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Multilinear fractional integral operators on generalized weighted Morrey spaces

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Abstract

Let $I_{\alpha,m}$ be multilinear fractional integral operator and let $(b_1,\dots,b_m)\in(BMO)^m$. In this paper, the estimates of $I_{\alpha,m}$, the m-linear commutators $I^{\Sigma b}_{\alpha,m}$ and the iterated commutators $I^{\Pi b}_{\alpha,m}$ on the generalized weighted Morrey spaces are established.

MSC: 42B35; 42B20

Keywords: multilinear fractional integral; generalized weighted Morrey space; commutator; Muckenhoupt weight

1 Introduction and results

The classical Morrey spaces were introduced by Morrey [1] in 1938, have been studied intensively by various authors, and together with weighted Lebesgue spaces play an important role in the theory of partial differential equations; they appeared to be quite useful in the study of local behavior of the solutions of elliptic differential equations and describe local regularity more precisely than Lebesgue spaces. See [2–4] for details. Moreover, various Morrey spaces have been defined in the process of this study. Mizuhara [5] introduced the generalized Morrey space M^p_φ; Komori and Shirai [6] defined the weighted Morrey spaces $L^{p,\kappa}(\omega)$; Guliyev [7] gave the concept of generalized weighted Morrey space $M^p_\varphi(\omega)$, which could be viewed as an extension of both M^p_φ and $L^{p,\kappa}(\omega)$. The boundedness of some operators on these Morrey spaces can be seen in [5–9].

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $(\mathbb{R}^n)^m=\mathbb{R}^n\times\cdots\times\mathbb{R}^n$ be the m-fold product space ($m\in\mathbb{N}$), and let $\vec{f}=(f_1,\dots,f_m)$ be a collection of m functions on $\mathbb{R}^n$. Given $\alpha\in(0,mn)$ and $(b_1,\dots,b_m)\in(BMO)^m$. We consider the multilinear fractional integral operators $I_{\alpha,m}$ defined by

$$I_{\alpha,m}(\vec{f})(x)=\int_{(\mathbb{R}^n)^m}\frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}\,dy_1\cdots dy_m. \tag{1.1}$$

The corresponding m-linear commutators $I^{\Sigma b}_{\alpha,m}$ and the iterated commutators $I^{\Pi b}_{\alpha,m}$ defined by, respectively,

$$I^{\Sigma b}_{\alpha,m}(\vec{f})(x)=\sum_{i=1}^m\int_{(\mathbb{R}^n)^m}\frac{(b_i(x)-b_i(y_i))\prod_{j=1}^m f_j(y_j)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}\,dy_1\cdots dy_m \tag{1.2}$$

and

$$I_{\alpha,m}^{\Pi b}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{i=1}^m (b_i(x) - b_i(y_i)) f_i(y_i)}{(|x-y_1| + \cdots + |x-y_m|)^{mn-\alpha}}\, dy_1 \cdots dy_m. \tag{1.3}$$

As is well known, multilinear fractional integral operator was first studied by Grafakos [10], subsequently, by Kenig and Stein [11], Grafakos and Kalton [12]. In 2009, Moen [13] introduced weight function $A_{\vec{P},q}$ and gave weighted inequalities for multilinear fractional integral operators; In 2013, Chen and Wu [14] obtained the weighted norm inequalities for the multilinear commutators $I_{\alpha,m}^{\Sigma b}$ and $I_{\alpha,m}^{\Pi b}$. More results of the weighted inequalities for multilinear fractional integral and its commutators can be found in [15–17].

The aim of the present paper is to investigate the boundedness of multilinear fractional integral operator and its commutator on the generalized weighted Morrey spaces. Our results can be formulated as follows.

Theorem 1.1 *Let $m \geq 2$ and let $0 < \alpha < mn$. Suppose $1/p = \sum_{i=1}^m 1/p_i$, $1/q_i = 1/p_i - \alpha/mn$, and $1/q = \sum_{i=1}^m 1/q_i = 1/p - \alpha/n$, $\vec{\omega} = (\omega_1, \ldots, \omega_m)$ satisfy the $A_{\vec{p},q}$ condition with $\omega_1^{q_1}, \ldots, \omega_m^{q_m} \in A_\infty$, and $\vec{\varphi}_k = (\varphi_{k1}, \ldots, \varphi_{km})$, $k = 1, 2$, satisfy the condition*

$$\int_s^\infty \frac{\operatorname{ess\,inf}_{r<t<\infty} \prod_{i=1}^m \varphi_{1i}(x,t)(\omega_i^{p_i}(B(x,t)))^{\frac{1}{p_i}}}{\prod_{i=1}^m (\omega_i^{p_i}(B(x,r)))^{\frac{1}{p_i}}} \frac{dr}{r^{1-\alpha}} \leq C\varphi_2(x,s), \tag{1.4}$$

where $\varphi_2 = \prod_{i=1}^m \varphi_{2i}$, $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i$. If $p_1, \ldots, p_m \in (1, \infty)$, then there exists a constant C independent of $\vec{f}$ such that

$$\|I_{\alpha,m}\vec{f}\|_{M_{\varphi_2}^q(\nu_{\vec{\omega}}^q)} \leq C \prod_{i=1}^m \|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i^{p_i})}; \tag{1.5}$$

If $p_1, \ldots, p_m \in [1, \infty)$, and $\min\{p_1, \ldots, p_m\} = 1$, then there exists a constant C independent of $\vec{f}$ such that

$$\|I_{\alpha,m}\vec{f}\|_{WM_{\varphi_2}^q(\nu_{\vec{\omega}}^q)} \leq C \prod_{i=1}^m \|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i^{p_i})}. \tag{1.6}$$

Theorem 1.2 *Let $m \geq 2$ and let $0 < \alpha < mn$. Suppose $p_1, \ldots, p_m \in (1, \infty)$ with $1/p = \sum_{i=1}^m 1/p_i$, $1/q_i = 1/p_i - \alpha/mn$ and $1/q = \sum_{i=1}^m 1/q_i = 1/p - \alpha/n$, $\vec{\omega} = (\omega_1, \ldots, \omega_m)$ satisfy the $A_{\vec{p},q}$ condition with $\omega_1^{p_1}, \ldots, \omega_m^{p_m} \in A_\infty$, $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i$, and $\vec{\varphi}_k = (\varphi_{k1}, \ldots, \varphi_{km})$, $k = 1, 2$, satisfy the condition*

$$\int_s^\infty \left(1 + \ln\frac{r}{s}\right)^m \frac{\operatorname{ess\,inf}_{r<t<\infty} \prod_{i=1}^m \varphi_{1i}(x,t)(\omega_i^{p_i}(B(x,t)))^{\frac{1}{p_i}}}{\prod_{i=1}^m (\omega_i^{p_i}(B(x,r)))^{\frac{1}{p_i}}} \frac{dr}{r^{1-\alpha}} \leq C\varphi_2(x,s), \tag{1.7}$$

where $\varphi_2 = \prod_{i=1}^m \varphi_{2i}$, $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i$. If $(b_1, \ldots, b_m) \in (BMO)^m$, then there exists a constant $C > 0$ independent of $\vec{f}$ such that

$$\big\|I_{\alpha,m}^{\Sigma b}(\vec{f})\big\|_{M_{\varphi_2}^q(\nu_{\vec{\omega}}^q)} \leq C \prod_{i=1}^m \|b_i\|_* \|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i^{p_i})}; \tag{1.8}$$

and

$$\left\| I_{\alpha,m}^{\Pi b}(\vec{f}) \right\|_{M^{q}_{\varphi_2}(\nu^{q}_{\vec{\omega}})} \le C \prod_{i=1}^{m} \|b_i\|_* \|f_i\|_{M^{p_i}_{\varphi_{1i}}(\omega_i^{p_i})}. \tag{1.9}$$

2 Definitions and preliminaries

A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B = B(x_0, r_B)$ denote the ball with the center x_0 and radius r_B. For any ball B and $\lambda > 0$, λB denotes the ball concentric with B whose radius is λ times as long. For a given weight function ω and a measurable set E, we also denote the Lebesgue measure of E by $|E|$ and set weighted measure $\omega(E) = \int_E \omega(x)\,dx$.

The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy-Littlewood maximal functions in [18]. A weight ω is said to belong to A_p for $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{1-p'}\,dx\right)^{p-1} \le C, \tag{2.1}$$

where p' is the dual of p such that $1/p + 1/p' = 1$. The class A_1 is defined by replacing the above inequality with

$$\frac{1}{|B|}\int_B w(y)\,dy \le C \cdot \operatorname*{ess\,inf}_{x\in B} w(x) \quad \text{for every ball } B \subset \mathbb{R}^n. \tag{2.2}$$

A weight ω is said to belong to A_∞ if there are positive numbers C and δ so that

$$\frac{\omega(E)}{\omega(B)} \le C\left(\frac{|E|}{|B|}\right)^{\delta} \tag{2.3}$$

for all balls B and all measurable $E \subset B$. It is well known that

$$A_\infty = \bigcup_{1\le p<\infty} A_p. \tag{2.4}$$

We need another weight class $A_{p,q}$ introduced by Muckenhoupt and Wheeden in [19]. A weight function ω belongs to $A_{p,q}$ for $1 < p < q < \infty$ if there is a constant $C > 0$ such that, for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B \omega(x)^q\,dx\right)^{1/q}\left(\frac{1}{|B|}\int_B \omega(x)^{-p'}\,dx\right)^{p'} \le C. \tag{2.5}$$

When $p = 1$, ω is in the class $A_{1,q}$ with $1 < q < \infty$ if there is a constant $C > 0$ such that, for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B \omega(x)^q\,dx\right)^{1/q}\left(\operatorname*{ess\,sup}_{x\in B} \frac{1}{\omega(x)}\right) \le C. \tag{2.6}$$

Let us recall the definition of multiple weights. For m exponents $p_1,\dots,p_m$, we write $\vec{p} = (p_1,\dots,p_m)$. Let $p_1,\dots,p_m \in [1,\infty)$, $1/p = \sum_{i=1}^m 1/p_i$, and let $q > 0$. Given $\vec{\omega} = (\omega_1,\dots,\omega_m)$, set $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i$. We say that $\vec{\omega}$ satisfies the $A_{\vec{p},q}$ condition if it satisfies

$$\sup_B \left(\frac{1}{|B|} \int_B \nu_{\vec{\omega}}(x)^q \, dx \right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|B|} \int_B \omega_i(x)^{-p_i'} \, dx \right)^{1/p_i'} \le C. \tag{2.7}$$

When $p_i = 1$, $(\frac{1}{|B|}\int_B \omega_i(x)^{-p_i'}(x)\,dx)^{1/p_i'}$ is understood as $(\inf_{x\in B}\omega_i(x))^{-1}$.

Lemma 2.1 [13, 14] *Let $0 < \alpha < mn$, and $p_1,\dots,p_m \in [1,\infty)$, let $1/p = \sum_{k=1}^m 1/p_k$, and let $1/q = 1/p - \alpha/n$. If $\vec{\omega} \in A_{\vec{p},q}$, then*

$$\nu_{\vec{\omega}}^q \in A_{mq} \quad \textit{and} \quad \omega_i^{-p_i'} \in A_{mp_i'} \quad \textit{for } i = 1,\dots,m, \tag{2.8}$$

where $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i$.

Lemma 2.2 [20] *Let $m \ge 2$, $q_1,\dots,q_m \in [1,\infty)$ and $q \in (0,\infty)$ with $1/q = \sum_{i=1}^m 1/q_i$. Assume that $\omega_1^{q_1},\dots,\omega_m^{q_m} \in A_\infty$ and $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i$. Then for any ball B, there exists a constant $C > 0$ such that*

$$\prod_{i=1}^m \left(\int_B \omega_i(x)^{q_i} \, dx \right)^{q/q_i} \le C \int_B \nu_{\vec{\omega}}(x)^q \, dx. \tag{2.9}$$

Let $1 \le p < \infty$, let φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, and let ω be a non-negative measurable function on $\mathbb{R}^n$. Following [7], we denote by $M_\varphi^p(\omega)$ the generalized weighted Morrey space and the space of all functions $f \in L_{\mathrm{loc}}^p(\omega)$ with finite norm

$$\|f\|_{M_\varphi^p(w)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} \left(\frac{1}{w(B(x,r))} \|f\|_{L^p(\omega,B(x,r))}^p \right)^{1/p}, \tag{2.10}$$

where

$$\|f\|_{L^p(\omega,B(x,r))} = \int_{B(x,r)} |f(y)|^p w(y) \, dy.$$

Furthermore, by $WM_\varphi^p(\omega)$ we denote the weak generalized weighted Morrey space of all function $f \in WM_\varphi^p(\omega)$ for which

$$\|f\|_{WM_\varphi^p(w)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} \left(\frac{1}{w(B(x,r))} \|f\|_{WL^p(\omega,B(x,r))}^p \right)^{1/p}, \tag{2.11}$$

where

$$\|f\|_{WL^p(\omega,B(x,r))} = \sup_{t>0} t\big(\omega\big(\{y \in B(x,r) : |f(y)| > t\}\big)\big)^{\frac{1}{p}}.$$

(1) If $\omega = 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M_\varphi^p(\omega) = L^{p,\lambda}$ is the classical Morrey space.
(2) If $\varphi(x,r) = \omega(B(x,r))^{\frac{\kappa-1}{p}}$, then $M_\varphi^p(\omega) = L^{p,\kappa}(\omega)$ is the weighted Morrey space.

(3) If $\varphi(x,r) = \nu(B(x,r))^{\frac{\kappa}{p}}\omega(B(x,r))^{-\frac{1}{p}}$, then $M_{\varphi}^{p}(\omega) = L^{p,\kappa}(\nu,\omega)$ is the two weighted Morrey space.

(4) If $\omega = 1$, then $M_{\varphi}^{p}(\omega) = M_{\varphi}^{p}$ is the generalized Morrey space.

(5) If $\varphi(x,r) = \omega(B(x,r))^{-\frac{1}{p}}$, then $M_{\varphi}^{p}(\omega) = L^{p}(\omega)$.

Let us recall the definition and some properties of *BMO*. A locally integrable function b is said to be in *BMO* if

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B \left|b(x) - b_B\right| dx = \|b\|_* < \infty,$$

where $b_B = |B|^{-1}\int_B b(y)\,dy$.

Lemma 2.3 (John-Nirenberg inequality; see [21]) *Let $b \in BMO$. Then for any ball $B \subset \mathbb{R}^n$, there exist positive constants C_1 and C_2 such that for all $\lambda > 0$,*

$$\left|\left\{x \in B : \left|b(x) - b_B\right| > \lambda\right\}\right| \leq C_1|B|\exp\left(-C_2\lambda/\|b\|_*\right). \tag{2.12}$$

By Lemma 2.3, it is easy to get the following.

Lemma 2.4 *Suppose $\omega \in A_\infty$ and $b \in BMO$. Then for any $p \geq 1$ we have*

$$\left(\frac{1}{\omega(B)}\int_B \left|b(x) - b_B\right|^p \omega(x)\,dx\right)^{1/p} \leq C\|b\|_*. \tag{2.13}$$

Lemma 2.5 [22] *Let $b \in BMO$, $1 \leq p < \infty$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{|B(x_0,r_1)|}\int_{B(x_0,r_1)} \left|b(y) - b_{B(x_0,r_2)}\right|^p dy\right)^{\frac{1}{p}} \leq C\|b\|_*\left(1 + \left|\ln\frac{r_1}{r_2}\right|\right), \tag{2.14}$$

where $C > 0$ is independent of f, x_0, r_1, and r_2.

By Lemma 2.4 and Lemma 2.5, it is easily to prove the following results.

Lemma 2.6 *Suppose $\omega \in A_\infty$ and $b \in BMO$. Then for any $1 \leq p < \infty$ and $r_1, r_2 > 0$, we have*

$$\left(\frac{1}{\omega(B(x_0,r_1))}\int_{B(x_0,r_1)} \left|b(x) - b_{B(x_0,r_2)}\right|^p \omega(x)\,dx\right)^{1/p} \leq C\|b\|_*\left(1 + \left|\ln\frac{r_1}{r_2}\right|\right). \tag{2.15}$$

We also need the following result.

Lemma 2.7 [23] *Let f be a real-valued nonnegative function and measurable on E. Then*

$$\left(\operatorname*{ess\,inf}_{x\in E} f(x)\right)^{-1} = \operatorname*{ess\,sup}_{x\in E}\frac{1}{f(x)}. \tag{2.16}$$

At the end of this section, we list some known results about weighted norm inequalities for the multilinear fractional integrals and their commutators.

Lemma 2.8 [13] *Let $m \geq 2$ and let $0 < \alpha < mn$. Suppose $1/p = 1/p_1 + \cdots + 1/p_m$, $1/q = 1/p - \alpha/n$, $\vec{\omega} = (\omega_1, \ldots, \omega_m)$ satisfies the $A_{\vec{p},q}$ condition. If $p_1, \ldots, p_m \in (1,\infty)$, then there exists* *a constant C independent of* $\vec{f} = (f_1, \dots, f_m)$ *such that*

$$\|I_{\alpha,m}\vec{f}\|_{L^q(\nu_{\vec{\omega}}^q)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i})}. \tag{2.17}$$

If $p_1, \dots, p_m \in [1, \infty)$, *and* $\min\{p_1, \dots, p_m\} = 1$, *then there exists a constant C independent of* $\vec{f}$ *such that*

$$\|I_{\alpha,m}\vec{f}\|_{WL^q(\nu_{\vec{\omega}}^q)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i})}, \tag{2.18}$$

where $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i$.

Lemma 2.9 [14] *Let* $m \ge 2$, *let* $0 < \alpha < mn$ *and let* $(b_1, \dots, b_m) \in (BMO)^m$. *For* $1 < p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, *and* $1/q = 1/p - \alpha/n$, *if* $\vec{\omega} \in A_{\vec{p},q}$, *then there exists a constant* $C > 0$ *such that*

$$\big\|I_{\alpha,m}^{\Sigma b}(\vec{f})\big\|_{L^q(\nu_{\vec{\omega}}^q)} \le C \prod_{i=1}^m \|b_i\|_* \|f_i\|_{L^{p_i}(\omega_i^{p_i})}; \tag{2.19}$$

and

$$\big\|I_{\alpha,m}^{\Pi b}(\vec{f})\big\|_{L^q(\nu_{\vec{\omega}}^q)} \le C \prod_{i=1}^m \|b_i\|_* \|f_i\|_{L^{p_i}(\omega_i^{p_i})}, \tag{2.20}$$

where $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i$.

3 Proof of Theorem 1.1

We first prove the following conclusions.

Theorem 3.1 *Let* $m \ge 2$ *and let* $0 < \alpha < mn$. *Suppose* $1/p = \sum_{i=1}^m 1/p_i$, $1/q_i = 1/p_i - \alpha/mn$, *and* $1/q = \sum_{i=1}^m 1/q_i = 1/p - \alpha/n$, $\vec{\omega} = (\omega_1, \dots, \omega_m)$ *satisfy the* $A_{\vec{P},q}$ *condition with* $\omega_1^{q_1}, \dots, \omega_m^{q_m} \in A_\infty$. *If* $p_1, \dots, p_m \in (1, \infty)$, *then there exists a constant C independent of* $\vec{f}$ *such that*

$$\begin{aligned}\|I_{\alpha,m}\vec{f}\|_{L^q(\nu_{\vec{\omega}}^q, B(x_0,s))} \le{}& C \prod_{i=1}^m \big(\omega_i^{q_i}\big(B(x_0,s)\big)\big)^{\frac{1}{q_i}} \\ &\times \int_{2s}^\infty \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i}, B(x_0,r))} \big(\omega_i^{p_i}\big(B(x_0,r)\big)\big)^{-\frac{1}{p_i}}\right) \frac{dr}{r^{1-\alpha}}.\end{aligned} \tag{3.1}$$

If $p_1, \dots, p_m \in [1, \infty)$, *and* $\min\{p_1, \dots, p_m\} = 1$, *then there exists a constant C independent of* $\vec{f}$ *such that*

$$\begin{aligned}\|I_{\alpha,m}\vec{f}\|_{WL^q(\nu_{\vec{\omega}}^q, B(x_0,s))} \le{}& C \prod_{i=1}^m \big(\omega_i^{q_i}\big(B(x_0,s)\big)\big)^{\frac{1}{q_i}} \\ &\times \int_{2s}^\infty \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i}, B(x_0,r))} \big(\omega_i^{p_i}\big(B(x_0,r)\big)\big)^{-\frac{1}{p_i}}\right) \frac{dr}{r^{1-\alpha}},\end{aligned} \tag{3.2}$$

where $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i$.

Proof We represent f_i as $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i \chi_{B(x_0,2s)}$, $i = 1, \dots, m$, and $\chi_{B(x_0,2s)}$ denotes the characteristic function of $B(x_0, 2s)$. Then

$$\begin{aligned}\prod_{i=1}^m f_i(y_i) &= \prod_{i=1}^m \bigl(f_i^0(y_i) + f_i^\infty(y_i)\bigr)\\ &= \sum_{\alpha_1,\dots,\alpha_m \in \{0,\infty\}} f_1^{\alpha_1}(y_1) \cdots f_m^{\alpha_m}(y_m)\\ &= \prod_{i=1}^m f_i^0(y_i) + \Sigma' f_1^{\alpha_1}(y_1) \cdots f_m^{\alpha_m}(y_m),\end{aligned}$$

where each term of Σ' contains at least one $\alpha_i \neq 0$. Since $I_{\alpha,m}$ is an m-linear operator,

$$\begin{aligned}\|I_{\alpha,m}\vec{f}\|_{L^q(\nu_{\vec{\omega}}^q, B(x_0,s))} &\le C \bigl\| I_{\alpha,m}\bigl(f_1^0, \dots, f_m^0\bigr)\bigr\|_{L^q(\nu_{\vec{\omega}}^q, B(x_0,s))}\\ &\quad + C\Sigma' \bigl\| I_{\alpha,m}\bigl(f_1^{\alpha_1}, \dots, f_m^{\alpha_m}\bigr)\bigr\|_{L^q(\nu_{\vec{\omega}}^q, B(x_0,s))}\\ &= J^{0,\dots,0} + \Sigma' J^{\alpha_1,\dots,\alpha_m}\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\|I_{\alpha,m}\vec{f}\|_{WL^q(\nu_{\vec{\omega}}^q, B(x_0,s))} &\le C \bigl\| I_{\alpha,m}\bigl(f_1^0, \dots, f_m^0\bigr)\bigr\|_{WL^q(\nu_{\vec{\omega}}^q, B(x_0,s))}\\ &\quad + C\Sigma' \bigl\| I_{\alpha,m}\bigl(f_1^{\alpha_1}, \dots, f_m^{\alpha_m}\bigr)\bigr\|_{WL^q(\nu_{\vec{\omega}}^q, B(x_0,s))}\\ &= K^{0,\dots,0} + \Sigma' K^{\alpha_1,\dots,\alpha_m}.\end{aligned} \tag{3.4}$$

Then by (2.17), if $1 < p_i < \infty$, $i = 1, \dots, m$, we get

$$J^{0,\dots,0} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i}, B(x_0,2s))}. \tag{3.5}$$

By (2.18), if $\min\{p_1, \dots, p_m\} = 1$, then

$$K^{0,\dots,0} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i}, B(x_0,2s))}. \tag{3.6}$$

Applying Hölder's inequality, for $1 \le p_i \le q_i < \infty$, $i = 1, \dots, m$, we have

$$\begin{aligned}1 &\le \left(\frac{1}{|B|}\int_B \omega_i(y_i)^{p_i}\, dy_i\right)^{\frac{1}{p_i}} \left(\frac{1}{|B|}\int_B \omega_i(y_i)^{-p_i'}\, dy_i\right)^{\frac{1}{p_i'}}\\ &\le \left(\frac{1}{|B|}\int_B \omega_i(y_i)^{q_i}\, dy_i\right)^{\frac{1}{q_i}} \left(\frac{1}{|B|}\int_B \omega_i(y_i)^{-p_i'}\, dy_i\right)^{\frac{1}{p_i'}}\end{aligned}$$

for any ball $B \subset \mathbb{R}^n$. Then

$$\bigl|B(x_0, 2s)\bigr|^{m-\frac{\alpha}{n}} \le \prod_{i=1}^m \bigl(\omega_i^{q_i}\bigl(B(x_0, 2s)\bigr)\bigr)^{\frac{1}{q_i}} \bigl(\omega_i^{-p_i'}\bigl(B(x_0, 2s)\bigr)\bigr)^{\frac{1}{p_i'}}.$$

Thus, for $1 \le p_i < \infty$,

$$\begin{aligned}
&\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,2s))} \\
&\quad \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,2s))} \cdot \left|B(x_0,2s)\right|^{m-\frac{\alpha}{n}} \int_{2s}^\infty \frac{dr}{r^{mn-\alpha+1}} \\
&\quad \le C \prod_{i=1}^m \left(\omega_i^{q_i}\left(B(x_0,2s)\right)\right)^{\frac{1}{q_i}} \|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,2s))} \left(\omega_i^{-p_i'}\left(B(x_0,2s)\right)\right)^{\frac{1}{p_i'}} \int_{2s}^\infty \frac{dr}{r^{mn-\alpha+1}} \\
&\quad \le C \prod_{i=1}^m \left(\omega_i^{q_i}\left(B(x_0,2s)\right)\right)^{\frac{1}{q_i}} \\
&\qquad \cdot \int_{2s}^\infty \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,r))} \left(\omega_i^{-p_i'}\left(B(x_0,r)\right)\right)^{\frac{1}{p_i'}}\right) \frac{dr}{r^{mn-\alpha+1}}.
\end{aligned}$$

From (2.7) and Lemma 2.2 we get

$$\begin{aligned}
\prod_{i=1}^m \left(\omega_i^{-p_i'}\left(B(x_0,r)\right)\right)^{\frac{1}{p_i'}} &\le C\left|B(x_0,r)\right|^{\frac{1}{q}+\sum_{i=1}^m \frac{1}{p_i'}} \left(\int_{B(x_0,r)} \nu_{\vec{\omega}}(x)^q \, dx\right)^{-\frac{1}{q}} \\
&\le C\left|B(x_0,r)\right|^{m-\frac{\alpha}{n}} \prod_{i=1}^m \left(\omega_i^{q_i}\left(B(x_0,r)\right)\right)^{-\frac{1}{q_i}}. \qquad (3.7)
\end{aligned}$$

Using Hölder's inequality,

$$\left(\frac{1}{|B|}\int_B \omega_i(y)^{p_i}\,dy\right)^{\frac{1}{p_i}} \le \left(\frac{1}{|B|}\int_B \omega_i(y)^{q_i}\,dy\right)^{\frac{1}{q_i}}.$$

Note that $1/q_i = 1/p_i - \alpha/mn$, then

$$\left(\omega_i^{q_i}\left(B(x_0,r)\right)\right)^{-\frac{1}{q_i}} \le Cr^{\alpha/m}\left(\omega_i^{p_i}\left(B(x_0,r)\right)\right)^{-\frac{1}{p_i}}. \qquad (3.8)$$

Then for $1 \le p_i < \infty$, $i = 1, \dots, m$,

$$\begin{aligned}
\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,2s))} &\le \prod_{i=1}^m \left(\omega_i^{q_i}\left(B(x_0,2s)\right)\right)^{\frac{1}{q_i}} \\
&\quad \times \int_{2s}^\infty \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,r))} \left(\omega_i^{p_i}\left(B(x_0,r)\right)\right)^{-\frac{1}{p_i}}\right) \frac{dr}{r^{1-\alpha}}. \qquad (3.9)
\end{aligned}$$

This gives $J^{0,\dots,0}$ and $K^{0,\dots,0}$ are majored by

$$\prod_{i=1}^m \left(\omega_i^{q_i}\left(B(x_0,2s)\right)\right)^{\frac{1}{q_i}} \cdot \int_{2s}^\infty \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,r))} \left(\omega_i^{p_i}\left(B(x_0,r)\right)\right)^{-\frac{1}{p_i}}\right) \frac{dr}{r^{1-\alpha}}. \qquad (3.10)$$

For the other term, let us first consider the case when $\alpha_1 = \cdots = \alpha_m = \infty$. For any $x \in B(x_0,s)$, $y \in B(x_0,2^{j+1}s) \setminus B(x_0,2^j s)$, we have $|x-y_i| \approx |x-y_j|$ for $i \neq j$. Then

$$\begin{aligned}
&\bigl|I_{\alpha,m}\bigl(f_1^\infty,\dots,f_m^\infty\bigr)(x)\bigr| \\
&\quad\le C\int_{(\mathbb{R}^n\setminus B(x_0,2s))^m}\frac{|f_1(y_1)\cdots f_m(y_m)|}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}\,dy_1\cdots dy_m \\
&\quad\le C\sum_{j=1}^\infty\int_{(B(x_0,2^{j+1}s)\setminus B(x_0,2^j s))^m}\frac{|f_1(y_1)\cdots f_m(y_m)|}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}\,dy_1\cdots dy_m \\
&\quad\le C\sum_{j=1}^\infty\prod_{i=1}^m\int_{B(x_0,2^{j+1}s)\setminus B(x_0,2^j s)}\frac{|f_i(y_i)|}{|x-y_i|^{n-\frac{\alpha}{m}}}\,dy_i \\
&\quad\le C\sum_{j=1}^\infty\prod_{i=1}^m\biggl(\bigl(2^{j+1}s\bigr)^{-n+\frac{\alpha}{m}}\int_{B(x_0,2^{j+1}s)}\bigl|f_i(y_i)\bigr|\,dy_i\biggr).
\end{aligned}$$

Applying Hölder's inequality, it can be found that $\sup_{x\in B(x_0,s)}|I_{\alpha,m}(f_1^\infty,\dots,f_m^\infty)(x)|$ is less than

$$C\sum_{j=1}^\infty\prod_{i=1}^m\bigl(\bigl(2^{j+1}s\bigr)^{-n+\frac{\alpha}{m}}\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,2^{j+1}s))}\bigl(\omega_i^{-p_i}\bigl(B\bigl(x_0,2^{j+1}s\bigr)\bigr)\bigr)^{\frac{1}{p_i}}\bigr).$$

Hence,

$$\begin{aligned}
&\sup_{x\in B(x_0,s)}\bigl|I_{\alpha,m}\bigl(f_1^\infty,\dots,f_m^\infty\bigr)(x)\bigr| \\
&\quad\le C\sum_{j=1}^\infty\int_{2^{j+1}s}^{2^{j+2}s}\bigl(2^{j+2}s\bigr)^{-nm+\alpha-1}\Biggl(\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,2^{j+1}s))}\bigl(\omega_i^{-p_i}\bigl(B\bigl(x_0,2^{j+1}s\bigr)\bigr)\bigr)^{\frac{1}{p_i}}\Biggr)\,dr \\
&\quad\le C\sum_{j=1}^\infty\int_{2^{j+1}s}^{2^{j+2}s}\Biggl(\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,2^{j+1}s))}\bigl(\omega_i^{-p_i}\bigl(B\bigl(x_0,2^{j+1}s\bigr)\bigr)\bigr)^{\frac{1}{p_i}}\Biggr)\frac{dr}{r^{mn-\alpha+1}} \\
&\quad\le C\int_{2s}^\infty\Biggl(\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,r))}\bigl(\omega_i^{-p_i}\bigl(B(x_0,r)\bigr)\bigr)^{\frac{1}{p_i}}\Biggr)\frac{dr}{r^{mn-\alpha+1}}.
\end{aligned}$$

Substituting (3.7) and (3.8) into the above, we obtain

$$\begin{aligned}
&\sup_{x\in B(x_0,s)}\bigl|T\bigl(f_1^\infty,\dots,f_m^\infty\bigr)(x)\bigr| \\
&\quad\le C\int_{2s}^\infty\Biggl(\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,r))}\bigl(\omega_i^{p_i}\bigl(B(x_0,r)\bigr)\bigr)^{-\frac{1}{p_i}}\Biggr)\frac{dr}{r^{1-\alpha}}. \qquad (3.11)
\end{aligned}$$

Using Hölder's inequality,

$$\biggl(\int_{B(x_0,2s)}\nu_{\vec{\omega}}(x)^q\,dx\biggr)^{\frac{1}{q}}\le C\prod_{i=1}^m\bigl(\omega_i^{q_i}\bigl(B(x_0,2s)\bigr)\bigr)^{\frac{1}{q_i}}. \qquad (3.12)$$

From (3.11) and (3.12) we know $J^{\infty,\dots,\infty}$ and $K^{\infty,\dots,\infty}$ are not greater than (3.10) for $1\le p_i<\infty$, $i=1,\dots,m$.

Now we consider the case where exactly τ of the α_i are ∞ for some $1 \leq \tau < m$. We only give the arguments for one of the cases. The rest is similar and can easily be obtained from the arguments below by permuting the indices. Then for any $x \in B(x_0, s)$,

$$\begin{aligned}
&\left|I_{\alpha,m}\left(f_1^\infty, \ldots, f_\tau^\infty, f_{\tau+1}^0, \ldots, f_m^0\right)(x)\right| \\
&\quad \leq C \int_{(\mathbb{R}^n \setminus B(x_0,2s))^\tau} \int_{(B(x_0,2s))^{m-\tau}} \frac{|f_1(y_1) \cdots f_m(y_m)|}{(|x-y_1| + \cdots + |x-y_m|)^{mn-\alpha}}\, dy_1 \cdots dy_m \\
&\quad \leq C \prod_{i=\tau+1}^m \int_{B(x_0,2s)} \left|f_i(y_i)\right| dy_i \\
&\qquad \times \sum_{j=1}^\infty \frac{1}{|B(x_0, 2^{j+1}s)|^{m-\alpha/n}} \int_{(B(x_0,2^{j+1}s)\setminus B(x_0,2^j s))^\tau} \left|f_1(y_1) \cdots f_\tau(y_\tau)\right| dy_1 \cdots dy_\tau \\
&\quad \leq C \prod_{i=\tau+1}^m \int_{B(x_0,2s)} \left|f_i(y_i)\right| dy_i \cdot \sum_{j=1}^\infty \frac{1}{|B(x_0, 2^{j+1}s)|^{m-\alpha/n}} \prod_{i=1}^\tau \int_{B(x_0,2^{j+1}s)\setminus B(x_0,2^j s)} \left|f_i(y_i)\right| dy_i \\
&\quad \leq C \sum_{j=1}^\infty \prod_{i=1}^m \left(2^{j+1}s\right)^{-n+\alpha/m} \int_{B(x_0,2^{j+1}s)} \left|f_i(y_i)\right| dy_i.
\end{aligned}$$

Similar to the estimates for $J^{\infty,\ldots,\infty}$, we get

$$\begin{aligned}
&\sup_{x \in B(x_0,s)} \left|I_{\alpha,m}\left(f_1^\infty, \ldots, f_\tau^\infty, f_{\tau+1}^0, \ldots, f_m^0\right)(x)\right| \\
&\quad \leq C \int_{2s}^\infty \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i}, B(x_0,r))} \left(\omega_i^{p_i}\left(B(x_0,r)\right)\right)^{-\frac{1}{p_i}}\right) \frac{dr}{r^{1-\alpha}}.
\end{aligned} \tag{3.13}$$

Then $J^{\infty,\ldots,\infty,0,\ldots,0}$ and $K^{\infty,\ldots,\infty,0,\ldots,0}$ are all less than

$$\prod_{i=1}^m \left(\omega_i^{q_i}\left(B(x_0, 2s)\right)\right)^{\frac{1}{q_i}} \cdot \int_{2s}^\infty \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i}, B(x_0,r))} \left(\omega_i^{p_i}\left(B(x_0,r)\right)\right)^{-\frac{1}{p_i}}\right) \frac{dr}{r^{1-\alpha}}. \tag{3.14}$$

Combining the above estimates, the proof of Theorem 3.1 is completed. □

Now, we can give the proof of Theorem 1.1. From the definition of generalized weighted Morrey space, the norm of $I_{\alpha,m}(\vec{f})$ on $M^q_{\varphi_2}(\nu^q_{\vec{\omega}})$ equals

$$\sup_{x \in R^n, r>0} \varphi_2(x,s)^{-1} \left(\frac{1}{\nu^q_{\vec{\omega}}(B(x,s))} \int_{B(x,s)} \left|I_{\alpha,m}(\vec{f})(y)\right|^q \nu^q_{\vec{\omega}}(y)\, dy\right)^{1/q}. \tag{3.15}$$

By Lemma 2.2 we have

$$\left(\int_{B(x,s)} \nu^q_{\vec{\omega}}(x)\, dx\right)^{-\frac{1}{q}} \leq C \prod_{i=1}^m \left(\int_{B(x,s)} \omega_i^{q_i}(x)\, dx\right)^{-\frac{1}{q_i}}. \tag{3.16}$$

Combining (3.1) and (3.16),

$$\begin{aligned}&\left(\frac{1}{\nu_{\vec{\omega}}^q(B(x,s))}\int_{B(x,s)}\left|I_{\alpha,m}(\vec{f})(y)\right|^q\nu_{\vec{\omega}}^q(y)\,dy\right)^{1/q}\\&\quad\leq\int_{2s}^{\infty}\left(\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,r))}\left(\omega_i^{p_i}\left(B(x_0,r)\right)\right)^{-\frac{1}{p_i}}\right)\frac{dr}{r^{1-\alpha}}.\end{aligned}\tag{3.17}$$

Since $f_i\in M_{\varphi_{1i}}^{p_i}(\omega_i^{p_i})$, from Lemma 2.7 and the fact $\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x,r))}$ are all non-decreasing functions of r, we get

$$\begin{aligned}\frac{\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x,r))}}{\operatorname{ess\,inf}_{r<t<\infty}\prod_{i=1}^m\varphi_{1i}(x,t)(\omega_i^{p_i}(B(x,t)))^{\frac{1}{p_i}}}&\leq\operatorname*{ess\,sup}_{0<r<t<\infty}\frac{\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x,r))}}{\prod_{i=1}^m\varphi_{1i}(x,t)(\omega_i^{p_i}(B(x,t)))^{\frac{1}{p_i}}}\\&\leq\operatorname*{ess\,sup}_{t>0,x\in\mathbb{R}^n}\frac{\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x,t))}}{\prod_{i=1}^m\varphi_{1i}(x,t)(\omega_i^{p_i}(B(x,t)))^{\frac{1}{p_i}}}\\&\leq C\prod_{i=1}^m\|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i^{p_i})}.\end{aligned}\tag{3.18}$$

Then

$$\begin{aligned}&\int_s^{\infty}\left(\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega^{p_i},B(x,r))}\left(\omega_i^{p_i}\left(B(x,r)\right)\right)^{-\frac{1}{p_i}}\right)\frac{dr}{r^{1-\alpha}}\\&\quad=\int_s^{\infty}\frac{\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x,r))}}{\operatorname{ess\,inf}_{r<t<\infty}\prod_{i=1}^m\varphi_{1i}(x,t)(\omega_i^{p_i}(B(x,t)))^{\frac{1}{p_i}}}\\&\qquad\times\frac{\operatorname{ess\,inf}_{r<t<\infty}\prod_{i=1}^m\varphi_{1i}(x,t)(\omega_i^{p_i}(B(x,t)))^{\frac{1}{p_i}}}{\prod_{i=1}^n(\omega_i^{p_i}(B(x,r)))^{\frac{1}{p_i}}}\frac{dr}{r^{1-\alpha}}\\&\quad\leq C\prod_{i=1}^m\|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i^{p_i})}\int_s^{\infty}\frac{\operatorname{ess\,inf}_{r<t<\infty}\prod_{i=1}^m\varphi_{1i}(x,t)\omega_i(B(x,t))^{\frac{1}{p_i}}}{\prod_{i=1}^n\omega_i(B(x,r))^{\frac{1}{p_i}}}\frac{dr}{r^{1-\alpha}}.\end{aligned}\tag{3.19}$$

By (1.4) we get

$$\int_s^{\infty}\left(\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x,r))}\left(\omega_i^{p_i}\left(B(x,r)\right)\right)^{-\frac{1}{p_i}}\right)\frac{dr}{r^{1-\alpha}}\leq C\varphi_2(x,s)\prod_{i=1}^m\|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i^{p_i})}.\tag{3.20}$$

Combining (3.15), (3.17), and (3.20), then

$$\|I_{\alpha,m}\vec{f}\|_{M_{\varphi_2}^q(\nu_{\vec{\omega}}^q)}\leq C\prod_{i=1}^m\|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i^{p_i})}.$$

This completes the proof of first part of Theorem 1.1.

Similarly, the norm of $I_{\alpha,m}(\vec{f})$ on $WM_{\varphi_2}^p(\nu_{\vec{\omega}}^q)$ equals

$$\sup_{x\in R^n,r>0}\varphi_2(x,s)^{-1}\left(\frac{1}{\nu_{\vec{\omega}}^q(B(x,s))}\left\|I_{\alpha,m}(\vec{f})\right\|_{WL^q(\nu_{\vec{\omega}}^q,B(x,s))}^q\right)^{1/q}.\tag{3.21}$$

Combining (3.2) and (3.16),

$$\left(\frac{1}{\nu_{\vec{\omega}}^{q}(B(x,s))}\big\|I_{\alpha,m}(\vec{f})\big\|^{q}_{WL^{q}(\nu_{\vec{\omega}}^{q},B(x,s))}\right)^{1/q} \leq C\int_{s}^{\infty}\left(\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x,r))}\big(\omega_i^{p_i}\big(B(x,r)\big)\big)^{-\frac{1}{p_i}}\right)\frac{dr}{r^{1-\alpha}}. \tag{3.22}$$

Substituting (3.20) into (3.22),

$$\left(\frac{1}{\nu_{\vec{\omega}}^{q}(B(x,s))}\big\|I_{\alpha,m}(\vec{f})\big\|^{q}_{WL^{q}(\nu_{\vec{\omega}}^{q},B(x,s))}\right)^{1/q} \leq C\varphi_2(x,s)\prod_{i=1}^{m}\|f_i\|_{M^{p_i}_{\varphi_{1i}}(\omega_i^{p_i})}. \tag{3.23}$$

Then

$$\|I_{\alpha,m}\vec{f}\|_{WM^{q}_{\varphi_2}(\nu_{\vec{\omega}}^{q})} \leq C\prod_{i=1}^{m}\|f_i\|_{M^{p_i}_{\varphi_{1i}}(\omega_i^{p_i})}.$$

This completes the proof of second part of Theorem 1.1.

4 Proof of Theorem 1.2

Theorem 4.1 *Let $m \geq 2$ and let $0 < \alpha < mn$. Suppose $1/p = \sum_{i=1}^{m} 1/p_i$, $1/q_i = 1/p_i - \alpha/mn$, and $1/q = \sum_{i=1}^{m} 1/q_i = 1/p - \alpha/n$, $\vec{\omega} = (\omega_1,\ldots,\omega_m)$ satisfy the $A_{\vec{p},q}$ condition with $\omega_1^{q_1},\ldots,\omega_m^{q_m} \in A_\infty$, $\nu_{\vec{\omega}} = \prod_{i=1}^{m}\omega_i$. If $p_1,\ldots,p_m \in (1,\infty)$, $(b_1,\ldots,b_m) \in (BMO)^m$, then there exists a constant C independent of $\vec{f}$ such that*

$$\begin{aligned}\big\|I_{\alpha,m}^{\Sigma b}\vec{f}\big\|_{L^{q}(\nu_{\vec{\omega}}^{q},B(x_0,s))} &\leq C\prod_{i=1}^{m}\|b_i\|_*\big(\omega_i^{q_i}\big(B(x_0,s)\big)\big)^{\frac{1}{q_i}} \\ &\quad\times\int_{2s}^{\infty}\left(1+\ln\frac{r}{s}\right)^{m}\left(\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,r))}\big(\omega_i^{p_i}\big(B(x_0,r)\big)\big)^{-\frac{1}{p_i}}\right)\frac{dr}{r^{1-\alpha}}\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\big\|I_{\alpha,m}^{\Pi b}\vec{f}\big\|_{L^{q}(\nu_{\vec{\omega}}^{q},B(x_0,s))} &\leq C\prod_{i=1}^{m}\|b_i\|_*\big(\omega_i^{q_i}\big(B(x_0,s)\big)\big)^{\frac{1}{q_i}} \\ &\quad\times\int_{2s}^{\infty}\left(1+\ln\frac{r}{s}\right)^{m}\left(\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,r))}\big(\omega_i^{p_i}\big(B(x_0,r)\big)\big)^{-\frac{1}{p_i}}\right)\frac{dr}{r^{1-\alpha}},\end{aligned} \tag{4.2}$$

where $\nu_{\vec{\omega}} = \prod_{i=1}^{m}\omega_i$.

Proof We will give the proof for $I_{\alpha,m}^{\Pi b}$ because the proof for $I_{\alpha,m}^{\Sigma b}$ is very similar but easier. Moreover, for simplicity of the expansion, we only present the case $m = 2$.

We represent f_i as $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i\chi_{B(x_0,2s)}$, $i = 1,2$, and $\chi_{B(x_0,2s)}$ denotes the characteristic function of $B(x_0,2s)$. Then

$$\begin{aligned}\|I_{\alpha,2}^{\Pi b}(\vec{f})\|_{L^q(\nu_{\vec{\omega}}^q,B(x_0,s))} &\le C\left(\int_{B(x_0,s)}\left|I_{\alpha,2}^{\Pi b}(f_1^0,f_2^0)(x)\right|^q\nu_{\vec{\omega}}^q(x)\,dx\right)^{\frac{1}{q}} \\ &\quad + C\left(\int_{B(x_0,s)}\left|I_{\alpha,2}^{\Pi b}(f_1^0,f_2^\infty)(x)\right|^q\nu_{\vec{\omega}}^q(x)\,dx\right)^{\frac{1}{q}} \\ &\quad + C\left(\int_{B(x_0,s)}\left|I_{\alpha,2}^{\Pi b}(f_1^\infty,f_2^0)(x)\right|^q\nu_{\vec{\omega}}^q(x)\,dx\right)^{\frac{1}{q}} \\ &\quad + C\left(\int_{B(x_0,s)}\left|I_{\alpha,2}^{\Pi b}(f_1^\infty,f_2^\infty)(x)\right|^q\nu_{\vec{\omega}}^q(x)\,dx\right)^{\frac{1}{q}} \\ &= I + II + III + IV. \end{aligned} \tag{4.3}$$

Since $I_{\alpha,2}^{\Pi b}$ bounded from $L^{p_1}(\omega_1^{p_1}) \times L^{p_2}(\omega_2^{p_2})$ to $L^q(\nu_{\vec{\omega}}^q)$, we get

$$\left(\int_{B(x_0,s)}\left|I_{\alpha,2}^{\Pi b}(f_1^0,f_2^0)(x)\right|^q\nu_{\vec{\omega}}^q(x)\,dx\right)^{\frac{1}{q}} \le C\prod_{i=1}^2\|b_i\|_*\|f_i\|_{L^{p_i}(\omega^{p_i},B(x_0,2s))}.$$

Then by (3.9) we get

$$\begin{aligned} I &\le C\prod_{i=1}^2\|b_i\|_*\left(\omega_i^{q_i}(B(x_0,s))\right)^{\frac{1}{q_i}} \\ &\quad \cdot \int_{2s}^\infty\left(\prod_{i=1}^2\|f_i\|_{L^{p_i}(\omega^{p_i},B(x_0,r))}\left(\omega_i^{p_i}(B(x_0,r))\right)^{-\frac{1}{p_i}}\right)\frac{dr}{r^{1-\alpha}}. \end{aligned} \tag{4.4}$$

Owing to the symmetry of II and III, we only estimate II. Taking $\lambda_i = (b_i)_{B(x_0,s)}$, then

$$\begin{aligned} I_{\alpha,2}^{\Pi b}(f_1^0,f_2^\infty)(x) &= \left(b_1(x)-\lambda_1\right)\left(b_2(x)-\lambda_2\right)I_{\alpha,2}\left(f_1^0,f_2^\infty\right)(x) \\ &\quad - \left(b_1(x)-\lambda_1\right)I_{\alpha,2}\left(f_1^0,(b_2-\lambda_2)f_2^\infty\right)(x) \\ &\quad - \left(b_2(x)-\lambda_2\right)I_{\alpha,2}\left((b_1-\lambda_1)f_1^0,f_2^\infty\right)(x) \\ &\quad + I_{\alpha,2}\left((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty\right)(x) \\ &= II_1 + II_2 + II_3 + II_4. \end{aligned} \tag{4.5}$$

Similar to the estimate of (3.13), for any $x \in B(x_0,s)$ we can deduce

$$\begin{aligned} &\sup_{x\in B(x_0,s)}\left|I_{\alpha,2}\left(f_1^0,f_2^\infty\right)(x)\right| \\ &\quad \le C\int_{2s}^\infty\left(\prod_{i=1}^2\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,r))}\left(\omega_i^{p_i}(B(x_0,r))\right)^{-\frac{1}{p_i}}\right)\frac{dr}{r^{1-\alpha}}. \end{aligned} \tag{4.6}$$

By Lemma 2.1 we know $\nu_{\vec{\omega}}^{q} \in A_\infty$. Applying Hölder's inequality and (2.13), we have

$$\begin{aligned}&\left(\int_{B(x_0,s)} \left|\left(b_1(x)-\lambda_1\right)\left(b_2(x)-\lambda_2\right)\right|^q \nu_{\vec{\omega}}^{q}(x)\,dx\right)^{\frac{1}{q}}\\&\quad\le C\prod_{i=1}^{2}\left(\int_{B(x_0,s)}\left|b_i(x)-\lambda_i\right|^{2q}\nu_{\vec{\omega}}^{q}(x)\,dx\right)^{\frac{1}{2q}} \le C\prod_{i=1}^{2}\|b_i\|_*\cdot\left(\nu_{\vec{\omega}}^{q}\left(B(x_0,s)\right)\right)^{\frac{1}{q}}.\end{aligned}\tag{4.7}$$

Then by (4.6), (4.7), and (3.12), we have

$$\begin{aligned}&\left(\int_{B(x_0,s)} |II_1|^q \nu_{\vec{\omega}}^{q}(x)\,dx\right)^{\frac{1}{q}}\\&\quad\le\left(\int_{B(x_0,s)} \left|\left(b_1(x)-\lambda_1\right)\left(b_2(x)-\lambda_2\right)\right|^q \nu_{\vec{\omega}}^{q}(x)\,dx\right)^{\frac{1}{q}}\sup_{x\in B(x_0,s)}\left|I_{\alpha,2}\left(f_1^0,f_2^\infty\right)(x)\right|\\&\quad\le C\prod_{i=1}^{2}\|b_i\|_*\left(\omega_i^{q_i}\left(B(x_0,s)\right)\right)^{\frac{1}{q_i}}\\&\qquad\cdot\int_{2s}^{\infty}\left(\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(\omega_i,B(x_0,r))}\left(\omega_i^{p_i}\left(B(x_0,r)\right)\right)^{-\frac{1}{p_i}}\right)\frac{dr}{r^{1-\alpha}}.\end{aligned}\tag{4.8}$$

For any $x\in B(x_0,s)$, we have

$$\begin{aligned}&\left|I_{\alpha,2}\left(f_1^0,(b_2-\lambda_2)f_2^\infty\right)(x)\right|\\&\quad\le C\int_{B(x_0,2s)}\int_{\mathbb{R}^n\setminus B(x_0,2s)}\frac{|f_1(y_1)(b_2(y_2)-\lambda_2)f_2(y_2)|}{(|x-y_1|+|x-y_2|)^{2n-\alpha}}\,dy_1\,dy_2\\&\quad\le C\sum_{j=1}^{\infty}\left(2^{j+1}s\right)^{-2n+\alpha}\int_{B(x_0,2s)}\left|f_1(y_1)\right|dy_1\int_{B(x_0,2^{j+1}s)}\left|\left(b_2(y_2)-\lambda_2\right)f_2(y_2)\right|dy_2.\end{aligned}\tag{4.9}$$

Note that

$$\int_{B(x_0,2s)}\left|f_1(y_1)\right|dy_1\le C\|f_1\|_{L^{p_1}(\omega_1^{p_1},B(x_0,2s))}\left(\omega_1^{-p_1'}\left(B(x_0,2s)\right)\right)^{\frac{1}{p_1'}}\tag{4.10}$$

and

$$\begin{aligned}&\int_{B(x_0,2^{j+1}s)}\left|\left(b_2(y_2)-\lambda_2\right)f_2(y_2)\right|dy_2\\&\quad\le C\|f_2\|_{L^{p_2}(\omega_2^{p_2},B(x_0,2^{j+1}s))}\left\|b_2(\cdot)-\lambda_2\right\|_{L^{p_2'}(\omega_2^{-p_2'},B(x_0,2^{j+1}s))}.\end{aligned}\tag{4.11}$$

Then

$$\begin{aligned}&\sup_{x\in B(x_0,s)}\left|I_{\alpha,2}\left(f_1^0,(b_2-\lambda_2)f_2^\infty\right)(x)\right|\\&\quad\le C\sum_{j=1}^{\infty}\left(2^{j+1}s\right)^{-2n+\alpha}\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,2^{j+1}s))}\end{aligned}$$

$$\begin{aligned}
&\times \left(\omega_1^{-p_1}\left(B\left(x_0, 2^{j+1}s\right)\right)\right)^{\frac{1}{p_1}} \left\|b_2(\cdot) - \lambda_2\right\|_{L^{p_2'}(\omega_2^{-p_2'}, B(x_0, 2^{j+1}s))} \\
&\leq C \int_{2s}^{\infty} \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(\omega_i^{p_i}, B(x_0,r))} \left(\omega_1^{-p_1'}\left(B(x_0,r)\right)\right)^{\frac{1}{p_1'}} \\
&\quad \times \left\|b_2(\cdot) - \lambda_2\right\|_{L^{p_2'}(\omega_2^{-p_2'}, B(x_0,r))} \frac{dr}{r^{2n+1-\alpha}}. \qquad (4.12)
\end{aligned}$$

From Lemma 2.1 we know $\omega_2^{-p_2'} \in A_{2p_2'}$, then by Lemma 2.4 we get

$$\begin{aligned}
&\left\|b_2(\cdot) - \lambda_2\right\|_{L^{p_2'}(\omega_2^{-p_2'}, B(x_0,r))} \\
&\quad \leq C\left(\int_{B(x_0,r)} \left|b_2(z) - \lambda_2\right|^{p_2'} \omega_2^{-p_2'}(z)\,dz\right)^{\frac{1}{p_2'}} \\
&\quad \leq C\left(1 + \left|\ln \frac{r}{s}\right|\right) \|b_2\|_* (\omega_2^{-p_2'}\left(B(x_0,r)\right)^{\frac{1}{p_2'}}. \qquad (4.13)
\end{aligned}$$

By (3.7) and (3.8) we have

$$\prod_{i=1}^{2} \left(\omega_i^{-p_i'}\left(B(x_0,r)\right)\right)^{\frac{1}{p_i'}} \leq C\left|B(x_0,r)\right|^2 \prod_{i=1}^{2} \left(\omega_i^{p_i}\left(B(x_0,r)\right)\right)^{-\frac{1}{p_i}}. \qquad (4.14)$$

From (4.12), (4.13), and (4.14) we can deduce

$$\begin{aligned}
&\sup_{x \in B(x_0,s)} \left|I_{\alpha,2}\left(f_1^0, (b_2 - \lambda_2) f_2^{\infty}\right)(x)\right| \\
&\quad \leq C\|b_2\|_* \int_{2s}^{\infty} \left(1 + \ln \frac{r}{s}\right) \left(\prod_{i=1}^{2} \|f_i\|_{L^{p_i}(\omega_i^{p_i}, B(x_0,r))} \left(\omega_i^{p_i}\left(B(x_0,r)\right)\right)^{-\frac{1}{p_i}}\right) \frac{dr}{r^{1-\alpha}}. \qquad (4.15)
\end{aligned}$$

Applying (2.13) and (3.12) we have

$$\begin{aligned}
\left(\int_{B(x_0,s)} \left|b_1(x) - \lambda_1\right|^q \nu_{\vec{\omega}}^q(x)\,dx\right)^{\frac{1}{q}} &\leq C\|b_1\|_* \left(\nu_{\vec{\omega}}^q\left(B(x_0,s)\right)\right)^{\frac{1}{q}} \\
&\leq C\|b_1\|_* \prod_{i=1}^{2} \left(\omega_i^{q_i}\left(B(x_0,r)\right)\right)^{\frac{1}{q_i}}. \qquad (4.16)
\end{aligned}$$

Then by (4.15) and (4.16),

$$\begin{aligned}
&\left(\int_{B(x_0,s)} |II_2|^q \nu_{\vec{\omega}}^q(x)\,dx\right)^{\frac{1}{q}} \\
&\quad \leq \left(\int_{B(x_0,s)} \left|b_1(x) - \lambda_1\right|^q \nu_{\vec{\omega}}^q(x)\,dx\right)^{\frac{1}{q}} \sup_{x \in B(x_0,s)} \left|I_{\alpha,2}\left(f_1^0, (b_2 - \lambda_2) f_2^{\infty}\right)(x)\right| \\
&\quad \leq C \prod_{i=1}^{2} \|b_i\|_* \omega_i\left(B(x_0,s)\right)^{\frac{1}{p_i}} \\
&\qquad \times \int_{2s}^{\infty} \left(1 + \ln \frac{r}{s}\right) \left(\prod_{i=1}^{2} \|f_i\|_{L^{p_i}(\omega_i^{p_i}, B(x_0,r))} \left(\omega_i^{p_i}\left(B(x_0,r)\right)\right)^{-\frac{1}{p_i}}\right) \frac{dr}{r^{1-\alpha}}. \qquad (4.17)
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
&\left(\int_{B(x_0,s)} |II_3|^p \nu_{\vec{\omega}}(x)\,dx\right)^{\frac{1}{p}} \\
&\quad\le C\prod_{i=1}^{2}\|b_i\|_*\omega_i\big(B(x_0,s)\big)^{\frac{1}{p_i}} \\
&\qquad\times\int_{2s}^{\infty}\left(1+\ln\frac{r}{s}\right)\left(\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,r))}\big(\omega_i^{p_i}\big(B(x_0,r)\big)\big)^{-\frac{1}{p_i}}\right)\frac{dr}{r^{1-\alpha}}. \qquad (4.18)
\end{aligned}$$

For any $x\in B(x_0,s)$, with the same method of estimate for (4.15) we have

$$\begin{aligned}
&\big|I_{\alpha,2}\big((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^{\infty}\big)(x)\big| \\
&\quad\le C\sum_{j=1}^{\infty}\big(2^{j+1}s\big)^{-2n+\alpha}\prod_{i=1}^{2}\int_{B(x_0,2^{j+1}s)}\big|\big(b_i(y_i)-\lambda_i\big)f_i(y_i)\big|\,dy_i \\
&\quad\le C\int_{2s}^{\infty}\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,r)}\cdot\big\|b_i(\cdot)-\lambda_i\big\|_{L^{p_i'}(\omega_i^{-p_i'},B(x_0,r))}\frac{dr}{r^{2n-\alpha+1}} \\
&\quad\le C\prod_{i=1}^{2}\|b_i\|_*\int_{2s}^{\infty}\left(1+\ln\frac{r}{s}\right)^2 \\
&\qquad\times\left(\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,r))}\big(\omega_i^{p_i}\big(B(x_0,r)\big)\big)^{-\frac{1}{p_i}}\right)\frac{dr}{r^{1-\alpha}}. \qquad (4.19)
\end{aligned}$$

Then

$$\begin{aligned}
&\left(\int_{B(x_0,s)} |II_4|^q \nu_{\vec{\omega}}^q(x)\,dx\right)^{\frac{1}{q}} \\
&\quad\le C\big(\nu_{\vec{\omega}}^q\big(B(x_0,s)\big)\big)^{\frac{1}{q}}\sup_{x\in B(x_0.s)}\big|I_{\alpha,2}\big((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^{\infty}\big)(x)\big| \\
&\quad\le C\prod_{i=1}^{2}\|b_i\|_*\omega_i\big(B(x_0,s)\big)^{\frac{1}{p_i}} \\
&\qquad\times\int_{2s}^{\infty}\left(1+\ln\frac{r}{s}\right)^2\left(\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,r))}\big(\omega_i^{p_i}\big(B(x_0,r)\big)\big)^{-\frac{1}{p_i}}\right)\frac{dr}{r^{1-\alpha}}. \qquad (4.20)
\end{aligned}$$

Then combining (4.8), (4.17), (4.18), and (4.20) we get

$$\begin{aligned}
&\left(\int_{B(x_0,s)} |II|^q \nu_{\vec{\omega}}^q(x)\,dx\right)^{\frac{1}{q}} \\
&\quad\le C\prod_{i=1}^{2}\|b_i\|_*\omega_i\big(B(x_0,s)\big)^{\frac{1}{p_i}} \\
&\qquad\times\int_{2s}^{\infty}\left(1+\ln\frac{r}{s}\right)^2\left(\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(\omega_i^{p_i},B(x_0,r))}\big(\omega_i^{p_i}\big(B(x_0,r)\big)\big)^{-\frac{1}{p_i}}\right)\frac{dr}{r^{1-\alpha}}. \qquad (4.21)
\end{aligned}$$

Finally, we still decompose $I^{\Pi b}_{\alpha,2}(f_1^\infty, f_2^\infty)(x)$ as follows:

$$\begin{aligned} I^{\Pi b}_{\alpha,2}(f_1^\infty, f_2^\infty)(x) &= \big(b_1(x) - \lambda_1\big)\big(b_2(x) - \lambda_2\big) I_{\alpha,2}\big(f_1^\infty, f_2^\infty\big)(x) \\ &\quad - \big(b_1(x) - \lambda_1\big) I_{\alpha,2}\big(f_1^\infty, (b_2 - \lambda_2) f_2^\infty\big)(x) \\ &\quad - \big(b_2(x) - \lambda_2\big) I_{\alpha,2}\big((b_1 - \lambda_1) f_1^\infty, f_2^\infty\big)(x) \\ &\quad + I_{\alpha,2}\big((b_1 - \lambda_1) f_1^\infty, (b_2 - \lambda_2) f_2^\infty\big)(x) \\ &= IV_1 + IV_2 + IV_3 + IV_4. \end{aligned} \tag{4.22}$$

Because each term IV_j is completely analogous to II_j, $j = 1, 2, 3, 4$, being slightly different, we get the following estimate without details:

$$\begin{aligned} &\left(\int_{B(x_0,s)} |IV|^q \nu_{\vec{\omega}}^q(x)\, dx \right)^{\frac{1}{q}} \\ &\quad \le C \prod_{i=1}^{2} \|b_i\|_* \omega_i\big(B(x_0, s)\big)^{\frac{1}{p_i}} \\ &\quad\quad \times \int_{2s}^{\infty} \left(1 + \ln\frac{r}{s}\right)^2 \left(\prod_{i=1}^{2} \|f_i\|_{L^{p_i}(\omega_i^{p_i}, B(x_0,r))} \big(\omega_i^{p_i}\big(B(x_0, r)\big)\big)^{-\frac{1}{p_i}} \right) \frac{dr}{r^{1-\alpha}}. \end{aligned} \tag{4.23}$$

Summing up the above estimates, (4.2) is proved for $m = 2$. □

In the following we give the proof of Theorem 1.2. From (3.16) and (4.2),

$$\begin{aligned} &\left(\frac{1}{\nu_{\vec{\omega}}^q(B(x,s))} \int_{B(x,s)} \big| I^{\Pi b}_{\alpha,m}(\vec{f})(y)\big|^q \nu_{\vec{\omega}}^q(y)\, dy \right)^{1/q} \\ &\quad \le C \prod_{i=1}^{m} \|b_i\|_* \int_{2s}^{\infty} \left(1 + \ln\frac{r}{s}\right)^m \\ &\quad\quad \times \left(\prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega_i^{p_i}, B(x_0,r))} \big(\omega_i^{p_i}\big(B(x_0, r)\big)\big)^{-\frac{1}{p_i}} \right) \frac{dr}{r^{1-\alpha}}. \end{aligned} \tag{4.24}$$

Since $\vec{\varphi}_k$, $k = 1, 2$, satisfy the condition (1.7), and $f_i \in M^{p_i}_{\varphi_{1i}}(\omega_i^{p_i})$, by (3.18) we get

$$\begin{aligned} &\int_{2s}^{\infty} \left(1 + \ln\frac{r}{s}\right)^m \left(\prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega_i^{p_i}, B(x_0,r))} \big(\omega_i^{p_i}\big(B(x_0, r)\big)\big)^{-\frac{1}{p_i}} \right) \frac{dr}{r^{1-\alpha}} \\ &\quad = \int_{s}^{\infty} \frac{\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i}, B(x_0,r))}}{\operatorname{ess\,inf}_{r<t<\infty} \prod_{i=1}^m \varphi_{1i}(x,t)(\omega_i^{p_i}(B(x,t)))^{\frac{1}{p_i}}} \\ &\quad\quad \times \left(1 + \ln\frac{r}{s}\right)^m \frac{\operatorname{ess\,inf}_{r<t<\infty} \prod_{i=1}^m \varphi_{1i}(x,t)(\omega_i^{p_i}(B(x,t)))^{\frac{1}{p_i}}}{\prod_{i=1}^m (\omega_i^{p_i}(B(x,t)))^{\frac{1}{p_i}}} \frac{dr}{r^{1-\alpha}} \\ &\quad \le C \prod_{i=1}^{m} \|f_i\|_{M^{p_i}_{\varphi_{1i}}(\omega_i^{p_i})} \int_{s}^{\infty} \left(1 + \ln\frac{r}{s}\right)^m \frac{\operatorname{ess\,inf}_{r<t<\infty} \prod_{i=1}^m \varphi_{1i}(x,t)(\omega_i^{p_i}(B(x,t)))^{\frac{1}{p_i}}}{\prod_{i=1}^m (\omega_i^{p_i}(B(x,t)))^{\frac{1}{p_i}}} \frac{dr}{r^{1-\alpha}} \\ &\quad \le C\varphi_2(x,s) \prod_{i=1}^{m} \|f_i\|_{M^{p_i}_{\varphi_{1i}}(\omega_i^{p_i})}. \end{aligned} \tag{4.25}$$

Combining (4.24) and (4.25), we have

$$\left\|I_{\alpha,m}^{\Pi b}(\vec{f})\right\|_{M_{\varphi_2}^{q}(\nu_{\vec{\omega}}^{q})} \le C\prod_{i=1}^{m}\|b_i\|_*\|f_i\|_{M_{\varphi_{1i}}^{p_i}(\omega_i^{p_i})}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper together. They also read and approved the final manuscript.

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