Positivity (2014) 18:359–373
DOI 10.1007/s11117-013-0248-2

Positivity

Compactness and sequential completeness in some spaces of operators

Marian Nowak

Received: 24 December 2012 / Accepted: 10 June 2013 / Published online: 3 July 2013

Abstract Let X be a completely regular Hausdorff space and $C_b(X)$ be the Banach lattice of all real-valued bounded continuous functions on X, endowed with the strict topologies β_σ, β_τ and β_t. Let $\mathcal{L}_{\beta_z,\xi}(C_b(X), E)$ $(z = \sigma, \tau, t)$ stand for the space of all (β_z, ξ)-continuous linear operators from $C_b(X)$ to a locally convex Hausdorff space (E, ξ), provided with the topology $\mathcal{T}_s$ of simple convergence. We characterize relative $\mathcal{T}_s$-compactness in $\mathcal{L}_{\beta_z,\xi}(C_b(X), E)$ in terms of the representing Baire vector measures. It is shown that if (E, ξ) is sequentially complete, then the spaces $(\mathcal{L}_{\beta_z,\xi}(C_b(X), E), \mathcal{T}_s)$ are sequentially complete whenever $z = \sigma$; $z = \tau$ and X is paracompact; $z = t$ and X is paracompact and Čech complete. Moreover, a Dieudonné–Grothendieck type theorem for operators on $C_b(X)$ is given.

Keywords Spaces of bounded continuous functions · Strict topologies · Dini topologies · Continuous linear operators · Topology of simple convergence · Baire measures · Banach lattice · Compactness · Sequential completeness

Mathematics Subject Classification (2010) 46G10 · 28A32 · 28A25 · 46A70

1 Introduction and terminology

For terminology concerning vector lattices we refer the reader to [1]. We denote by $\sigma(L, K)$, $\tau(L, K)$ and $\beta(L, K)$ the weak topology, the Mackey topology and the strong topology on L, with respect to a dual pair $\langle L, K\rangle$.

M. Nowak (✉)
Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra,
ul. Szafrana 4A, 65516 Zielona Góra, Poland
e-mail: M.Nowak@wmie.uz.zgora.pl

From now on we assume that X is a completely regular Hausdorff space. Let $C_b(X)$ be the Banach lattice of all real-valued bounded continuous functions on X, endowed with the uniform norm $\|\cdot\|$. Then the Banach dual $C_b(X)'$ of $C_b(X)$ with the natural order ($\Phi_1 \le \Phi_2$ if $\Phi_1(u) \le \Phi_2(u)$ for each $0 \le u \in C_b(X)$) is a Dedekind complete Banach lattice. By $C_b(X)''$ we will denote the Banach bidual of $C_b(X)$.

Let $\mathcal{B}$ be the algebra of Baire sets in X, which is the algebra generated by the class $\mathcal{Z}$ of all zero-sets of functions of $C_b(X)$. Let $M(X)$ stand for the space of all Baire measures on $\mathcal{B}$. Then $M(X)$ with the norm $\|\mu\| = |\mu|(X)$ (= the total variation of μ) and the natural order ($\mu_1 \le \mu_2$ if $\mu_1(A) \le \mu_2(A)$ for all $A \in \mathcal{B}$) is a Dedekind complete Banach lattice (see [20, p. 114, p. 122]). Due to the Alexandrov representation theorem (see [19], [20, Theorem 5.1]) $C_b(X)'$ can be identified with $M(X)$ through the lattice isomorphism $M(X) \ni \mu \mapsto \Phi_\mu \in C_b(X)'$, where $\Phi_\mu(u) = \int_X u d\mu$ for all $u \in C_b(X)$, and $\|\Phi_\mu\| = \|\mu\|$.

The strict topologies β_σ, β_τ and β_t on $C_b(X)$ are of importance in the topological measure theory (see [18], [20] for more details). Note that in [18] $\beta_\sigma, \beta_\tau, \beta_t$ are denoted by β_1, β, β_0 respectively. It is well known that β_z ($z = \sigma, \tau, t$) is a locally convex-solid topology (see [20, Theorem 11.6]), and $\beta_t \subset \beta_\tau \subset \beta_\sigma \subset \mathcal{T}_{\|\cdot\|}$. Recall that β_σ is a σ-Dini topology (resp. β_τ is a Dini topology), that is $u_n \to 0$ in β_σ whenever $u_n(x) \downarrow 0$ for all $x \in X$ (resp. $u_\alpha \to 0$ in β_τ whenever $u_\alpha(x) \downarrow 0$ for all $x \in X$) (see [18, Theorem 6.2], [20, Theorems 11.16 and 11.28]). β_t is the finest locally convex topology on $C_b(X)$ that agrees with the compact-open topology η on each set $B_r = \{u \in C_b(X), \|u\| \le r\}$, $r > 0$ (see [20, Theorem 10.5]). Moreover, $(C_b(X), \beta_z)$ (for $z = \sigma$; $z = \tau$ whenever X is paracompact; $z = t$ whenever X is paracompact and Čech complete) is a strongly Mackey space, that is, every relatively countably $\sigma(C_b(X)'_{\beta_z}, C_b(X))$-compact subset of $C_b(X)'_{\beta_z}$ is β_z-equicontinuous (see [20, Theorems 11.5, 12.22 and 12.9], [18, Theorem 4.5]). We have (see [20, Theorem 11.8], [18, Theorem 4.3]):

$$(C_b(X), \beta_z)' = \{\Phi_\mu : \mu \in M_z(X)\} = L_z(C_b(X)) \ \ (z = \sigma, \tau, t), \tag{1.1}$$

where $M_\sigma(X)$, $M_\tau(X)$, $M_t(X)$ are subspaces of $M(X)$ of all σ-additive τ-additive and tight Baire measures, respectively. $L_\sigma(C_b(X))$, $L_\tau(C_b(X))$ and $L_t(C_b(X))$ are subspaces of $C_b(X)'$ of all σ-additive, τ-additive and tight functionals, respectively.

From now on we assume that (E, ξ) is a locally convex Hausdorff space (briefly, lcHs). Let $\mathcal{P}_\xi$ stand for a directed family of seminorms on E that generates ξ.

Following the definitions of σ-additive, τ-additive and tight functionals on $C_b(X)$ one can distinguish the corresponding classes of linear operators on $C_b(X)$.

Definition 1.1 A linear operator $T : C_b(X) \to E$ is said to be:

(i) *σ-additive* if $T(u_n) \to 0$ for ξ whenever (u_n) is a sequence in $C_b(X)$ such that $u_n(x) \downarrow 0$ for all $x \in X$.

(ii) τ-additive if $T(u_\alpha) \to 0$ for ξ whenever (u_α) is a net in $C_b(X)$ such that $u_\alpha(x) \downarrow 0$ for all $x \in X$.

(iii) *tight* if $T(u_\alpha) \to 0$ for ξ whenever $\sup_\alpha \|u_\alpha\| < \infty$ and $u_\alpha \to 0$ uniformly on compact sets in X.

By $\mathcal{L}_{\|\cdot\|,\xi}(C_b(X), E)$ (resp. $\mathcal{L}_{\beta_z,\xi}(C_b(X), E)$ for $z = \sigma, \tau, t$) we will denote the space of all $(\|\cdot\|, \xi)$-continuous (resp. (β_z, ξ)-continuous) linear operators $T : C_b(X) \to E$. Let $W(C_b(X), E)$ be the space of all weakly compact operators from the Banach space $C_b(X)$ to (E, ξ). Then

$$\mathcal{L}_{\beta_t,\xi}(C_b(X), E) \subset \mathcal{L}_{\beta_\tau,\xi}(C_b(X), E) \subset \mathcal{L}_{\beta_\sigma,\xi}(C_b(X), E) \subset \mathcal{L}_{\|\cdot\|,\xi}(C_b(X), E)$$

and

$$W(C_b(X), E) \subset \mathcal{L}_{\|\cdot\|,\xi}(C_b(X), E).$$

By $\mathcal{T}_s$ we will denote the topology of simple convergence on $\mathcal{L}_{\|\cdot\|,\xi}(C_b(X), E)$. Then $\mathcal{T}_s$ is generated by the family $\{q_{p,u} : p \in \mathcal{P}_\xi, u \in C_b(X)\}$ of seminorms, where

$$q_{p,u}(T) := p(T(u)) \quad for \quad T \in \mathcal{L}_{\|\cdot\|,\xi}(C_b(X), E).$$

Graves and Ruess [6, Theorem 7] characterized relative compactness in $ca(\Sigma, E)$ (= the space of all E-valued countably additive measures on a σ-algebra Σ) in the topology $\mathcal{T}_s$ of simple convergence (convergence on each $A \in \Sigma$) in terms of the properties of the integration operators from $\mathcal{S}(\Sigma)$ to E and from $L(\Sigma)$ to E. In [12, Theorem 3.2] (resp. [14, Theorem 3.4]) we study relative $\mathcal{T}_s$-compactness in the space $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$ of all $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-continuous linear operators from $B(\Sigma)$ to E (resp. in the space $\mathcal{L}_{\tau,\xi}(L^\infty(\mu), E)$ of all $(\tau(L^\infty(\mu), L^1(\mu)), \xi)$-continuous linear operators from $L^\infty(\mu)$ to E).

In this paper we study topological properties of the spaces $\mathcal{L}_{\beta_z,\xi}(C_b(X), E), \mathcal{T}_s)$ for $z = \sigma, \tau, t$. We characterize relative $\mathcal{T}_s$-compactness in $\mathcal{L}_{\beta_z,\xi}(C_b(X), E)$ in terms of the corresponding Baire and Borel vector measures (see Theorems 3.2, 4.2, and 5.7 below). It is shown that if (E, ξ) is a sequentially complete lcHs, then the space $(\mathcal{L}_{\beta_z,\xi}(C_b(X), E), \mathcal{T}_s)$ is sequentially complete whenever $z = \sigma$; $z = \tau$ and X is paracompact; $z = t$ and X is paracompact and Čech complete (see Corollaries 3.4, 4.5 and 5.4 below). Moreover, we derive a Dieudonné–Grothendieck type theorem for tight and weakly compact operators on $C_b(X)$ (see Theorem 5.8 below).

2 Representation of continuous operators on $C_b(X)$

Let $B(\mathcal{B})$ denote the Banach lattice of all functions $u : X \to \mathbb{R}$ that are uniform limits of sequences of $\mathcal{B}$-simple functions, provided with the uniform norm $\|\cdot\|$.

It is well known that $C_b(X) \subset B(\mathcal{B})$ (see [2, Lemma 1.2]) and one can embed isometrically $B(\mathcal{B})$ in $C_b(X)''$ by the mapping $\pi : B(\mathcal{B}) \to C_b(X)''$, where for each $u \in B(\mathcal{B})$,

$$\pi(u)(\Phi_\mu) = \int_X u\,d\mu \quad \text{for all} \quad \mu \in M(X).$$

Assume that (E, ξ) is a locally convex Hausdorff space. By $(E, \xi)'$ or E'_ξ we denote the topological dual of (E, ξ). Then the space $E''_\xi = (E'_\xi, \beta(E'_\xi, E))'$ is the bidual of (E, ξ). Let $\mathcal{E}_\xi$ stand for the set of all ξ-equicontinuous subsets of E'_ξ. Note that ξ is the topology of uniform convergence on all sets $\mathcal{A} \in \mathcal{E}_\xi$, i.e., ξ is generated by the family of seminorms $\{p_\mathcal{A} : \mathcal{A} \in \mathcal{E}_\xi\}$, where

$$p_\mathcal{A}(e) = \sup\{|e'(e)| : e' \in \mathcal{A}\} \quad \text{for} \quad e \in E.$$

Let ξ_ε stand for the topology on E''_ξ of uniform convergence on all sets $\mathcal{A} \in \mathcal{E}_\xi$, i.e., ξ_ε is generated by the family of seminorms $\{q_\mathcal{A} : \mathcal{A} \in \mathcal{E}_\xi\}$, where

$$q_\mathcal{A}(e'') = \sup\{|e''(e')| : e' \in \mathcal{A}\} \quad \text{for} \quad e'' \in E''_\xi,$$

(see [5, Chapter 8.7]).

Let $i : E \to E''_\xi$ stand for the canonical embedding, i.e., $i(e)(e') = e'(e)$ for $e \in E$ and $e' \in E'_\xi$. Moreover, let $j : i(E) \to E$ denote the left inverse of i, that is, $j \circ i = id_E$. Note that j is $(\sigma(i(E), E'_\xi), \sigma(E, E'_\xi))$-continuous.

Assume that $T : C_b(X) \to E$ is $(\|\cdot\|, \xi)$-continuous linear operator. Let $T' : E'_\xi \to C_b(X)'$ and $T'' : C_b(X)'' \to E''_\xi$ denote the conjugate and the biconjugate of T, respectively. Let

$$\hat{T} := T'' \circ \pi : B(\mathcal{B}) \to E''_\xi.$$

Since the topology $(\mathcal{T}_{\|\cdot\|_{C_b(X)}})_\varepsilon$ on $C_b(X)''$ coincides with $\|\cdot\|_{C_b(X)''}$-topology, in view of [5, Proposition 8.7.2] T'' is $(\|\cdot\|_{C_b(X)''}, \xi_\varepsilon)$-continuous. Then $\hat{T}$ is $(\|\cdot\|, \xi_\varepsilon)$-continuous. For $A \in \mathcal{B}$ let us put

$$\hat{m}_T(A) := \hat{T}(\mathbb{1}_A).$$

Then

$$\hat{m}_T : \mathcal{B} \longrightarrow E''_\xi$$

is a ξ_ε-bounded measure, called the representing measure for T. For each $e' \in E'_\xi$ let

$$(\hat{m}_T)_{e'}(A) := \hat{m}_T(A)(e') \quad \text{for all} \quad A \in \mathcal{B}.$$

From the general properties of the operator $\hat{T}$ it follows immediately that

$$\hat{T}(C_b(X)) \subset i(E) \quad \text{and} \quad T(u) = j(\hat{T}(u)) \quad \text{for all} \quad u \in C_b(X).$$

The next theorem gives a characterization of $(\|\cdot\|, \xi)$-continuous linear operators $T : C_b(X) \to E$ in terms of their representing measures (see [13, Theorem 2.1]).

Theorem 2.1 *Let $T : C_b(X) \longrightarrow E$ be a $(\|\cdot\|, \xi)$-continuous linear operator. Then the following statements hold:*

(i) $(\hat{m}_T)_{e'} \in M(X)$ *for each* $e' \in E'_\xi$.
(ii) *The mapping* $E'_\xi \ni e' \mapsto (\hat{m}_T)_{e'} \in M(X)$ *is* $(\sigma(E'_\xi, E), \sigma(M(X), C_b(X)))$-*continuous.*
(iii) *For each* $e' \in E'_\xi$,

$$\hat{T}(u)(e') = e'(T(u)) = \int_X u d(\hat{m}_T)_{e'} \quad \text{for all} \quad u \in C_b(X).$$

Conversely, let $\hat{m} : \mathcal{B} \to E''_\xi$ *be a vector measure satisfying* (i) *and* (ii). *Then there exists a unique* $(\|\cdot\|, \xi)$-*continuous linear operator* $T : C_b(X) \to E$ *such that* (iii) *holds and* $\hat{m}(A) = (T'' \circ \pi)(\mathbb{1}_A)$ *for all* $A \in \mathcal{B}$.

In consequence, the vector measure $\hat{m} : \mathcal{B} \to E''_\xi$ *satisfying* (i), (ii) *and* (iii) *is uniquely determined by a* $(\|\cdot\|, \xi)$-*continuous linear operator* $T : C_b(X) \to E$.

In view of Theorem 2.1 and (1.1) we have

Corollary 2.2 *Let* $T : C_b(X) \to E$ *be a* $(\|\cdot\|, \xi)$-*continuous linear operator, and* $z = \sigma, \tau, t$. *Then for each* $e' \in E'_\xi$ *the following statements are equivalent:*

(i) $e' \circ T \in C_b(X)'_{\beta_z}$.
(ii) $(\hat{m}_T)_{e'} \in M_z(X)$.

Note that a subset $\mathcal{K}$ of $\mathcal{L}_{\beta_z,\xi}(C_b(X), E)$ is (β_z, ξ)-equicontinuous ($z = \sigma, \tau, t$) if and only if for each $\mathcal{A} \in \mathcal{E}_\xi$, the set $\{e' \circ T : T \in \mathcal{K}, e' \in \mathcal{A}\}$ in $C_b(X)'_{\beta_z}$ is β_z-equicontinuous.

The following result will be of importance (see [17, Theorem 2]).

Theorem 2.3 *Let* $\mathcal{K}$ *be a* $\mathcal{T}_s$-*compact subset of* $\mathcal{L}_{\beta_z,\xi}(C_b(X), E)$ *for* $z = \sigma, \tau, t$. *If* C *is a* $\sigma(E'_\xi, E)$-*closed and* ξ-*equicontinuous subset of* E'_ξ, *then* $\{e' \circ T : T \in \mathcal{K}, e' \in C\}$ *is a* $\sigma(C_b(X)'_{\beta_z}, C_b(X))$-*compact set in* $C_b(X)'_{\beta_z}$.

Assume now that $T : C_b(X) \to E$ is a weakly compact operator, that is, T maps bounded sets in the Banach space $C_b(X)$ into relatively $\sigma(E, E'_\xi)$-compact sets in E (hence T is $(\|\cdot\|, \xi)$-continuous). Then by the Gantmacher type theorem (see [5, Corollary 9.3.2]) we have

$$T''(C_b(X)'') \subset i(E).$$

Let us put

$$\widetilde{T} := j \circ T'' \circ \pi : B(\mathcal{B}) \longrightarrow E$$

and

$$m_T(A) := \widetilde{T}(\mathbb{1}_A) \quad \text{for} \quad A \in \mathcal{B}.$$

Note that

$$\widetilde{T} = j \circ \hat{T} \quad \text{and} \quad m_T = j \circ \hat{m}_T : \mathcal{B} \longrightarrow E.$$

Then for each $e' \in E'_\xi$ we have

$$(\hat{m}_T)_{e'}(A) = (e' \circ m_T)(A) \quad \text{for each} \quad A \in \mathcal{B}.$$

It follows that for each $\mathcal{A} \in \mathcal{E}_\xi$ and $A \in \mathcal{B}$ we have

$$q_{\mathcal{A}}(\hat{m}_T(A)) = p_{\mathcal{A}}(m_T(A)). \tag{2.1}$$

For terminology and basic results concerning integration with respect to vector measures we refer to [7,10,15,16]. Recall that a vector measure $m : \mathcal{B} \to E$ is said to be ξ-strongly bounded if $m(A_n) \to 0$ in ξ for each pairwise disjoint sequence (A_n) in $\mathcal{B}$.

The following Alexandrov type theorem for weakly compact operators on $C_b(X)$ is of importance (see [13, Theorems 4.1 and 4.2]).

Theorem 2.4 *Assume that (E, ξ) is a quasicomplete lcHs. Then for a weakly compact operator $T : C_b(X) \to E$ the following statements hold:*

(i) $m_T : \mathcal{B} \to E$ *is ξ-strongly bounded.*
(ii) $\hat{m}_T : \mathcal{B} \to E''_\xi$ *is ξ_ε-strongly bounded.*
(iii) $T(u) = \int_X u dm_T$ *for all* $u \in C_b(X)$.

3 Topological properties of the space $\mathcal{L}_{\beta_\sigma,\xi}(C_b(X), E)$

We start with a characterization of (β_σ, ξ)-equicontinuous sets in $\mathcal{L}_{\beta_\sigma,\xi}(C_b(X), E)$.

Proposition 3.1 *For a subset $\mathcal{K}$ of $\mathcal{L}_{\beta_\sigma,\xi}(C_b(X), E)$ the following statement are equivalent:*

(i) *$\mathcal{K}$ is (β_σ, ξ)-equicontinuous.*
(ii) *$\mathcal{K}$ is uniformly σ-additive, i.e., $T(u_n) \to 0$ in ξ uniformly for $T \in \mathcal{K}$ whenever $u_n(x) \downarrow 0$ for all $x \in X$.*
(iii) *The set $\{\hat{m}_T(A) : T \in \mathcal{K}, A \in \mathcal{B}\}$ is ξ_ε-bounded in E''_ξ and $\hat{m}_T(Z_n) \to 0$ in ξ_ε uniformly for $T \in \mathcal{K}$ whenever $Z_n \downarrow \emptyset$, $Z_n \in \mathcal{Z}$.*

Moreover, if (E, ξ) is a quasicomplete lcHs and $\mathcal{K} \subset \mathcal{L}_{\beta_\sigma,\xi}(C_b(X), E) \cap W(C_b(X), E)$, then each of the statements (i)–(iii) *is equivalent to the following:*

(iv) *$\int_X u_n dm_T \to 0$ in ξ uniformly for $T \in \mathcal{K}$ whenever $u_n(x) \downarrow 0$ for $x \in X$.*
(v) *The set $\{m_T(A) : T \in \mathcal{K}, A \in \mathcal{B}\}$ is ξ-bounded in E and $m_T(Z_n) \to 0$ in ξ uniformly for $T \in \mathcal{K}$ whenever $Z_n \downarrow \emptyset$, $Z_n \in \mathcal{Z}$.*

Proof (i)$\Longrightarrow$(ii) Assume that $\mathcal{K}$ is (β_σ, ξ)-equicontinuous. Let $p \in \mathcal{P}_\xi$ and let $\varepsilon > 0$ be given. Then there is a β_σ-neighborhood V of 0 in $C_b(X)$ such that for each $T \in \mathcal{K}$ we have $p(T(u)) \leq \varepsilon$ for all $u \in V$. Assume that (u_n) is a sequence in $C_b(X)$ such that $u_n(x) \downarrow 0$ for all $x \in X$. Then $u_n \to 0$ for β_σ because β_σ is a σ-Dini topology. Choose $n_\varepsilon \in \mathbb{N}$ such that $u_n \in V$ for $n \geq n_\varepsilon$. Hence $\sup_{T\in\mathcal{K}} p(T(u_n)) \leq \varepsilon$ for $n \geq n_\varepsilon$.

(ii)$\Longrightarrow$(iii) Assume that $\mathcal{K}$ is uniformly σ-additive, and let (u_n) be a sequence in $C_b(X)$ such that $u_n(x) \downarrow 0$ for all $x \in X$. Then for each $\mathcal{A} \in \mathcal{E}_\xi$, we have

$$\sup_{T\in\mathcal{K}} p_{\mathcal{A}}(T(u_n)) = \sup_{T\in\mathcal{K}} (\sup\{|e'(T(u_n))| : e' \in \mathcal{A}\}) \to 0.$$

This means that the set $\{e' \circ T : T \in \mathcal{K}, e' \in \mathcal{A}\}$ in $C_b(X)'_{\beta_\sigma}$ is uniformly σ-additive. Assume that $Z_n \downarrow \emptyset$, $Z_n \in \mathcal{Z}$. In view of [20, Theorem 11.14] we get

$$\sup_{T\in\mathcal{K}} q_{\mathcal{A}}(\hat{m}_T(Z_n)) = \sup_{T\in\mathcal{K}} (\sup\{|(\hat{m}_T)_{e'}(Z_n)| : e' \in \mathcal{A}\}) \to 0.$$

This means that $\hat{m}_T(Z_n) \to 0$ in ξ_ε uniformly for $T \in \mathcal{K}$. Moreover, we have

$$\sup\{|(\hat{m}_T)_{e'}(A)| : T \in\mathcal{K}, e' \in \mathcal{A}, A \in \mathcal{B}\} \leq \sup\{|(\hat{m}_T)_{e'}|(X) : T \in \mathcal{K}, e' \in \mathcal{A}\} < \infty.$$

It follows that

$$\sup\{q_{\mathcal{A}}(\hat{m}_T(A)) : T \in \mathcal{K}, A \in \mathcal{B}\} < \infty,$$

i.e., the set $\{\hat{m}_T(A) : T \in \mathcal{K}, A \in \mathcal{B}\}$ is ξ_ε-bounded in E''_ξ.

(iii)$\Longrightarrow$(i) Assume that $\{\hat{m}_T(A) : T \in \mathcal{K}, A \in \mathcal{B}\}$ is ξ_ε-bounded in E''_ξ and $\hat{m}_T(Z_n) \to 0$ in ξ_ε uniformly for $T \in \mathcal{K}$ whenever $Z_n \downarrow \emptyset$, $Z \in \mathcal{Z}$. It follows that for each $\mathcal{A} \in \mathcal{E}_\xi$, we have

$$\sup\{|(\hat{m}_T)_{e'}|(X) : T \in \mathcal{K}, e' \in \mathcal{A}\} \leq 4 \sup\{|(\hat{m}_T)_{e'}(A)| : T \in \mathcal{K}, e' \in \mathcal{A}, A \in \mathcal{B}\} < \infty,$$

and moreover, for each sequence (Z_n) in $\mathcal{Z}$ such that $Z_n \downarrow \emptyset$, we have

$$\sup_{T\in\mathcal{K}} q_{\mathcal{A}}(\hat{m}_T(Z_n)) \to 0, \quad \text{i.e.,} \quad \sup\{|(\hat{m}_T)_{e'}(Z_n)| : T \in \mathcal{K}, e' \in \mathcal{A}\} \to 0.$$

By [20, Theorem 11.14], we obtain that the set $\{e' \circ T : T \in \mathcal{K}, e' \in \mathcal{A}\}$ in $C_b(X)'_{\beta_\sigma}$ is β_σ-equicontinuous. This means that the set $\mathcal{K}$ is (β_σ, ξ)-equicontinuous.

Assume that (E, ξ) is a quasicomplete lcHs and $\mathcal{K}$ is a subset of $\mathcal{L}_{\beta_\sigma,\xi}(C_b(X), E) \cap W(C_b(X), E)$. Then in view of (2.1) and Theorem 2.4 we obtain that (ii)$\Longleftrightarrow$(iv) and (iii)$\Longleftrightarrow$(v). □

Now we can state a characterization of relatively $\mathcal{T}_s$-compact sets in the space $\mathcal{L}_{\beta_\sigma,\xi}(C_b(X), E)$.

Theorem 3.2 *Let $\mathcal{K}$ be a subset of $\mathcal{L}_{\beta_\sigma,\xi}(C_b(X), E)$. Then the following statements are equivalent:*

(i) *$\mathcal{K}$ is relatively $\mathcal{T}_s$-compact.*
(ii) *$\mathcal{K}$ is (β_σ, ξ)-equicontinuous and for each $u \in C_b(X)$, the set $\{T(u) : T \in \mathcal{K}\}$ is relatively ξ-compact in E.*
(iii) *$\mathcal{K}$ is uniformly σ-additive and for each $u \in C_b(X)$, the set $\{T(u) : T \in \mathcal{K}\}$ is relatively ξ-compact in E.*
(iv) *The following conditions hold:*
  (a) *$\{\hat{m}_T(A) : T \in \mathcal{K}, A \in \mathcal{B}\}$ is ξ_ε-bounded in E''_ξ.*
  (b) *$\hat{m}_T(Z_n) \to 0$ in ξ_ε uniformly for $T \in \mathcal{K}$ whenever $Z_n \downarrow \emptyset$, $Z_n \in \mathcal{Z}$.*
  (c) *For each $u \in C_b(X)$, the set $\{T(u) : T \in \mathcal{K}\}$ is relatively ξ-compact in E.*

Proof (i)$\Longleftrightarrow$(ii) See [13, Theorem 3.3].
(ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv) It follows from Proposition 3.1. □

The following Banach–Steinhaus type theorem for σ-additive operators $T : C_b(X) \to E$ will be useful (see [13, Corollary 3.7]).

Proposition 3.3 *Let $T_n : C_b(X) \to E$ be σ-additive operators for $n \in \mathbb{N}$. Assume that $T(u) = \xi - \lim T_n(u)$ exists for all $u \in C_b(X)$. Then*

(i) *$T : C_b(X) \to E$ is a σ-additive operator.*
(ii) *The family $\{T_n : n \in \mathbb{N}\}$ is uniformly σ-additive.*

As a consequence of Proposition 3.3 we get:

Corollary 3.4 *Assume that (E, ξ) is a sequentially complete lcHs. Then the space $(\mathcal{L}_{\beta_\sigma,\xi}(C_b(X), E), \mathcal{T}_s)$ is sequentially complete.*

Proof Let (T_n) be a $\mathcal{T}_s$-Cauchy sequence in $\mathcal{L}_{\beta_\sigma,\xi}(C_b(X), E)$. Then for each $u \in C_b(X)$, $(T_n(u))$ is a ξ-Cauchy sequence in E, and hence $T(u) = \xi - \lim T_n(u)$ exists. By Proposition 3.3 the operator $T : C_b(X) \to E$ is σ-additive, i.e., $T \in \mathcal{L}_{\beta_\sigma,\xi}(C_b(X), E)$ and $T_n \to T$ in $\mathcal{T}_s$, as desired. □

4 Topological properties of the space $\mathcal{L}_{\beta_\tau,\xi}(C_b(X), E)$

Now arguing as in the proof of Proposition 3.1 and using [20, Theorem 11.24] and the fact that β_τ is a Dini topology, we can obtain the following characterization of (β_τ, ξ)-continuous subsets of $\mathcal{L}_{\beta_\tau,\xi}(C_b(X), E)$.

Proposition 4.1 *For a subset $\mathcal{K}$ of $\mathcal{L}_{\beta_\tau,\xi}(C_b(X), E)$ the following statements are equivalent:*

(i) *$\mathcal{K}$ is (β_τ, ξ)-equicontinuous.*
(ii) *$\mathcal{K}$ is uniformly τ-additive, i.e., $T(u_\alpha) \to 0$ in ξ uniformly for $T \in \mathcal{K}$ whenever $u_\alpha(x) \downarrow 0$ for all $x \in X$.*
(iii) *The set $\{\hat{m}_T(A) : T \in \mathcal{K}, A \in \mathcal{B}\}$ is ξ_ε-bounded in E''_ξ and $\hat{m}_T(Z_\alpha) \to 0$ in ξ_ε uniformly for $T \in \mathcal{K}$ whenever $Z_\alpha \downarrow \emptyset$, $Z_\alpha \in \mathcal{Z}$.*

Moreover, if (E,ξ) *is a quasicomplete lcHs and* $\mathcal{K} \subset \mathcal{L}_{\beta_\tau,\xi}(C_b(X),E) \cap W(C_b(X),E)$*, then each of the statements* (i)–(iii) *is equivalent to the following:*

(iv) $\int_X u_\alpha \, dm_T \to 0$ *in* ξ *uniformly for* $T \in \mathcal{K}$ *whenever* $u_\alpha(x) \downarrow 0$ *for* $x \in X$.
(v) *The set* $\{m_T(A) : T \in \mathcal{K}, A \in \mathcal{B}\}$ *is* ξ*-bounded in* E *and* $m_T(Z_\alpha) \to 0$ *in* ξ *uniformly for* $T \in \mathcal{K}$ *whenever* $Z_\alpha \downarrow \emptyset$, $Z_\alpha \in \mathcal{Z}$.

It is known that if X is paracompact, then $(C_b(X),\beta_\tau)$ is a strongly Mackey space (see [20, Theorem 12.22]). Now we are ready to present a characterization of relatively $\mathcal{T}_s$-compact sets in the space $\mathcal{L}_{\beta_\tau,\xi}(C_b(X),E)$.

Theorem 4.2 *Assume that* X *is paracompact. Then for a subset* $\mathcal{K}$ *of* $\mathcal{L}_{\beta_\tau,\xi}(C_b(X),E)$ *the following statements are equivalent:*

(i) $\mathcal{K}$ *is relatively* $\mathcal{T}_s$*-compact.*
(ii) $\mathcal{K}$ *is* (β_τ,ξ)*-equicontinuous and for each* $u \in C_b(X)$*, the set* $\{T(u) : T \in \mathcal{K}\}$ *is relatively* ξ*-compact in* E.
(iii) $\mathcal{K}$ *is uniformly* τ*-additive and for each* $u \in C_b(X)$*, the set* $\{T(u) : T \in \mathcal{K}\}$ *is relatively* ξ*-compact in* E.
(iv) *The following conditions hold:*
  (a) $\{\hat{m}_T(A) : T \in \mathcal{K}, A \in \mathcal{B}\}$ *is* ξ_ε*-bounded in* E''_ξ.
  (b) $\hat{m}_T(Z_\alpha) \to 0$ *in* ξ_ε *uniformly for* $T \in \mathcal{K}$ *whenever* $Z_\alpha \downarrow \emptyset$, $Z_\alpha \in \mathcal{Z}$.
  (c) *For each* $u \in C_b(X)$*, the set* $\{T(u) : T \in \mathcal{K}\}$ *is relatively* ξ*-compact in* E.

Proof (i)$\Longrightarrow$(ii) Assume that $\mathcal{K}$ is relatively $\mathcal{T}_s$-compact. Let W be an absolutely convex and ξ-closed neighborhood of 0 for ξ in E. Then the polar W^0 of W with respect to the dual pair $\langle E, E'_\xi\rangle$ is a $\sigma(E'_\xi, E)$-closed and ξ-equicontinuous subset of E'_ξ (see [1, Theorem 9.21]). Hence in view of Theorem 2.3 the set $H = \{e' \circ T : T \in \mathcal{K}, e' \in W^0\}$ in $C_b(X)'_{\beta_\tau}$ is relatively $\sigma(C_b(X)'_{\beta_\tau}, C_b(X))$-compact. Since $(C_b(X),\beta_\tau)$ is a strongly Mackey space, the set H is β_τ-equicontinuous. It follows that there exists a β_τ-neighborhood V of 0 in $C_b(X)$ such that $H \subset V^0$, where V^0 is the polar of V with respect to the dual pair $\langle C_b(X), C_b(X)'_{\beta_\tau}\rangle$. It follows that for each $T \in \mathcal{K}$ we have that $\{e' \circ T : e' \in W^0\} \subset V^0$, i.e., if $e' \in W^0$, then $|e'(T(u))| \leq 1$ for all $u \in V$. This means that for each $T \in \mathcal{K}$ we have that $W^0 \subset T(V)^0$. Hence $T(V) \subset T(V)^{00} \subset W^{00} = W$ for each $T \in \mathcal{K}$, i.e., $\mathcal{K}$ is (β_τ,ξ)-equicontinuous. Clearly, for each $u \in C_b(X)$, the set $\{T(u) : T \in \mathcal{K}\}$ is relatively ξ-compact in E.

(ii)$\Longrightarrow$(i) It follows from [3, Chap. 3, §3.4, Corollary 1].

(ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv) It follows from Proposition 4.1. □

Now we will need the following result.

Proposition 4.3 *Assume that* X *is paracompact. Then for a linear operator* $T : C_b(X) \to E$ *the following statements are equivalent:*

(i) $e' \circ T \in L_\tau(C_b(X))$ *for each* $e' \in E'_\xi$.
(ii) T *is* (β_τ,ξ)*-continuous.*
(iii) T *is* τ*-additive.*

Proof (i)$\Longrightarrow$(ii) Assume that $e' \circ T \in L_\tau(C_b(X)) = C_b(X)'_{\beta_\tau}$ for each $e' \in E'_\xi$. Then T is $(\sigma(C_b(X), M_\tau(X)), \sigma(E, E'_\xi))$-continuous (see [1, Theorem 9.26]). Hence T is $(\tau(C_b(X), M_\tau(X)), \tau(E, E'_\xi))$-continuous (see [1, Ex.11, p. 149]). Since $\beta_\tau = \tau(C_b(X), M_\tau(X))$ (see [20, Theorem 12.22]) and $\xi \subset \tau(E, E'_\xi)$, T is (β_τ, ξ)-continuous.

(ii)$\Longrightarrow$(iii) Assume that T is (β_τ, ξ)-continuous and let (u_α) be a net in $C_b(X)$ such that $u_\alpha(x) \downarrow 0$ for all $x \in X$. Then $u_\alpha \to 0$ for β_τ because β_τ is a Dini topology. It follows that $T(u_\alpha) \to 0$ for ξ.

(iii)$\Longrightarrow$(i) It is obvious. □

As a consequence of Proposition 4.3 we can derive the following Banach-Steinhaus type theorem for τ-additive operators $T : C_b(X) \to E$.

Corollary 4.4 *Assume that X is paracompact. Let $T_n : C_b(X) \to E$ be τ-additive operators for $n \in \mathbb{N}$. Assume that $T(u) = \xi - \lim T_n(u)$ exists for all $u \in C_b(X)$. Then*

(i) *T is a τ-additive operator.*
(ii) *The family $\{T_n : n \in \mathbb{N}\}$ is uniformly τ-additive.*

Proof For each $e' \in E'_\xi$ we have $(e' \circ T)(u) = \lim(e' \circ T_n)(u)$ for all $u \in C_b(X)$, and it follows that $(e' \circ T_n)$ is a $\sigma(C_b(X)'_{\beta_\tau}, C_b(X))$-Cauchy sequence in $C_b(X)'_{\beta_\tau}$. Since X is normal and metacompact (see [20, §2]), the space $(C_b(X)'_{\beta_\tau}, \sigma(C_b(X)'_{\beta_\tau}, C_b(X)))$ is sequentially complete (see [20, Theorem 14.12], [18, Theorem 8.7], [11]). Hence for $e' \in E'_\xi$ there exists $\Phi_{e'} \in C_b(X)'_{\beta_\tau}$ such that $\Phi_{e'}(u) = \lim(e' \circ T_n)(u)$ for all $u \in C_b(X)$. It follows that $e' \circ T = \Phi_{e'} \in C_b(X)'_{\beta_\tau} = L_\tau(C_b(X))$, and by Proposition 4.3 we have that T is τ-additive and $T_n \to T$ for $\mathcal{T}_s$. Since $\{T_n : n \in \mathbb{N}\} \cup \{T\}$ is a $\mathcal{T}_s$-compact subset of $\mathcal{L}_{\beta_\tau,\xi}(C_b(X), E)$, by Theorem 4.2 the set $\{T_n : n \in \mathbb{N}\}$ is uniformly τ-additive. □

Corollary 4.5 *Assume that X is paracompact and (E, ξ) is a sequentially complete lcHs. Then the space $(\mathcal{L}_{\beta_\tau,\xi}(C_b(X), E), \mathcal{T}_s)$ is sequentially complete.*

5 Topological properties of the space $\mathcal{L}_{\beta_t,\xi}(C_b(X), E)$

Recall that X is said to be Čech complete if it is a G_δ subset of its Stone–Čech compactification βX (see [20, §2, p. 106–107]). It is known that if X is paracompact and Čech complete, then the space $(C_b(X), \beta_t)$ is strongly Mackey (see [20, Theorem 12.9]). Hence using Theorem 2.3 and arguing as in the proof of Theorem 4.2, we can state the following characterization of relatively $\mathcal{T}_s$-compact sets in $\mathcal{L}_{\beta_t,\xi}(C_b(X), E)$.

Theorem 5.1 *Assume that X is paracompact and Čech complete. Then for a subset $\mathcal{K}$ of $\mathcal{L}_{\beta_t,\xi}(C_b(X), E)$ the following statements are equivalent:*

(i) *$\mathcal{K}$ is relatively $\mathcal{T}_s$-compact.*
(ii) *$\mathcal{K}$ is (β_t, ξ)-equicontinuous and for each $u \in C_b(X)$, the set $\{T(u) : T \in \mathcal{K}\}$ is relatively ξ-compact in E.*

We will need the following characterization of (β_t, ξ)-continuous operators $T : C_b(X) \to E$.

Theorem 5.2 *Assume that X is paracompact and Čech complete. Then for a linear operator $T : C_b(X) \to E$ the following statements are equivalent:*

(i) $e' \circ T \in L_t(C_b(X))$ *for each* $e' \in E'_\xi$.
(ii) T *is* (β_t, ξ)*-continuous.*
(iii) T *is tight.*

Proof (i)$\Longrightarrow$(ii) Assume that $e' \circ T \in L_t(C_b(X), E) = C_b(X)'_{\beta_t}$ for each $e' \in E'_\xi$. Then T is $(\sigma(C_b(X), M_t(X)), \sigma(E, E'_\xi))$-continuous (see [1, Theorem 9.26]). Hence T is $(\tau(C_b(X), M_t(X)), \tau(E, E'_\xi))$-continuous (see [1, Ex. 11, p. 149]). Since $\beta_t = \tau(C_b(X), M_t(X))$ and $\xi \subset \tau(E, E'_\xi)$, T is (β_t, ξ)-continuous

(ii)$\Longrightarrow$(iii) Assume that T is (β_t, ξ)-continuous, and let (u_α) be a net in $C_b(X)$ such that $\sup_\alpha \|u_\alpha\| = r < \infty$ and $u_\alpha \to 0$ for the compact-open topology η on $C_b(X)$. Since $\eta\big|_{B_r} = \beta_t\big|_{B_r}$ $(B_r = \{u \in C_b(X) : \|u\| \le r\})$, we have that $u_\alpha \to 0$ for β_t. Hence $T(u_\alpha) \to 0$ for ξ.

(iii)$\Longrightarrow$(i) It is obvious. □

It is known that if X is paracompact, then X is metacompact and normal (see [20, §2]). Hence in view of ([20, Theorem 14.12], [11]), we conclude that if X is paracompact and Čech complete, then the space $(C_b(X)'_{\beta_t}, \sigma(C_b(X)'_{\beta_t}, C_b(X)))$ is sequentially complete. Now we can state the following Banach-Steinhaus type theorem for tight operators $T : C_b(X) \to E$.

Corollary 5.3 *Assume that X is paracompact and Čech complete. Let $T_n : C_b(X) \to E$ be tight operators for $n \in \mathbb{N}$. Assume that $T(u) = \xi - \lim T_n(u)$ exists for all $u \in C_b(X)$. Then*

(i) T *is a tight operator.*
(ii) *The family* $\{T_n : n \in \mathbb{N}\}$ *is uniformly tight, i.e.,* $T_n(u_\alpha) \xrightarrow[\alpha]{} 0$ *in* ξ *uniformly for* $n \in \mathbb{N}$ *whenever* $\sup_\alpha \|u_\alpha\| < \infty$ *and* $u_\alpha \to 0$ *uniformly on compact sets in* X.

Proof Arguing as in the Proof of Corollary 4.4 and using Theorem 5.2 we see that $T : C_b(X) \to E$ is a tight operator. Since $\{T_n : n \in \mathbb{N}\} \cup \{T\}$ is a $\mathcal{T}_s$-compact subset of $\mathcal{L}_{\beta_t,\xi}(C_b(X), E)$, by Theorem 5.1 the family $\{T_n : n \in \mathbb{N}\}$ is (β_t, ξ)-equicontinuous. Let $p \in \mathcal{P}_\xi$ and $\varepsilon > 0$ be given. Then there exists a neighborhood V of 0 for β_t such that $\sup_n p(T_n(u)) \le \varepsilon$ for all $u \in V$. Assume that $\sup_\alpha \|u_\alpha\| < \infty$ and $u_\alpha \to 0$ for η. Then $u_\alpha \to 0$ for β_t, and hence there exists α_0 such that $u_\alpha \in V$ for $\alpha \ge \alpha_0$. Hence $\sup_n p(T_n(u_\alpha)) \le \varepsilon$ for $\alpha \ge \alpha_0$. □

Corollary 5.4 *Assume that X is paracompact and Čech complete, and (E, ξ) is a sequentially complete lcHs. Then the space $(\mathcal{L}_{\beta_t,\xi}(C_b(X), E), \mathcal{T}_s)$ is sequentially complete.*

Let $\mathcal{B}a$ (resp. $\mathcal{B}o$) denote the σ-algebra of Baire sets (resp. Borel sets) in X. By $B(\mathcal{B}a)$ (resp. $B(\mathcal{B}o)$) we denote the Banach lattice of all bounded $\mathcal{B}a$-measurable (resp. $\mathcal{B}o$-measurable) functions $u : X \to \mathbb{R}$, provided with the uniform norm $\|\cdot\|$.

Let $m : \mathcal{B}o \to E$ be a ξ-countably additive measure. For $p \in \mathcal{P}_\xi$ we define a semivariation $\|m\|_p$ of m by

$$\|m\|_p(A) := \sup\{|e' \circ m|(A) : e' \in V_p^o\} \quad \text{for} \quad A \in \mathcal{B}o,$$

where V_p^o is the polar of $V_p = \{e \in E : p(e) \leq 1\}$ in the duality $\langle E, E'_\xi \rangle$.

We say that m is *inner regular* by compact sets (resp. *outer regular* by open sets) if for each $A \in \mathcal{B}o$, $p \in \mathcal{P}_\xi$ and $\varepsilon > 0$ there exists a compact set K in X, $K \subset A$ such that $\|m\|_p(A \setminus K) \leq \varepsilon$ (resp. there exists an open set U in X, $A \subset U$ such that $\|m\|_p(U \setminus A) \leq \varepsilon$).

Now we present a characterization of tight and weakly compact operators on $C_b(X)$.

Theorem 5.5 *Assume that (E, ξ) is a quasicomplete lcHs. Let $T : C_b(X) \to E$ be a weakly compact operator. Then the following statements are equivalent:*

(i) *T is (β_t, ξ)-continuous.*
(ii) *T is tight.*
(iii) *$e' \circ T \in L_t(C_b(X))$ for each $e' \in E'_\xi$.*
(iv) *$e' \circ m_T \in M_t(X)$ for each $e' \in E'_\xi$.*
(v) *m_T can be uniquely extended to a ξ-countably additive Borel measure $\widetilde{m}_T : \mathcal{B}o \to E$ which is inner regular by compact sets and outer regular by open sets, and*

$$T(u) = \int_X u \, dm_T = \int_X u d\widetilde{m}_T \quad \textit{for all} \quad u \in C_b(X).$$

Proof (i)$\Longrightarrow$(ii) See the proof of implication (i)$\Longrightarrow$(ii) of Theorem 5.2.

(ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv) It is obvious.

(iv)$\Longrightarrow$(v) Assume that $e' \circ m_T \in M_t(X) \subset M_\sigma(X)$ for each $e' \in E'_\xi$. Since m_T is ξ-strongly bounded and $e' \circ m_T : \mathcal{B} \to E$ is countably additive (see [20, p. 118]), by the Kluvanek Extension Theorem (see [9, Theorem of Extension], [15, Corollary 2]) m_T can be extended to a ξ-countably additive measure $\overline{m}_T : \mathcal{B}a \to E$, The uniqueness of this extension follows from the uniqueness of the extension of $e' \circ m_T$ from $\mathcal{B}$ to $\mathcal{B}a$ for each $e' \in E'_\xi$ (see [20, §6, pp. 117-118]).

Hence by [8, Theorem 4] $\overline{m}_T$ can be uniquely extended to a ξ-countably additive Borel measure $\widetilde{m}_T : \mathcal{B}o \to E$ which is inner regular by compact sets and outer regular by open sets. Since $C_b(X) \subset B(\mathcal{B}) \subset B(\mathcal{B}a) \subset B(\mathcal{B}o)$, we have that

$$T(u) = \int_X u dm_T = \int u d\widetilde{m}_T \quad \text{for all} \quad u \in C_b(X).$$

(v)$\Longrightarrow$(i) It follows from [8, Theorem 4]. □

Now assume that $T : C_b(X) \to E$ is a (β_t, ξ)-continuous and weakly compact operator. Then by Theorem 5.5, for each $e' \in E'_\xi$ we have

$$(e' \circ T)(u) = \int_X u d(e' \circ m_T) = \int_X u d(\widetilde{e' \circ m_T}) = \int_X u d(e' \circ \widetilde{m}_T) \qquad (5.1)$$

for all $u \in C_b(X)$, where $\widetilde{e' \circ m_T}$ denotes the compact-regular Borel measure that uniquely extends a tight Baire measure $e' \circ m_T$. Hence

$$e' \circ \widetilde{m}_T = \widetilde{e' \circ m_T} \quad \text{for each} \quad e' \in E'_\xi. \qquad (5.2)$$

Proposition 5.6 *Assume that (E, ξ) is a quasicomplete lcHs. For a subset $\mathcal{K}$ of $\mathcal{L}_{\beta_t,\xi}(C_b(X), E) \cap W(C_b(X), E)$ the following statements are equivalent:*

(i) *$\mathcal{K}$ is (β_t, ξ)-equicontinuous.*

(ii) *The following conditions hold:*
  (a) $\sup_{T\in\mathcal{K}} \|\widetilde{m}_T\|_p(X) < \infty$ *for each $p \in \mathcal{P}_\xi$.*
  (b) *The family $\{\widetilde{m}_T : T \in \mathcal{K}\}$ of Borel measures is uniformly tight (i.e., for each $p \in \mathcal{P}_\xi$ and $\varepsilon > 0$ there exists a compact set K in X such that* $\sup_{T\in\mathcal{K}} \|\widetilde{m}_T\|_p(X \setminus K) \le \varepsilon$).

Proof (i)$\Longrightarrow$(ii) Assume that T is (β_t, ξ)-continuous. Let $p \in \mathcal{P}_\xi$. Then $V_p^o \in \mathcal{E}_\xi$ and it follows that the set $\{e' \circ T : T \in \mathcal{K}, e' \in V_p^o\}$ in $C_b(X)'_{\beta_t}$ is β_t-equicontinuous. Hence in view of (5.1) and (5.2) by [18, Theorem 5.1] we have that

$$\sup_{T\in\mathcal{K}} \|\widetilde{m}_T\|_p(X) = \sup\{|e' \circ \widetilde{m}_T|(X) : T \in \mathcal{K}, e' \in V_p^o\} < \infty,$$

and the family $\{e' \circ \widetilde{m}_T : T \in \mathcal{K}, e' \in V_p^o\}$ of compact regular scalar Borel measures is uniformly tight, i.e., for each $\varepsilon > 0$ there exists a compact set K in X such that $\sup\{|e' \circ \widetilde{m}_T|(X \setminus K) : T \in \mathcal{K}, e' \in V_p^o\} \le \varepsilon$. It follows that $\sup_{T\in\mathcal{K}} \|\widetilde{m}_T\|_p(X \setminus K) \le \varepsilon$, as desired.

(ii)$\Longrightarrow$(i) Assume that (ii) holds. Then for each $p \in \mathcal{P}_\xi$ we see that

$$\sup\{|e' \circ \widetilde{m}_T|(X) : T \in \mathcal{K}, e' \in V_p^o\} < \infty$$

and the family $\{e' \circ \widetilde{m}_T : T \in \mathcal{K}, e' \in V_p^o\}$ is uniformly tight. Then by (5.1) and [18, Theorem 5.1], we conclude that the family $\{e' \circ T : T \in \mathcal{K}, e' \in V_p^o\}$ in $C_b(X)'_{\beta_t}$ is β_t-equicontinuous. It follows that the family $\mathcal{K}$ is (β_t, ξ)-equicontinuous. □

As a consequence of Theorem 5.1 and Proposition 5.6 we have:

Theorem 5.7 *Assume that X is Čech complete and paracompact and (E, ξ) is a quasicomplete lcHs. Then for a subset $\mathcal{K}$ of $\mathcal{L}_{\beta_t,\xi}(C_b(X), E) \cap W(C_b(X), E)$ the following statements are equivalent:*

(i) *$\mathcal{K}$ is relatively $\mathcal{T}_s$-compact in $\mathcal{L}_{\beta_t,\xi}(C_b(X), E)$.*

(ii) *$\mathcal{K}$ is (β_t, ξ)-equicontinuous and for each $u \in C_b(X)$, the set $\{\int_X u d\widetilde{m}_T : T \in \mathcal{K}\}$ is relatively ξ-compact in E.*

(iii) *The following conditions hold:*

(a) $\sup_{T \in \mathcal{K}} \|\widetilde{m}_T\|_p(X) < \infty$ *for each $p \in \mathcal{P}_\xi$.*

(b) *The family $\{\widetilde{m}_T : T \in \mathcal{K}\}$ is uniformly tight.*

(c) *For each $u \in C_b(X)$, the set $\{\int_X u d\widetilde{m}_T : T \in \mathcal{K}\}$ is relatively ξ-compact in E.*

Assume that X is locally compact. Then β_t is the original topology β of Buck (see [4]) and is generated by the family of seminorms $\{p_v : v \in C_0(X)\}$, where

$$p_v(u) = \sup\{|u(x)v(x)| : x \in X\} \quad \text{for} \quad u \in C_b(X),$$

and $C_0(X)$ denotes the space of continuous functions on X vanishing at infinity (see [20, Theorem 10.3] for more details). Then $\beta_t = \beta_\tau$ (see [20, Theorem 10.14]).

Now we are ready to derive a Dieudonné–Grothendieck type theorem for tight and weakly compact operators on $C_b(X)$ (see [16, Chapter 5.2]).

Theorem 5.8 *Assume that X is locally compact and (E, ξ) is a quasicomplete lcHs. Let $T_n : C_b(X) \to E$ be tight and weakly compact operators for $n \in \mathbb{N}$. Assume that $\xi - \lim \widetilde{m}_{T_n}(A)$ exists for each open Baire set A. Then*

(i) *$T(u) = \xi - \lim T_n(u)$ exists for each $u \in C_b(X)$.*

(ii) *$T : C_b(X) \to E$ is a tight and weakly compact operator.*

Proof In view of [16, Theorem 5.2.23] there exists a unique ξ-countably additive measure $\widetilde{m} : \mathcal{B}o \to E$ which is inner regular by compact sets and outer regular by open sets and such that

$$\int_X u d\widetilde{m} = \xi - \lim \int_X u d\widetilde{m}_{T_n}$$

for all $u \in B(\mathcal{B}o)$. Let

$$T_{\widetilde{m}}(u) = \int_X u d\widetilde{m} \quad \text{for all} \quad u \in B(\mathcal{B}o).$$

Since $\widetilde{m}$ is ξ-countably additive, $\widetilde{m}$ is ξ-strongly bounded and it follows that the integration operator $T_{\widetilde{m}} : B(\mathcal{B}o) \to E$ is weakly compact (see [7, Theorem 7]). Define $T = T_{\widetilde{m}}|_{C_b(X)} : C_b(X) \to E$. Then T is weakly compact, and by Theorem 5.5 T is tight, as desired. □

Acknowledgments The author wishes to thank the referee for useful suggestions that have improved the paper.

References

1. Aliprantis, C.D., Burkinshaw, O.: Positive Operators. Academic Press, New York (1985)
2. Aguayo, J., Sánchez, J.: Weakly compact operators and the strict topologies. Bull. Aust. Math. Soc. **39**, 353–359 (1989)
3. Bourbaki, N.: Elements of Mathematics, Topological Vector Spaces, Chap. 1–5. Springer, Berlin (1987)
4. Buck, R.C.: Bounded continuous functions on a locally compact space. Michigan Math. J. **5**, 95–104 (1958)
5. Edwards, R.E.: Functional Analysis, Theory and Applications. Holt, Rinehart and Winston, New York (1965)
6. Graves, W.H., Ruess, W.: Compactness in spaces of vector-valued measures and a natural Mackey topology for spaces of bounded measurable functions. Contemp. Math. **2**, 189–203 (1980)
7. Hoffmann-Jörgensen, J.: Vector measures. Math. Scand. **28**, 5–32 (1971)
8. Khurana, S.S.: Vector measures on topological spaces. Georgian Math. J. **14**(4), 687–698 (2007)
9. Kluvanek, I., The extension and closure of vector measures. In: Vector and Operator Valued Measures and Applications (Proceedings of Symposium on Snowbird Resort, Alta, Utah, 1972), pp. 175–198. Academic Press, New York (1973)
10. Lewis, D.R.: Integration with respect to vector measures. Pac. J. Math. **33**(1), 157–165 (1970)
11. Moran, W.: Measures on metacompact spaces. Proc. Lond. Math. Soc. **20**, 507–524 (1970)
12. Nowak, M.: Vector measures and Mackey topologies. Indag. Math. **23**, 113–122 (2012)
13. Nowak, M.: Vector measures and strict topologies. Topol. Appl. **159**(5), 1421–1432 (2012)
14. Nowak, M.: Absolutely continuous on function spaces and vector measures. Positivity. doi:10.1007/s11117-012-0187-3
15. Panchapagesan, T.V.: Applications of a theorem of Grothendieck to vector measures. J. Math. Anal. Appl. **214**, 89–101 (1997)
16. Panchapagesan, T.V.: The Bartle-Dunford-Schwartz Integral. Monografie Matematyczne, vol. 69. Birkhäuser, Verlag AG (2008)
17. Schaefer, H., Xiao-Dong, Z.: On the Vitali-Hahn-Saks theorem. In: Operator Theory: Advances and Applications, vol. 75. Birkhäuser, Basel, pp. 289–297 (1995)
18. Sentilles, F.D.: Bounded continuous functions on a completely regular spaces. Trans. Am. Math. Soc. **168**, 311–336 (1972)
19. Varadarajan, V.S.: Measures on topological spaces. Mat. Sbornik (N.S.), 55:(97), (1961), 35–100; Am. Math. Soc. Transl. **48**(2), 161–228 (1965)
20. Wheeler, R.: A survey of Baire measures and strict topologies. Expositiones Math. **1**, 97–190 (1983)