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Research Article

Moment Inequality for φ-Mixing Sequences and Its Applications

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Firstly, the maximal inequality for φ-mixing sequences is given. By using the maximal inequality, we study the convergence properties for φ-mixing sequences. The Hájek-Rényi-type inequality, strong law of large numbers, strong growth rate and the integrability of supremum for φ-mixing sequences are obtained.

1. Introduction

Let $\{X_n,\ n \geq 1\}$ be a random variable sequence defined on a fixed probability space $(\Omega, \mathcal{F}, P)$ and $S_n = \sum_{i=1}^{n} X_i$ for each $n \geq 1$. Let n and m be positive integers. Write $\mathcal{F}_n^m = \sigma(X_i,\ n \leq i \leq m)$. Given σ-algebras $\mathcal{B}, \mathcal{R}$ in $\mathcal{F}$, let

$$\varphi(\mathcal{B}, \mathcal{R}) = \sup_{A \in \mathcal{B}, B \in \mathcal{R}, P(A) > 0} |P(B \mid A) - P(B)|. \tag{1.1}$$

Define the φ-mixing coefficients by

$$\varphi(n) = \sup_{k \geq 1} \varphi\left(\mathcal{F}_1^k, \mathcal{F}_{k+n}^{\infty}\right), \quad n \geq 0. \tag{1.2}$$

Definition 1.1. A random variable sequence $\{X_n,\ n \geq 1\}$ is said to be a φ-mixing random variable sequence if $\varphi(n) \downarrow 0$ as $n \to \infty$.

The concept of φ-mixing random variables was introduced by Dobrushin [1] and many applications have been found. See, for example, Dobrushin [1], Utev [2], and Chen [3] for central limit theorem, Herrndorf [4] and Peligrad [5] for weak invariance principle, Sen [6, 7] for weak convergence of empirical processes, Iosifescu [8] for limit theorem, Peligrad [9] for Ibragimov-Iosifescu conjecture, Shao [10] for almost sure invariance principles, Hu and Wang [11] for large deviations, and so forth. When these are compared with the corresponding results of independent random variable sequences, there still remains much to be desired. The main purpose of this paper is to study the maximal inequality for φ-mixing sequences, by which we can get the Hájek-Rényi-type inequality, strong law of large numbers, strong growth rate, and the integrability of supremum for φ-mixing sequences.

Throughout the paper, C denotes a positive constant which may be different in various places. The main results of this paper depend on the following lemmas.

Lemma 1.2 (see Lu and Lin [12]). *Let $\{X_n,\ n \geq 1\}$ be a sequence of φ-mixing random variables. Let $X \in L_p(\mathcal{F}_1^k)$, $Y \in L_q(\mathcal{F}_{k+n}^\infty)$, $p \geq 1$, $q \geq 1$, and $1/p + 1/q = 1$. Then*

$$|EXY - EXEY| \leq 2(\varphi(n))^{1/p}(E|X|^p)^{1/p}(E|Y|^q)^{1/q}. \tag{1.3}$$

Lemma 1.3 (see Shao [10]). *Let $\{X_n,\ n \geq 1\}$ be a φ-mixing sequence. Put $T_a(n) = \sum_{i=a+1}^{a+n} X_i$. Suppose that there exists an array $\{C_{a,n}\}$ of positive numbers such that*

$$ET_a^2(n) \leq C_{a,n} \quad \text{for every } a \geq 0,\ n \geq 1. \tag{1.4}$$

Then for every $q \geq 2$, there exists a constant C depending only on q and $\varphi(\cdot)$ such that

$$E\left(\max_{1\leq j\leq n}|T_a(j)|^q\right) \leq C\left[C_{a,n}^{q/2} + E\left(\max_{a+1\leq i\leq a+n}|X_i|^q\right)\right] \tag{1.5}$$

for every $a \geq 0$ and $n \geq 1$.

Lemma 1.4 (see Hu et al. [13]). *Let $b_1, b_2, \ldots$ be a nondecreasing unbounded sequence of positive numbers and let $\alpha_1, \alpha_2, \ldots$ be nonnegative numbers. Let r and C be fixed positive numbers. Assume that for each $n \geq 1$,*

$$E\left(\max_{1\leq l\leq n}|S_l|\right)^r \leq C\sum_{l=1}^{n}\alpha_l, \tag{1.6}$$

$$\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^r} < \infty, \tag{1.7}$$

then

$$\lim_{n\to\infty}\frac{S_n}{b_n} = 0 \quad a.s., \tag{1.8}$$

and with the growth rate

$$\frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right) \quad a.s., \tag{1.9}$$

where

$$\beta_n = \max_{1\le k\le n} b_k v_k^{\delta/r}, \quad \forall 0<\delta<1, \quad v_n = \sum_{k=n}^{\infty}\frac{\alpha_k}{b_k^r}, \quad \lim_{n\to\infty}\frac{\beta_n}{b_n}=0,$$

$$E\left(\max_{1\le l\le n}\left|\frac{S_l}{b_l}\right|^r\right) \le 4C\sum_{l=1}^{n}\frac{\alpha_l}{b_l^r} < \infty, \tag{1.10}$$

$$E\left(\sup_{l\ge 1}\left|\frac{S_l}{b_l}\right|^r\right) \le 4C\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^r} < \infty.$$

If further we assume that $\alpha_n > 0$ *for infinitely many* n*, then*

$$E\left(\sup_{l\ge 1}\left|\frac{S_l}{\beta_l}\right|^r\right) \le 4C\sum_{l=1}^{\infty}\frac{\alpha_l}{\beta_l^r} < \infty. \tag{1.11}$$

Lemma 1.5 (see Fazekas and Klesov [14] and Hu [15]). *Let* $b_1, b_2, \ldots$ *be a nondecreasing unbounded sequence of positive numbers and let* $\alpha_1, \alpha_2, \ldots$ *be nonnegative numbers. Denote* $\Lambda_k = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ *for* $k \ge 1$*. Let* r *be a fixed positive number satisfying* (1.6)*. If*

$$\sum_{l=1}^{\infty}\Lambda_l\left(\frac{1}{b_l^r}-\frac{1}{b_{l+1}^r}\right) < \infty, \tag{1.12}$$

$$\frac{\Lambda_n}{b_n^r} \text{ is bounded}, \tag{1.13}$$

then (1.8)–(1.11) *hold.*

2. Maximal Inequality for φ-Mixing Sequences

Theorem 2.1. *Let* $\{X_n, n \ge 1\}$ *be a sequence of* φ*-mixing random variables satisfying* $\sum_{n=1}^{\infty}\varphi^{1/2}(n) < \infty$*. Assume that* $EX_n = 0$ *and* $EX_n^2 < \infty$ *for each* $n \ge 1$*. Then there exists a constant* C *depending only on* $\varphi(\cdot)$ *such that for any* $n \ge 1$ *and* $a \ge 0$

$$E\left(\sum_{i=a+1}^{a+n} X_i\right)^2 \le C\sum_{i=a+1}^{a+n} EX_i^2, \tag{2.1}$$

$$E\left(\max_{1\le j\le n}\left|\sum_{i=a+1}^{a+j} X_i\right|^2\right) \le C\sum_{i=a+1}^{a+n} EX_i^2. \tag{2.2}$$

In particular,

$$E\left(\sum_{i=1}^{n}X_i\right)^2 \le C\sum_{i=1}^{n}EX_i^2, \tag{2.3}$$

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}X_i\right|^2\right) \le C\sum_{i=1}^{n}EX_i^2, \tag{2.4}$$

where C may be different in various places.

Proof. By Lemma 1.2 for $p = q = 2$, we can see that

$$\begin{aligned}
E\left(\sum_{i=a+1}^{a+n}X_i\right)^2 &= \sum_{i=a+1}^{a+n}EX_i^2 + 2\sum_{a+1\le i<j\le a+n}E(X_iX_j)\\
&\le \sum_{i=a+1}^{a+n}EX_i^2 + 4\sum_{a+1\le i<j\le a+n}\varphi^{1/2}(j-i)\left(EX_i^2\right)^{1/2}\left(EX_j^2\right)^{1/2}\\
&\le \sum_{i=a+1}^{a+n}EX_i^2 + 2\sum_{k=1}^{n-1}\sum_{i=a+1}^{a+n-k}\varphi^{1/2}(k)\left(EX_i^2 + EX_{k+i}^2\right)\\
&\le \left(1 + 4\sum_{k=1}^{\infty}\varphi^{1/2}(k)\right)\sum_{i=a+1}^{a+n}EX_i^2\\
&\doteq C\sum_{i=a+1}^{a+n}EX_i^2,
\end{aligned} \tag{2.5}$$

which implies (2.1). By (2.1) and Lemma 1.3 (take $q = 2$), we can get

$$\begin{aligned}
E\left(\max_{1\le j\le n}\left|\sum_{i=a+1}^{a+j}X_i\right|^2\right) &\le C\left[\sum_{i=a+1}^{a+n}EX_i^2 + E\left(\max_{a+1\le i\le a+n}X_i^2\right)\right]\\
&\le C\sum_{i=a+1}^{a+n}EX_i^2.
\end{aligned} \tag{2.6}$$

The proof is completed. □

3. Hájek-Rényi-Type Inequality for φ-Mixing Sequences

In this section, we will give the Hájek-Rényi-type inequality for φ-mixing sequences, which can be applied to obtain the strong law of large numbers and the integrability of supremum.

Theorem 3.1. *Let $\{X_n, n \ge 1\}$ be a sequence of φ-mixing random variables satisfying $\sum_{n=1}^{\infty}\varphi^{1/2}(n) < \infty$ and let $\{b_n, n \ge 1\}$ be a nondecreasing sequence of positive numbers. Then for* *any* $\varepsilon > 0$ *and any integer* $n \geq 1$,

$$P\left\{\max_{1\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\right| \geq \varepsilon\right\} \leq \frac{4C}{\varepsilon^2}\sum_{j=1}^{n}\frac{\operatorname{Var}(X_j)}{b_j^2}, \tag{3.1}$$

where C *is defined in* (2.4) *in Theorem 2.1.*

Proof. Without loss of generality, we assume that $b_n \geq 1$ for all $n \geq 1$. Let $\alpha = \sqrt{2}$. For $i \geq 0$, define

$$A_i = \left\{1 \leq k \leq n : \alpha^i \leq b_k < \alpha^{i+1}\right\}. \tag{3.2}$$

For $A_i \neq \emptyset$, we let $v(i) = \max\{k : k \in A_i\}$ and t_n be the index of the last nonempty set A_i. Obviously, $A_i A_j = \emptyset$ if $i \neq j$ and $\sum_{i=0}^{t_n} A_i = \{1, 2, \ldots, n\}$. It is easily seen that $\alpha^i \leq b_k \leq b_{v(i)} < \alpha^{i+1}$ if $k \in A_i$ and $\{X_n - EX_n,\ n \geq 1\}$ is also a sequence of φ-mixing random variables. By Markov's inequality and (2.4), we have

$$\begin{aligned}
&P\left\{\max_{1\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\right| \geq \varepsilon\right\} \\
&\quad = P\left\{\max_{0\leq i\leq t_n, A_i\neq\emptyset}\ \max_{k\in A_i}\left|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\right| \geq \varepsilon\right\} \\
&\quad \leq \sum_{i=0, A_i\neq\emptyset}^{t_n} P\left\{\frac{1}{\alpha^i}\max_{1\leq k\leq v(i)}\left|\sum_{j=1}^{k}(X_j - EX_j)\right| \geq \varepsilon\right\} \\
&\quad \leq \frac{1}{\varepsilon^2}\sum_{i=0, A_i\neq\emptyset}^{t_n}\frac{1}{\alpha^{2i}}E\left\{\max_{1\leq k\leq v(i)}\left|\sum_{j=1}^{k}(X_j - EX_j)\right|^2\right\} \\
&\quad \leq \frac{C}{\varepsilon^2}\sum_{i=0, A_i\neq\emptyset}^{t_n}\frac{1}{\alpha^{2i}}\sum_{j=1}^{v(i)}\operatorname{Var}(X_j) \\
&\quad \leq \frac{C}{\varepsilon^2}\sum_{j=1}^{n}\operatorname{Var}(X_j)\sum_{i=0, A_i\neq\emptyset, v(i)\geq j}^{t_n}\frac{1}{\alpha^{2i}}.
\end{aligned} \tag{3.3}$$

Now we estimate $\sum_{i=0, A_i\neq\emptyset, v(i)\geq j}^{t_n}(1/\alpha^{2i})$. Let $i_0 = \min\{i : A_i \neq \emptyset,\ v(i) \geq j\}$. Then $b_j \leq b_{v(i_0)} < \alpha^{i_0+1}$ follows from the definition of $v(i)$. Therefore,

$$\sum_{i=0, A_i\neq\emptyset, v(i)\geq j}^{t_n}\frac{1}{\alpha^{2i}} < \sum_{i=i_0}^{\infty}\frac{1}{\alpha^{2i}} = \frac{1}{1 - 1/\alpha^2}\frac{1}{\alpha^{2i_0}} < \frac{\alpha^2}{1 - 1/\alpha^2}\frac{1}{b_j^2} = \frac{4}{b_j^2}. \tag{3.4}$$

Thus, (3.1) follows from (3.3) and (3.4) immediately. □

Theorem 3.2. *Let $\{X_n,\ n \geq 1\}$ be a sequence of φ-mixing random variables satisfying $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ and let $\{b_n,\ n \geq 1\}$ be a nondecreasing sequence of positive numbers. Then for any $\varepsilon > 0$ and any positive integers $m < n$,*

$$P\left(\max_{m\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\right| \geq \varepsilon\right) \leq \frac{4C}{\varepsilon^2}\left(\sum_{j=1}^{m}\frac{\operatorname{Var}(X_j)}{b_m^2} + 4\sum_{j=m+1}^{n}\frac{\operatorname{Var}(X_j)}{b_j^2}\right), \tag{3.5}$$

where C is defined in (3.1).

Proof. Observe that

$$\max_{m\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\right| \leq \left|\frac{1}{b_m}\sum_{j=1}^{m}(X_j - EX_j)\right| + \max_{m+1\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=m+1}^{k}(X_j - EX_j)\right|, \tag{3.6}$$

thus

$$\begin{aligned} P&\left(\max_{m\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\right| \geq \varepsilon\right) \\ &\leq P\left(\left|\frac{1}{b_m}\sum_{j=1}^{m}(X_j - EX_j)\right| \geq \frac{\varepsilon}{2}\right) + P\left(\max_{m+1\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=m+1}^{k}(X_j - EX_j)\right| \geq \frac{\varepsilon}{2}\right) \doteq I + II. \end{aligned} \tag{3.7}$$

For I, by Markov's inequality and (2.3), we have

$$I \leq \frac{4}{\varepsilon^2 b_m^2}E\left(\sum_{j=1}^{m}(X_j - EX_j)\right)^2 \leq \frac{4C}{\varepsilon^2}\sum_{j=1}^{m}\frac{\operatorname{Var}(X_j)}{b_m^2}. \tag{3.8}$$

For II, we will apply Theorem 3.1 to $\{X_{m+i},\ 1 \leq i \leq n-m\}$ and $\{b_{m+i},\ 1 \leq i \leq n-m\}$. Noting that

$$\max_{m+1\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=m+1}^{k}(X_j - EX_j)\right| = \max_{1\leq k\leq n-m}\left|\frac{1}{b_{m+k}}\sum_{j=1}^{k}(X_{m+j} - EX_{m+j})\right|, \tag{3.9}$$

thus, by Theorem 3.1, we get

$$II \leq \frac{4C}{(\varepsilon/2)^2}\sum_{j=1}^{n-m}\frac{\operatorname{Var}(X_{m+j})}{b_{m+j}^2} = \frac{16C}{\varepsilon^2}\sum_{j=m+1}^{n}\frac{\operatorname{Var}(X_j)}{b_j^2}. \tag{3.10}$$

Therefore, the desired result (3.5) follows from (3.7)–(3.10) immediately. □

Theorem 3.3. *Let $\{X_n,\ n \geq 1\}$ be a sequence of φ-mixing random variables satisfying $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ and let $\{b_n,\ n \geq 1\}$ be a nondecreasing sequence of positive numbers. Denote $T_n = \sum_{i=1}^{n}(X_i - EX_i)$ for $n \geq 1$. Assume that*

$$\sum_{j=1}^{\infty} \frac{\operatorname{Var}(X_j)}{b_j^2} < \infty, \tag{3.11}$$

then for any $r \in (0,2)$,

$$E\left(\sup_{n\geq 1}\left|\frac{T_n}{b_n}\right|^r\right) \leq 1 + \frac{4Cr}{2-r}\sum_{j=1}^{\infty}\frac{\operatorname{Var}(X_j)}{b_j^2} < \infty, \tag{3.12}$$

where C is defined in (3.1). *Furthermore, if $\lim_{n\to\infty} b_n = +\infty$, then*

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{j=1}^{n}(X_j - EX_j) = 0 \quad a.s. \tag{3.13}$$

Proof. By the continuity of probability and Theorem 3.1, we get

$$\begin{aligned}
E\left(\sup_{n\geq 1}\left|\frac{T_n}{b_n}\right|^r\right) &= \int_0^{\infty} P\left(\sup_{n\geq 1}\left|\frac{T_n}{b_n}\right|^r > t\right)dt \\
&= \int_0^{1} P\left(\sup_{n\geq 1}\left|\frac{T_n}{b_n}\right|^r > t\right)dt + \int_1^{\infty} P\left(\sup_{n\geq 1}\left|\frac{T_n}{b_n}\right|^r > t\right)dt \\
&\leq 1 + \int_1^{\infty} P\left(\sup_{n\geq 1}\left|\frac{T_n}{b_n}\right| > t^{1/r}\right)dt \\
&\leq 1 + \int_1^{\infty} \lim_{N\to\infty} P\left(\max_{1\leq n\leq N}\left|\frac{T_n}{b_n}\right| > t^{1/r}\right)dt \\
&\leq 1 + 4C\sum_{j=1}^{\infty}\frac{\operatorname{Var}(X_j)}{b_j^2}\int_1^{\infty} t^{-2/r}dt \\
&= 1 + \frac{4Cr}{2-r}\sum_{j=1}^{\infty}\frac{\operatorname{Var}(X_j)}{b_j^2} < \infty.
\end{aligned} \tag{3.14}$$

Observe that

$$P\left(\bigcup_{n=m}^{\infty}\left(\left|\frac{T_n}{b_n}\right| > \varepsilon\right)\right) = P\left(\bigcup_{N=m}^{\infty}\left(\max_{m\leq n\leq N}\left|\frac{T_n}{b_n}\right| > \varepsilon\right)\right) = \lim_{N\to\infty} P\left(\max_{m\leq n\leq N}\left|\frac{T_n}{b_n}\right| > \varepsilon\right). \tag{3.15}$$

By Theorem 3.2, we have that

$$P\left(\max_{m\le n\le N}\left|\frac{T_n}{b_n}\right|>\varepsilon\right)\le\frac{4C}{\varepsilon^2}\left(\sum_{j=1}^{m}\frac{\operatorname{Var}(X_j)}{b_m^2}+4\sum_{j=m+1}^{N}\frac{\operatorname{Var}(X_j)}{b_j^2}\right). \tag{3.16}$$

Hence, by (3.11) and Kronecker's Lemma, it follows that

$$\lim_{m\to\infty}P\left(\bigcup_{n=m}^{\infty}\left(\left|\frac{T_n}{b_n}\right|>\varepsilon\right)\right)=0,\quad \forall\varepsilon>0, \tag{3.17}$$

which is equivalent to

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{j=1}^{n}(X_j-EX_j)=0\quad \text{a.s.} \tag{3.18}$$

So the desired results are proved. □

4. Strong Law of Large Numbers and Growth Rate for φ-Mixing Sequences

Theorem 4.1. *Let $\{X_n,\ n\ge 1\}$ be a sequence of mean zero φ-mixing random variables satisfying $\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$ and let $\{b_n,\ n\ge 1\}$ be a nondecreasing unbounded sequence of positive numbers. Assume that*

$$\sum_{n=1}^{\infty}\frac{EX_n^2}{b_n^2}<\infty, \tag{4.1}$$

then

$$\lim_{n\to\infty}\frac{S_n}{b_n}=0\quad a.s., \tag{4.2}$$

and with the growth rate

$$\frac{S_n}{b_n}=O\left(\frac{\beta_n}{b_n}\right)\quad a.s., \tag{4.3}$$

where

$$\beta_n = \max_{1\le k\le n} b_k v_k^{\delta/2}, \quad \forall\, 0<\delta<1, \quad v_n = \sum_{k=n}^{\infty} \frac{\alpha_k}{b_k^2}, \quad \lim_{n\to\infty} \frac{\beta_n}{b_n} = 0,$$
$$\alpha_k = CEX_k^2, \quad k \ge 1, \text{ where } C \text{ is defined in (2.4)}, \tag{4.4}$$

$$E\left(\max_{1\le l\le n}\left|\frac{S_l}{b_l}\right|^2\right) \le 4\sum_{l=1}^{n}\frac{\alpha_l}{b_l^2} < \infty,$$
$$E\left(\sup_{l\ge 1}\left|\frac{S_l}{b_l}\right|^2\right) \le 4\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^2} < \infty. \tag{4.5}$$

If further we assume that $\alpha_n > 0$ *for infinitely many* n, *then*

$$E\left(\sup_{l\ge 1}\left|\frac{S_l}{\beta_l}\right|^2\right) \le 4\sum_{l=1}^{\infty}\frac{\alpha_l}{\beta_l^2} < \infty. \tag{4.6}$$

Proof. By (2.4) in Theorem 2.1, we have

$$E\left(\max_{1\le k\le n}|S_k|^2\right) \le C\sum_{i=1}^{n} EX_i^2 = \sum_{k=1}^{n}\alpha_k. \tag{4.7}$$

It follows by (4.1) that

$$\sum_{n=1}^{\infty}\frac{\alpha_n}{b_n^2} = C\sum_{n=1}^{\infty}\frac{EX_n^2}{b_n^2} < \infty. \tag{4.8}$$

Thus, (4.2)–(4.6) follow from (4.7), (4.8), and Lemma 1.4 immediately. We complete the proof of the theorem. □

Theorem 4.2. *Let* $\{X_n,\ n \ge 1\}$ *be a sequence of* φ*-mixing random variables with* $\sum_{n=1}^{\infty}\varphi^{1/2}(n) < \infty$. $1 \le p < 2$. *Denote* $Q_n = \max_{1\le k\le n} EX_k^2$ *for* $n \ge 1$ *and* $Q_0 = 0$. *Assume that*

$$\sum_{n=1}^{\infty}\frac{Q_n}{n^{2/p}} < \infty, \tag{4.9}$$

then

$$\lim_{n\to\infty}\frac{1}{n^{1/p}}\sum_{i=1}^{n}(X_i - EX_i) = 0 \quad a.s., \tag{4.10}$$

and with the growth rate

$$\frac{1}{n^{1/p}}\sum_{i=1}^{n}(X_i - EX_i) = O\left(\frac{\beta_n}{n^{1/p}}\right) \quad a.s., \tag{4.11}$$

where

$$\beta_n = \max_{1\le k\le n} k^{1/p} v_k^{\delta/2}, \quad \forall 0<\delta<1, \quad v_n = \sum_{k=n}^{\infty}\frac{\alpha_k}{k^{2/p}}, \quad \lim_{n\to\infty}\frac{\beta_n}{n^{1/p}} = 0,$$
$$\alpha_k = C(kQ_k - (k-1)Q_{k-1}), \quad k \ge 1, \text{ where } C \text{ is defined in (2.4)}, \tag{4.12}$$

$$E\left(\max_{1\le l\le n}\left|\frac{S_l}{l^{1/p}}\right|^2\right) \le 4\sum_{l=1}^{n}\frac{\alpha_l}{l^{2/p}} < \infty, \tag{4.13}$$

$$E\left(\sup_{l\ge 1}\left|\frac{S_l}{l^{1/p}}\right|^2\right) \le 4\sum_{l=1}^{\infty}\frac{\alpha_l}{l^{2/p}} < \infty. \tag{4.14}$$

If further we assume that $\alpha_n > 0$ for infinitely many n, then

$$E\left(\sup_{l\ge 1}\left|\frac{S_l}{\beta_l}\right|^2\right) \le 4\sum_{l=1}^{\infty}\frac{\alpha_l}{\beta_l^2} < \infty. \tag{4.15}$$

In addition, for any $r \in (0,2)$,

$$E\left(\sup_{l\ge 1}\left|\frac{S_l}{l^{1/p}}\right|^r\right) \le 1 + \frac{4r}{2-r}\sum_{l=1}^{\infty}\frac{\alpha_l}{l^{2/p}} < \infty. \tag{4.16}$$

Proof. Assume that $EX_n = 0$, $b_n = n^{1/p}$, and $\Lambda_n = \sum_{l=1}^{n}\alpha_l$, $n \ge 1$. By (2.4) in Theorem 2.1, we can see that

$$E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}X_i\right|^2\right) \le C\sum_{i=1}^{n}EX_i^2 \le CnQ_n = \sum_{k=1}^{n}\alpha_k. \tag{4.17}$$

It is a simple fact that $\alpha_k \ge 0$ for all $k \ge 1$. It follows by (4.9) that

$$\begin{aligned}\sum_{l=1}^{\infty}\Lambda_l\left(\frac{1}{b_l^2} - \frac{1}{b_{l+1}^2}\right) &= C\sum_{l=1}^{\infty} lQ_l\left(\frac{1}{l^{2/p}} - \frac{1}{(l+1)^{2/p}}\right) \\ &\le \frac{2C}{p}\sum_{l=1}^{\infty}\frac{Q_l}{l^{2/p}} < \infty.\end{aligned} \tag{4.18}$$

That is to say (1.12) holds. By Remark 2.1 in Fazekas and Klesov [14], (1.12) implies (1.13). By Lemma 1.5, we can obtain (4.10)–(4.15) immetiately. By (4.14), it follows that

$$\begin{aligned} E\left(\sup_{l\geq 1}\left|\frac{S_l}{l^{1/p}}\right|^r\right) &= \int_0^\infty P\left(\sup_{l\geq 1}\left|\frac{S_l}{l^{1/p}}\right|^r > t\right)dt \\ &\leq 1 + \int_1^\infty P\left(\sup_{l\geq 1}\left|\frac{S_l}{l^{1/p}}\right| > t^{1/r}\right)dt \\ &\leq 1 + E\left(\sup_{l\geq 1}\left|\frac{S_l}{l^{1/p}}\right|^2\right)\int_1^\infty t^{-2/r}dt \\ &\leq 1 + \frac{4r}{2-r}\sum_{l=1}^\infty \frac{\alpha_l}{l^{2/p}} < \infty. \end{aligned} \tag{4.19}$$

The proof is completed. □

Remark 4.3. By using the maximal inequality, we get the Hájek-Rényi-type inequality, the strong law of large numbers and the strong growth rate for φ-mixing sequences. In addition, we get some new bounds of probability inequalities for φ-mixing sequences, such as (3.1), (3.5), (3.12), (4.5)–(4.6), and (4.13)–(4.16).

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