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Functional equations in paranormed spaces

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Abstract
In this paper, we prove the Hyers-Ulam stability of various functional equations in paranormed spaces.

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Keywords: Hyers-Ulam stability; paranormed space; functional equation

1 Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [1] and Steinhaus [2] independently, and since then several generalizations and applications of this notion have been investigated by various authors (see [3–7]). This notion was defined in normed spaces by Kolk [8].

We recall some basic facts concerning Fréchet spaces.

Definition 1.1 ([9]) Let X be a vector space. A paranorm $P : X \to [0,\infty)$ is a function on X such that

(1) $P(0) = 0$;
(2) $P(-x) = P(x)$;
(3) $P(x + y) \leq P(x) + P(y)$ (triangle inequality);
(4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).

The pair (X,P) is called a *paranormed space* if P is a *paranorm* on X.

The paranorm is called *total* if, in addition, we have

(5) $P(x) = 0$ implies $x = 0$.

A *Fréchet space* is a total and complete paranormed space.

The stability problem of functional equations originated from a question of Ulam [10] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [12] for additive mappings and by Th. M. Rassias [13] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th. M. Rassias' theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach.

In 1990 during the 27th International Symposium on Functional Equations, Th. M. Rassias [15] asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991 Gajda [16], following the same approach as in Th. M. Rassias [13], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [16], as well as by Th. M. Rassias and Šemrl [17] that one cannot prove a Th. M. Rassias' type theorem when $p = 1$ (cf. the books of P. Czerwik [18], D. H. Hyers, G. Isac and Th. M. Rassias [19]).

In 1982 J. M. Rassias [20] followed the innovative approach of the Th. M. Rassias' theorem [13] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p+q \neq 1$. Găvruta [14] provided a further generalization of Th. M. Rassias' theorem.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [21] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [22] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [23] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [24–33]).

In [34], Jun and Kim considered the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{1.1}$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [35], Lee *et al.* considered the following quartic functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{1.2}$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation*, and every solution of the quartic functional equation is said to be a *quartic mapping*.

Throughout this paper, assume that (X, P) is a Fréchet space and that $(Y, \|\cdot\|)$ is a Banach space.

In this paper, we prove the Hyers-Ulam stability of the Cauchy additive functional equation, the quadratic functional equation, the cubic functional equation (1.1) and the quartic functional equation (1.2) in paranormed spaces.

2 Hyers-Ulam stability of the Cauchy additive functional equation

In this section, we prove the Hyers-Ulam stability of the Cauchy additive functional equation in paranormed spaces.

Note that $P(2x) \leq 2P(x)$ for all $x \in Y$.

Theorem 2.1 *Let r, θ be positive real numbers with $r > 1$, and let $f : Y \to X$ be an odd mapping such that*

$$P\big(f(x+y) - f(x) - f(y)\big) \leq \theta\left(\|x\|^r + \|y\|^r\right) \tag{2.1}$$

for all $x, y \in Y$. Then there exists a unique Cauchy additive mapping $A : Y \to X$ such that

$$P\big(f(x) - A(x)\big) \le \frac{2\theta}{2^r - 2}\|x\|^r \tag{2.2}$$

for all $x \in Y$.

Proof Letting $y = x$ in (2.1), we get

$$P\big(f(2x) - 2f(x)\big) \le 2\theta\|x\|^r$$

for all $x \in Y$. So

$$P\left(f(x) - 2f\left(\frac{x}{2}\right)\right) \le \frac{2}{2^r}\theta\|x\|^r$$

for all $x \in Y$. Hence

$$\begin{aligned} P\left(2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right) &\le \sum_{j=l}^{m-1} P\left(2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right) \\ &\le \frac{2}{2^r}\sum_{j=l}^{m-1}\frac{2^j}{2^{rj}}\theta\|x\|^r \end{aligned} \tag{2.3}$$

for all nonnegative integers m and l with $m > l$ and all $x \in Y$. It follows from (2.3) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in Y$. Since X is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : Y \to X$ by

$$A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in Y$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.3), we get (2.2).

It follows from (2.1) that

$$\begin{aligned} P\big(A(x+y) - A(x) - A(y)\big) &= \lim_{n\to\infty} P\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right) \\ &\le \lim_{n\to\infty} 2^n P\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right) \\ &\le \lim_{n\to\infty}\frac{2^n\theta}{2^{nr}}\big(\|x\|^r + \|y\|^r\big) = 0 \end{aligned}$$

for all $x, y \in Y$. Hence $A(x+y) = A(x) + A(y)$ for all $x, y \in Y$ and so the mapping $A : Y \to X$ is Cauchy additive.

Now, let $T : Y \to X$ be another Cauchy additive mapping satisfying (2.2). Then we have

$$\begin{aligned} P\big(A(x) - T(x)\big) &= P\left(2^n\left(A\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right)\right) \\ &\le 2^n P\left(A\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right) \end{aligned}$$

$$\leq 2^n\left(P\left(A\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right)+P\left(T\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right)\right)$$
$$\leq \frac{4\cdot 2^n}{(2^r-2)2^{nr}}\theta\|x\|^r,$$

which tends to zero as $n\to\infty$ for all $x\in Y$. So we can conclude that $A(x)=T(x)$ for all $x\in Y$. This proves the uniqueness of A. Thus the mapping $A:Y\to X$ is a unique Cauchy additive mapping satisfying (2.2). □

Theorem 2.2 *Let r be a positive real number with $r<1$, and let $f:X\to Y$ be an odd mapping such that*

$$\|f(x+y)-f(x)-f(y)\|\leq P(x)^r+P(y)^r \tag{2.4}$$

for all $x,y\in X$. Then there exists a unique Cauchy additive mapping $A:X\to Y$ such that

$$\|f(x)-A(x)\|\leq \frac{2}{2-2^r}P(x)^r \tag{2.5}$$

for all $x\in X$.

Proof Letting $y=x$ in (2.4), we get

$$\|2f(x)-f(2x)\|\leq 2P(x)^r$$

and so

$$\left\|f(x)-\frac{1}{2}f(2x)\right\|\leq P(x)^r$$

for all $x\in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f\left(2^lx\right)-\frac{1}{2^m}f\left(2^mx\right)\right\| &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^jx\right)-\frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\|\\ &\leq \sum_{j=l}^{m-1}\frac{2^{rj}}{2^j}P(x)^r \end{aligned}\tag{2.6}$$

for all nonnegative integers m and l with $m>l$ and all $x\in X$. It follows from (2.6) that the sequence $\{\frac{1}{2^n}f(2^nx)\}$ is a Cauchy sequence for all $x\in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^nx)\}$ converges. So one can define the mapping $A:X\to Y$ by

$$A(x):=\lim_{n\to\infty}\frac{1}{2^n}f\left(2^nx\right)$$

for all $x\in X$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (2.6), we get (2.5).

It follows from (2.4) that

$$\begin{aligned}\|A(x+y)-A(x)-A(y)\| &= \lim_{n\to\infty}\frac{1}{2^n}\left\|f\left(2^n(x+y)\right)-f\left(2^nx\right)-f\left(2^ny\right)\right\|\\ &\leq \lim_{n\to\infty}\frac{2^{nr}}{2^n}\left(P(x)^r+P(y)^r\right)=0\end{aligned}$$

for all $x, y \in X$. Thus $A(x+y) = A(x) + A(y)$ for all $x, y \in X$ and so the mapping $A : X \to Y$ is Cauchy additive.

Now, let $T : X \to Y$ be another Cauchy additive mapping satisfying (2.5). Then we have

$$\begin{aligned} \|A(x) - T(x)\| &= \frac{1}{2^n}\|A(2^n x) - T(2^n x)\| \\ &\le \frac{1}{2^n}\big(\|A(2^n x) - f(2^n x)\| + \|T(2^n x) - f(2^n x)\|\big) \\ &\le \frac{4 \cdot 2^{nr}}{(2-2^r)2^n} P(x)^r, \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. Thus the mapping $A : X \to Y$ is a unique Cauchy additive mapping satisfying (2.5). □

3 Hyers-Ulam stability of the quadratic functional equation

In this section, we prove the Hyers-Ulam stability of the quadratic functional equation in paranormed spaces.

Note that $P(2x) \le 2P(x)$ for all $x \in Y$.

Theorem 3.1 *Let r, θ be positive real numbers with $r > 2$, and let $f : Y \to X$ be a mapping satisfying $f(0) = 0$ and*

$$P\big(f(x+y) + f(x-y) - 2f(x) - 2f(y)\big) \le \theta\big(\|x\|^r + \|y\|^r\big) \tag{3.1}$$

for all $x, y \in Y$. Then there exists a unique quadratic mapping $Q_2 : Y \to X$ such that

$$P\big(f(x) - Q_2(x)\big) \le \frac{2\theta}{2^r - 4}\|x\|^r \tag{3.2}$$

for all $x \in Y$.

Proof Letting $y = x$ in (3.1), we get

$$P\big(f(2x) - 4f(x)\big) \le 2\theta\|x\|^r$$

for all $x \in Y$. So

$$P\left(f(x) - 4f\left(\frac{x}{2}\right)\right) \le \frac{2}{2^r}\theta\|x\|^r$$

for all $x \in Y$. Hence

$$\begin{aligned} P\left(4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right) &\le \sum_{j=l}^{m-1} P\left(4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right) \\ &\le \frac{2}{2^r}\sum_{j=l}^{m-1}\frac{4^j}{2^{rj}}\theta\|x\|^r \end{aligned} \tag{3.3}$$

for all nonnegative integers m and l with $m > l$ and all $x \in Y$. It follows from (3.3) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in Y$. Since X is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q_2 : Y \to X$ by

$$Q_2(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in Y$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.3), we get (3.2).

It follows from (3.1) that

$$\begin{aligned}
&P\big(Q_2(x+y) + Q_2(x-y) - 2Q_2(x) - 2Q_2(y)\big) \\
&\quad = \lim_{n\to\infty} P\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)\right) \\
&\quad \le \lim_{n\to\infty} 4^n P\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right) \\
&\quad \le \lim_{n\to\infty} \frac{4^n\theta}{2^{nr}}\big(\|x\|^r + \|y\|^r\big) = 0
\end{aligned}$$

for all $x, y \in Y$. Hence $Q_2(x+y) + Q_2(x-y) = 2Q_2(x) + 2Q_2(y)$ for all $x, y \in Y$ and so the mapping $Q_2 : Y \to X$ is quadratic.

Now, let $T : Y \to X$ be another quadratic mapping satisfying (3.2). Then we have

$$\begin{aligned}
P\big(Q_2(x) - T(x)\big) &= P\left(4^n\left(Q_2\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right)\right) \\
&\le 4^n P\left(Q_2\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right) \\
&\le 4^n\left(P\left(Q_2\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right) + P\left(T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right)\right) \\
&\le \frac{4 \cdot 4^n}{(2^r - 4)2^{nr}}\theta\|x\|^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in Y$. So we can conclude that $Q_2(x) = T(x)$ for all $x \in Y$. This proves the uniqueness of Q_2. Thus the mapping $Q_2 : Y \to X$ is a unique quadratic mapping satisfying (3.2). □

Theorem 3.2 *Let r be a positive real number with $r < 2$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\big\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\big\| \le P(x)^r + P(y)^r \tag{3.4}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q_2 : X \to Y$ such that

$$\big\|f(x) - Q_2(x)\big\| \le \frac{2}{4 - 2^r}P(x)^r \tag{3.5}$$

for all $x \in X$.

Proof Letting $y = x$ in (3.4), we get

$$\|4f(x) - f(2x)\| \leq 2P(x)^r$$

and so

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{1}{2}P(x)^r$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f(2^j x) - \frac{1}{4^{j+1}}f(2^{j+1}x)\right\| \\ &\leq \frac{1}{2}\sum_{j=l}^{m-1}\frac{2^{rj}}{4^j}P(x)^r \end{aligned} \tag{3.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q_2 : X \to Y$ by

$$Q_2(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.5).

It follows from (3.4) that

$$\begin{aligned}&\|Q_2(x+y) + Q_2(x-y) - 2Q_2(x) - 2Q_2(y)\| \\ &\quad= \lim_{n\to\infty}\frac{1}{4^n}\|f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y)\| \\ &\quad\leq \lim_{n\to\infty}\frac{2^{nr}}{4^n}\left(P(x)^r + P(y)^r\right) = 0\end{aligned}$$

for all $x, y \in X$. Thus $Q_2(x + y) + Q_2(x - y) = 2Q_2(x) + 2Q_2(y)$ for all $x, y \in X$ and so the mapping $Q_2 : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (3.5). Then we have

$$\begin{aligned}\|Q_2(x) - T(x)\| &= \frac{1}{4^n}\|Q_2(2^n x) - T(2^n x)\| \\ &\leq \frac{1}{4^n}\left(\|Q_2(2^n x) - f(2^n x)\| + \|T(2^n x) - f(2^n x)\|\right) \\ &\leq \frac{4 \cdot 2^{nr}}{(4-2^r)4^n}P(x)^r,\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q_2(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q_2. Thus the mapping $Q_2 : X \to Y$ is a unique quadratic mapping satisfying (3.5). □

4 Hyers-Ulam stability of the cubic functional equation

In this section, we prove the Hyers-Ulam stability of the cubic functional equation in paranormed spaces.

Note that $P(2x) \leq 2P(x)$ for all $x \in Y$.

Theorem 4.1 *Let r, θ be positive real numbers with $r > 3$, and let $f : Y \to X$ be a mapping such that*

$$P\left(\frac{1}{2}f(2x+y) + \frac{1}{2}f(2x-y) - f(x+y) - f(x-y) - 6f(x)\right) \leq \theta\left(\|x\|^r + \|y\|^r\right) \tag{4.1}$$

for all $x, y \in Y$. Then there exists a unique cubic mapping $C : Y \to X$ such that

$$P\big(f(x) - C(x)\big) \leq \frac{\theta}{2^r - 8}\|x\|^r \tag{4.2}$$

for all $x \in Y$.

Proof Letting $y = 0$ in (4.1), we get

$$P\big(f(2x) - 8f(x)\big) \leq \theta\|x\|^r$$

for all $x \in Y$. So

$$P\left(f(x) - 8f\left(\frac{x}{2}\right)\right) \leq \frac{1}{2^r}\theta\|x\|^r$$

for all $x \in Y$. Hence

$$\begin{aligned} P\left(8^l f\left(\frac{x}{2^l}\right) - 8^m f\left(\frac{x}{2^m}\right)\right) &\leq \sum_{j=l}^{m-1} P\left(8^j f\left(\frac{x}{2^j}\right) - 8^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right) \\ &\leq \frac{1}{2^r}\sum_{j=l}^{m-1}\frac{8^j}{2^{rj}}\theta\|x\|^r \end{aligned} \tag{4.3}$$

for all nonnegative integers m and l with $m > l$ and all $x \in Y$. It follows from (4.3) that the sequence $\{8^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in Y$. Since X is complete, the sequence $\{8^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $C : Y \to X$ by

$$C(x) := \lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right)$$

for all $x \in Y$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.3), we get (4.2).

It follows from (4.1) that

$$\begin{aligned} &P\left(\frac{1}{2}C(2x+y) + \frac{1}{2}C(2x-y) - C(x+y) - C(x-y) - 6C(x)\right) \\ &\quad = \lim_{n\to\infty} P\left(8^n\left(\frac{1}{2}f\left(\frac{2x+y}{2^n}\right) + \frac{1}{2}f\left(\frac{2x-y}{2^n}\right) - f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x-y}{2^n}\right) - 6f\left(\frac{x}{2^n}\right)\right)\right) \end{aligned}$$

$$\begin{aligned}&\leq \lim_{n\to\infty} 8^n P\left(\frac{1}{2}f\left(\frac{2x+y}{2^n}\right)+\frac{1}{2}f\left(\frac{2x-y}{2^n}\right)-f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x-y}{2^n}\right)-6f\left(\frac{x}{2^n}\right)\right)\\&\leq \lim_{n\to\infty}\frac{8^n\theta}{2^{nr}}\left(\|x\|^r+\|y\|^r\right)=0\end{aligned}$$

for all $x, y \in Y$. Hence

$$\frac{1}{2}C(2x+y)+\frac{1}{2}C(2x-y)=C(x+y)+C(x-y)+6C(x)$$

for all $x, y \in Y$ and so the mapping $C : Y \to X$ is cubic.

Now, let $T : Y \to X$ be another cubic mapping satisfying (4.2). Then we have

$$\begin{aligned}P\bigl(C(x)-T(x)\bigr) &= P\left(8^n\left(C\left(\frac{x}{2^n}\right)-T\left(\frac{x}{2^n}\right)\right)\right)\\&\leq 8^n P\left(C\left(\frac{x}{2^n}\right)-T\left(\frac{x}{2^n}\right)\right)\\&\leq 8^n\left(P\left(C\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right)+P\left(T\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right)\right)\\&\leq \frac{2\cdot 8^n}{(2^r-8)2^{nr}}\theta\|x\|^r,\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in Y$. So we can conclude that $C(x) = T(x)$ for all $x \in Y$. This proves the uniqueness of C. Thus the mapping $C : Y \to X$ is a unique cubic mapping satisfying (4.2). □

Theorem 4.2 *Let r be a positive real number with $r < 3$, and let $f : X \to Y$ be a mapping such that*

$$\left\|\frac{1}{2}f(2x+y)+\frac{1}{2}f(2x-y)-f(x+y)-f(x-y)-6f(x)\right\| \leq P(x)^r+P(y)^r \tag{4.4}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\left\|f(x)-C(x)\right\| \leq \frac{1}{8-2^r}P(x)^r \tag{4.5}$$

for all $x \in X$.

Proof Letting $y = 0$ in (4.4), we get

$$\left\|8f(x)-f(2x)\right\| \leq P(x)^r$$

and so

$$\left\|f(x)-\frac{1}{8}f(2x)\right\| \leq \frac{1}{8}P(x)^r$$

for all $x \in X$. Hence

$$\left\|\frac{1}{8^l}f\left(2^l x\right)-\frac{1}{8^m}f\left(2^m x\right)\right\| \leq \sum_{j=l}^{m-1}\left\|\frac{1}{8^j}f\left(2^j x\right)-\frac{1}{8^{j+1}}f\left(2^{j+1}x\right)\right\| \leq \frac{1}{8}\sum_{j=l}^{m-1}\frac{2^{rj}}{8^j}P(x)^r \tag{4.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.6) that the sequence $\{\frac{1}{8^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{8^n}f(2^n x)\}$ converges. So one can define the mapping $C : X \to Y$ by

$$C(x) := \lim_{n\to\infty} \frac{1}{8^n} f\left(2^n x\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.6), we get (4.5).

It follows from (4.4) that

$$\begin{aligned}
&\left\| \frac{1}{2}C(2x+y) + \frac{1}{2}C(2x-y) - C(x+y) - C(x-y) - 6C(x) \right\| \\
&\quad = \lim_{n\to\infty} \frac{1}{8^n} \left\| \frac{1}{2} f\left(2^n(2x+y)\right) + \frac{1}{2} f\left(2^n(2x-y)\right) \right. \\
&\qquad \left. - f\left(2^n(x+y)\right) - f\left(2^n(x-y)\right) - 6f\left(2^n x\right) \right\| \\
&\quad \le \lim_{n\to\infty} \frac{2^{nr}}{8^n} \left(P(x)^r + P(y)^r\right) = 0
\end{aligned}$$

for all $x, y \in X$. Thus

$$\frac{1}{2}C(2x+y) + \frac{1}{2}C(2x-y) = C(x+y) + C(x-y) + 6C(x)$$

for all $x, y \in X$ and so the mapping $C : X \to Y$ is cubic.

Now, let $T : X \to Y$ be another cubic mapping satisfying (4.5). Then we have

$$\begin{aligned}
\left\| C(x) - T(x) \right\| &= \frac{1}{8^n} \left\| C\left(2^n x\right) - T\left(2^n x\right) \right\| \\
&\le \frac{1}{8^n} \left(\left\| C\left(2^n x\right) - f\left(2^n x\right) \right\| + \left\| T\left(2^n x\right) - f\left(2^n x\right) \right\| \right) \\
&\le \frac{2 \cdot 2^{nr}}{(8 - 2^r)8^n} P(x)^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $C(x) = T(x)$ for all $x \in X$. This proves the uniqueness of C. Thus the mapping $C : X \to Y$ is a unique cubic mapping satisfying (4.5). □

5 Hyers-Ulam stability of the quartic functional equation

In this section, we prove the Hyers-Ulam stability of the quartic functional equation in paranormed spaces.

Note that $P(2x) \le 2P(x)$ for all $x \in Y$.

Theorem 5.1 *Let r, θ be positive real numbers with $r > 4$, and let $f : Y \to X$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&P\left(\frac{1}{2} f(2x+y) + \frac{1}{2} f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x) + 3f(y) \right) \\
&\quad \le \theta \left(\|x\|^r + \|y\|^r \right) \qquad (5.1)
\end{aligned}$$

for all $x, y \in Y$. Then there exists a unique quartic mapping $Q_4 : Y \to X$ such that

$$P\big(f(x) - Q_4(x)\big) \le \frac{\theta}{2^r - 16}\|x\|^r \tag{5.2}$$

for all $x \in Y$.

Proof Letting $y = 0$ in (4.1), we get

$$P\big(f(2x) - 16f(x)\big) \le \theta\|x\|^r$$

for all $x \in Y$. So

$$P\left(f(x) - 16f\left(\frac{x}{2}\right)\right) \le \frac{1}{2^r}\theta\|x\|^r$$

for all $x \in Y$. Hence

$$\begin{aligned} &P\left(16^l f\left(\frac{x}{2^l}\right) - 16^m f\left(\frac{x}{2^m}\right)\right) \\ &\quad\le \sum_{j=l}^{m-1} P\left(16^j f\left(\frac{x}{2^j}\right) - 16^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right) \\ &\quad\le \frac{1}{2^r}\sum_{j=l}^{m-1}\frac{16^j}{2^{rj}}\theta\|x\|^r \end{aligned} \tag{5.3}$$

for all nonnegative integers m and l with $m > l$ and all $x \in Y$. It follows from (5.3) that the sequence $\{16^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in Y$. Since X is complete, the sequence $\{16^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q_4 : Y \to X$ by

$$Q_4(x) := \lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right)$$

for all $x \in Y$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.3), we get (5.2).

It follows from (5.1) that

$$\begin{aligned} &P\left(\frac{1}{2}Q_4(2x+y) + \frac{1}{2}Q_4(2x-y) - 2Q_4(x+y) - 2Q_4(x-y) - 12Q_4(x) + 3Q_4(y)\right) \\ &\quad= \lim_{n\to\infty} P\left(16^n\left(\frac{1}{2}f\left(\frac{2x+y}{2^n}\right) + \frac{1}{2}f\left(\frac{2x-y}{2^n}\right) - 2f\left(\frac{x+y}{2^n}\right)\right.\right. \\ &\qquad\left.\left. - 2f\left(\frac{x-y}{2^n}\right) - 12f\left(\frac{x}{2^n}\right) + 3f\left(\frac{y}{2^n}\right)\right)\right) \\ &\quad\le \lim_{n\to\infty} 16^n P\left(\frac{1}{2}f\left(\frac{2x+y}{2^n}\right) + \frac{1}{2}f\left(\frac{2x-y}{2^n}\right) - 2f\left(\frac{x+y}{2^n}\right)\right. \\ &\qquad\left. - 2f\left(\frac{x-y}{2^n}\right) - 12f\left(\frac{x}{2^n}\right) + 3f\left(\frac{y}{2^n}\right)\right) \\ &\quad\le \lim_{n\to\infty}\frac{16^n\theta}{2^{nr}}\left(\|x\|^r + \|y\|^r\right) = 0 \end{aligned}$$

for all $x, y \in Y$. Hence

$$\frac{1}{2}Q_4(2x+y) + \frac{1}{2}Q_4(2x-y) = 2Q_4(x+y) + 2Q_4(x-y) + 12Q_4(x) - 3Q_4(y)$$

for all $x, y \in Y$ and so the mapping $Q_4 : Y \to X$ is quartic.

Now, let $T : Y \to X$ be another quartic mapping satisfying (5.2). Then we have

$$\begin{aligned} P\big(Q_4(x) - T(x)\big) &= P\left(16^n\left(Q_4\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right)\right) \\ &\leq 16^n P\left(Q_4\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right) \\ &\leq 16^n\left(P\left(Q_4\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right) + P\left(T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right)\right) \\ &\leq \frac{2\cdot 16^n}{(2^r-16)2^{nr}}\theta\|x\|^r, \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in Y$. So we can conclude that $Q_4(x) = T(x)$ for all $x \in Y$. This proves the uniqueness of Q_4. Thus the mapping $Q_4 : Y \to X$ is a unique quartic mapping satisfying (5.2). □

Theorem 5.2 *Let r be a positive real number with $r < 4$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\|\frac{1}{2}f(2x+y) + \frac{1}{2}f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x) + 3f(y)\right\| \\ \leq P(x)^r + P(y)^r \tag{5.4}$$

for all $x, y \in X$. Then there exists a unique quartic mapping $Q_4 : X \to Y$ such that

$$\big\|f(x) - Q_4(x)\big\| \leq \frac{1}{16-2^r}P(x)^r \tag{5.5}$$

for all $x \in X$.

Proof Letting $y = 0$ in (5.4), we get

$$\big\|16f(x) - f(2x)\big\| \leq P(x)^r$$

and so

$$\left\|f(x) - \frac{1}{16}f(2x)\right\| \leq \frac{1}{16}P(x)^r$$

for all $x \in X$. Hence

$$\left\|\frac{1}{16^l}f\left(2^l x\right) - \frac{1}{16^m}f\left(2^m x\right)\right\| \leq \sum_{j=l}^{m-1}\left\|\frac{1}{16^j}f\left(2^j x\right) - \frac{1}{16^{j+1}}f\left(2^{j+1}x\right)\right\| \leq \frac{1}{16}\sum_{j=l}^{m-1}\frac{2^{rj}}{16^j}P(x)^r \tag{5.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (5.6) that the sequence $\{\frac{1}{16^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{16^n} f(2^n x)\}$ converges. So one can define the mapping $Q_4 : X \to Y$ by

$$Q_4(x) := \lim_{n\to\infty} \frac{1}{16^n} f\left(2^n x\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.6), we get (5.5).

It follows from (5.4) that

$$\begin{aligned}
&\left\| \frac{1}{2} Q_4(2x+y) + \frac{1}{2} Q_4(2x-y) - 2Q_4(x+y) - 2Q_4(x-y) - 12Q_4(x) + 3Q_4(y) \right\| \\
&\quad = \lim_{n\to\infty} \frac{1}{16^n} \left\| \frac{1}{2} f\left(2^n(2x+y)\right) + \frac{1}{2} f\left(2^n(2x-y)\right) - 2f\left(2^n(x+y)\right) \right. \\
&\qquad \left. - 2f\left(2^n(x-y)\right) - 12f\left(2^n x\right) + 3f\left(2^n y\right) \right\| \\
&\quad \le \lim_{n\to\infty} \frac{2^{nr}}{16^n} \left(P(x)^r + P(y)^r\right) = 0
\end{aligned}$$

for all $x, y \in X$. Thus

$$\frac{1}{2} Q_4(2x+y) + \frac{1}{2} Q_4(2x-y) = 2Q_4(x+y) + 2Q_4(x-y) + 12Q_4(x) - 3Q_4(y)$$

for all $x, y \in X$ and so the mapping $Q_4 : X \to Y$ is quartic.

Now, let $T : X \to Y$ be another quartic mapping satisfying (5.5). Then we have

$$\begin{aligned}
\left\| Q_4(x) - T(x) \right\| &= \frac{1}{16^n} \left\| Q_4\left(2^n x\right) - T\left(2^n x\right) \right\| \\
&\le \frac{1}{16^n} \left(\left\| Q_4\left(2^n x\right) - f\left(2^n x\right) \right\| + \left\| T\left(2^n x\right) - f\left(2^n x\right) \right\| \right) \\
&\le \frac{2 \cdot 2^{nr}}{(16 - 2^r) 16^n} P(x)^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q_4(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q_4. Thus the mapping $Q_4 : X \to Y$ is a unique quartic mapping satisfying (5.5). □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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