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# The landscape of M-theory compactifications on seven-manifolds with $G_2$ holonomy

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ABSTRACT: We study the physics of globally consistent four-dimensional  $\mathcal{N} = 1$  supersymmetric M-theory compactifications on  $G_2$  manifolds constructed via twisted connected sum; there are now perhaps fifty million examples of these manifolds. We study a rich example that exhibits  $U(1)^3$  gauge symmetry and a spectrum of massive charged particles that includes a trifundamental. Applying recent mathematical results to this example, we compute the form of membrane instanton corrections to the superpotential and spacetime topology change in a compact model; the latter include both the (non-isolated)  $G_2$  flop and conifold transitions. The conifold transition spontaneously breaks the gauge symmetry to  $U(1)^2$ , and associated field theoretic computations of particle charges make correct predictions for the topology of the deformed  $G_2$  manifold. We discuss physical aspects of the abelian  $G_2$  landscape broadly, including aspects of Higgs and Coulomb branches, membrane instanton corrections, and some general aspects of topology change.

KEYWORDS: Superstring Vacua, M-Theory

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## 1 Introduction

The landscape of four-dimensional string compactifications with  $\mathcal{N} = 1$  supersymmetry is vast. There are a variety of corners of the landscape, and while certain special corners are well-controlled and amenable to detailed calculations, it is often true that much less can be said about physics in the broader regions. This can be true for a number of reasons:

- (1) The theory could be strongly coupled.
- (2) The theory could be at small volume.
- (3) The relevant mathematical tools might not be adequately developed.

Of course, the extent to which these are drawbacks for an understanding of any particular region of the landscape is time-dependent. However, it is sometimes the case that there are techniques that allow for control of theories at strong coupling or small volume that coincide with available mathematics. For example, the existence of construction techniques and knowledge of the relevant moduli spaces in the case of Calabi-Yau varieties allows for the study of many aspects of F-theory, despite the fact that the theory is inherently strongly coupled.

Four-dimensional  $\mathcal{N} = 1$  compactifications of M-theory with non-abelian gauge symmetry are faced with all of these issues: (1) the theory is a strongly coupled limit of the type IIA superstring, (2) the existence of non-abelian gauge symmetry requires taking a

singular limit of the compactification manifold so that there is no large-volume approximation, and (3) relatively little is known about the relevant seven-manifolds (i.e., manifolds with  $G_2$  holonomy) compared to, for example, Calabi-Yau threefolds. Though the list of such seven-manifolds has historically been rather sparse, the situation has improved in recent years due to the Kovalev “twisted connected sum” (TCS) construction [1] which has been generalized and corrected in recent years [2–5]. The list of TCS examples is now large enough to warrant speaking of a landscape of four-dimensional  $\mathcal{N} = 1$  M-theory compactifications on seven-manifolds with  $G_2$  holonomy, which we will refer to as the “abelian  $G_2$  landscape” since these are compactifications on smooth manifolds and hence have no non-abelian gauge symmetry.

How large is the abelian  $G_2$  landscape? Saying something quantitative requires being more specific about what one means. Drawing a sharp analogy to type IIB vacua, one could mean the number of de Sitter vacua and its dependence on the choice of M-theory flux. However, making this analogy reliably within the supergravity approximation would require restricting attention to vacua which do not exhibit non-abelian gauge symmetry. Moreover, even in that approximation the one-instanton effects from wrapped membranes likely play an important role in moduli stabilization, and though we will discuss progress in this direction it is not yet possible to say whether the known instanton corrections are the leading instanton corrections. The comparison to type IIB flux vacua is further complicated by the fact that a classical flux superpotential may not play as significant of a role in M-theory, since all geometric moduli may in principle be stabilized by non-perturbative effects.

Perhaps a coarser comparison is more appropriate to estimate the size of the abelian  $G_2$  landscape: how many suitable compactification manifolds exist, and how does this compare to the number of analogous manifolds used for type IIB compactifications? This is really where the recent gains [1, 2, 5] have been made. Since it is very familiar to physicists, it is worth comparing to the Kreuzer-Skarke classification [6] of Calabi-Yau threefold hypersurfaces in certain four-dimensional toric varieties. In this case, there are four-dimensional toric varieties associated to any reflexive polytope via its triangulations, and there are 473,800,776 such polytopes. However, the Calabi-Yau hypersurfaces associated to this list exhibit only 30,108 distinct Hodge pairs  $(h^{1,1}, h^{2,1})$ ; though there is other topological data that may distinguish the Calabi-Yau hypersurfaces, many of them may be different realizations of the same Calabi-Yau. The heuristic lesson is that the many different Calabi-Yau “building blocks” do not necessarily give rise to distinct Calabi-Yau manifolds. By comparison, the TCS construction of  $G_2$  manifolds uses a pair of suitable “building blocks” and if there is a “matching pair” of building blocks then a TCS  $G_2$  manifold can be constructed from them, though  $G_2$  manifolds constructed from the same matching pair may be topologically equivalent. In fact there are now [3, 5] at least fifty million such matching pairs, and it stands to reason that the abelian  $G_2$  landscape is now quite large.

Aside from the “landscape” implied by a large number of example compactifications, there is also evidence for a stronger notion of the word, as some topology changing transitions between branches of  $G_2$  moduli space are known, and we will study the physics of one such example. Given these two facts, it is natural to wonder whether a version of

Reid’s fantasy [7] for Calabi-Yau threefolds also holds for  $G_2$  manifolds; perhaps the many  $G_2$  moduli spaces now known by the TCS construction are part of one large connected irreducible  $G_2$  landscape.

The purpose of this paper, which is complementary to our work [8] on singular limits of  $G_2$  compactifications, is to introduce the TCS construction into the physics literature, study a rich example in detail, and to discuss what can be said broadly about the physics of the abelian  $G_2$  landscape using currently available mathematical results. In section 2 we will review  $G_2$  manifolds,  $G_2$ -structures, and the TCS construction. We encourage the reader to read the following outline carefully, since it also serves as a summary of our results.

In section 3 we will study a TCS  $G_2$  compactification and three branches of its moduli space. This globally consistent compact model exhibits abelian gauge symmetry and massive charged particles, a limit in moduli in which some particles become massless, non-perturbative instanton corrections to the superpotential, spontaneous symmetry breaking, and spacetime topology change via a non-isolated flop or conifold transition.

We begin in section 3.1 by introducing a TCS  $G_2$  manifold studied in [5] that we call  $X$ , focusing on one of the building blocks of that manifold; since  $b_2(X) = 3$ , the gauge symmetry of M-theory compactified on  $X$  is  $U(1)^3$ . We review the construction of that building block presented in [3], but perform new computations of topological intersections. We show that these intersections in the building block determine two-cycle and five-cycle intersections in  $X$ , which in turn determine the charges of massive particles arising from M2-branes wrapped on two-cycles.

There are 24 different massive charged particles, since there are 24 rigid holomorphic curves in the building block that become two-cycles in  $X$ . In section 3.2 we compute their charges, which happen to include a trifundamental. By a result of [5], to each of these rigid holomorphic curves in the building block there is an associated rigid associative submanifold in  $X$  diffeomorphic to  $S^2 \times S^1$ ; an M2-brane instanton wrapped on such a cycle is expected to generate a non-perturbative correction to the superpotential [9]. We explicitly compute the form of the non-perturbative superpotential in our example, which happens to be intricate. There are 24 rigid associatives in six different homology classes with four representatives each. These generate a six term non-perturbative superpotential, each with a prefactor 4 that is the  $G_2$  analog of a Gromov-Witten invariant, and the total non-perturbative superpotential depends on three moduli fields. This superpotential is a generalized racetrack.

In section 3.3 we study limits in  $G_2$  moduli space in which some of the charged particles become massless. This is achieved using calibrated geometry and a specific property of the calibrated three-cycles, namely, that they contain non-trivial two-cycles. The limit shrinks four three-cycles and two-cycles of the same respective classes to zero size, yielding four circles of conifold points. We call this  $G_2$  limit  $X_c$ , and M-theory compactified on  $X_c$  has  $U(1)^3$  gauge symmetry with instanton corrections and four massless particles with the same charge, as well as a spectrum of massive particles. Relative to  $X_c$ ,  $X$  was a small resolution of the four circles of conifold points; the other small resolution gives a  $G_2$  manifold  $X_s$  related to  $X$  via a (non-isolated) flop transition, and M-theory on  $X$

and  $X_s$  have similar physics. The circles of conifold points can also be smoothed by a deformation within  $G_2$  moduli to give a  $G_2$  manifold  $X_d$ , where M-theory on  $X_d$  has  $U(1)^2$  gauge symmetry. The gauge symmetry has been spontaneously broken, and a simple field theoretic prediction related to the charges of the Higgs fields correctly predicts aspects of the topology of  $X_d$ , and the associated spectrum of massive particles for M-theory on  $X_d$ . These constructions are analogous to the original flop [10–12] and conifold [13] transitions for Calabi-Yau threefolds, as well as their recent extension to transitions for Calabi-Yau fourfolds [14].<sup>1</sup>

In section 4 we broadly discuss the physics of the abelian  $G_2$  landscape as determined by the topology of the known TCS  $G_2$  manifolds. For technical reasons, nearly all of the known examples have  $b_2 = 0$  and so do not exhibit even abelian gauge symmetry, and therefore they must be Higgs branches from the field theory viewpoint if singular limits with non-abelian gauge symmetry exist for them. As we will have already shown in section 3, however, examples with abelian gauge symmetry do exist and may arise in one of a few different ways that we review. We discuss membrane instanton corrections to the superpotential and potential implications for moduli stabilization; a number of known examples exhibit more than 40 such corrections, and generalized racetracks are to be expected (as seen in section 3). In [5] a number of general statements are made about the topology of possible  $G_2$  transitions; we comment on the associated physical implications. We also discuss some common model-building assumptions in light of the existence of these new vacua.

In section 5 we conclude, briefly discussing needed mathematical progress that would be physically useful, as well as future physical prospects, including for abelian de Sitter vacua.

## 2 $G_2$ manifolds from twisted connected sums

In this section we review Kovalev’s construction [1] for obtaining compact  $G_2$  manifolds from twisted connection sums. We will review those results that would not be prudent to review in the middle of the physics discussions of sections 3 or 4, for reasons of length or relevance.

The basic construction glues appropriate matching pairs of “building blocks”, each comprised of an algebraic threefold times a circle, to give a  $G_2$  manifold. In Kovalev’s original work, the building blocks were constructed from Fano threefolds having a K3 surface in their anticanonical class. Recently it has been shown [5] by Corti, Haskins, Nordström, and Pacini (CHNP) that weak-Fano threefolds — which require only  $-K \cdot C \geq 0$  for  $K$  the canonical class and  $C$  any holomorphic curve rather than the strict inequality characteristic of Fano threefolds — can also serve as appropriate building blocks. While a seemingly small change, this adapted construction increases the number of matching pairs (and thus  $G_2$  manifolds) by orders of magnitude, from hundreds or thousands to tens of millions.

For further details on the content of this section, we refer the reader to [5].

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<sup>1</sup>Note that our transitions are different from the  $G_2$  flop [15] and  $G_2$  conifold [16] transitions which have previously been studied in a non-compact setting, since those involved isolated singularities.

**General aspects of  $G_2$  manifolds.** Before presenting the TCS construction, let us review some basic facts about  $G_2$  manifolds in general, as well as some conventions we will use throughout.

A  $G_2$ -structure on a seven-manifold  $X$  is a principal subbundle of the frame bundle of  $X$  that has structure group  $G_2$ . Practically, each  $G_2$  structure is characterized by a three-form  $\Phi$  and a metric  $g_\Phi$  such that every tangent space of  $X$  admits an isomorphism with  $\mathbb{R}^7$  that identifies  $g_\Phi$  with  $g_0 \equiv dx_1^2 + \dots + dx_7^2$  and  $\Phi$  with

$$\Phi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}, \quad (2.1)$$

where  $dx_{ijk} \equiv dx_i \wedge dx_j \wedge dx_k$ . Note that the subgroup of  $GL(7, \mathbb{R})$  which preserves  $\Phi_0$  is the exceptional Lie group  $G_2$  [17]. The three-form  $\Phi$ , sometimes called the  $G_2$ -form, determines an orientation, the Riemannian metric  $g_\Phi$ , and a Hodge star  $\star_\Phi$  which we will often shorten to  $\star$ . We will refer to the pair  $(\Phi, g_\Phi)$  as a  $G_2$ -structure.

For a seven-manifold  $X$  with a  $G_2$ -structure  $(\Phi, g_\Phi)$  and associated Levi-Civita connection  $\nabla$ , the *torsion* of the  $G_2$ -structure is  $\nabla\Phi$ , and when  $\nabla\Phi = 0$  the  $G_2$  structure is said to be *torsion-free*. The following are equivalent:

- $Hol(g_\Phi) \subseteq G_2$
- $\nabla\Phi = 0$ , and
- $d\Phi = d\star\Phi = 0$ .

The triple  $(X, \Phi, g_\Phi)$  is called a  $G_2$ -manifold if  $(\Phi, g_\Phi)$  is a torsion-free  $G_2$ -structure on  $X$ . Then by the above equivalence, the metric  $g_\Phi$  has  $Hol(g_\Phi) \subseteq G_2$  and  $g_\Phi$  is Ricci-flat. For a compact  $G_2$ -manifold  $X$ ,  $Hol(g_\Phi) = G_2$  if and only if  $\pi_1(X)$  is finite [18]. In this case the moduli space of metrics with holonomy  $G_2$  is a smooth manifold of dimension  $b_3(X)$ .

Calibrated geometry will be important in our work. In the absence of explicit metric knowledge, as is typically the case for compact Calabi-Yau or  $G_2$  manifolds, the volumes of certain cycles can nevertheless be computed via calibrated geometry as developed in the seminal work of Harvey and Lawson [19]. Their fundamental observation is the following. Let  $X$  be a Riemannian manifold and  $\alpha$  a closed  $p$ -form such that  $\alpha|_\xi \leq vol_\xi$  for all oriented tangent  $p$ -planes  $\xi$  on  $X$ . Then any compact oriented  $p$ -dimensional submanifold  $T$  of  $X$  with the property that  $\alpha|_T = vol_T$  is a minimum volume representative of its homology class, that is

$$vol(T) = \int_T \alpha = \int_{T'} \alpha \leq vol(T') \quad (2.2)$$

for any  $T'$  such that  $[T - T'] = 0$  in  $H_p(X, \mathbb{R})$ . Note in particular the useful fact that  $vol(T)$  is computed precisely by  $\int_T \alpha$ , even though one may not know the metric on  $X$ .

If  $X$  is a Calabi-Yau threefold, the Kähler form  $\omega$  and the holomorphic three-form  $\Omega$  are calibration forms for two-cycles and three-cycles; they calibrate holomorphic curves and special Lagrangian submanifolds. Note, therefore, in M-theory compactifications on  $X$  the presence of calibrated two-cycles allows for control over massive charged particle states obtained from wrapped M2-branes. This computes particle masses as a function of moduli.

If  $X$  is a  $G_2$  manifold,  $\Phi$  and  $\star\Phi$  are calibration forms which calibrate so-called associative three-cycles and coassociative four-cycles, respectively. This allows for control over topological defects obtained from wrapping M2-branes and M5-branes on calibrated three-cycles and four-cycles; these are instantons, domain walls, and strings. Note the absence of calibrated two-cycles, however.

**$G_2$  structures on product manifolds.** Let  $V$  be a Kähler manifold of complex dimension 3 with Kähler form  $\omega$ , and suppose  $V$  has a nowhere-vanishing holomorphic 3-form  $\Omega$  satisfying the basic Calabi-Yau condition that  $\Omega \wedge \bar{\Omega}$  is a constant times the volume form  $\frac{1}{3!}\omega \wedge \omega \wedge \omega$ . (Notice that we are not insisting that  $V$  be compact.) Multiplying  $\Omega$  by a suitable real constant if necessary, we may assume that

$$\frac{i}{8}\Omega \wedge \bar{\Omega} = \frac{1}{3!}\omega \wedge \omega \wedge \omega. \tag{2.3}$$

Then the product manifold  $S^1 \times V$  has a natural  $G_2$  structure whose  $G_2$ -form is

$$\Phi := d\varphi \wedge \omega + \text{Re}(\Omega), \tag{2.4}$$

where  $\varphi$  is an angular coordinate on the circle.<sup>2</sup>

To see this, we let  $z_1, z_2, z_3$  be complex coordinates on  $V$  for which  $\Omega = dz_1 \wedge dz_2 \wedge dz_3$  and  $\omega = \frac{i}{2}(dz_1 \wedge \bar{dz}_1 + dz_2 \wedge \bar{dz}_2 + dz_3 \wedge \bar{dz}_3)$ . We let  $\varphi = x_1$ ,  $z_1 = x_2 + ix_3$ ,  $z_2 = x_4 + ix_5$ ,  $z_3 = x_6 + ix_7$ . Then a brief calculation gives

$$\begin{aligned} \Omega &= (dx_2 + idx_3) \wedge (dx_4 + idx_5) \wedge (dx_6 + idx_7) \\ \text{Re}(\Omega) &= dx_{246} - dx_{257} - dx_{347} - dx_{356} \\ \omega &= dx_2 \wedge dx_3 + dx_4 \wedge dx_5 + dx_6 \wedge dx_7 \\ d\varphi \wedge \omega &= dx_{123} + dx_{145} + dx_{167}. \end{aligned} \tag{2.5}$$

It follows that  $d\varphi \wedge \omega + \text{Re}(\Omega)$  is a  $G_2$ -form. Of course, the holonomy on the product manifold  $S^1 \times V$  is actually a subgroup of  $SU(3)$  rather than being all of  $G_2$ .

A variant of this construction leads to the “barely  $G_2$  manifolds” studied by Joyce [20, 21] and Harvey-Moore [9]: if  $V$  has an anti-holomorphic involution  $\alpha$  which maps  $\omega \mapsto -\omega$  and  $\Omega \rightarrow \bar{\Omega}$ , then  $(-1, \alpha)$  preserves the  $G_2$  form on  $S^1 \times V$  and so leads to a  $G_2$  structure on the quotient  $(S^1 \times V)/(-1, \alpha)$ . If  $\alpha$  has no fixed points, then this quotient is again a seven-manifold. (This is a case with holonomy contained in  $SU(3) \rtimes \mathbb{Z}_2$  rather than being all of  $G_2$ .)

Finally, one of the key building blocks for the TCS construction is a  $G_2$  manifold  $S^1 \times V$  in which  $V$  is itself the product of  $\mathbb{C}^*$  with a K3 surface  $S$ . To define the Calabi-Yau structure on  $V$ , we must specify both a Ricci-flat Kähler form  $\omega_S$  and a holomorphic 2-form  $\Omega_S$  on the K3 surface  $S$ , and for this purpose we use the normalization  $\Omega_S \wedge \bar{\Omega}_S = 2\omega_S \wedge \omega_S$  which implies

$$\text{Re}(\Omega_S) \wedge \text{Re}(\Omega_S) = \text{Im}(\Omega_S) \wedge \text{Im}(\Omega_S) = \omega_S \wedge \omega_S. \tag{2.6}$$

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<sup>2</sup>Notice that the phase of  $\Omega$  can be varied, which varies the  $G_2$  structure on  $S^1 \times V$ .



This is the normalization familiar in hyperKähler geometry, because in this case the triple  $(\omega_S, \text{Re}(\Omega_S), \text{Im}(\Omega_S))$  is an orthogonal basis of the space of self-dual harmonic 2-forms on  $S$ , and all basis elements have the same norm in  $H^2(S, \mathbb{R})$ . In fact, given any rotation in  $\text{SO}(3)$ , we can change the complex structure on  $S$  without changing the underlying Ricci-flat metric in such a way as to apply the given rotation to the basis  $(\omega_S, \text{Re}(\Omega_S), \text{Im}(\Omega_S))$ .

We now choose a complex linear coordinate  $z = e^{t+i\theta}$  on  $\mathbb{C}^*$  and define, on  $V = \mathbb{C}^* \times S$ ,

$$\begin{aligned}\omega &= \frac{i dz \wedge \bar{d}z}{2z\bar{z}} + \omega_S = dt \wedge d\theta + \omega_S \\ \Omega &= -i \frac{dz}{z} \wedge \Omega_S = (d\theta - i dt) \wedge \Omega_S,\end{aligned}\tag{2.7}$$

so that

$$\text{Re}(\Omega) = d\theta \wedge \text{Re}(\Omega_S) + dt \wedge \text{Im}(\Omega_S).\tag{2.8}$$

Such a  $V$ , equipped with  $\omega$  and  $\Omega$ , is called a *Calabi-Yau cylinder*, and the map  $\xi : V \rightarrow \mathbb{R}$  defined by  $\xi(z, x) = \log |z|$  is called the *cylinder projection*.<sup>3</sup>

For a Calabi-Yau cylinder  $V$ , the three-form

$$\Phi = d\varphi \wedge dt \wedge d\theta + d\varphi \wedge \omega_S + d\theta \wedge \text{Re}(\Omega_S) + dt \wedge \text{Im}(\Omega_S).\tag{2.9}$$

on  $S^1 \times V$  defines a  $G_2$  structure<sup>4</sup> with a very interesting property which is the basis of the TCS construction. Because the Ricci-flat metric on  $S$  is hyperKähler, we can change the complex structure on  $S$  (without changing the underlying Ricci flat metric) to obtain a new K3 surface  $\Sigma$  with Kähler form  $\omega_\Sigma$  and holomorphic 2-form  $\Omega_\Sigma$  such that  $\omega_\Sigma = \text{Re}(\Omega_S)$ ,  $\text{Re}(\Omega_\Sigma) = \omega_S$ , and  $\text{Im}(\Omega_\Sigma) = -\text{Im}(\Omega_S)$ . Then if we send  $(\varphi, t, \theta, S)$  to  $(\theta, -t, \varphi, \Sigma)$ , the  $G_2$  structure is unchanged!

**Preliminaries for twisted connected sums.** We will need the notion of an *asymptotically cylindrical* (ACyl) Calabi-Yau threefold, but before giving the detailed definitions we'd like to state the basic idea. The TCS construction of compact  $G_2$  manifolds utilizes two complex threefolds which “asymptote” in a particular way that allows for a particular gluing procedure. The complex threefold will be an ACyl Calabi-Yau threefold, which is defined to asymptote to a Calabi-Yau cylinder.

Let  $V$  be a complete, but not necessarily compact, Calabi-Yau threefold on which a Ricci-flat Kähler form  $\omega$  and a holomorphic three-form  $\Omega$  have been specified. We say  $V$  is an *asymptotically cylindrical* (ACyl) Calabi-Yau threefold if there is a compact set  $K \subset V$ , a Calabi-Yau cylinder  $V_\infty$  with cylinder projection  $\xi_\infty : V_\infty \rightarrow \mathbb{R}$ , and a diffeomorphism  $\eta : \xi_\infty^{-1}(0, \infty) \rightarrow V \setminus K$  such that  $\forall k \geq 0$ , some  $\lambda > 0$ , and as  $t \rightarrow \infty$

$$\begin{aligned}\eta^* \omega - \omega_\infty &= d\rho, & \text{for some } \rho \text{ such that } |\nabla^k \rho| &= O(e^{-\lambda t}) \\ \eta^* \Omega - \Omega_\infty &= d\zeta, & \text{for some } \zeta \text{ such that } |\nabla^k \zeta| &= O(e^{-\lambda t})\end{aligned}\tag{2.10}$$

<sup>3</sup>In earlier papers, the term “Calabi-Yau cylinder” was used for only half of this space, namely,  $\xi^{-1}(0, \infty)$ .

<sup>4</sup>To verify the normalization condition, we define  $\omega_0 = \frac{i dz \wedge \bar{d}z}{2z\bar{z}}$  so that  $\omega = \omega_0 + \omega_S$  and compute:

$$\frac{i}{8} \Omega \wedge \bar{\Omega} = \frac{i}{8} \frac{dz}{z} \wedge \frac{\bar{d}z}{\bar{z}} \wedge \Omega_S \wedge \bar{\Omega}_S = \frac{1}{4} \omega_0 \wedge \Omega_S \wedge \bar{\Omega}_S = \frac{1}{2} \omega_0 \wedge \omega_S \wedge \omega_S = \frac{1}{6} \omega \wedge \omega \wedge \omega.$$



where  $\nabla$  and  $|\cdot|$  are defined using the Calabi-Yau metric  $g_\infty$  on  $V_\infty$ . We refer to  $V_\infty = \mathbb{R}^+ \times S^1 \times S$  as the *asymptotic end* of  $V$  and the associated hyperKähler K3 surface  $S$  as the *asymptotic K3 surface* of  $V$ .

Since the TCS is a powerful construction technique for building compact  $G_2$  manifolds from elementary parts, it is of fundamental importance to be able to construct the parts themselves. Namely, we would like to have a theorem specifying how to construct ACyl Calabi-Yau threefolds from Kähler threefolds in a simple way. This is as follows [3]. Let  $Z$  be a closed Kähler threefold with a morphism  $f : Z \rightarrow \mathbb{P}^1$  that has a reduced smooth K3 fiber  $S$  with class  $[S] = -[K_Z]$ , and let  $V = Z \setminus S$ . If  $\Omega_S$  is a nowhere vanishing  $(2,0)$ -form on  $S$  and the Kähler form  $\omega_S$  is the restriction of a Kähler class on  $Z$ , then  $V$  has a metric that makes it into an ACyl Calabi-Yau threefold which asymptotes to a Calabi-Yau cylinder satisfying (2.7).

In addition to this, Corti Haskins Nordström and Pacini [5] make some additional assumptions that simplify the calculation of topological invariants for their  $G_2$  manifolds. To this end, let  $Z$  be a nonsingular algebraic 3-fold  $Z$  together with a projective morphism  $f : Z \rightarrow \mathbb{P}^1$ . Such a  $Z$  is a *building block* if:

- (i) the anticanonical class  $-K_Z \in H^2(Z)$  is primitive.
- (ii)  $S = f^*(\infty)$  is a nonsingular K3 surface in the anticanonical class.
- (iii) The cokernel of the restriction map  $H^2(Z, \mathbb{Z}) \rightarrow H^2(S, \mathbb{Z})$  is torsion-free.
- (iv) The group  $H^3(Z)$ , and thus also  $H^4(Z)$ , is torsion-free.

The original blocks of Kovalev [1] were Fano threefolds, while Kovalev-Lee [2] utilized building blocks with non-symplectic involutions on the K3s. The broadest class of building blocks to date utilize weak-Fano three-folds due to CHNP [3, 5]. These will be reviewed momentarily, but let us first introduce the TCS construction since it does not require a specific type of building block.

**Compact  $G_2$  manifolds from twisted connected sums.** We now review Kovalev’s twisted connected sum construction for compact  $G_2$  manifolds. The basic idea is to glue two ACyl Calabi-Yau threefolds in a particular way which ensures the existence of a  $G_2$  metric. To do so, the ACyl Calabi-Yau threefolds must be compatible.

Let  $V_\pm$  be a pair of asymptotically cylindrical Calabi-Yau threefolds with Kähler forms  $\omega_\pm$  and holomorphic three-forms  $\Omega_\pm$  specified. Then by definition  $V_\pm$  asymptotes to one end of a Calabi-Yau cylinder, i.e.,  $V_{\infty,\pm} = \mathbb{R}^+ \times S_\pm^1 \times S_\pm$  where  $S_\pm$  is the asymptotic hyperKähler K3 surface of  $V_\pm$ . Of course, these are real six-manifolds, and we must add a seventh dimension and glue appropriately. To add the seventh dimension, define the seven-manifolds  $M_\pm = S_\pm^1 \times V_\pm$  and let  $\theta_\pm$  be the standard coordinate on the  $S^1$ . Since  $V_\pm$  asymptotes to a Calabi-Yau cylinder,  $M_\pm$  asymptotes to a circle product with a Calabi-Yau cylinder. Now suppose that there exists a diffeomorphism  $r : S_+ \rightarrow S_-$ , preserving the

Ricci-flat metric, such that

$$\begin{aligned}
 r^*(\omega_{S_-}) &= \text{Re}(\Omega_{S_+}) \\
 r^*(\text{Re}(\Omega_{S_-})) &= \omega_{S_+} \\
 r^*(\text{Im}(\Omega_{S_-})) &= -\text{Im}(\Omega_{S_+})
 \end{aligned}
 \tag{2.11}$$

Then we can glue the seven-manifolds  $M_{\pm}$  in their asymptotic regions as follows: on the region in  $\mathbb{R}^+$  defined by  $t \in (T, T + 1)$  consider the diffeomorphism

$$\begin{aligned}
 F : \quad M_+ \cong S_-^1 \times \mathbb{R}^+ \times S_+^1 \times S_+ &\longrightarrow S_+^1 \times \mathbb{R}^+ \times S_-^1 \times S_- \cong M_-, \\
 (\theta_-, t, \theta_+, x) &\longmapsto (\theta_+, T + 1 - t, \theta_-, r(x))
 \end{aligned}
 \tag{2.12}$$

There are  $G_2$  structures on these asymptotic regions; see e.g. [5] for their detailed structure, since they won't be critical for us. By truncating each  $M_{\pm}$  at  $t = T + 1$  we obtain a pair of compact seven-manifolds  $M_{\pm}(T)$  with boundaries  $S_+^1 \times S_-^1 \times S_{\pm}$ ; then they can be glued together with the diffeomorphism  $F$  to form a *twisted connected sum* seven-manifold  $M_r = M_+(T) \cup_F M_-(T)$ . This is a compact seven-manifold which admits a closed  $G_2$  structure that is determined by the  $G_2$  structures on  $M_{\pm}$ ; however, they are not a priori torsion-free. This leads to:

**Theorem 1** [Kovalev's Theorem] *Let  $(V_{\pm}, \omega_{\pm}, \Omega_{\pm})$  be two ACyl Calabi-Yau three-folds with asymptotic ends of the form  $\mathbb{R}^+ \times S^1 \times S_{\pm}$  for a pair of hyperKähler K3 surfaces  $S_{\pm}$ , and suppose that there exists a diffeomorphism  $r : S_+ \rightarrow S_-$  preserving the Ricci-flat metrics and satisfying (2.11). Define the twisted connected sum  $M_r$  as above with closed  $G_2$  structure  $\Phi_{T,r}$ . Then for sufficiently large  $T$  there is a torsion-free perturbation of  $\Phi_{T,r}$  within its cohomology class; call this torsion-free  $G_2$  structure  $\Phi$ .*

Since a torsion-free  $G_2$  structure determines a metric with holonomy exactly  $G_2$ ,  $(M_r, \Phi)$  is a compact seven-manifold with holonomy  $G_2$ .

We call a diffeomorphism  $r : S_+ \rightarrow S_-$  preserving the Ricci-flat metrics and satisfying 2.11 a *Donaldson matching*.<sup>5</sup>

In summary, to build twisted connected sum  $G_2$  manifolds, one needs only appropriate building blocks  $Z_{\pm}$  and to choose an appropriate Donaldson matching on associated asymptotic hyperKähler K3 surfaces  $S_{\pm}$ . One aspect of the recent progress [5] by Corti, Haskins, Nordström, and Pacini was to greatly enlarge the known set of building blocks compared to those originally considered by Kovalev [1], who utilized *building blocks of Fano type* built from Fano threefolds, and by Kovalev-Lee [2], who utilized *building blocks of non-symplectic type*; the new larger class of building blocks of [5] utilize weak-Fano threefolds.

Let us explain how the authors of [5] obtain ACyl Calabi-Yau threefolds from semi-Fano threefolds. First, a *weak Fano threefold* is a nonsingular projective complex threefold  $Y$  such that the anticanonical class  $-K_Y$  satisfies  $-K_Y \cdot C \geq 0$  for any compact algebraic curve  $C \subset Y$ , and furthermore  $(-K_Y)^3 > 0$ . Since the latter is an even integer, the *anticanonical*

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<sup>5</sup>This is usually called a “hyper-Kähler rotation” in the literature, but in fact it is a very particular type of hyper-Kähler rotation and we prefer a different name. According to §11.9 of [18], the use of such a diffeomorphism for a gluing construction of  $G_2$  manifolds was first proposed by Donaldson.

degree  $(-K_Y)^3$  can be defined in terms of the genus  $g_Y$  of  $Y$  as  $(-K_Y)^3 = 2g_Y - 2$ . There is a key fact about any smooth weak Fano threefold that is important for constructing building blocks: a general divisor in the anticanonical class is a non-singular K3 surface. CHNP make the additional assumption that the linear system  $|-K_Y|$  contains two non-singular members  $S_0$  and  $S_\infty$  which intersect transversally. Most weak Fano threefolds satisfy this assumption.

A *building block of semi-Fano type* is defined as followed. Let  $Y$  be a semi-Fano threefold with torsion-free  $H^3(Y)$ ,  $|S_0, S_\infty| \subset |-K_Y|$  a generic pencil with smooth base locus  $C$ , and take  $S \in |S_0, S_\infty|$  generic. Furthermore, let  $Z$  be the blow-up of  $Y$  at  $C$ . Then  $S$  is a smooth K3 surface and its proper transform in  $Z$  is isomorphic to  $S$ . The pair  $(Z, S)$  constructed in this way is called a semi-Fano building block. Then the image of  $H^2(Z, \mathbb{Z}) \rightarrow H^2(S, \mathbb{Z})$  equals that of  $H^2(Y, \mathbb{Z}) \rightarrow H^2(S, \mathbb{Z})$ , and furthermore the latter map is injective; this will be important in a theorem on the cohomology of the  $G_2$  manifold which we will review in the next section. Other relevant technical statements and remarks can be found in section 3 of [5].

**Topology of the twisted connected sum  $G_2$  manifolds.** Studying the topology of a TCS  $G_2$  manifold  $X$  is critical for understanding the physics of the associated M-theory vacuum. As  $X$  is constructed from elementary building blocks, its topology is determined by the topology of the building blocks and the gluing map. Though most of the discussion holds for general building blocks, we will occasionally comment on results specific to the use of building blocks of semi-Fano type. For more details see section 4 of [5].

The fundamental group and the Betti numbers were computed for early examples of  $G_2$  manifolds, but thanks to [5], it is now possible to compute the full integral cohomology for many twisted connected sums, including the torsional components of  $H^3(X, \mathbb{Z})$  and  $H^4(X, \mathbb{Z})$ , as well as the first Pontryagin class  $p_1$ . If it weren't for a general observation that we will discuss, the explicit knowledge of  $p_1$  in examples would play a critical role [22] in determining the quantization of M-theory flux in those examples.

The second integral cohomology of a twisted connected sum  $G_2$  manifold  $X$  is given by

$$H^2(X, \mathbb{Z}) = (N_+ \cap N_-) \oplus K_+ \oplus K_- \tag{2.13}$$

where  $N_\pm$  is the image of  $H^2(Z_\pm, \mathbb{Z})$  in  $H^2(S_\pm, \mathbb{Z})$ , and  $K_\pm := \ker(\rho_\pm)$  where we have

$$\rho_\pm : H^2(V_\pm, \mathbb{Z}) \longrightarrow H^2(S_\pm, \mathbb{Z}) \tag{2.14}$$

being the natural restriction maps.<sup>6</sup> Intuitively, the contributions of  $K_\pm$  to  $H^2(X, \mathbb{Z})$  are non-trivial classes on the ACyl Calabi-Yau threefolds  $V_\pm$  which restrict trivially to the K3 surfaces  $S_\pm$ , and therefore the gluing map (which twists the classes of the K3 surfaces according to the Donaldson matching  $r$ ) will not affect elements of  $K_\pm$ ; they become non-trivial in  $X$ , as well. Alternatively, classes which restrict non-trivially to the K3's are

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<sup>6</sup>There is a neighborhood of  $S$  in  $Z$  which is diffeomorphic to a product of  $S$  with a disk. The cohomology groups of all the nearby K3 surfaces to  $S$  in this neighborhood can be identified with those of  $S$ , so restricting a cohomology class to any one of those nearby K3 surfaces gives a cohomology class on  $S$  itself.

subject to the gluing map, and therefore only classes in the intersection  $N_+ \cap N_-$ , become non-trivial classes in  $H^2(X, \mathbb{Z})$ .

For our purposes we will not need to know the full third cohomology of  $X$ , instead only that it contains three-forms related to the building blocks

$$H^3(X, \mathbb{Z}) \supset H^3(Z_+, \mathbb{Z}) \oplus H^3(Z_-, \mathbb{Z}) \oplus K_+ \oplus K_- \quad (2.15)$$

and we refer the reader to Theorem 4.9 of [5] for the full result. This is an interesting result: for any  $\alpha_{\pm} \in K_{\pm}$  on one of the building blocks, we have an associated non-trivial two-form and threeform on  $X$ , arising as [5]

$$\alpha_{\pm} \in K_{\pm} \quad \longleftrightarrow \quad \alpha_{\pm} \in H^2(X, \mathbb{Z}) \quad \text{and} \quad \alpha_{\pm} \wedge d\theta_{\mp} \in H^3(X, \mathbb{Z}) \quad (2.16)$$

where again  $\theta_{\mp}$  is the coordinate on  $S^1_{\mp}$ . So it is precisely clear how a non-trivial two-form on a building block can give both non-trivial two-forms and three-forms on  $X$ .

Finally, we will utilize some results of [5] regarding the existence of associative submanifolds, both rigid and not. There are two known ways in which to obtain associative submanifolds in TCS  $G_2$  manifolds. Recall that  $V$  is an ACyl Calabi-Yau threefold  $V$  of a TCS  $G_2$  building block. The first result is that if  $L \subset V$  is a compact special Lagrangian submanifold with  $b_1(L) = 0$  and  $L$  is non-trivial in the relative homology  $H_3(V, S^1 \times S)$ , then there is a small deformation of  $L$  in  $X$  which is an associative threefold. This associative is not rigid. The second result is that if  $C$  is a rigid holomorphic curve in  $V$ , then a small deformation of  $S^1 \times C$  in  $X$  is a rigid associative. This latter result is significant. It gives the first construction technique for compact rigid associative submanifolds in compact  $G_2$  manifolds, and therefore it is now possible to compute the form of membrane instanton corrections to the superpotential in examples; see the following.

### 3 A rich example

In this section we study an explicit example of [5], performing a number of new computations necessary to uncover interesting physical aspects of this M-theory vacuum, as well as studying topology changing transitions to other  $G_2$  manifolds and M-theory vacua.

For the  $G_2$  manifold  $X$  that we study, we will show that M-theory on  $X$  yields an  $\mathcal{N} = 1$  supersymmetric four-dimensional supergravity theory at low energies with  $U(1)^3$  gauge symmetry and a spectrum of massive charged particles including trifundamentals. Vacua exhibiting  $U(1)^3$  gauge symmetry and trifundamental matter were also recently discovered [23] among F-theory compactifications. Using the topological progress of [5], we also compute the form of membrane instanton corrections to the superpotential (for the first time in a compact model) and  $G_2$  topology changing transitions. The transitions of  $X$  to other  $G_2$  manifolds include both the (non-isolated)  $G_2$  flop and  $G_2$  conifold transitions; in the former *both* two-cycles and three-cycles collapse and re-emerge in a different topology, while in the latter two-cycles and three-cycles collapse, but only a three-cycle emerges after deformation. Physically, the non-isolated conifold transition breaks the  $U(1)^3$  gauge symmetry of M-theory on  $X$  to  $U(1)^2$  in the usual way.

Before delving into details, we'd like to state the basic mathematical idea that gives rise to the interesting physics. We will study an example from [5] which utilizes one building block with  $K \neq 0$ , which is known to fit into a matching pair giving rise to a  $G_2$  manifold with  $H^2(X, \mathbb{Z}) = K \cong \mathbb{Z}^3$ , and therefore the associated M-theory vacuum exhibits  $U(1)^3$  gauge symmetry. We will compute the topological intersections of two-cycles with five-cycles in the  $G_2$  manifold to determine the charges of massive particles on this M-theory vacuum; in this example these intersections are conveniently related to intersections in the algebraic threefold of the building block. Given the homology classes of some rigid holomorphic curves we will determine the homology classes of their associated rigid associative threefolds; this determines the moduli dependence of some instanton corrections to the superpotential, and we find a six term generalized racetrack in three moduli. We will then study topology change in detail, where the non-isolated  $G_2$  flop and conifold transitions occur via movement in  $G_2$  moduli and can be understood in terms of induced flop and conifold transitions in the building block.

As a brief physical review, consider a compactification of M-theory on a smooth  $G_2$  manifold  $X$  at large volume.<sup>7</sup> Its metric is determined by a torsion-free  $G_2$  form  $\Phi \in H^3(X)$ , and  $\Psi \equiv \star\Phi$  is the dual four-form. This compactification gives a four-dimensional  $\mathcal{N} = 1$  theory with an associated massless effective action obtained from Kaluza-Klein reduction of 11-dimensional supergravity; see [26] for more details. It exhibits

$$\begin{aligned}
 & b_2(X) \text{ abelian vector multiplets from } C_3\text{-reduction along } \sigma \in H^2(X) \\
 & \qquad \qquad \qquad \text{and} \\
 & b_3(X) \text{ neutral chiral multiplets from } \int_T(\Phi + iC_3) \text{ for all } T \in H_3(X),
 \end{aligned}$$

where  $C_3$  is the M-theory three-form. We emphasize that the gauge group is  $G = U(1)^{b_2(X)}$ ; it is an abelian theory without any massless charged particles; however, massive charged particles can arise from M2-branes wrapped on two-cycles.

### 3.1 The $G_2$ manifold and relevant building block

We wish to study an example from [5] where  $b_2(X) \neq 0$  arises from the fact that one of the building blocks has  $K \neq 0$ . While we will focus mostly on a particular building block of that type since it gives rise to the physics we are interested in, it is worth noting that it does form a matching pair with building blocks from example 7.1 of [5]; the associated  $G_2$  manifolds have  $H^2(X, \mathbb{Z}) = K_+$ , where  $K_+$  is the  $K$  lattice of the building block we study in detail. From now on we will drop  $\pm$  subscripts, focusing only on the building block of interest.

As discussed, one way to obtain a building block with  $K \neq 0$  is to blow up an algebraic threefold along a non-generic (rather than generic) anticanonical pencil. The example we use is example 4.8 of [3], and the threefold we begin with is the simplest one,  $Y = \mathbb{P}^3$ . Consider the non-generic pencil  $|S_0, S_\infty| \subset |\mathcal{O}(4)|$  with  $S_0$  the tetrahedron

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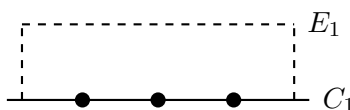
<sup>7</sup>Outside of the strict large volume approximation, M-theory compactifications with non-abelian gauge sectors are sometimes well approximated by a combined supergravity and super Yang-Mills action; see [24, 25].

$S = \{x_1x_2x_3x_4 = 0\}$  and  $S_\infty$  a generic non-singular quartic surface which meets all coordinate planes  $x_i = 0$  transversely. The base locus of the pencil is the union of four non-singular curves  $C_i := \{x_i = 0\} \cap S_\infty$  where since

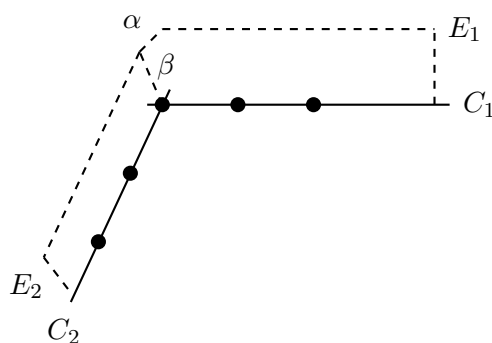
$$\chi(TC_i) = \int_{C_i} c_1(TC_i) = \int_{4H^2} (-H) = -4 = 2 - 2g \tag{3.1}$$

we see that  $C_i$  is a genus 3 curve.  $Z$  is obtained from  $Y$  by blowing up the base curves  $C_i$  one at a time, and the associated ACyl Calabi-Yau threefold is  $V = Z \setminus S$ .

For simplicity let  $Z$  be obtained by blowing up along  $C_1, C_2, C_3, C_4$  in that order. Associated to these blow-ups are four exceptional divisors  $E_i$ , giving  $h^2(Z) = h^2(Y) + 4 = 5$ . After the first blowup, the geometry appears as



where the dashed box denotes the exceptional divisor  $E_1$  obtained by blowing up along  $C_1$  and the three dots represent the intersections of  $C_1$  with  $C_2, C_3,$  and  $C_4$ , four times each. Blowing up again, we obtain



where now we see  $E_1$  and  $E_2$ , the exceptional divisors of the consecutive blow-ups along  $C_1$  and  $C_2$ .  $E_1$  and  $E_2$  are fibrations over  $C_1$  and  $C_2$  with generic fibers being curves of class  $\gamma_1$  and  $\gamma_2$ . The dot at the intersection of  $C_1$  and  $C_2$  represents their four intersection points, and the additional dots on  $C_1$  and  $C_2$  represent their four intersections with  $C_3$  and  $C_4$  respectively. The jagged dashed curve represents the inverse image of  $C_1 \cdot C_2$ , which is a singular curve that is a reducible variety with two components  $\alpha$  and  $\beta$ ; these are curves of class  $\gamma_1 - \gamma_2$  and  $\gamma_2$ , respectively, In fact since  $C_1 \cdot C_2$  occurs at four points there are actually four rigid holomorphic curves of class  $\gamma_1 - \gamma_2$ . We have shown these images primarily to demonstrate the appearance of such holomorphic curves, but also to give intuition for the geometry. Blow-ups three and four proceed in a similar fashion, but are harder to draw.

After performing all of the blow-ups, we would like to know the effective curves in  $Z$  and their intersections with divisors.  $E_i$  is an exceptional divisor that is fibered over  $C_i$  with generic fiber a  $\mathbb{P}^1$   $\gamma_i$  that moves in families. Above the points  $C_i \cdot C_{j>i}$  the fiber is a rigid holomorphic curve of class  $\gamma_i - \gamma_{j>i}$ ; given the six possible choices of  $i, j$  and the

fact that  $C_i \cdot C_j$  is a set of four points, this yields curves in six homology classes with four representatives each, for a total of 24 rigid holomorphic curves. Note that none of these curve classes can be written as a positive linear combination of two others.

How do the exceptional divisors  $E_i$  intersect the curves  $\gamma_j$ ? Choose a general fiber in  $E_j$ ; this is a curve of class  $\gamma_j$  and it clearly does not intersect any exceptional divisor  $E_{i \neq j}$ . On the other hand, since the rigid curves of class  $\gamma_2 - \gamma_1$  are contained in  $E_1$  and are transverse to  $E_2$ , so  $E_2 \cdot (\gamma_1 - \gamma_2) = 1$  and therefore  $E_2 \cdot \gamma_2 = -1$ . However, computing  $E_1 \cdot \gamma_1$  cannot be done by counting points and must be done indirectly. To do so we use a few simple facts. First,  $\gamma_1$  is a rational curve contained in  $E_1$ , and therefore  $-\chi(\gamma_1) = 2g - 2 = -2$ . Alternatively  $-\chi(\gamma_1) = -\int_{\gamma_1} c_1(\gamma_1) = (K_{E_1} + \gamma_1) \cdot \gamma_1 = K_{E_1} \cdot \gamma_1$  where the last equality holds because  $\gamma_1$  moves in  $E_1$ . Letting our blow-up be  $\pi : Z \rightarrow Y$ , then  $K_Z = \pi^*K_Y + E_1 + E_2 + E_3 + E_4$  and adjunction therefore gives  $K_{E_1} = (K_Z + E_1)|_{E_1} = (\pi^*(K_Y) + 2E_1 + E_2 + E_3 + E_4)|_{E_1}$ . Putting it all together

$$-2 = -\chi(\gamma_1) = (\pi^*(K_Y) + 2E_1 + E_2 + E_3 + E_4)|_{E_1} \cdot \gamma_1 = 2E_1 \cdot \gamma_1 \quad (3.2)$$

The last equality holds because  $E_{j>1} \cdot \gamma_1 = 0$  from above and also since a generic canonical divisor in  $Y$  misses a generic point in  $C_1$  and therefore the rational curve of class  $\gamma_1$  above such a point in the blowup  $Z$ . We therefore obtain  $E_1 \cdot \gamma_1 = -1$  and overall have

$$E_i \cdot \gamma_j = -\delta_{ij} \quad (3.3)$$

which we will use to compute physically relevant intersections in a moment.

Before doing so, we compute  $K$  in order to determine the number of  $U(1)$  symmetries and their generators (in cohomology). Recalling that  $V = Z \setminus S$  and  $K = \ker(\rho)$  with

$$\rho : H^2(V) \rightarrow H^2(S), \quad (3.4)$$

the restriction map and where  $b_2(V) = 4$  since we've subtracted out  $S$ . Now, since each  $E_i$  is a fibration over a curve  $C_i$  which itself is a curve in  $S$  of class  $H|_S$ , then  $\rho(E_i) = H|_S$  and therefore  $\rho(E_i - E_j) = 0 \in H^2(S)$ ; i.e.,  $E_i - E_j \in K$  and in fact we will choose a basis  $E_1 - E_2$ ,  $E_1 - E_3$  and  $E_1 - E_4$  for  $K$ ; call these  $D_1$ ,  $D_2$ , and  $D_3$  respectively. We also see that  $K$  is rank three, and since the second cohomology of a TCS  $G_2$  manifold  $H^2(X, \mathbb{Z}) \supset K$  then we have at least three  $U(1)$  symmetries (and in fact choosing the other building block as in [5] we will have precisely three.) By choosing generators of  $K$  we have chosen a basis for the three associated  $U(1)$ 's in the M-theory compactification. One way to see this is to note that  $D_i \times S^1$  now are three non-trivial five-cycles in  $X$  which have dual non-trivial two forms, which give rise to  $U(1)$  symmetries. We would like to compute the intersection of five-cycles with two-cycles since these determine the charges of massive particles in  $G_2$  compactifications if positive volume cycles exist. For example, for us the positive two-cycles are the ones obtained from holomorphic curves in the building block.

Since the two-cycles and associated five-cycles in  $X$  we are studying come "from one end" of the  $G_2$  manifold, i.e., from one building block rather than from the intersection  $N_+ \cap N_-$  of the  $N$  lattices of the two different building blocks, we can compute the intersections of these five-cycles and two-cycles one one end. These intersections in  $X$  are determined by



intersections of the relevant divisors and curves in  $V$ . Additionally, since the divisors we are interested in generate  $K$  they do not intersect  $S$ , and therefore any intersection with a curve  $\gamma$  happens away from  $S$ , so that in all

$$(D_i \times S^1) \cdot_X \gamma = D_i \cdot_V \gamma = D_i \cdot_Z \gamma \tag{3.5}$$

and thus we simply need to compute intersections in  $Z$ .

### 3.2 Massive charged particles and instanton corrections

Let us now study the charged particles in the theory. These arise from M2-branes wrapped two-cycles in  $X$ ; since  $E_i$  contains  $\gamma_i$  and only differences of the  $E_i$ 's are in  $K$ , curves in  $Z$  of class  $\gamma_i$  do not become two-cycles in  $X$ . The rigid holomorphic curves of class  $\gamma_i - \gamma_{j>1}$  do become two-cycles in  $X$ , however. An M2-brane on a curve  $\gamma$  gives a particle of charge  $(D_i \times S^1) \cdot_X \gamma = D_i \cdot_Z \gamma =: Q_i$  under  $U(1)_i$ . Using (3.3) and naming the particles arising from an M2-brane on a two-cycle of class  $\gamma_i - \gamma_j$  to be  $\Psi_{ij}^k$  with  $k = 1, \dots, 4$ , we compute the charges

	$Q_1$	$Q_2$	$Q_3$
$\Psi_{12}^k$	-2	-1	-1
$\Psi_{13}^k$	-1	-2	-1
$\Psi_{14}^k$	-1	-1	-2
$\Psi_{23}^k$	1	-1	0
$\Psi_{24}^k$	1	0	-1
$\Psi_{34}^k$	0	1	-1

for these massive particles. As the particles are massive, they necessarily arise as vector pairs so that for any  $\Psi$  there is another chiral multiplet  $\bar{\Psi}$  with opposite charge, so that the superpotential  $W$  contains a term  $m_\Psi \Psi \bar{\Psi}$ . These latter fields arise from anti M2-branes. Since these particles arise from rigid holomorphic curves in one of the building blocks, there are representatives of this two-cycle class within a compact rigid associative submanifold.

There are 24 rigid holomorphic curves in  $V = Z \setminus S$  in six different homology classes  $\gamma_i - \gamma_{j>i}$ , with four representatives of each class. As discussed in section 2, to each such curve there is a compact rigid associative in  $X$ , giving 24 compact rigid associatives in  $X$ , also in six different homology classes  $T_i - T_{j>i} \in H_3(X, \mathbb{Z})$  since the curves come in six different classes. M2-branes wrapped on rigid associative cycles are expected to generate instanton corrections [9] to the superpotential;<sup>8</sup> for these rigid associatives we have constructed the associated superpotential takes the form

$$W \supset 4(A_1 e^{-\Phi_1} + A_2 e^{-\Phi_2} + A_3 e^{-\Phi_3} + A_4 e^{\Phi_1 - \Phi_2} + A_5 e^{\Phi_1 - \Phi_3} + A_6 e^{\Phi_2 - \Phi_3}) \tag{3.6}$$

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<sup>8</sup>Determining whether or not the instanton corrects the superpotential requires a careful analysis of instanton zero modes. The rigidity condition ensures the absence of deformation modes that would otherwise kill the superpotential correction, but it may be the case in some models that the Wilson line modulini associated to the  $S^1$  in the rigid associative also kill the superpotential correction. Whether or not this is the case (in particular whether the modulini are lifted by interactions) seems to be model dependent, and we leave this analysis to future work.

where the factor of 4 is because there are four rigid associatives (and therefore four instanton corrections) per class and  $\Phi_1$ ,  $\Phi_2$ , and  $\Phi_3$  are the moduli associated to  $T_1 - T_2$ ,  $T_1 - T_3$ , and  $T_1 - T_4$ , respectively. While there may be other rigid associatives which also give rise to instanton corrections, we see at the very least that any TCS  $G_2$  manifold constructed from this building block realizes a six term generalized racetrack in four different moduli fields.

### 3.3 Massless limits, topology change, and the Higgs mechanism

In this section we will study singular limits in  $G_2$  moduli space and topology changing transitions. In the singular limit the massive charged particles of the last section will become massless. In one transition we will perform a non-isolated  $G_2$  flop to another branch of the moduli space in which these particles are massive; in another we will perform a non-isolated  $G_2$  conifold transition in which one of the  $U(1)$  symmetries is broken and there are particles charged under the remaining  $U(1)$  symmetries. More complicated conifolds also exist in this example. (Non-compact realizations of *isolated*  $G_2$  flop and conifold transitions were studied in [15, 24] and [16], respectively.)

We discuss a potential technical obstruction before turning to details. Taking a singular limit in which particles become massless requires gaining some control over two-cycles: given the lack of a calibration form for two-cycles, how might one do this? Our basic idea begins with noting that the rigid associative threefolds in  $X$  appeared because of the existence of rigid holomorphic curves in the building block, and in this example these curves also became non-trivial two-cycles in  $X$ . If one flopped a curve in the threefold building block then sometimes the topology of  $X$  itself changes. Since these curves sit inside rigid associative threefolds, it is natural to expect that by sending the associative to zero volume by tuning in  $G_2$  moduli, the curve within it might also collapse. However, since the rigid associatives of [5] are diffeomorphic to  $S^2 \times S^1$ , and *a priori* the  $S^1$  rather than the  $S^2$  might collapse. So one would like evidence in moduli that when the rigid associative threefold collapses, the  $S^2$  within it also collapses. We will argue that this should be expected when the (non-isolated)  $G_2$  flop and conifold transitions arise from transitions on one of the building blocks.

Before studying degenerations of the  $G_2$  manifold, we would like to understand degenerations and topology change in the building block. First note that the four successive blow-ups of the last section together gave a birational map  $Z \rightarrow Y$  that was not crepant; i.e., the canonical class of the variety changed in the process and therefore  $Y$  should be viewed as an auxiliary variety useful for constructing  $Z$ , but not related to  $V$  in moduli in the way we would like.

Instead, the variety related to  $Z$  which we would like to consider is  $P$ , the variety obtained via blowing down all of 24 rigid holomorphic curves, which therefore has 24 conifold points.  $Z$  can be obtained via a sequence of blow-ups along divisors  $X_i \equiv \{x_i = 0\}$

$$Z = P_{4321} \xrightarrow{\pi_1} P_{432} \xrightarrow{\pi_2} P_{43} \xrightarrow{\pi_3} P_4 \xrightarrow{\pi_4} P \tag{3.7}$$

where  $\pi_i$  is the blow-up along  $X_i$ .  $P$  is simply the total space of the pencil  $|S_0, S_\infty|$

$$P = \{(x, \lambda) \mid x \in S_\lambda, \text{ and } S_\lambda \in |S_0, S_\infty|\} \subset \mathbb{P}^3 \times \mathbb{P}^1 \tag{3.8}$$

with parameter  $\lambda$ . Note here that each successive blowup adds holomorphic curves in reverse order: in  $Z \rightarrow Y$  the successive blow-ups yielded 0, 4, 8, and 12 holomorphic curves respectively, whereas in  $Z \rightarrow P$  the successive blow-ups yield 12, 8, 4 and 0 holomorphic curves, respectively. This is a simple consequence of blowing up along divisors  $X_i$  rather than curves  $C_i$ . For example,  $\pi_4$  blows up along  $X_4$  which contains 12 conifold points, coming in three sets of four where  $x_4 = x_j = 0$  for  $j = 1, 2, 3$ . Therefore  $\pi_4$  produces 12 curves, and the next blowup  $\pi_3$  along  $X_3$  resolves the 8 conifold points at  $x_3 = x_k = 0$  for  $k = 1, 2$ , producing 8 curves, etc. The last blow-up giving rise to curves is  $\pi_2$  which produces the curves in class  $\gamma_1 - \gamma_2$  in  $Z$ .

To study topology change, we first blow down to the singular variety

$$Z \xrightarrow{\pi} P_{43} \tag{3.9}$$

where  $\pi = \pi_1 \circ \pi_2$ . This map blows down the four rigid holomorphic curves in class  $\gamma_1 - \gamma_2$  to four conifold points, which are the only singularities in  $P_{43}$ . From  $P_{43}$ , we may perform another small resolution of the conifold points (i.e., not  $\pi$ ) which flops the curve  $\gamma_1 - \gamma_2$ , or we may deform the conifold points.

The other small resolution proceeds in the usual way. The divisor we blow up along,  $X_2$ , is a non-Cartier Weil divisor that passes through the conifold points. One such point is locally of the form

$$yx_2 = zw \tag{3.10}$$

and there are two resolutions of the conifold corresponding to blowing up along  $x_2 = z = 0$  or  $x_2 = w = 0$ ; though codimension two in the local  $\mathbb{C}^4$  ambient space, this blow-up is codimension one in the hypersurface, i.e., along the divisor  $X_2$ .

The deformation is more subtle, but can be understood by first thinking of a deformation of  $P$  and relating it to  $P_{43}$ . Recalling that  $P$  is of the form  $(x, \lambda)$ , for  $\lambda = 0$  this is  $(S_0, \lambda)$  where  $S_0 = \{x_1x_2x_3x_4\} = 0$ , we can perform a deformation  $P \xrightarrow{\pi_\epsilon} P_\epsilon$  by deforming  $S_0 \rightarrow S_{0,\epsilon}$  where

$$S_{0,\epsilon} = \{(x_1x_2 + \epsilon Q_2)x_3x_4 = 0\} \tag{3.11}$$

in terms of a quadric  $Q_2$  in  $x_1, x_2, x_3, x_4$ . For simplicity also define  $Q \equiv x_1x_2 + \epsilon Q_2$  and note that we recover  $S_0$  in the  $\epsilon \rightarrow 0$  limit. Then  $P_\epsilon$  is just the total space of the pencil  $|S_{0,\epsilon}, S_\infty|$

$$P_\epsilon = \{(x, \lambda) \mid x \in |S_{0,\epsilon}, S_\infty|\} \subset \mathbb{P}^3 \times \mathbb{P}^1 \tag{3.12}$$

and  $P_\epsilon$  has 20 conifold points instead of 24; there are 8 at  $\{Q = x_3 = 0\} \cap S_\infty$ , 8 at  $\{Q = x_4 = 0\} \cap S_\infty$ , and 4 at  $\{x_3 = x_4 = 0\} \cap S_\infty$ , so the deformation smoothed four conifold points.

The reason for deforming  $P$  in this way — that is, “picking” out the  $x_1$  and  $x_2$  coordinates to put in  $Q$  rather than some other set — is that the four conifold points that were lost were those resolved by the third of the four blow-ups, i.e.,  $\pi_2$ . This means that we can blow up  $P_\epsilon$  in the same order, to give new deformed varieties as

$$P_{43,\epsilon} \xrightarrow{\pi_3} P_{4,\epsilon} \xrightarrow{\pi_4} P_\epsilon, \tag{3.13}$$

with  $P_{43,\epsilon}$  smooth. This is because the 12 conifold points  $\{Q = x_4 = 0\} \cup \{x_3 = x_4 = 0\}$  are resolved by  $\pi_4$  and the 8 conifold points  $\{Q = x_3 = 0\}$  are resolved by  $\pi_3$ . Then the map

$$P_{43} \xrightarrow{\pi_3} P_4 \xrightarrow{\pi_4} P \xrightarrow{\pi_\epsilon} P_\epsilon \xrightarrow{\pi_4^{-1}} P_{4,\epsilon} \xrightarrow{\pi_3^{-1}} P_{43,\epsilon} \quad (3.14)$$

means that there is a deformation of  $P_{43}$ , which has four conifold points, to  $P_{43,\epsilon}$  which is smooth.

In summary, we have a map  $Z \rightarrow P_{43}$  which blows down four rigid rational curves of class  $\gamma_1 - \gamma_2$ , yielding four conifold points. There is another small resolution of  $Z' \rightarrow P_{43}$  where  $Z$  and  $Z'$  are related via a flop transition, where the curves of class  $\gamma_1 - \gamma_2$  are flopped. There is also a deformation  $P_{43} \rightarrow P_{43,\epsilon}$  which deforms the four conifold points in  $P_{43}$ , so that  $Z$  and  $Z'$  are related to  $P_{43,\epsilon}$  via a conifold transition.

Now we must demonstrate how this transition in the building block affects the topology of the  $G_2$  manifold  $X$  and discuss how one might induce this transition via movement in  $G_2$  moduli.

First, we determine topology of the  $G_2$  manifolds that would be produced in the transition, should such transitions exist in moduli space. Denote the  $G_2$  manifolds obtained via the other small resolution and deformation of  $P_{43}$  as  $X_s$  and  $X_d$ , respectively. Since the building blocks of the two small resolutions of  $P_{43}$  are related by a flop of four rigid holomorphic curves away from the neck,  $K$  does not change and therefore  $b_2(X) = b_2(X_s)$ . Since the three-cycles that appear in the two small resolutions are in one to one correspondence with the appearance of two-cycles, which are the same in number, we also have  $b_3(X) = b_3(X_s)$ . Now consider the deformation. Since there are four conifold points in  $P_{43}$ , the deformation to the smooth manifold  $P_{43,\epsilon}$  produces three-spheres which are expected to be<sup>9</sup> special Lagrangian. While we will say more in the physics discussion momentarily,  $\dim(K_s) = 2$  where  $K_s$  is  $K$ -lattice of the building block associated to  $P_{43,\epsilon}$ . Therefore  $b_2(X) = b_2(X_d) + 1$ . However, though a three-sphere appears in the deformation, recall that a three-cycle diffeomorphic to  $S^2 \times S^1$  vanishes in the blow-down to  $P_{43}$ ; therefore  $b_3(X) = b_3(X_d)$ . In summary, the  $G_2$  manifolds have Betti numbers related by

$$b_2(X) = b_2(X_s) = b_2(X_d) + 1 \quad b_3(X) = b_3(X_s) = b_3(X_d) \quad (3.15)$$

for the non-isolated flop and conifold transitions, respectively.

Now we argue that such topological transitions should actually exist via movement in  $G_2$  moduli. Kovalev's theorem guarantees the existence of a torsion-free  $G_2$  structure  $\Phi$  that is a small deformation of the natural  $G_2$  structure  $\Phi_{T,r}$  on the twisted connected sum, and moreover  $[\Phi] = [\Phi_{T,r}]$ . Now,  $H^3(X, \mathbb{Z})$  and  $H^2(X, \mathbb{Z})$  both contain  $K_+ \oplus K_-$ , and in fact recall from section 2 that for  $\alpha_\pm \in K_\pm$  we have

$$\alpha_\pm \neq 0 \in H^2(X, \mathbb{Z}) \quad \text{and} \quad \alpha_\pm \wedge d\theta_\mp \neq 0 \in H^3(X, \mathbb{Z}). \quad (3.16)$$

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<sup>9</sup>Technically, even in the Calabi-Yau case, this is only known for non-compact examples. The existence of special Lagrangian representatives of the new class associated to the three-spheres of deformation is a common assumption in the literature that we also make.

Therefore choosing an integral basis of  $H^3(X, \mathbb{Z})$  in which to expand  $\Phi$  we see

$$\Phi = \sum_{i=1}^{rk(K_+)} \phi_i \alpha_+^i \wedge d\theta_- + \dots \tag{3.17}$$

for the  $G_2$ -form. One might also think of this suggestively as

$$\Phi = \left( \sum_{i=1}^{rk(K_+)} \phi_i \alpha_+^i \right) \wedge d\theta_- + \dots \tag{3.18}$$

and we wish to integrate this three-form over one of these associative threefolds diffeomorphic to  $S^2 \times S^1$ . Now if we integrate the nearby form  $\Phi_{T,r}$ , the integral  $\int_{S^1} d\theta_- = 2\pi R_-$  factors out and does not depend on moduli. We anticipate that passing from  $\Phi_{T,r}$  to  $\Phi$  changes this behavior somewhat, but the change should not be large. In particular, if we are close to the point in the moduli space of the building block where the singularity appears, and if  $R_-$  is large, we would expect small corrections so that the integral of  $S^1$  remains positive even when the transition point in moduli is reached.<sup>10</sup> If so, then if we vary  $\phi_i$  such that a rigid associative vanishes, the  $S^1$  stays at finite volume and therefore the  $S^2$  must vanish. So we expect to be able to control the two-cycles via their relation to these calibrated three-cycles. This is our argument in favor of the existence of the non-isolated  $G_2$  flop.<sup>11</sup>

To argue for the existence of the non-isolated  $G_2$  conifold, we must also be able to control the three-cycles produced in the deformation of  $P_{43}$  in  $G_2$  moduli. As discussed above, in the building block the deformation is expected to produce four special Lagrangian three-spheres, which of course have  $b_1 = 0$ . Since the three-spheres are not cycles in  $S_s \times S^1$  but are in  $H_3(V_s, \mathbb{Z})$ , then by the long exact sequence in relative homology

$$\dots \rightarrow H_3(S_s \times S^1, \mathbb{Z}) \rightarrow H_3(V_s, \mathbb{Z}) \rightarrow H_3(V_s, S_s \times S^1) \rightarrow H_2(S_s \times S^1, \mathbb{Z}) \rightarrow \dots \tag{3.19}$$

they are non-trivial in  $H_3(V_s, \mathbb{Z})$ . So these three-spheres are expected to satisfy the conditions of the theorem of [5] discussed in section 2, and therefore a small deformation of any one of them is expected to give an associative in the  $G_2$  manifold  $X_s$ . Such associatives can be used to control the deformation of the  $G_2$  conifold in  $G_2$  moduli.

In summary, we have argued for the existence of non-isolated flop and conifold transitions beginning with the TCS  $G_2$  manifold  $X$ . The topology change from  $X$  to the other small resolution  $X_s$  and the deformation  $X_d$  is given in (3.15). We have explained in each case how movement in  $G_2$  moduli could cause associative submanifolds in the  $G_2$  manifolds  $X$ ,  $X_s$ , and  $X_d$  to vanish or grow in such a way that flop and/or conifold transitions may be induced by the one in the building block.

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<sup>10</sup>Of course, this statement should be mathematically proven, if possible!

<sup>11</sup>CHNP point out that a more complete mathematical treatment of the non-isolated  $G_2$  flop and  $G_2$  conifold transitions would involve proving that the singular space has an appropriate metric, which is a limit of metrics on nearby nonsingular spaces. This same point can be made about the flop [10–12] and conifold [13] transitions for Calabi-Yau threefolds, where the metric is known for local models [27] but not for global models. (Metrics are also known for local models of the isolated  $G_2$  flop and conifold transitions [28–30].)

### 3.4 Physics of the topology change

We now discuss the physics of M-theory on the branches of  $G_2$  moduli related by topology change, using  $X$ ,  $X_c$ ,  $X_s$  and  $X_d$  for the original manifold, the singular limit with circles of conifolds, the other  $G_2$  small resolution, and the  $G_2$  deformation, respectively.

The “ $G_2$  blow-down”  $X \rightarrow X_c$  is a limit in which the volumes of the four rigid associatives associated to rigid curves of class  $\gamma_1 - \gamma_2$  vanish; accordingly, the massive chiral multiplet made of  $\Psi_{14}^i$  and  $\bar{\Psi}_{14}^i$  becomes massless in the limit. At large volume there are instanton corrections as given in (3.6), and as the limit is approached there may be instanton corrections that are subleading at large volume that become important.

M-theory on  $X_c$  therefore has massless particles charged under  $U(1)^3$  with charges

	$Q_1$	$Q_2$	$Q_3$
$\Psi_{12}^k$	-2	-1	-1
$\bar{\Psi}_{12}^k$	2	1	1

in addition to massive particles with the same charge as the massive particles for M-theory on  $X$  which didn't become massless.

M-theory on  $X_s$ , obtained via a  $G_2$  small resolution from  $X_c$  via associative threefolds as discussed in the last section, has  $U(1)^3$  gauge symmetry and a spectrum of charged particles identical to that of M-theory on  $X$ ; though the curve classes flopped as  $\gamma_1 - \gamma_2 \mapsto \gamma_2 - \gamma_1$ , anti M2-branes on curves of the latter class in  $X_s$  have the same charges as M2-branes on curves of the former class in  $X$ , so the overall set of particle charges remains the same. M-theory on  $X_s$  also exhibits a non-perturbative superpotential generated by membrane instantons, but the rigid associatives associated to the flopped rigid holomorphic curve has a relative sign, so that the non-perturbative superpotential is identical to (3.6) except for the replacement  $\Phi_1 \mapsto -\Phi_1$ .

M-theory on  $X_d$  is slightly more complicated, and so we will devote a few paragraphs to some details that we have not yet discussed. Since  $b_2(X_d) = b_2(X) - 1 = 2$ , we know that that M-theory on  $X_d$  exhibits  $U(1)^2$  gauge symmetry rather than the  $U(1)^3$  of M-theory on  $X_c$ .

First, a simple field theoretic argument for what must be true of M-theory on  $X_d$ . Note that since the only massless charged fields at the singular point  $X_c$  have charges  $\pm(-2, -1, -1)$  under  $U(1)^3$ , these must be the fields which spontaneously break one of the  $U(1)$  symmetries. These fields are uncharged under the combinations  $\tilde{Q}_1 \equiv Q_1 - 2Q_2$  and  $\tilde{Q}_2 \equiv Q_1 - 2Q_3$ , and therefore these must be the two  $U(1)$  symmetries which exist for M-theory on  $X_d$  (up to redefinition). On  $X$  these  $U(1)$ 's have generators  $E_1 - E_2 - 2(E_1 - E_3) = 2E_3 - E_2 - E_1 \equiv 2E_3 - E$  and similarly  $2E_4 - E_2 - E_1 = 2E_4 - E$ , respectively; we note that both generators have a common term  $E$ .

This field theoretic argument must match the topology of  $X_d$ , since the latter determines the particle charges. How does one see this? Since the intersection theory was originally determined by blowing up along curves rather than divisors, we will do the same here. If we perform the deformation of the pencil in  $\mathbb{P}^3$  as discussed

$$|S_0, S_\infty| \rightarrow |S_{0,\epsilon}, S_\infty| \tag{3.20}$$

then its base locus is now a union of three curves instead of four; two of them are again  $C_3$  and  $C_4$ , but  $C_1$  and  $C_2$  have been replaced by  $C_Q \equiv \{S_\infty = Q = 0\}$ . So the base locus of  $|S_{0,\epsilon}, S_\infty|$  is the union of  $C_3, C_4$ , and  $C_Q$  with classes  $[C_3] = [C_4] = H^2$  and  $[C_Q] = 2H^2$ . Blow up along the curves of the base locus sequentially in the order  $C_Q, C_3, C_4$ . Now we have three exceptional divisors,  $E_Q, E_3$  and  $E_4$ , respectively, which restrict to curves of class  $H, H$ , and  $2H$  in  $S$ , respectively. The  $K$ -lattice of the deformation  $K_d$  is therefore generated by  $2E_3 - E_Q$  and  $2E_4 - E_Q$ , and therefore these are the generators of  $U(1)^2$  for M-theory on  $X_d$ ; note that they look identical to what the field theoretical answer required, but let us compute particle charges as a rigorous check. Letting  $\gamma_Q, \gamma_3$  and  $\gamma_4$  be the class of the generic fiber of  $E_Q, E_3$ , and  $E_4$ , the particles come from M2-branes wrapped on 8 rigid holomorphic curves of class  $\gamma_Q - \gamma_3$ , 8 of class  $\gamma_Q - \gamma_4$ , and 4 of class  $\gamma_3 - \gamma_4$ . The intersection theory is computed as before with the result  $E_Q \cdot \gamma_Q = E_3 \cdot \gamma_3 = E_4 \cdot \gamma_4 = -1$ , as are the particle charges via the intersections of the particle curves with the  $U(1)$  generating divisors within the building block. For example, under the  $U(1)$  of  $2E_3 - E_Q$  the particles on curves of class  $\gamma_Q - \gamma_3$  have charge 3.

A short computation shows that the topological calculation of particle charges matches the field theoretic expectation from the previous paragraph. The result is that the generators of  $K_d$  (and thus of  $U(1)$ 's)  $2E_3 - E_Q$  and  $2E_4 - E_Q$  correspond precisely to  $\tilde{Q}_1$  and  $\tilde{Q}_2$ . M-theory on  $X_d$  exhibits massive particles with charge

	$\tilde{Q}_1$	$\tilde{Q}_2$
$\Psi_{Q3}^j$	3	1
$\Psi_{Q4}^j$	1	3
$\Psi_{34}^k$	-2	2

and their conjugates, where  $j = 1 \dots 8$  and  $k = 1 \dots 4$ .

It is satisfying that the field theory prediction of particle charges after  $U(1)^3 \rightarrow U(1)^2$  symmetry breaking matched the topological computation after the non-isolated  $G_2$  conifold transition.

#### 4 The landscape of M-theory on $G_2$ manifolds

Having reviewed the twisted connected sum construction and studied a rich example with many interesting physical features, in this section we would like to study some aspects of the associated landscape of (abelian) four-dimensional  $\mathcal{N} = 1$  compactifications. Some of the general statements about the physics of  $G_2$  compactifications that were introduced in the example will be reintroduced for the sake of completeness.

What is a coarse measure of the size of the (known) abelian  $G_2$  landscape and how has it grown in recent years? We can be more precise than we were in the introduction, since we have introduced the TCS construction. The earliest examples of compact  $G_2$  manifolds due to Joyce [20, 21] were relatively few in number. The original TCS examples [1] utilized building blocks of Fano type, and the number of such examples is determined in part by the number of smooth Fano threefolds; these have been classified and there are precisely 105 deformation families. By contrast, [5] also constructs  $G_2$  manifolds using semi-Fano



building blocks, which utilize weak-Fano threefolds; there are at least hundreds of thousands of deformation families of smooth weak-Fano threefolds. These give rise to [5] at least 50 million matching pairs, arising only from ACyl Calabi-Yau threefolds from semi-Fano building blocks of rank at most two or from toric semi-Fano threefolds; given the limited nature of this search, many more can probably be obtained by considering higher rank building blocks. These 50 million matching pairs from semi-Fano building blocks each give rise to a TCS  $G_2$  manifold and, therefore, a four-dimensional M-theory vacuum, some of which may be equivalent in cases where the same  $G_2$  manifold arises from different building blocks. It is noteworthy that the number of  $G_2$  building blocks is within a factor of ten of the number of Kreuzer-Skarke “Calabi-Yau building blocks,” i.e., the 500 million reflexive four-dimensional polytopes.

The number of abelian  $G_2$  compactifications is already quite large and via systematic application of known construction techniques it will likely continue to grow in the coming years. Given this large number of examples and the already existing evidence for topology change, it seems reasonable to wonder whether  $G_2$  moduli space is connected, as Reid has conjectured [7] for Calabi-Yau threefolds. Such a property would strengthen the meaning of the abelian  $G_2$  landscape, as the associated vacua would form a connected moduli space of a single theory.

Finally, though it will not be critical in the following, it is worth mentioning that TCS  $G_2$  manifolds often have large numbers of moduli. For example, table 5 of [5] lists a few dozen  $G_2$  manifolds with  $47 \leq b_3(X) \leq 155$ , taking many different values in this range; models with only a handful of moduli are scarce.

#### 4.1 Higgs branches, Coulomb branches, and gauge enhancement

In [8] we will study a number of different ways in which one might take a singular limits of a  $G_2$  compactification in order to obtain non-abelian gauge enhancement or massless charged matter in the theory<sup>12</sup> If an M-theory compactification on a  $G_2$  manifold  $X$  admits a limit in which non-abelian gauge enhancement occurs, then a natural question is whether the the vacuum is on a Higgs branch or a Coulomb branch.

In this section we will not look in detail at Higgsing from a non-abelian theory, instead speaking of “Higgs branches” and “Coulomb branches” loosely according to the value of  $b_2(X)$ . For now, let us be slightly more precise. Suppose there existed a singular limit of  $X$  which realizes a gauge sector with gauge group  $G \times U(1)^k$ , where  $G$  is the nonabelian part (with finite center). If smoothing the manifold back to  $X$  (or another member in the same family of  $G_2$  manifolds as  $X$ ) Higgses this theory in a standard way, then an upper bound on the number of  $U(1)$ ’s is set by the dimension of the maximal torus of the gauge theory on the singular space; that is

$$b_2(X) \leq rk(G) + k. \tag{4.1}$$

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<sup>12</sup>For some earlier work on non-abelian gauge symmetries in  $G_2$  compactifications, see [24, 31] or the review [32]. For the relationship between these constructions and chiral type IIa constructions with intersecting D6-branes, see [33–35].

Certainly if a gauge enhanced singular limit exists and  $b_2(X) = 0$  then the vacuum obtained from M-theory on  $X$  is on a Higgs branch; conversely  $b_2(X) \neq 0$  is necessary for this vacuum to be on a Coulomb branch. That said, if  $G$  is finite but  $k \neq 0$  then  $b_2(X) = k$  and the terminology is slightly ambiguous since there still are long range forces but none of them arise from Cartan  $U(1)$ 's of a non-abelian  $G$ .

Despite these caveats, we will loosely call these vacua with  $b_2(X) = 0$  Higgs branches and vacua with  $b_2(X) \neq 0$  Coulomb branches. Vacua of the latter type are particularly useful since, for example, particle charges can be computed, and if they are the charges of massive W-bosons of a spontaneously broken gauge theory then the charges are intimately related to gauge enhancement in a singular limit.

**Higgs branches: their prevalence and drawbacks.** It turns out that *nearly all* of the known examples are on Higgs branches; i.e., they have  $b_2(X) = 0$ . This follows from topological properties of the building blocks used to construct most TCS  $G_2$  manifolds. Since for a semi-Fano building block the map  $\rho : H^2(V) \rightarrow H^2(S)$  is injective,  $K = 0$ ; then for any TCS  $G_2$  manifold built out of two semi-Fano building blocks

$$H^2(X, \mathbb{Z}) = N_+ \cap N_-. \tag{4.2}$$

However in order to construct a TCS  $G_2$  manifold from the building blocks, one must also solve the matching problem; i.e., there must exist a Donaldson matching  $r : S_+ \rightarrow S_-$ . This problem is much easier to solve if  $N_+ \cap N_- = 0$ , in which case we have  $H^2(X, \mathbb{Z}) = 0$ . M-theory on such an  $X$  is on a Higgs branch if a singular limit with non-abelian gauge symmetry exists.

We can be slightly more specific. A manifold  $X$  is said to be 2-connected if  $\pi_1(X) = \pi_2(X) = 0$ ; then we also have  $H^1(X) = H^2(X) = 0$ . A smooth 2-connected seven-manifold — and therefore a smooth 2-connected  $G_2$  manifold — is classified up to almost-diffeomorphism<sup>13</sup> by the pair of non-negative integers  $(b_4(X), \text{div } p_1(X))$ , where  $\text{div } p_1(X)$  measures the divisibility of the first Pontryagin class [36]. It so happens that the discussed 50 million matching pairs [3, 5] give rise to 2-connected  $G_2$  manifolds. Therefore since  $H^2(X) = 0$  for all of these manifolds, any of the associated M-theory vacua are on Higgs branches. Though (as we saw in section 3) there are known vacua with  $b_2(X) \neq 0$ , which are on Coulomb branches if some of the associated  $U(1)$ 's embed into a non-abelian group in a singular limit, essentially the entire known  $G_2$  landscape is comprised of Higgs branches.

In practice, studying singular limits of  $G_2$  compactifications which exhibit massless charged matter or non-abelian gauge enhancement is much more difficult when approaching from Higgs branches rather than from Coulomb branches. One physical reason is that the Higgs vacuum does not exhibit any charged particles since the gauge symmetry is completely broken; therefore there is no charge to “measure” (as we did in section 3) via the intersection theory of non-trivial two-cycles and five-cycles in  $X$ . Moreover, it is conceivable that progress could be made in  $G_2$  Higgs vacua similar to the recent progress in

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<sup>13</sup>An almost-diffeomorphism is an invertible map which is smooth except possibly at a finite number of points. The classification given in [36] was recently sharpened to a diffeomorphism classification by introducing additional invariants [37].

F-theory Higgs vacua. For example, in the latter case it is known [38–40] how to recover the spectrum of massive W-bosons in a completely Higgsed theory via the study of an elliptic fibration. It is possible that some  $G_2$  manifolds admit a similar elliptic fibration [40].

**Coulomb branches: their scarcity and utility.** Though most TCS  $G_2$  vacua are on Higgs branches, vacua on Coulomb branches do exist. In fact, most of the original examples of Joyce [20, 21] were Coulomb branches due to  $H^2(X)$  being non-trivial. Those examples were seven-manifolds with ADE singularities, and upon smoothing to the  $G_2$  manifold many expected features occur. In [8] we will study the physics of some Joyce manifolds, where the simplest cases involve moving from a non-abelian theory  $G$  to  $U(1)^{rk(G)}$  via adjoint breaking. We will also find a number of topological defects appeared in the Joyce manifolds, for example the 't Hooft-Polyakov monopoles characteristic of symmetry breaking to Coulomb branches.

To obtain a TCS  $G_2$  vacuum on a Coulomb branch it is necessary to study examples with non-trivial  $H^2(X)$ ; since

$$H^2(X, \mathbb{Z}) = (N_+ \cap N_-) \oplus K_+ \oplus K_- \tag{4.3}$$

some combination of  $K_{\pm}$  and  $N_+ \cap N_-$  must be non-trivial. One option is to use so-called non-perpendicular orthogonal gluing, in which case  $N_+ \cap N_-$  is non-trivial; the drawback of this option is that it can be difficult to find a Donaldson matching. The other option is to use a building block with  $K \neq 0$ ; but then the building block cannot be taken from the large collection of semi-Fano blocks which all have  $K = 0$ .

In [5] two possible methods were suggested for constructing building blocks with  $K \neq 0$ , and thus  $G_2$  compactifications on Coulomb branches. The first is to construct new building blocks obtained by blowing up a non-generic anticanonical (AC) pencil in a toric semi-Fano 3-fold, which *is not* a “semi-Fano building block” since the latter assumes a blow-up of a generic AC pencil. Ref. [3, 5] identified some examples of this type and computed properties of the associated lattices  $K$ . Given that the basic object of this approach is a toric variety associated to a three-dimensional reflexive polytope, it would be interesting to study whether the “non-generic AC” pencil construction of building blocks with  $K \neq 0$  can be systematized, and if so what the associated physics is on the Coulomb branch. The other suggestion of [5] for constructing building blocks with  $K \neq 0$  is to use one of the 74 non-symplectic type building blocks introduced in [2]. In section 3 we studied an example of the former type and saw that many physical effects can be computed.

Though they are relatively scarce (for technical reasons) in the class of known TCS  $G_2$  manifold examples, those which describe M-theory vacua on Coulomb branches, or more generally vacua with  $U(1)$  symmetries and massive charged particles are practically useful and physically interesting. Suppose M-theory on  $X$  yielded a Coulomb branch vacuum. Then massive W-bosons from M2-branes wrapped on two-cycles are part of the charged particle spectrum of the theory and it is clear what to look for: limits in  $G_2$  moduli space in which those two-cycles go to zero volume.

Though there is no calibration form for two-cycles in a  $G_2$  manifold, one can imagine cases where two-cycles are contained in an associative threefold or coassociative fourfold;

the natural question in such a case is whether some degenerations (zero volume limits) of the associative or coassociative submanifolds give rise to collapses of the two-cycles they contain. This can be one handle on obtaining non-abelian gauge enhancement or massless charged particles, as we saw explicitly in the example of section 3. These two-cycle volumes were directly controllable via associative threefolds; we will discuss this idea further in our work [8].

## 4.2 Membrane instantons, $G_2$ transitions, and fluxes

The results of [5] have a number of other implications for the physics of M-theory compactifications on TCS  $G_2$  manifolds, as we will discuss in this section.

**Instantons and rigid associatives.** Instanton effects arising from wrapped branes and strings can generate non-perturbative corrections to the scalar potential that play an important role in moduli stabilization. In M-theory compactifications these may arise from wrapped M2-brane or M5-brane instanton corrections to the superpotential. While M5-brane instantons play a major role in M-theory compactifications on Calabi-Yau fourfolds to three dimensions [41], for example by providing effects which lift Coulomb branches, these corrections don't exist in  $G_2$  compactifications since seven-manifolds with holonomy precisely  $G_2$  have  $b_6(X) = 0$ ; that is, there are no cycles on which to wrap M5-brane instantons.

In contrast, in  $G_2$  compactifications an M2-brane instanton may generate a superpotential correction if it is wrapped on a rigid supersymmetric (i.e., associative) three-cycle [9]. These instanton corrections to the superpotential  $W$  take the heuristic form

$$Ae^{-\Phi} \tag{4.4}$$

for the  $G_2$  modulus  $\Phi$  associated to the rigid associative three-cycle wrapped by the instanton. While it is not yet possible to make complete statements about the structure of the instanton prefactor  $A$  due to the absence of a microscopic description for instanton zero modes, in the analogous D-brane instanton cases in F-theory and/or type IIA the prefactor  $A$  may contain chiral matter insertions or an intricate geometric moduli dependence (see e.g. [42] and [43, 44]). Note that the absolute value of the prefactor should be given by the  $\eta$  function of an appropriate Dirac operator [9].

Twisted connected sum  $G_2$  compactifications are currently the only  $G_2$  compactifications where one may concretely study instanton corrections to the superpotential, since the first compact rigid associative cycles in a  $G_2$  manifold were constructed in [5], and this construction is specific to twisted connected sum  $G_2$  manifolds. The relevant theorem is that if  $C$  is a rigid holomorphic curve in  $V$ , then a small deformation of  $S^1 \times C$  in  $X$  is a rigid associative. An M2-brane on this rigid associative corrects the superpotential in models where interactions lift Wilson line moduli; see the footnote in section 3.2.

While this gives a method for identifying compact rigid associatives in TCS  $G_2$  manifolds, there may exist others rigid associatives that are of a different type. We emphasize this point because it means that while current techniques allow for the identification of

some instanton corrections, it is not yet possible to say whether these are all of the corrections, or even the leading corrections. Thus, explicit  $G_2$  moduli stabilization via instantons is still out of reach.

That issue aside, how many instanton corrections exist in known examples? For some of the examples in [5] the number  $a_0$  of rigid associatives associated to rigid holomorphic curves in the building blocks was computed, with  $a_0$  ranging from 0 to 66, taking a variety of values in between. In the example of M-theory on  $X$  that we studied in section 3,  $a_0 = 24$  due to the existence of 24 rigid holomorphic curves in one of the building blocks; after the studied non-isolated  $G_2$  flop (conifold) transition the geometry had  $a_0 = 24$  ( $a_0 = 20$ ). Note that some rigid associatives may be in the same homology class (as in the example), in which case there is an associated multiplicity factor in front of the instanton correction, which should be thought of as the M-theory on  $G_2$  analog of Gromov-Witten invariant prefactors of worldsheet instantons on Calabi-Yau threefolds; recall, for example, that there are 2875 lines in the quintic which give rise to instanton corrections from string worldsheets, though all in the same homology class.

If some number of these rigid associatives are in different homology classes, however, the superpotential takes the form of a racetrack or a generalized racetrack with multiple terms, i.e.,

$$W_{\text{inst}} = \sum_i A_i e^{-\Phi_i} \tag{4.5}$$

where  $\Phi_I$  is the chiral multiplet modulus associated to the rigid associative via Kaluza-Klein reduction. More specifically in the example we studied the superpotential took the form

$$W = 4(A_1 e^{-\Phi_1} + A_2 e^{-\Phi_2} + A_3 e^{-\Phi_3} + A_4 e^{\Phi_1 - \Phi_2} + A_5 e^{\Phi_1 - \Phi_3} + A_6 e^{\Phi_2 - \Phi_3}) + \dots \tag{4.6}$$

which is a six-term generalized racetrack. In a singular limit of  $X$  there may be additional terms of this structure in the superpotential due to a confining hidden gauge sector; see the studies [45, 46] which utilize hidden sectors. Based on this evidence, it seems that racetracks or generalized racetracks occur frequently.

**$G_2$  transitions.** Given the existence of flop and conifold transitions for string compactifications on Calabi-Yau threefolds, it is natural to wonder about the possibility of topology changing transitions in  $G_2$  compactifications of M-theory. The existence of such a transition would require two topologically distinct families of  $G_2$  manifolds which give the same singular space in some limit of their respective moduli spaces. The transition would occur by taking the limit of one of the families, and then passing to the other family via the intermediate singular space. This is the natural analog for  $G_2$  manifolds of a flop or conifold transition, as already explored in an example in section 3.

In general there are still difficulties with establishing the existence of  $G_2$  transitions for TCS  $G_2$  manifolds, partly because of difficulties in controlling the sizes of the corrections to  $\Phi_{T,r}$  as moduli are varied, but there is an interesting and natural possibility for realizing these transitions given that the building blocks are composed of algebraic threefolds. Namely, if the algebraic threefold of the building block can *itself* undergo a transition and

on both sides of the  $G_2$  transition the TCS construction can be used to construct topologically distinct  $G_2$  manifolds, then one might study whether the associated transition between  $G_2$  manifolds exists via movement in  $G_2$  moduli. This was precisely what we did in section 3, utilizing the fact that two-cycle volumes should be controlled via related associative submanifolds. We found that there should be (non-isolated)  $G_2$  flop and conifold transitions related to flop and conifold transitions in a building block.

In [5] a number of interesting general observations were made in about  $G_2$  transitions which are induced by conifold transitions in the building blocks. Suppose there is a conifold transition  $F \rightarrow \tilde{X} \rightarrow Y$  between a smooth Fano  $F$  and a smooth semi-Fano  $Y$  via an intermediate singular threefold  $\tilde{X}$ . Suppose further that one is able to use the associated building blocks  $(Z_Y, S_Y)$  and  $(Z_F, S_F)$  to construct TCS  $G_2$  manifolds  $X_Y$  and  $X_F$ . Then it is natural to wonder whether there is a  $G_2$  transition from  $X_F$  to  $X_Y$  associated to the threefold transition from  $F$  to  $Y$ . In [5] it is observed that

- 1)  $b_2(Y) > b_2(F)$
- 2)  $b_3(Y) \leq b_3(F)$ , and in fact it typically is a strict inequality
- 3)  $Y$ , but not  $F$ , contains compact rigid rational curves which do not intersect smooth anticanonical divisors and give rise to compact rigid rational curves in the associated ACyl  $CY_3$   $Z_Y \setminus S_Y$ .

At the level of constructing associated  $G_2$  manifolds, the authors note that 1) implies that solving the matching problem for building blocks constructed from  $Y$  is more difficult than for those constructed from  $F$ ; that 2) implies that  $b_3(X_Y) \leq b_3(X_F)$ , i.e., the number of  $G_2$  moduli often changes; and that 3) implies that the rigid rational curves of  $Y$  give rise to compact rigid associatives in  $X_Y$  which do not exist in  $X_F$ .

We would like to note that each of these observations has interesting physical consequences in the associated M-theory compactifications. The associated physical statements are:

- 1) Since changing  $b_2$  of the building blocks does not necessarily change  $b_2$  of the associated  $G_2$  manifolds, such a  $G_2$  transition could in principle be a Higgs-Higgs transition or a Higgs-Coulomb transition (if gauge symmetry exists on the singular space at all), whereas for a conifold transition in string theory it is a Higgs-Coulomb transition.
- 2) A change in the number of moduli has implications for moduli stabilization, but there are also corresponding instantons (which do not necessarily correct  $W$ ), domain walls, and axion strings that appear in the compactification on  $F$  due to wrapped M-branes.
- 3) There are instanton corrections to the superpotential for M-theory on  $Y$  that do not exist for M-theory on  $F$ ; this is similar to behavior elsewhere in the landscape, for example in the Higgs-Coulomb transition that may arise for three-dimensional M-theory compactifications on elliptically fibered Calabi-Yau fourfolds.

Again, we emphasize that these are physical statements following from the topology of potential TCS  $G_2$  transitions which may be induced by transitions in the algebraic building



blocks; the topological statements are true, but there may not exist  $G_2$  metrics throughout the proposed transition. It would be interesting to study whether they exist in broad classes of examples.

**Flux and fluxless compactifications.** In M-theory compactifications it is possible to turn on four-form flux  $G_4$ . Consider M-theory on a manifold  $X$ . For the theory to be well-defined, the flux must satisfy the quantization condition [22]

$$\left[ \frac{G_4}{2\pi} \right] - \frac{p_1(X)}{4} \in H^4(X, \mathbb{Z}) \quad (4.7)$$

where  $p_1(X)$  is the first Pontryagin class of  $X$ . This flux quantization condition has an interesting corollary: since this specific combination of four-forms must be integral, if  $p_1(X)/4$  is not integral then choosing such a compactification manifold  $X$  *requires*  $G_4 \neq 0$ . This is a well-known phenomenon in F-theory, where the elliptically fibered Calabi-Yau fourfold of the related M-theory compactification sometimes requires that flux be turned on.

What about for M-theory compactifications on a  $G_2$  manifold  $X$ ? In [5]  $p_1(X)$  was computed for the first time in terms of data of the building blocks; thus, in concrete examples one can now check whether  $p_1(X)/4$  is integral. While the precise knowledge of  $p_1(X)$  is convenient, it is not necessary to answer the question of whether flux must be turned on, since it is known from [47] that  $p_1(X)/4$  is integral and thus one can always consistently choose to set  $G_4 = 0$  in any  $G_2$  compactification. However, if  $G_4 \neq 0$  there is a perturbative flux superpotential and moduli stabilization is qualitatively different; one may also argue that this is more generic.

**Common model-building assumptions in light of TCS  $G_2$  manifolds.** In studying the landscape scenarios are often put forth for moduli stabilization and supersymmetry breaking based on sound theoretical arguments and calculations, but before large classes of examples exist; once they do exist, though, it is interesting to re-evaluate the scenario.

A well-known example is the large number of type IIB flux vacua, where this large number arises from a large number of possible integral Ramond-Ramond fluxes that may be chosen to stabilize the complex structure of the Calabi-Yau  $X$ . Though the general calculations are sound, typically quoted flux vacuum counts (e.g.  $10^{500}$ ) exist for large  $h^{2,1}(X) \gtrsim 100$ , and integral fluxes have never been constructed for Calabi-Yau manifolds with such large Hodge numbers for reasons of computational complexity. If this obstacle were removed, it would be nice to have an explicit example which confirms the assumptions and results of the proposed scenario.

Similarly, scenarios have been proposed for moduli stabilization and supersymmetry breaking (as well as phenomenology) in  $G_2$  compactifications of M-theory. For example, in one scenario known at the  $G_2$ -MSSM (see e.g. the review [48]) at least three important assumptions are made:

- 1) The M-theory compactification is fluxless, i.e.,  $G_4$  is cohomologically trivial.
- 2) The primary source of moduli stabilization and supersymmetry breaking is from a strongly coupled hidden sector, which generates a non-perturbative superpotential



$w_{np}$  containing terms of the form

$$Ae^{-n_i\Phi_i} \quad n_i \in \mathbb{Z} \quad (4.8)$$

where  $\Phi_i$  are the metric moduli of the  $G_2$  compactification and  $A$  is determined by dimensional transmutation of the confining gauge theory. If this term drives moduli stabilization, membrane instanton corrections to the superpotential must be subleading.

- 3) The visible sector is an  $SU(5)$  GUT broken to the MSSM via Wilson lines.

We would like to discuss some of these assumptions in light of the existence of TCS  $G_2$  compactifications and the associated physics discussed in this section. We will address each in turn.

The first assumption is always possible, since (as discussed) the flux quantization condition never forces the introduction of  $G_4$ -flux in a  $G_2$  compactification of M-theory [47], but setting  $G_4 = 0$  is also a non-generic choice, since it is choosing the origin out of an entire vector space (the non-torsional part of  $H^4(X, \mathbb{Z})$ ). Interestingly, the absence of a flux superpotential — and therefore the choice  $G_4 = 0$  — is critical in the moduli stabilization scenario of [48]. It would be interesting to understand the extent to which fluxes might alter the results of [48], or whether the existence of de Sitter vacua depends in important ways on the choice of flux or fluxless compactifications; the latter dependence is plausible due to the fundamentally different structure of the scalar potential in the two cases.

The second assumption is the one deserving the most scrutiny in light of the recent progress. Since it has now been shown that examples often exhibit many instanton corrections to the superpotential and in the only explicitly computed example we found the intricate form (3.6), it is reasonable to expect that, at least in some cases, these effects will compete with the non-perturbative superpotential of the confining hidden sector utilized in [48]. In a number of examples of [5] there are over 40 cycles which support M2-brane instanton corrections to the superpotential. Though (as discussed) these may be in the same homology class and thus generate an exponentially suppressed correction in the same  $G_2$  modulus, *at least one* instanton generated superpotential term exists in all of these compactifications, and perhaps more if the rigid associatives are homologically distinct; the example we studied is an existence proof of the latter possibility. It would be interesting to understand how the scenario [48] changes when taking into account instanton corrections; if it is a single new term it could, together with the confining contribution, give a standard racetrack, whereas if there are multiple distinct instanton corrections it would be a generalized racetrack with many exponentially suppressed terms.

Not enough is known about singular limits of TCS  $G_2$  manifolds to evaluate the third assumption formally, beyond the typical arguments made from heterotic / M-theory duality, since relatively little is known about singular limits of compact  $G_2$  manifolds as we will discuss in [8]. Phenomenologically, it is an assumption.

## 5 Conclusions

In this paper we have studied M-theory compactifications to four dimensions on  $G_2$  manifolds constructed via twisted connected sum. There are now perhaps fifty million examples.

We have shown that recent topological progress [5] in TCS  $G_2$  manifolds now allows for interesting physical quantities to be computed in the associated M-theory vacua on a TCS  $G_2$  manifold  $X$ . These include the  $U(1)$  symmetries of the vacuum, the charges of massive particles, the structure of some membrane instanton corrections to the superpotential, spacetime topology change, and spontaneous symmetry breaking in a  $G_2$  conifold transition.

However, it is physically critical to understand singular limits of these manifolds and their associated M-theory vacua. In our view, the most important mathematical progress that would aid future physical progress is to have a better understanding of singularities that develop upon movement in  $G_2$  moduli space, both in general and in the twisted connected sum construction, since they are necessary for realizing non-abelian gauge sectors or massless charged matter, and therefore realistic vacua. In related work [8] we will address a number of physical issues related to such degenerations and will conjecture that the right approach will be to move to a wall in a “cone of effective associatives.” In particular, as we will discuss in [8] a critical physical issue for understanding non-abelian gauge enhancement is to have some control over intersections of two-cycles with five-cycles and limits in which they degenerate. These degenerations are difficult to study since there are no calibration forms for two-cycles; instead it would be useful to have techniques to identify those cases in which a two-cycle within an associative (coassociative) submanifold vanishes as the associative (coassociative) itself vanishes. We saw such a phenomenon in the example of section 3 due to a particular factorization property which holds for certain cycles in TCS  $G_2$  manifolds. While two-cycles are more difficult to study than three- and four-cycles in a  $G_2$  manifold, it would be important understand and control them further since they determine the particle physics of these M-theory vacua.

It seems reasonable to hope that the singularities needed for non-abelian gauge symmetry can eventually be engineered in the context of the TCS construction, either by finding a singular ACyl Calabi-Yau threefold with a singular curve not extending to the boundary, or by extending the TCS construction to allow the K3 surfaces along the neck to have rational double points. We leave this to future work.

Our work is also a first step towards the explicit construction of de Sitter vacua in fluxless  $G_2$  compactifications of M-theory, as we have computed the form of membrane instanton corrections to the superpotential; see section 3.2 for a discussion of instanton zero modes. These instantons play an even more significant role for moduli stabilization than their type IIB ED3-instanton counterparts, since in a smooth  $G_2$  compactification the fields which may be identified as moduli are the metric and axion moduli which give  $b_3(X)$  massless uncharged chiral supermultiplets, and these are the fields that appear in membrane instanton corrections to the superpotential. Therefore, membrane instanton corrections may in principle stabilize all moduli, potentially giving rise to de Sitter vacua.

Physically, completing such a program requires having “enough” instantons to stabilize all moduli, and furthermore one must be able to guarantee that these are the leading

instanton corrections. Mathematically, this requires the construction of “enough” associative submanifolds, ensuring that they are also leading. While not completely precise, a rough way to think of the “leading” associatives is as follows. Let  $T_i$  be an integral basis for  $H^3(X, \mathbb{Z})$ . Then any rigid associative  $M$  can be expanded in this basis as  $M = m_i T_i$ . The leading instantons arise from instantons closer to the origin where  $m_i = 0 \forall i$ , so one algorithm would be to find all rigid associatives in all homology classes in an appropriately sized box around the origin.

While such vacua are not realistic, giving rise to universes with axions and perhaps massive charged particles and photons but no non-abelian gauge interactions, they nevertheless would be de Sitter vacua. This may be the most direct route to realizing de Sitter vacua in M-theory. If the TCS construction can be extended to include singular limits carrying non-abelian gauge fields, those de Sitter vacua could be quite realistic.

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## References

- [1] A. Kovalev, *Twisted connected sums and special Riemannian holonomy*, *J. Reine Angew. Math.* **565** (2003) 125 [[math/0012189](#)] [[INSPIRE](#)].
- [2] A. Kovalev and N.-H. Lee, *K3 surfaces with non-symplectic involution and compact irreducible  $G_2$ -manifolds*, *Math. Proc. Cambridge Philos. Soc.* **151** (2011) 193 [[arXiv:0810.0957](#)].
- [3] A. Corti, M. Haskins, J. Nordström and T. Pacini, *Asymptotically cylindrical Calabi-Yau 3-folds from weak Fano 3-folds*, *Geom. Topol.* **17** (2013) 1955.
- [4] M. Haskins, H.-J. Hein and J. Nordström, *Asymptotically cylindrical Calabi-Yau manifolds*, to appear in *J. Diff. Geom.* [arXiv:1212.6929](#).
- [5] A. Corti, M. Haskins, J. Nordström and T. Pacini,  *$G_2$ -manifolds and associative submanifolds via semi-Fano 3-folds*, to appear in *Duke Math. J.* [arXiv:1207.4470](#) [[INSPIRE](#)].
- [6] M. Kreuzer and H. Skarke, *Complete classification of reflexive polyhedra in four-dimensions*, *Adv. Theor. Math. Phys.* **4** (2002) 1209 [[hep-th/0002240](#)] [[INSPIRE](#)].
- [7] M. Reid, *The moduli space of 3-folds with  $K = 0$  may nevertheless be irreducible*, *Math. Ann.* **278** (1987) 329.

- [8] J. Halverson and D.R. Morrison, *On gauge enhancement and singular limits in  $G_2$  compactifications of M-theory*, to appear.
- [9] J.A. Harvey and G.W. Moore, *Superpotentials and membrane instantons*, [hep-th/9907026](#) [[INSPIRE](#)].
- [10] E. Witten, *Phases of  $N = 2$  theories in two-dimensions*, *Nucl. Phys. B* **403** (1993) 159 [[hep-th/9301042](#)] [[INSPIRE](#)].
- [11] P.S. Aspinwall, B.R. Greene and D.R. Morrison, *Multiple mirror manifolds and topology change in string theory*, *Phys. Lett. B* **303** (1993) 249 [[hep-th/9301043](#)] [[INSPIRE](#)].
- [12] P.S. Aspinwall, B.R. Greene and D.R. Morrison, *Calabi-Yau moduli space, mirror manifolds and space-time topology change in string theory*, *Nucl. Phys. B* **416** (1994) 414 [[hep-th/9309097](#)] [[INSPIRE](#)].
- [13] B.R. Greene, D.R. Morrison and A. Strominger, *Black hole condensation and the unification of string vacua*, *Nucl. Phys. B* **451** (1995) 109 [[hep-th/9504145](#)] [[INSPIRE](#)].
- [14] K. Intriligator, H. Jockers, P. Mayr, D.R. Morrison and M.R. Plesser, *Conifold Transitions in M-theory on Calabi-Yau Fourfolds with Background Fluxes*, *Adv. Theor. Math. Phys.* **17** (2013) 601 [[arXiv:1203.6662](#)] [[INSPIRE](#)].
- [15] M. Atiyah, J.M. Maldacena and C. Vafa, *An M-theory flop as a large- $N$  duality*, *J. Math. Phys.* **42** (2001) 3209 [[hep-th/0011256](#)] [[INSPIRE](#)].
- [16] M. Atiyah and E. Witten, *M theory dynamics on a manifold of  $G_2$  holonomy*, *Adv. Theor. Math. Phys.* **6** (2003) 1 [[hep-th/0107177](#)] [[INSPIRE](#)].
- [17] R.L. Bryant, *Metrics with exceptional holonomy*, *Ann. Math.* **126** (1987) 525.
- [18] D. Joyce, *Compact Manifolds with Special Holonomy*, *Oxford Mathematical Monographs*, Oxford University Press, Oxford U.K. (2000).
- [19] R. Harvey and H.B. Lawson Jr., *Calibrated geometries*, *Acta Math.* **148** (1982) 47.
- [20] D.D. Joyce, *Compact Riemannian 7-manifolds with holonomy  $G_2$ . I*, *J. Diff. Geom.* **43** (1996) 291.
- [21] D.D. Joyce, *Compact Riemannian 7-manifolds with holonomy  $G_2$ . II*, *J. Diff. Geom.* **43** (1996) 329.
- [22] E. Witten, *On flux quantization in M-theory and the effective action*, *J. Geom. Phys.* **22** (1997) 1 [[hep-th/9609122](#)] [[INSPIRE](#)].
- [23] M. Cvetič, D. Klevers, H. Piragua and P. Song, *Elliptic fibrations with rank three Mordell-Weil group: F-theory with  $U(1) \times U(1) \times U(1)$  gauge symmetry*, *JHEP* **03** (2014) 021 [[arXiv:1310.0463](#)] [[INSPIRE](#)].
- [24] B.S. Acharya, *On Realizing  $N = 1$  super Yang-Mills in M-theory*, [hep-th/0011089](#) [[INSPIRE](#)].
- [25] T. Friedmann and E. Witten, *Unification scale, proton decay and manifolds of  $G_2$  holonomy*, *Adv. Theor. Math. Phys.* **7** (2003) 577 [[hep-th/0211269](#)] [[INSPIRE](#)].
- [26] G. Papadopoulos and P.K. Townsend, *Compactification of  $D = 11$  supergravity on spaces of exceptional holonomy*, *Phys. Lett. B* **357** (1995) 300 [[hep-th/9506150](#)] [[INSPIRE](#)].
- [27] P. Candelas and X.C. de la Ossa, *Comments on conifolds*, *Nucl. Phys. B* **342** (1990) 246 [[INSPIRE](#)].
- [28] A. Brandhuber, J. Gomis, S.S. Gubser and S. Gukov, *Gauge theory at large- $N$  and new  $G_2$  holonomy metrics*, *Nucl. Phys. B* **611** (2001) 179 [[hep-th/0106034](#)] [[INSPIRE](#)].

- [29] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Cohomogeneity one manifolds of spin(7) and  $G_2$  holonomy*, *Phys. Rev. D* **65** (2002) 106004 [[hep-th/0108245](#)] [[INSPIRE](#)].
- [30] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *M theory conifolds*, *Phys. Rev. Lett.* **88** (2002) 121602 [[hep-th/0112098](#)] [[INSPIRE](#)].
- [31] B.S. Acharya, *M theory, Joyce orbifolds and super Yang-Mills*, *Adv. Theor. Math. Phys.* **3** (1999) 227 [[hep-th/9812205](#)] [[INSPIRE](#)].
- [32] B.S. Acharya and S. Gukov, *M theory and singularities of exceptional holonomy manifolds*, *Phys. Rept.* **392** (2004) 121 [[hep-th/0409191](#)] [[INSPIRE](#)].
- [33] M. Cvetič, G. Shiu and A.M. Uranga, *Chiral type-II orientifold constructions as M-theory on  $G_2$  holonomy spaces*, [hep-th/0111179](#) [[INSPIRE](#)].
- [34] M. Cvetič, G. Shiu and A.M. Uranga, *Chiral four-dimensional  $N = 1$  supersymmetric type 2A orientifolds from intersecting D6 branes*, *Nucl. Phys. B* **615** (2001) 3 [[hep-th/0107166](#)] [[INSPIRE](#)].
- [35] M. Cvetič, G. Shiu and A.M. Uranga, *Three family supersymmetric standard-like models from intersecting brane worlds*, *Phys. Rev. Lett.* **87** (2001) 201801 [[hep-th/0107143](#)] [[INSPIRE](#)].
- [36] D.L. Wilkens, *Closed  $(s - 1)$ -connected  $(2s + 1)$ -manifolds,  $s = 3, 7$* , *Bull. London Math. Soc.* **4** (1972) 27.
- [37] D. Crowley and J. Nordström, *The classification of 2-connected 7-manifolds*, [arXiv:1406.2226](#).
- [38] A. Grassi, J. Halverson and J.L. Shaneson, *Matter From Geometry Without Resolution*, *JHEP* **10** (2013) 205 [[arXiv:1306.1832](#)] [[INSPIRE](#)].
- [39] A. Grassi, J. Halverson and J.L. Shaneson, *Non-Abelian Gauge Symmetry and the Higgs Mechanism in F-theory*, *Commun. Math. Phys.* **336** (2015) 1231 [[arXiv:1402.5962](#)] [[INSPIRE](#)].
- [40] A. Grassi, J. Halverson and J.L. Shaneson, *Geometry and Topology of String Junctions*, [arXiv:1410.6817](#) [[INSPIRE](#)].
- [41] E. Witten, *Nonperturbative superpotentials in string theory*, *Nucl. Phys. B* **474** (1996) 343 [[hep-th/9604030](#)] [[INSPIRE](#)].
- [42] R. Blumenhagen, M. Cvetič and T. Weigand, *Spacetime instanton corrections in 4D string vacua: The Seesaw mechanism for D-brane models*, *Nucl. Phys. B* **771** (2007) 113 [[hep-th/0609191](#)] [[INSPIRE](#)].
- [43] M. Cvetič, R. Donagi, J. Halverson and J. Marsano, *On Seven-Brane Dependent Instanton Prefactors in F-theory*, *JHEP* **11** (2012) 004 [[arXiv:1209.4906](#)] [[INSPIRE](#)].
- [44] M. Kerstan and T. Weigand, *Fluxed M5-instantons in F-theory*, *Nucl. Phys. B* **864** (2012) 597 [[arXiv:1205.4720](#)] [[INSPIRE](#)].
- [45] B.S. Acharya, K. Bobkov, G. Kane, P. Kumar and D. Vaman, *An M-theory Solution to the Hierarchy Problem*, *Phys. Rev. Lett.* **97** (2006) 191601 [[hep-th/0606262](#)] [[INSPIRE](#)].
- [46] B.S. Acharya, K. Bobkov, G.L. Kane, P. Kumar and J. Shao, *Explaining the Electroweak Scale and Stabilizing Moduli in M-theory*, *Phys. Rev. D* **76** (2007) 126010 [[hep-th/0701034](#)] [[INSPIRE](#)].
- [47] C. Beasley and E. Witten, *A note on fluxes and superpotentials in M-theory compactifications on manifolds of  $G_2$  holonomy*, *JHEP* **07** (2002) 046 [[hep-th/0203061](#)] [[INSPIRE](#)].
- [48] B.S. Acharya, G. Kane and P. Kumar, *Compactified String Theories – Generic Predictions for Particle Physics*, *Int. J. Mod. Phys. A* **27** (2012) 1230012 [[arXiv:1204.2795](#)] [[INSPIRE](#)].