26, 27]. Despite the area-law behavior of the Wilson loop in MN background [4], the presence of massless fluctuations casts some shadow on the holographic interpretation as a dual to a confining theory such as $\mathcal{N} = 1$ SYM.

6 Conclusions and summary

Let us conclude by summarizing our results and draw lines for future investigation.

We have shown that all RG flows described by the PT ansatz [5] in Type IIB supergravity enjoy exact $SU(2) \times \widetilde{SU}(2)$ symmetry in that not only the metric but also background p -forms are invariant under diffeomorphisms generated by the six Killing vectors. We have then identified a very general ansatz for the combined fluctuations of metric and p -forms that diagonalizes the resulting equations for the bulk massless vectors. Although derived in the context of holography, our ansatz is expected to have much wider applicability in any flux compactification with isometry. Restricting our attention to the case of backgrounds invariant under world-sheet parity Ω , we have illustrated our procedure in the case of MN solution [4] for wrapped D5-branes. The spectrum of massless vector harmonics in this background — very much as the spectrum of massless scalars and transverse traceless fluctuations of the metric [26, 27] — is continuous and has no mass gap. This drawback might be related to the impossibility of fully decoupling KK states from the desired physical modes, which survive in the deep IR. In particular the very presence of an exact $SU(2) \times \widetilde{SU}(2)$ symmetry is a remnant of the breaking of $\mathcal{N} = 4$ to $\mathcal{N} = 1^*$ with common mass for the three chiral multiplets. Since this symmetry is an exact symmetry of any RG flows described by PT ansatz [5], not excluding KS solution [3], we should conclude

⁷The very same equation governs the dynamics of the transverse modes of the supersymmetry partners of the graviton e.g. gravitino, graviphoton, ... as originally shown in [19].

that holographic SYM is still undelivered [44] at least in a top-down approach. The strictly 5-d bottom-up approach embodied by Holographic QCD [37–41] seems more promising in this respect.

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