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On the uniqueness of Cournot equilibrium in case of concave integrated price flexibility

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Abstract We consider a class of homogeneous Cournot oligopolies with concave integrated price flexibility and convex cost functions. We provide new results about the semi-uniqueness and uniqueness of (Cournot) equilibria for the oligopolies that satisfy these conditions. The condition of concave integrated price flexibility is implied by (but does not imply) the log-concavity of a continuous decreasing price function. So, our results generalize previous results for decreasing log-concave price functions and convex cost functions. We also discuss the particular type of quasi-concavity that characterizes the conditional revenue and profit functions of the firms in these oligopolies and we point out an error of the literature on the equilibrium uniqueness in oligopolies with log-concave price functions. Finally, we explain how the condition of concave integrated price flexibility relates to other conditions on the price and aggregate revenue functions usually considered in the literature, e.g., their concavity.

Keywords Equilibrium (semi-)uniqueness · Generalized convexity · Log-concavity · Oligopoly · Price flexibility · Price function

1 Introduction

Cournot equilibrium is one of the oldest equilibrium notions in mathematical economics. Its definition refers to an abstract representation of an oligopoly, which can be described as follows. A set $N := \{1, \dots, n\}$ of firms produce a homogeneous good. Each firm $i \in N$ chooses a quantity of production x_i from its own feasible production set X_i, where $X_i \subseteq \mathbb{R}$, facing a production cost $c_i(x_i)$, where $c_i : X_i \to \mathbb{R}$. The aggregate production

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$y = \sum_{i \in N} x_i$, which is an element of (the Minkowski-sum) $Y := \sum_{l \in N} X_l$, is sold to the market at the price $p(y)$, given by a function $p : Y \to \mathbb{R}$. Each firm $i \in N$ receives a profit $\pi_i(\mathbf{x}) := p(\sum_{l \in N} x_l)x_i - c_i(x_i)$. So this defines firm i's *profit function* $\pi_i : X_1 \times \cdots \times X_n \to \mathbb{R}$. We refer to p as the *price function* and to c_i as the *cost function* of firm i. Given this oligopolistic structure, $\mathbf{x}$ is called a *(Cournot) equilibrium* if, for all $i \in N$, $x_i \in \text{argmax}_{z_i \in X_i} \pi_i(x_1, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_n)$.

The conceptual definition of a Cournot equilibrium implies neither its existence nor its uniqueness, however both the problem of the existence and that of the uniqueness of an equilibrium are of interest in economics. The two problems are not unrelated to one another. In particular, when the existence problem is solved the uniqueness problem reduces to the problem of equilibrium semi-uniqueness, that is, the problem of the existence of at most one equilibrium. For a review of many results on these topics see also [3,7].

In this article we are concerned with equilibrium semi-uniqueness in Cournot oligopolies with convex cost functions and concave integrated price flexibility. Henceforth, we shall assume that either $X_i = \mathbb{R}_+$ or $X_i = [0, m_i]$ where $m_i > 0$. In case $X_i = [0, m_i]$, we say that firm i has a *capacity constraint*. Our main concern is semi-uniqueness and not uniqueness because in our case each conditional profit function of each firm[1] is quasi-concave and it is not difficult to see that under additional continuity and compactness conditions various well-known theorems (as, e.g., that of Nikaido-Isoda or Kakutani fixed point theorem) guarantee the existence of an equilibrium.

To the best of our knowledge there are no general semi-uniqueness results for Cournot oligopolies with convex cost functions and concave integrated price flexibility. Important results, such as those in [5] for concave aggregate revenue and those in [1] for log-concave price functions do not cover our results because there are price functions with strictly concave integrated price flexibility (like $p(y) = e^{-\sqrt{1+y}}$) which are not log-concave (as in [1]) and to which there is not associated a concave aggregate revenue function (as in [5]). Our results are complementary to those in [5] because there exist price functions without concave integrated price flexibility, to which there is associated a strictly concave aggregate revenue function. However, we shall prove that all decreasing continuous log-concave price functions have concave integrated price flexibility.

The organisation of the article is as follows. In Sect. 2 the notion of concave integrated price flexibility is formally defined and compared with other convexity notions. In Sect. 3 we study the *Nash-sum*, i.e., the function $\sigma : E \to \mathbb{R}$ defined by $\sigma(\mathbf{e}) := \sum_{l=1}^{n} e_l$ where E is the of equilibria. There we provide sufficient conditions for the Nash-sum to be injective and sufficient conditions for it to be constant. In Sect. 4 we identify a class of oligopolies with concave integrated price flexibility for which the Nash-sum is injective and constant, which implies that an oligopoly in this class has at most one equilibrium; also the existence problem will be considered. Section 5 reconsiders the case of log-concave price functions. Section 6 concludes and an Appendix contains some results on (generalized) convexity.

2 Integrated price flexibility

Consider a price function $p : Y \to \mathbb{R}$. If p is differentiable at $y \in Y$ and $p(y) \neq 0$, the *price flexibility* of p at y is defined by

[1] I.e., the profit function of a firm as a function of its own production.

$$\eta_p(y) := Dp(y)\frac{y}{p(y)},$$

i.e., as the elasticity of p at y.[2]

For a positive continuous price function $p : Y \to \mathbb{R}$, the function $L_p : Y \to \mathbb{R}$ is defined by

$$L_p(y) := y \ln p(y) - \int_0^y \ln p(\xi)\, d\xi. \tag{1}$$

We call this function the *integrated price flexibility* of p. We use this terminology because for each $y \in Y$ at which p is differentiable, also L_p is differentiable and

$$DL_p(y) = \eta_p(y). \tag{2}$$

The concavity of the integrated price flexibility is the property we address in this article. Fact F (in the Appendix) guarantees that a (strictly) decreasing continuous log-concave[3] price function p has a (strictly) concave integrated price flexibility L_p. Facts A and B1 guarantee that if p is positive and continuous, then the (strict) concavity of L_p implies the (strict) decreasingness of p. Just to provide an example, consider the price function given by $p(y) = \delta e^{-(\alpha+\beta y)^\gamma}$ with positive parameters. For this function $D\eta_p(y) = -\frac{\beta\gamma(\alpha+\beta\gamma y)}{(\alpha+\beta y)^{2-\gamma}} < 0$ holds and therefore p has a strictly concave integrated price flexibility.

It is good to compare the property of the concavity of the integrated price flexibility with three other properties of p that play an important role in oligopoly theory: the concavity of p, the log-concavity of p and the concavity of the *aggregate revenue function* r associated to p, where $r : Y \to \mathbb{R}$ is defined by $r(y) := p(y)y$. For simplicity, we shall do this for the context where the price function p is *positive, decreasing and differentiable*; the domain Y of p may be compact or not.

First note that $\eta_p \leq 0$, $Dr = p \cdot (\eta_p + 1)$ and that the (strict) concavity of the integrated price flexibility of p is equivalent to the (strict) decreasingness of η_p. Historically, oligopolies with a concave price function belong to the first types studied. A positive concave price function p is log-concave and has a concave associated aggregate revenue r (and a strictly concave associated aggregate revenue r if p is strictly concave). The log-concavity of p and the concavity of r have received special attention in the literature (e.g., [1,5,7]). Concave aggregate revenue functions are compatible with other types of price functions, like the strictly convex ones. However, Fact H shows that the log-concavity of p and the concavity of r are quite incompatible when $Y = \mathbb{R}_+$. As the log-concavity of p implies the decreasingness of η_p, the log-concavity of p implies the concavity of the integrated price flexibility.[4] Therefore, also concave price functions have concave integrated price flexibility. Table 1 illustrates.

It turns out that with our method we can obtain equilibrium semi-uniqueness results also for price functions that are unbounded at 0, like $p(y) = 1/y$ $(y \neq 0)$ (and, for example,[5]

[2] In this article Df is the symbol we use for the derivative of a real function f of one real variable; D^+f (D^-f) will be used for right (left) derivatives. Furthermore when f is a real function of more than one real variable, then $D_i f$ ($D_i^+ f$, $D_i^- f$) will be used for the (right, left) derivative with respect to the ith variable.

[3] Log-concavity of a real-valued function presupposes that this function is positive.

[4] And sufficient for the strict concavity of the integrated price flexibility is the log-concavity and the strict decreasingness of p; see Fact F.

[5] Clearly 137 is assigned arbitrarily because the profit functions, and therefore also the set of equilibria, do not depend on the value of p at 0.

Table 1 Various concrete price functions

$p \setminus$ Property	p log-concave	r concave	η_p decreasing
$\frac{1}{1+y}$	No	Yes	Yes
$e^{-\sqrt{1+y}}$	No	No	Yes
$1+\frac{1}{1+y}$	No	Yes	No if $Y=\mathbb{R}_+$
$\begin{cases} 3 \ (y \in [0,1]) \\ -y^2+2y+2 \ (y \in [1,2]) \end{cases}$	Yes	Yes	Yes

$p(0) = 137$). In order to handle them we consider the following variant of L_p. For $q \in Y \setminus \{0\}$ and a price function $p : Y \to \mathbb{R}$ that is positive on $Y \setminus \{0\}$ and continuous on $Y \setminus \{0\}$, the function $\tilde{L}_p : Y \setminus \{0\} \to \mathbb{R}$ is defined by

$$\tilde{L}_p(y) \ := \ y \ln p(y) - \int_q^y \ln p(\xi)\, d\xi. \tag{3}$$

Also this function will be called *integrated price flexibility* of p (with respect to q). This function depends on q, but various of its properties, e.g., its (strict) concavity, do not depend on q. We say that p has *concave* (resp. *strictly concave*) *integrated price flexibility* if there exists a q such that $\tilde{L}_p$ is concave (resp. strictly concave).

3 Nash-sums

The result in the following lemma gives very weak sufficient conditions in order that $\mathbf{0}$ be not an equilibrium when there exist at least two equilibria. In this lemma we assume that each cost function c_i is convex. We recall that this implies that c_i is left and right differentiable at each interior point of its domain and that c_i may not be right differentiable at 0, but its right derivative exists at 0 as an element of $\mathbb{R} \cup \{-\infty\}$. If firm i has a capacity constraint m_i, then c_i may not be left differentiable at m_i but its left derivative exists as an element of $\mathbb{R} \cup \{+\infty\}$. We write $\overline{\mathbb{R}} \ := \ \mathbb{R} \cup \{-\infty, +\infty\}$.

Lemma 1 *Suppose each cost function is convex, and*

- *p is left and right differentiable at each point of* $\mathrm{Int}(Y)$,
- $D^+p(y) \leq D^-p(y) \leq 0 \ (y \in \mathrm{Int}(Y))$,
- *in case $Y = [0, m]$, the (left) derivative $D^-p(m)$ of p at m exists as an element of $\overline{\mathbb{R}}$ with* $D^-p(m) \ \leq 0$.[6]

Also suppose p is strictly decreasing on $Y \setminus \{0\}$ or each cost function is strictly convex. If there exists more than one equilibrium, then $\mathbf{0}$ is not an equilibrium. $\diamond$

Proof By way of contradiction, suppose $\mathbf{a} = \mathbf{0}$ and $\mathbf{b}$ are two distinct equilibria. Let $y_b = \sum_{l \in N} b_l$. Fix $i \in N$ with $b_i > 0$. By Fact A (in the Appendix), p is decreasing on $Y \setminus \{0\}$. Therefore $\overline{p}(0) \ := \ \lim_{y \downarrow 0} p(y) \in \mathbb{R} \cup \{+\infty\}$ is well-defined. Now the contradiction

$$\begin{aligned} \overline{p}(0) - D^+c_i(0) &\leq 0 \leq D^-p(y_b)b_i + p(y_b) - D^-c_i(b_i) \\ &\leq p(y_b) - D^-c_i(b_i) < \overline{p}(0) - D^+c_i(0) \end{aligned}$$

6 But p need not to be differentiable at m, even not continuous. For example for $p : [0, 1] \to \mathbb{R}$ defined by $p(y) = 1 + \sqrt{1-y} \ (0 \leq y < 1), \ \ p(1) = 0$, we have $D^-p(1) = -\infty$.

follows: the first (second) inequality holds as **a** (**b**) is an equilibrium,[7] the third holds as $D^- p(y_b) \leq 0$ and the fourth as p is decreasing on $Y \setminus \{0\}$, c_i is convex, and either p is even strictly decreasing on $Y \setminus \{0\}$ or c_i is strictly convex. □

Theorem 1 *Suppose each cost function is convex, and*

- *p is left and right differentiable at each point of* $\text{Int}(Y)$,
- $D^- p(y) \leq D^+ p(y) \leq 0$ $(y \in \text{Int}(Y))$,
- *in case* $Y = [0, m]$, *the (left) derivative* $D^- p(m)$ *of p at m exists as an element of* $\overline{\mathbb{R}}$ *with* $D^- p(m) \leq 0$.

Each of the following conditions is sufficient for the Nash-sum σ to be injective: (I) $D^+ p(y) < 0$ $(y \in \text{Int}(Y))$. *(II) Each cost function is strictly convex.* ⋄

Proof By way of contradiction, suppose **a** and **b** are distinct equilibria with $\sigma(\mathbf{a}) = \sigma(\mathbf{b}) =: y$. This implies that $y \in \text{Int}(Y)$. (An obvious variant of) Fact A implies that p is decreasing on $Y \setminus \{0\}$ and strictly decreasing if condition I holds. Fix i such that $a_i < b_i$. As **a**, **b** are equilibria, it follows that $D_i^- \pi_i(\mathbf{b}) \geq 0 \geq D_i^+ \pi_i(\mathbf{a})$, i.e.,

$$D^- p(y) b_i + p(y) - D^- c_i(b_i) \geq 0 \geq D^+ p(y) a_i + p(y) - D^- c_i(a_i).$$

So the inequality $D^- p(y) b_i - D^+ p(y) a_i \geq D^- c_i(b_i) - D^+ c_i(a_i)$ holds. If condition I holds, then the left hand side of the last inequality is an element of $]-\infty, 0[$ and the right hand side is an element of $[0, +\infty]$; if condition II holds, then the left hand side of the last inequality is an element of $]-\infty, 0]$ and the right hand side is an element of $]0, +\infty]$, which is a contradiction. □

Theorem 2 *Suppose each cost function is convex and increasing, p is positive on $Y \setminus \{0\}$, decreasing on $Y \setminus \{0\}$ and continuous on $Y \setminus \{0\}$ with concave integrated price flexibility $\tilde{L}_p$. Each of the following conditions is sufficient for the Nash-sum σ to be constant: (I) The integrated price flexibility is strictly concave. (II) All cost functions are strictly convex.* ⋄

Proof By way of contradiction, suppose I or II holds and **a**, **b** are equilibria with

$$y_a := \sum_{l \in N} a_l < \sum_{l \in N} b_l =: y_b.$$

Let $J := \{l \in N \mid a_l < b_l\}$, $\tilde{y}_a := \sum_{l \in J} a_l$, $\tilde{y}_b := \sum_{l \in J} b_l$, $s := \#J$. (Here s is the number of elements of J.) Note that $\mathbf{b} \neq \mathbf{0}$, $y_b > 0$, $s \geq 1$, $\tilde{y}_a \leq y_a$, $\tilde{y}_b \leq y_b$, $\tilde{y}_a < \tilde{y}_b$, $\tilde{y}_b - \tilde{y}_a \geq y_b - y_a$. By Fact B2 we see that $p \in \mathcal{D}_{\leq}$ and that in case I even $p \in \mathcal{D}_{<}$. So Lemma 1 applies and therefore $y_a \neq 0$.

As **a** and **b** are equilibria, $D_i^+ \pi_i(\mathbf{a}) \leq 0 \leq D_i^- \pi_i(\mathbf{b})$ $(i \in J)$, i.e.,

$$D^+ p(y_a) a_i + p(y_a) - D^+ c_i(a_i) \leq 0 \leq D^- p(y_b) b_i + p(y_b) - D^- c_i(b_i) \ (i \in J). \tag{4}$$

We now prove that

$$D^+ p(y_a) \tilde{y}_a + s p(y_a) \geq D^- p(y_b) \tilde{y}_b + s p(y_b) \ (> \text{in case I}). \tag{5}$$

If $D^- p(y_b) = -\infty$, then (5) holds. Now suppose $D^- p(y_b) \neq -\infty$ and hence p is left differentiable at y_b. First we prove that

$$D^- p(y_b) \tilde{y}_b + s p(y_b) \geq 0. \tag{6}$$

[7] It may be good to note that the arithmetic operations here make sense in $\overline{\mathbb{R}}$, so for instance undefined operations like $+\infty - \infty$ do not occur.

Well, as $b_l > 0$ for every $l \in J$ and $\mathbf{b}$ is an equilibrium, we have $D^- p(y_b) b_l + p(y_b) - D^- c_l(b_l) \geq 0$ for every $l \in J$, and hence $D^- p(y_b) b_l + p(y_b) \geq D^- c_l(b_l)$ for every $l \in J$ from which it follows (as cost functions are increasing),

$$D^- p(y_b)\, \tilde{y}_b + sp(y_b) = \sum_{l \in J} \left(D^- p(y_b)\, b_l + p(y_b) \right) \geq \sum_{l \in J} D^- c_l(b_l) \geq 0.$$

So we have $D^- p(y_b)(y_b - (y_b - \tilde{y}_b)) + sp(y_b) \geq 0$. Noting that $y_b - \tilde{y}_b \in [0, y_a]$, Fact C implies

$$D^+ p(y_a)(y_a - (y_b - \tilde{y}_b)) + sp(y_a) \geq D^- p(y_b)\tilde{y}_b + sp(y_b) \ (> \text{in case I}).$$

As $D^+ p(y_a) \leq 0$ and $y_a - (y_b - \tilde{y}_b) \geq \tilde{y}_a$, (5) follows.

As each c_i is convex (and strictly convex in case II), (5) implies

$$D^+ p(y_a)\tilde{y}_a + sp(y_a) - \sum_{i \in J} D^+ c_i(a_i) > D^- p(y_b)\tilde{y}_b + sp(y_b) - \sum_{i \in J} D^- c_i(b_i),$$

which in turn implies the existence of $i \in J$ such that

$$D^+ p(y_a) a_i + p(y_a) - D^+ c_i(a_i) > D^- p(y_b) b_i + p(y_b) - D^- c_i(b_i),$$

which is in contradiction with (4). □

4 Uniqueness

4.1 Semi-uniqueness

Theorem 3 below, our main result, provides a semi-uniqueness result for a new class of price functions. The oligopolies in Theorem 3 are the same as in Theorem 2, but with an extra differentiability assumption for p.

Theorem 3 *Suppose each cost function is convex and increasing, p is positive on $Y \setminus \{0\}$, decreasing on $Y \setminus \{0\}$ and continuous on $Y \setminus \{0\}$ with concave integrated price flexibility $\tilde{L}_p$. If p is differentiable on the interior of its domain, then each of the following conditions is sufficient for the existence of at most one equilibrium: (I) The integrated price flexibility is strictly concave. (II) All cost functions are strictly convex.* ⋄

Proof By Theorem 2 the Nash-sum is constant. By Theorem 1 the Nash-sum is injective. It follows that there exists at most one equilibrium. □

Concerning the explicit monotonicity property of p in Theorem 3 it is interesting to note that the concavity of $\tilde{L}_p$ does not imply the decreasingness of p (consider for example $p(y) = \ln(y + 1)$). But by Facts A and B1, a *positive* continuous p with concave integrated price flexibility L_p is decreasing.

4.2 Existence and uniqueness

In order to avoid further technicalities we now investigate the problem of the existence of a unique Cournot equilibrium assuming the differentiability of p everywhere.

For $i \in N$ and $k \in \sum_{l \in N \setminus \{i\}} X_l$, firm i's *conditional revenue function* $r_i^{(k)} : X_i \to \mathbb{R}$ is defined by $r_i^{(k)}(x_i) := p(x_i + k)x_i$ and firm i's *conditional profit function* $\pi_i^{(k)} : X_i \to \mathbb{R}$ is defined by $\pi_i^{(k)} := r_i^{(k)} - c_i$.

Theorem 4 *Suppose cost functions are convex, increasing and continuous, p is positive, differentiable and has concave integrated price flexibility L_p. Then profit functions are continuous and conditional profit functions are quasi-concave.* $\diamond$

Proof The continuity of the profit functions is clear. In order to prove the quasi-concavity of the conditional profit functions we fix $i \in N$ and $k \in \sum_{l \in N \setminus \{i\}} X_l$. Note that $r_i^{(k)}$ is differentiable and $Dr_i^{(k)}(x_i) = Dp(x_i + k)x_i + p(x_i + k)$. Let $I := \{x \in X_i \mid Dr_i^{(k)}(x_i) > 0\}$. As $p(k) > 0$, we have $0 \in I$. Fact D implies that I is convex. Let $a, b \in I$ with $a < b$. Fact C implies $Dr_i^{(k)}(b) \leq Dr_i^{(k)}(a)$. This implies that $r_i^{(k)}$ is concave on I. As $Dr_i^{(k)}(x_i) \leq 0$ $(x_i \in X_i \setminus I)$, $r_i^{(k)}$ is decreasing on the interval $X_i \setminus I$. By assumption c_i is convex. If $I = X_i$, then $r_i^{(k)} - c_i$ is concave and the proof is complete. Now suppose $I \neq X_i$. As I is convex, it follows that I is bounded; let $d := \sup I$. As the restriction $(r_i^{(k)} - c_i) \upharpoonright I$ is concave and continuous, also the function $(r_i^{(k)} - c_i) \upharpoonright [0, d]$ is concave and continuous. The set of maximisers of this function is non-empty and compact. Let z be the greatest one. Noting that $r_i^{(k)} - c_i$ is concave on $[0, z]$, decreasing on $X_i \setminus [0, z]$ and continuous, it follows that $\pi_i^{(k)}$ is quasi-concave. $\square$

As noted in the Introduction, an oligopoly with quasi-concave conditional profit functions has an equilibrium under additional (standard) assumptions. Such an assumption is: *profit functions are continuous, and if there is a firm without capacity constraint, then p is decreasing and for each firm i without capacity constraint $p(x)x - c_i(x) \leq -c_i(0)$ for x large enough.*

This condition, together with Theorems 3 and 4 and Facts A and B1, implies:

Corollary 1 *Suppose each cost function is convex, increasing and continuous, p is positive, differentiable with concave integrated price flexibility and if there is a firm without capacity constraint, then for each firm i without capacity constraint $p(x)x - c_i(x) \leq -c_i(0)$ for x large enough.*

Each of the following additional conditions is sufficient for the existence of a unique equilibrium: (I) The integrated price flexibility is strictly concave. (II) All cost functions are strictly convex. $\diamond$

5 Log-concave price functions

Consider a continuous (strictly) decreasing log-concave price function p. So L_p has (strictly) concave integrated price flexibility L_p by fact F and Theorem 3 implies:

Corollary 2 *Suppose the price function is log-concave, decreasing and differentiable and each cost function is convex and increasing. Then each of the following conditions is sufficient for the existence of at most one equilibrium:* (I) *The price function is strictly decreasing.* (II) *All cost functions are strictly convex.* $\diamond$

Related to Corollary 2 is Theorem 2.3 in [1] for a **duopoly**:
Suppose no duopolist has a capacity constraint, the price function is log-concave and strictly decreasing and each cost function is convex and strictly increasing. If for each duopolist i it holds that $p(x)x - c_i(x) < 0$ for x large enough, then there exists a unique equilibrium.

Note that in this 'theorem' it is not assumed that the price function is differentiable. The next example, where all conditions of Theorem 2.3 in [1] hold, shows that this theorem is false:

Example 1 Consider the following duopoly:

$$p(y) = \begin{cases} 51 - y \text{ if } y \in [0,1], \\ 99 - 49y \text{ if } y \in [1,2], \\ e^{-100(y-2)} \text{ if } y \in [2, +\infty[, \end{cases} \quad c_1(x_1) = 49x_1, \quad c_2(x_2) = 49x_2.$$

It is straightforward (using the symmetry of the situation) to verify that $(\frac{9}{20}, \frac{11}{20})$ and $(\frac{11}{20}, \frac{9}{20})$ are equilibria. ◇

As in Example 1 also all conditions of Corollary 2(I) hold, with the exception of the differentiability of p, it follows that Corollary 2 does not hold anymore when we omit the assumption that the price function is differentiable.

6 Concluding remarks

As we have explained in the Introduction, and shown in the body of the article, our results extend (and correct) a result in [1]. It should be clear that, e.g., the results in [1] are not sufficient to conclude that the triopoly given by

- $N = \{1, 2, 3\}$,
- $p(y) = e^{-\sqrt{1+y}}$,
- and $X_i = \mathbb{R}_+$, $c_i(x_i) = \frac{x_i}{100}$ for each firm $i \in N$,

has a unique Cournot equilibrium; also, the results in [3] (in particular Theorem 6.1) does not prove this fact.[8] So, to the best of our knowledge, the results presented in this article are new.

Our results build on—and extend—a technique developed in [6]. We conjecture that this technique can be also used to simplify and shorten the proof of important equilibrium semi-uniqueness results for oligopolies with convex cost functions and aggregate revenue functions such as that in Lemma 5 of [5] (which neither implies nor is implied by the results of this article, as it is clear from the table in Sect. 2).

It is good to remark that in this article, unlike in [5], we cannot guarantee equilibrium semi-uniqueness when the cost functions are not increasing. The reason is that in the proof of Theorem 2 we utilize a particular type of quasi-concavity of the conditional revenue functions which implies the crucial inequality (6). As it can be seen from the proof of Theorem 4 this particular quasi-concavity requires that, at each Cournot equilibrium $\mathbf{e}$, the conditional revenue of a firm i that produces $e_i > 0$ must be either concave and increasing, or concave and increasing up to some point $x_i \geq e_i$ where the conditional revenue is maximized and decreasing beyond that point, whence (6) follows easily. But this does not keep on holding—and hence inequality (6) does not generally hold true—if the cost function of firm i is convex but not increasing. It is good to note that also the conditional profit function of firm i is concave and increasing up to e_i and decreasing beyond that point, thus inheriting the same type of quasi-concavity of the conditional revenue function; also this fact does not keep on holding when the cost functions are convex but not increasing.

This particular type of quasi-concavity has a strong connection with the notion of semi-convexity provided in [4]. We think that this type of generalized convexity could have many

8 The reason is that, adopting the terminology and the definitions of [3], there does not exist a pair (α, β) such that $\alpha \leq 1$, $\beta \leq 1$, $\Delta^{p}_{\alpha,\beta}(12) \leq 0$, $(\alpha + \beta)\, Dp\,(12) - D^2 c_i\,(12) \leq 0$ and that (i) either $\alpha + \beta = 1$ or that (ii) $\alpha = 1$ and $\beta < 0$.

natural applications in economics and, to the best of our knowledge, this is the first article dealing with economic applications that explicitly refers to such a generalized convexity notion.

Finally, a large part of the literature on the semi-uniqueness and uniqueness of Cournot equilibrium takes into consideration oligopolies without capacity constraints (or better, firms' feasible production sets are assumed to be upper unbounded). As explained in [2], the assumption of upper unboundedness of the feasible productions sets of the firms is not an inconsequential assumption and influences some results of the literature on Cournot equilibrium, in particular when one considers the dynamics. Clearly, the equilibrium semi-uniqueness and uniqueness results of this article do not depend on the upper unboundedness of the feasible productions sets of the firms because we allow both upper bounded and upper unbounded feasible productions sets (note that when we provide sufficient conditions for the existence of a Cournot equilibrium we impose the upper boundedness of the profit functions, but the feasible production sets can be either bounded or not).

Appendix

The presentation here will be self-contained. Consider a function $p : Y \to \mathbb{R}$ where $Y = \mathbb{R}_+$ or $Y = [0, m]$ with $m > 0$. Define $r : Y \to \mathbb{R}$ by $r(y) := p(y)y$. When p is positive and continuous, the function $L_p : Y \to \mathbb{R}$ is defined by (1) and when p is positive on $Y \setminus \{0\}$ and continuous on $Y \setminus \{0\}$, the function $\tilde{L}_p : Y \to \mathbb{R}$ is defined by (3). We write $p \in \mathcal{D}_{\leq}$ $(\mathcal{D}_{<})$ if

- p is left and right differentiable at each interior point of its domain and in case $Y = [0, m]$, the (left) derivative $D^- p(m)$ of p at m exists as an element of $\overline{\mathbb{R}}$[9];
- $D^+ p(y) \leq D^- p(y)$ $(y \in \mathrm{Int}(Y))$;
- $D^- p \leq (<)\ 0$.

The following results for p, L_p, $\tilde{L}_p$ and r are presented here as *Facts*:

A. If $p \in \mathcal{D}_{\leq}$ $(\mathcal{D}_{<})$, then p is (strictly) decreasing on $Y \setminus \{0\}$.

B1. Suppose p is positive and continuous and L_p is (strictly) concave. Then $p \in \mathcal{D}_{\leq}$ $(p \in \mathcal{D}_{<})$.

B2. Suppose p is positive on $Y \setminus \{0\}$, continuous on $Y \setminus \{0\}$ and decreasing on $Y \setminus \{0\}$ and $\tilde{L}_p$ is (strictly) concave. Then $p \in \mathcal{D}_{\leq}$ $(p \in \mathcal{D}_{<})$.

C. Suppose p is positive on $Y \setminus \{0\}$, continuous on $Y \setminus \{0\}$, decreasing on $Y \setminus \{0\}$ and $\tilde{L}_p$ is (strictly) concave. Let $y_1, y_2 \in Y$ with $0 < y_1 < y_2$, $k \in [0, y_1]$ and $s \in \mathbb{R}$. Then

$$\begin{aligned} &D^- p(y_2)(y_2 - k) + sp(y_2) \geq 0 \\ &\quad \Rightarrow\ D^+ p(y_1)(y_1 - k) + sp(y_1) \geq (>)\ D^- p(y_2)(y_2 - k) + sp(y_2). \end{aligned}$$

D. Suppose p is positive, differentiable with concave integrated price flexibility L_p. Then, for each $k \in Y$, the set $H_k := \{x \in (Y - \{k\}) \cap Y \mid Dp(x + k)x + p(x + k) > 0\}$ is convex.

E. Suppose p is log-concave and strictly decreasing. Then $D^- p < 0$.

F. If p is log-concave, continuous and (strictly) decreasing, then p has (strictly) concave integrated price flexibility L_p.

[9] This defines $D^+ p : \mathrm{Int}(Y) \to \mathbb{R}$ and $D^- p : Y \setminus \{0\} \to \mathbb{R}$.

G. If r is (strictly) concave, then $p \in \mathcal{D}_{\leq}$ $(\mathcal{D}_{<})$.
H. Suppose $Y = \mathbb{R}_{+}$. If p is log-concave on $\mathbb{R}_{++}$ and non-constant on $Y \setminus \{0\}$, then r is not concave.

Proof of Fact A As p is right and left differentiable on $\text{Int}(Y)$, p is continuous on $\text{Int}(Y)$. Together with $D^{+}p(y) \leq (<)\ 0$ $(y \in \text{Int}(Y))$, this implies that p is (strictly) decreasing on $\text{Int}(Y)$. So, if $Y = \mathbb{R}_{+}$, we are done. Now suppose $Y = [0, m]$. As $D^{-}p(m) \leq 0$, p is even (strictly) decreasing on $]0, m]$. □

Proof of Fact B1 With the exception of $D^{-}p \leq (<)\ 0$ this Fact is a direct consequence of the (strict) concavity of L_p. The important observation here is to realize that $D^{+}L_p(0) = 0$ which can be proved with the rule of de l'Hôpital:

$$D^{+}L_p(0) = \lim_{h\downarrow 0} \frac{L_p(h) - L_p(0)}{h} = \lim_{h\downarrow 0} \frac{h \ln p(h) - \int_0^h \ln p(\xi)\, d\xi}{h}$$
$$= \ln p(0) - \lim_{h\downarrow 0} \frac{\int_0^h \ln p(\xi)\, d\xi}{h} = \ln p(0) - \lim_{h\downarrow 0} \frac{\ln p(h)}{1} = 0.$$

Let $y \neq 0$. As L_p is (strictly) concave, $\frac{yD^{-}p(y)}{p(y)} = D^{-}L_p(y) \leq (<)\ D^{+}L_p(0) = 0$. Thus $D^{-}p(y) \leq (<)\ 0$. □

Proof of Fact B2 Having the proof of Fact B1, we only have to prove that $D^{-}p < 0$ in case $\tilde{L}_p$ is strictly concave. So suppose there exists $y_0 \in Y \setminus \{0\}$ with $D^{-}p(y_0) \geq 0$. As p is decreasing on $Y \setminus \{0\}$, it follows that $D^{-}p \leq 0$. So $D^{-}p(y_0) = 0$ and therefore $D^{-}\tilde{L}_p(y_0) = 0$. But $D^{-}\tilde{L}_p$ is strictly decreasing and $D^{-}\tilde{L}_p \leq 0$, which is impossible. □

Proof of Fact C By contradiction, suppose there exist y_1, y_2, k, s as above and

$$D^{-}p(y_2)(y_2 - k) + sp(y_2) \geq 0,$$
$$D^{+}p(y_1)(y_1 - k) + sp(y_1) < (\leq)\ D^{-}p(y_2)(y_2 - k) + sp(y_2).$$

Since p is positive and decreasing on $[y_1, y_2]$, $0 < \frac{1}{p(y_1)} \leq \frac{1}{p(y_2)}$; so

$$D^{+}p(y_1)\frac{y_1 - k}{p(y_1)} + s < (\leq)\ \frac{1}{p(y_1)}(D^{-}p(y_2)(y_2 - k) + sp(y_2))$$
$$\leq \frac{1}{p(y_2)}(D^{-}p(y_2)(y_2 - k) + sp(y_2)) = D^{-}p(y_2)\frac{y_2 - k}{p(y_2)} + s,$$

and hence

$$D^{+}p(y_1)\frac{y_1 - k}{p(y_1)} < (\leq)\ D^{-}p(y_2)\frac{y_2 - k}{p(y_2)}. \tag{7}$$

As $\tilde{L}_p$ is (strictly) concave, we have $D^{+}\tilde{L}_p(y_1) \geq (>)\ D^{-}\tilde{L}_p(y_2)$. As p is decreasing on $Y \setminus \{0\}$, this leads to

$$0 \geq D^{+}p(y_1)\frac{y_1}{p(y_1)} \geq (>)\ D^{-}p(y_2)\frac{y_2}{p(y_2)}.$$

This implies (noting that $y_1 > 0$, $0 \leq \frac{y_1 - k}{y_1} \leq \frac{y_2 - k}{y_2}$ and $\frac{y_2 - k}{y_2} > 0$)

$$\frac{y_1 - k}{y_1} D^{+}p(y_1)\frac{y_1}{p(y_1)} \geq (>)\ \frac{y_2 - k}{y_2} D^{-}p(y_2)\frac{y_2}{p(y_2)},$$

which is a contradiction with (7). □

Proof of Fact D For $x \in (Y - \{k\}) \cap Y$ denote $Dp(x+k)x + p(x+k)$ by $g(x)$. It is sufficient to prove that $H_k \setminus \{0\}$ is convex. We provide the proof by contradiction. So suppose $x_1, x_2 \in H_k \setminus \{0\}$ with $x_1 < x_2$ and $g(x_3) \le 0$ where $x_3 = tx_1 + (1-t)x_2$ for some $t \in]0, 1[$. We have $g(x_3) \le 0 < \frac{1}{5} g(x_2) < g(x_2)$. As g is a derivative, g is also a Darboux-function. Therefore there exists $x_4 \in]x_3, x_2[$ with $g(x_4) = \frac{1}{5} g(x_2) > 0$. So we have

$$Dp(x_2 + k)(x_2 + k - k) + p(x_2 + k) > 0,\ 0 < x_4 + k < x_2 + k,\ k \in [0, x_4 + k].$$

By Facts A and B1, p is decreasing. Fact C implies that $Dp(x_4 + k)x_4 + p(x_4 + k) \ge Dp(x_2 + k)x_2 + p(x_2 + k)$. It follows that $0 < g(x_2) \le g(x_4) = \frac{1}{5} g(x_2)$, which is impossible. □

Proof of Fact E By way of contradiction, assume that there exists $y_0 \in Y \setminus \{0\}$ with $D^- p(y_0) \ge 0$. As p is strictly decreasing, $D^- p/p \le 0$ holds. This implies $D^- p(y_0) \le 0$. So $D^- p(y_0) = 0$. As p is log-concave, $D^- p/p$ is decreasing. As $D^- p(y_0)/p(y_0) = 0$, it follows that $D^- p/p$ vanishes on the proper interval $]0, y_0]$, and so does also $D^- p$. As p is also continuous on this interval, it follows that p is constant on this interval and therefore not strictly decreasing on this interval, which is a contradiction with the strict decreasingness of p. □

Proof of Fact F First we note that sufficient for the (strict) concavity of a function f defined on a real interval is that f is continuous, its left derivative $D^- f$ is well defined and (strictly) decreasing, and real-valued on the interior of its domain. Well, L_p has all these properties. In fact, we only have to prove that $D^- L_p$ is (strictly) decreasing. As p is (strictly) decreasing (using Fact E) we have $\frac{D^- p}{p} \le (<) 0$, so $D^- L_p \le (<) 0$. As p is log-concave, $\frac{D^- p}{p}$ is decreasing and therefore also $D^- L_p$ is (strictly) decreasing. □

Proof of Fact G As r is a (strictly) concave function it holds that r is left and right differentiable at each interior point of its domain. As $p(y) = r(y)/y$ $(y \ne 0)$, this implies also that p is left and right differentiable at each point of $\mathrm{Int}(Y)$. As r is (strictly) concave, it holds for all $y \in \mathrm{Int}(Y)$ that $D^+ r(y) \le D^- r(y) \le (<) \frac{r(y) - r(0)}{y - 0}$. So $D^+ p(y)y + p(y) \le D^- p(y)y + p(y) \le (<) p(y)$ and thus $D^+ p(y) \le D^- p(y) \le (<) 0$. In case $Y = [0, m]$, the proof of $D^- p(m) \le (<) 0$ is similar. □

Proof of Fact H By way of contradiction, assume that r is concave. As r is concave, $r(0) = 0$, $r \ge 0$ and r is somewhere positive, it follows that $\lim_{y \to +\infty} r(y)$ exists as an element of $\overline{\mathbb{R}}$ and

$$\lim_{y \to +\infty} r(y) > 0.$$

By Facts A and G, p is decreasing on $\mathbb{R}_{++}$. So there exists $z > 0$ with $D^+ p(z) < 0$. Let $\alpha := \ln p(z)$ and $\beta := D^+ \ln p(z) = \frac{D^+ p(z)}{p(z)} < 0$. Let $l : \mathbb{R} \to \mathbb{R}$ be defined by $l(y) := \alpha + \beta(y - z)$. Since l is affine, $l(z) = \ln p(z)$ and $\ln p$ is concave on $\mathbb{R}_{++}$, it follows that $\ln p(y) \le l(y)$ $(y \in \mathbb{R}_{++})$. Therefore, $r(y) = p(y)y \le e^{\alpha - \beta z + \beta y} y$ $(y \in \mathbb{R}_{++})$, whence a contradiction because $\lim_{y \to +\infty} r(y) = 0$. □

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