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General split equality problems in Hilbert spaces

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Abstract

A new convex feasibility problem, the split equality problem (SEP), has been proposed by Moudafi and Byrne. The SEP was solved through the ACQA and ARCQA algorithms. In this paper the SEPs are extended to infinite-dimensional SEPs in Hilbert spaces and we established the strong convergence of a proposed algorithm to a solution of general split equality problems (GSEPs).

Keywords: general split equality problem; strong convergence; the minimum norm solution

1 Introduction

In the present paper, we are concerned with the general split equality problem (GSEP) which is formulated as finding points x and y with the property:

$$x \in \bigcap_{i=1}^{\infty} C_i \text{ and } y \in \bigcap_{j=1}^{\infty} Q_j, \text{ such that } Ax = By, \tag{1.1}$$

where C_i and Q_j are two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, H_3 also is a Hilbert space, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators.

It generalizes the split equality problem (SEP), which is to find $x \in C$, $y \in Q$ such that $Ax = By$ [1], as well as the split feasibility problem (SFP). When $B = I$, the SEP becomes a SFP. As we know, the SEP has received much attention due to its applications in image reconstruction, signal processing, and intensity-modulated radiation therapy, see for instance [2–5].

To solve the SEP, Byrne and Moudafi put forward the alternating CQ-algorithm (ACQA) and the relaxed alternating CQ-algorithm (RACQA). For an exhaustive study of ACQA and RACQA, see for instance [6, 7]. The approximate SEP (ASEP), which is only to find approximate solutions to SEP, is also proposed and solved through the simultaneous iterative algorithm (SSEA), the relaxed SSEA (RSSEA) and the perturbed SSEA (PSSEA) by Byrne and Moudafi, see for example [1, 8].

This paper aims at a study of an iterative algorithm improved by Eslamian [9] for the GSEP in the Hilbert space. We show the strong convergence of the presented algorithms to a solution of the GSEP, and we obtain an algorithm which strongly converges to the minimum norm solution of the GSEP.

2 Preliminaries

For the sake of simplicity, we will denote by H a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $T : H \mapsto H$ be an operator on H. Recall that T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in H$. A typical example of nonexpansivity is the orthogonal projection P_C from H onto a nonempty closed convex subset $C \subseteq H$ defined by $\|x - P_C x\| = \min \|x - y\|$, $y \in C$. It is well known that $P_C x$ is characterized by the relation

$$P_C x \in C, \quad \langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.1 *Let $S = C \times Q$ in $R^N \times R^M = R^I$, where $I = N + M$. Define*

$$G = \begin{bmatrix} A & -B \end{bmatrix}, \qquad w = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and so} \quad G^*G = \begin{bmatrix} A^*A & -A^*B \\ -B^*A & B^*B \end{bmatrix},$$

then $w^* = \left[\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right]$ solves the SEP if and only if w^* solves the fixed point equation $P_S(I - \gamma G^*G)w^* = w^*$.

Lemma 2.2 *Let H be a Hilbert space. Then for any given sequence $\{x_n\}$ in H, any given sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$ and for any positive integer i, j with $i < j$,*

$$\left\|\sum_{n=1}^{\infty} \lambda_n x_n\right\|^2 \leq \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j \|x_i - x_j\|^2.$$

Lemma 2.3 *Let H be a Hilbert space. For every x and y in H, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

Lemma 2.4 *Let C be a nonempty closed convex subset of H, and let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Then T is demiclosed on C, that is, if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.5 *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n$ where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that*

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 *Let $\{t_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{t_{n_j}\}_{j\geq 0}$ of $\{t_n\}$ such that*

$$\{t_{n_j}\} < \{t_{n_j+1}\} \quad \text{for all } j \geq 0.$$

Also consider the sequence of the integers $\{\tau(n)\}_{n\geq n_0}$ defined by

$$\tau(n) = \max\{k \leq n | t_k < t_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ *is a nondecreasing sequence verifying* $\lim_{n\to\infty}\tau(n)=\infty$, *and for all* $n\geq n_0$, *the following two estimates hold*:

$$t_{\tau(n)}\leq t_{\tau(n)+1},\qquad t_n\leq t_{\tau(n)+1}.$$

3 Main results

Let $C_i\subseteq R^N$ and let $Q_i\subseteq R^M$ be closed, nonempty convex sets, and let A, B be $J\times N$ and $J\times M$ real matrices, respectively. Let $S_i=C_i\times Q_i$. Define

$$G=\begin{bmatrix}A & -B\end{bmatrix},\qquad w=\begin{bmatrix}x\\ y\end{bmatrix},$$

then

$$G^*G=\begin{bmatrix}A^*A & -A^*B\\ -B^*A & B^*B\end{bmatrix}.$$

The problem (1.1) can also be formulated as finding $w\in S=\bigcap_{i=1}^{+\infty}S_i$ with $Gw=0$ or with minimizing the function $\|Gw\|$ over $w\in S$ [1].

Proposition 3.1 $w^*=\left[\begin{smallmatrix}x^*\\ y^*\end{smallmatrix}\right]$ *solves the GSEP* (1.1) *if and only if*

$$w^*\in\bigcap_{i=1}^{+\infty}P_{S_i}\bigl(I-\lambda_{n,i}G^*G\bigr)w^*.$$

Proof Assume that there exists w^* satisfying $w^*\in\bigcap_{i=1}^{+\infty}P_{S_i}(I-\lambda_{n,i}G^*G)w^*$, then for any $i\in[1,+\infty)$, we have $w^*=P_{S_i}(I-\lambda_{n,i}G^*G)w^*$. We use x and y to express $w^*=P_{S_i}(I-\lambda_{n,i}G^*G)w^*$:

$$x^*=P_{C_i}\bigl(x^*-\lambda_{n,i}A^*\bigl(Ax^*-By^*\bigr)\bigr),\tag{3.1}$$

$$y^*=P_{C_i}\bigl(y^*+\lambda_{n,i}B^*\bigl(Ax^*-By^*\bigr)\bigr).\tag{3.2}$$

By Lemma 2.1, for any $i\in[1,+\infty)$, there exist $x^*\in C_i$ and $y^*\in Q_i$, such that $Ax^*=By^*$. Therefore, there exist $x^*\in\bigcap_{i=1}^{+\infty}C_i$ and $y^*\in\bigcap_{i=1}^{+\infty}Q_i$, such that $Ax^*=By^*$, that is to say, w^* solves GSEP (1.1).

Assume that w^* solves GSEP (1.1), such that $Gw^*=0$, that is, for any $i\in[1,+\infty)$, we have $x^*\in C_i$ and $y^*\in Q_i$, such that $Ax^*=By^*$. Substituting $Ax^*=By^*$ into (3.1) and (3.2), we obtain for any $i\in[1,+\infty)$, $w^*=P_{S_i}(I-\lambda_{n,i}G^*G)w^*$. Therefore, w^* solves $w^*\in\bigcap_{i=1}^{+\infty}P_{S_i}(I-\lambda_{n,i}G^*G)w^*$. □

Theorem 3.2 *Assume that the GSEP has a nonempty solution set* Ω. *Suppose that* f *is a self* k*-contraction mapping of* H, $k\in(0,1)$, *and let* $\{w_n\}$ *be a sequence generated by*

$$w_{n+1}=\alpha_nw_n+\beta_nf(w_n)+\sum_{i=1}^{\infty}\gamma_{n,i}P_{S_i}\bigl(I-\lambda_{n,i}G^*G\bigr)w_n,\quad n\geq 0,\tag{3.3}$$

where $\alpha_n+\beta_n+\sum_{i=1}^{\infty}\gamma_{n,i}=1$. *If the sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{n,i}\}$, *and* $\{\lambda_{n,i}\}$ *satisfy the following conditions*:

(i) $\lim_{n\to\infty}\beta_n=0$ *and* $\sum_{n=0}^{\infty}\beta_n=\infty$,

(ii) $\liminf_{n\to\infty}\alpha_n\gamma_{n,i} > 0$, *for each* $i \in N$,

(iii) $\{\lambda_{n,i}\} \subset (0, \frac{2}{L})$, *for each* $i \in N$, *where* $L = \rho(G^*G)$,

then the sequence $\{w_n\}$ *strongly converges to* w^*, *where* $w^* = P_\Omega f(w^*)$, $w^* = \left[\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right]$.

Proof We first prove that $\{w_n\}$ is bounded. Let $z \in \Omega$; actually, by Lemma 2.1, $z \in \Omega$ equals the fixed point equation $z = P_{S_i}(I - \lambda_{n,i}G^*G)z$. Note that for each $i \in N$, $\{\lambda_{n,i}\} \subset (0, \frac{2}{L})$, where $L = \rho(G^*G)$, then the operator $P_{S_i}(I - \lambda_{n,i}G^*G)$ is nonexpansive. We also know that f is a k-contraction mapping, then

$$\begin{aligned}
\|w_{n+1} - z\| &= \left\|\alpha_n w_n + \beta_n f(w_n) + \sum_{i=1}^{\infty}\gamma_{n,i}P_{S_i}\bigl(I - \lambda_{n,i}G^*G\bigr)w_n - z\right\| \\
&\leq \alpha_n\|w_n - z\| + \beta_n\bigl\|f(w_n) - z\bigr\| + \sum_{i=1}^{\infty}\gamma_{n,i}\bigl\|P_{S_i}\bigl(I - \lambda_{n,i}G^*G\bigr)w_n - z\bigr\| \\
&\leq \alpha_n\|w_n - z\| + \beta_n\bigl\|f(w_n) - z\bigr\| \\
&\quad + \sum_{i=1}^{\infty}\gamma_{n,i}\bigl\|P_{S_i}\bigl(I - \lambda_{n,i}G^*G\bigr)w_n - P_{S_i}\bigl(I - \lambda_{n,i}G^*G\bigr)z\bigr\| \\
&\leq \alpha_n\|w_n - z\| + \beta_n\bigl\|f(w_n) - z\bigr\| + \sum_{i=1}^{\infty}\gamma_{n,i}\|w_n - z\| \\
&= (1 - \beta_n)\|w_n - z\| + \beta_n\bigl\|f(w_n) - z\bigr\| \\
&\leq (1 - \beta_n)\|w_n - z\| + \beta_n\bigl\|f(w_n) - f(z)\bigr\| + \beta_n\bigl\|f(z) - z\bigr\| \\
&\leq (1 - \beta_n)\|w_n - z\| + \beta_n k\|w_n - z\| + \beta_n\bigl\|f(z) - z\bigr\| \\
&= \bigl(1 - (1 - k)\beta_n\bigr)\|w_n - z\| + (1 - k)\beta_n\frac{1}{1 - k}\bigl\|f(z) - z\bigr\| \\
&\leq \max\left\{\|w_n - z\|, \frac{1}{1 - k}\bigl\|f(z) - z\bigr\|\right\}.
\end{aligned}$$

Then, from the upper deduction we have $\|w_n - z\| \leq \max\{\|w_{n-1} - z\|, \frac{1}{1-k}\|f(z) - z\|\}$ and

$$\begin{aligned}
\|w_{n+1} - z\| &\leq (1 - \beta_n)\|w_n - z\| + \beta_n\bigl\|f(w_n) - z\bigr\| \\
&\leq \max\left\{\|w_n - z\|, \frac{1}{1 - k}\bigl\|f(z) - z\bigr\|\right\} \\
&\leq \cdots \\
&\leq \max\left\{\|w_0 - z\|, \frac{1}{1 - k}\bigl\|f(z) - z\bigr\|\right\}.
\end{aligned}$$

We can conclude that $\{w_n\}$, $\{f(w_n)\}$ are bounded.

Furthermore, from (3.3) and Lemma 2.2 we get

$$\begin{aligned}
\|w_{n+1} - z\|^2 &= \left\|\alpha_n w_n + \beta_n f(w_n) + \sum_{i=1}^{\infty}\gamma_{n,i}P_{S_i}\bigl(I - \lambda_{n,i}G^*G\bigr)w_n - z\right\|^2 \\
&= \left\|\alpha_n(w_n - z) + \beta_n\bigl(f(w_n) - z\bigr) + \sum_{i=1}^{\infty}\gamma_{n,i}\bigl(P_{S_i}\bigl(I - \lambda_{n,i}G^*G\bigr)w_n - z\bigr)\right\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\|w_n - z\|^2 + \beta_n\big\|f(w_n) - z\big\|^2 + \sum_{i=1}^{\infty}\gamma_{n,i}\big\|P_{S_i}\big(I - \lambda_{n,i}G^*G\big)w_n - z\big\|^2 \\
&\quad - \alpha_n\gamma_{n,i}\big\|P_{S_i}\big(I - \lambda_{n,i}G^*G\big)w_n - w_n\big\| \\
&\leq \alpha_n\|w_n - z\|^2 + \beta_n\big\|f(w_n) - z\big\|^2 + \sum_{i=1}^{\infty}\gamma_{n,i}\|w_n - z\|^2 \\
&\quad - \alpha_n\gamma_{n,i}\big\|P_{S_i}\big(I - \lambda_{n,i}G^*G\big)w_n - w_n\big\| \\
&= (1 - \beta_n)\|w_n - z\|^2 + \beta_n\big\|f(w_n - z)\big\|^2 \\
&\quad - \alpha_n\gamma_{n,i}\big\|P_{S_i}\big(I - \lambda_{n,i}G^*G\big)w_n - w_n\big\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\alpha_n\gamma_{n,i}\big\|P_{S_i}\big(I - \lambda_{n,i}G^*G\big)w_n - w_n\big\|^2 \\
&\quad \leq \|w_n - z\|^2 - \|w_{n+1} - z\|^2 + \beta_n\big\|f(w_n) - z\big\|^2.
\end{aligned} \tag{3.4}$$

In order to show that $\{w_n\} \to w^*$, we consider two cases.

Case 1: Suppose that $\{\|w_n - w^*\|\}$ is a monotone sequence. Since $\|w_n - w^*\|$ is bounded, $\|w_n - w^*\|$ is convergent. Take the limit on both sides for (3.4), because $\lim_{n\to\infty}\beta_n = 0$ and $\liminf_{n\to\infty}\alpha_n\gamma_{n,i} > 0$, and we get $\lim_{n\to\infty}\|P_{S_i}(I - \lambda_{n,i}G^*G)w_n - w_n\| = 0$, $\forall i \in N$.

We first prove there exists a unique $w^* \in \Omega$, such that $w^* = P_\Omega f(w^*)$. Since P_Ω is nonexpansive and f is a self k-contraction mapping, we get

$$\big\|P_\Omega(f)(w_1) - P_\Omega(f)(w_2)\big\| \leq \big\|f(w_1) - f(w_2)\big\| \leq k\|w_1 - w_2\|;$$

therefore, there exists a unique $w^* \in \Omega$, such that $w^* = P_\Omega f(w^*)$.

Next, we show that $\{w_n\} \to w^*$. Using Lemma 2.3, we get

$$\begin{aligned}
\big\|w_{n+1} - w^*\big\|^2 &= \left\|\alpha_n w_n + \beta_n f(w_n) + \sum_{i=1}^{\infty}\gamma_{n,i}P_S\big(I - \lambda_{n,i}G^*G\big)w_n - w^*\right\|^2 \\
&= \left\|\alpha_n\big(w_n - w^*\big) + \beta_n\big(f(w_n) - w^*\big) + \sum_{i=1}^{\infty}\gamma_{n,i}\big(P_S\big(I - \lambda_{n,i}G^*G\big)w_n - w^*\big)\right\|^2 \\
&\leq \left\|\alpha_n\big(w_n - w^*\big) + \sum_{i=1}^{\infty}\gamma_{n,i}\big(P_S\big(I - \lambda_{n,i}G^*G\big)w_n - w^*\big)\right\|^2 \\
&\quad + 2\beta_n\big\langle f(w_n) - w^*, w_{n+1} - w^*\big\rangle \\
&\leq (1 - \beta_n)^2\big\|w_n - w^*\big\|^2 + 2\beta_n\big\langle f(w_n) - f\big(w^*\big), w_{n+1} - w^*\big\rangle \\
&\quad + 2\beta_n\big\langle f\big(w^*\big) - w^*, w_{n+1} - w^*\big\rangle \\
&\leq (1 - \beta_n)^2\big\|w_n - w^*\big\|^2 + 2\beta_n k\big\|w_n - w^*\big\|\,\big\|w_{n+1} - w^*\big\| \\
&\quad + 2\beta_n\big\langle f\big(w^*\big) - w^*, w_{n+1} - w^*\big\rangle \\
&\leq (1 - \beta_n)^2\big\|w_n - w^*\big\|^2 + \beta_n k\big\{\big\|w_n - w^*\big\|^2 + \big\|w_{n+1} - w^*\big\|^2\big\} \\
&\quad + 2\beta_n\big\langle f\big(w^*\big) - w^*, w_{n+1} - w^*\big\rangle.
\end{aligned}$$

By induction, we obtain

$$\begin{aligned}\|w_{n+1}-w^*\|^2 &\le \frac{(1-\beta_n)^2+\beta_n k}{1-\beta_n k}\|w_n-w^*\|^2+\frac{2\beta_n}{1-\beta_n k}\langle f(w^*)-w^*, w_{n+1}-w^*\rangle \\ &= \frac{1-2\beta_n+\beta_n k}{1-\beta_n k}\|w_n-w^*\|^2+\frac{\beta_n^2}{1-\beta_n k}\|w_n-w^*\|^2 \\ &\quad+\frac{2\beta_n}{1-\beta_n k}\langle f(w^*)-w^*, w_{n+1}-w^*\rangle \\ &\le \left(1-\frac{2(1-k)\beta_n}{1-\beta_n k}\right)\|w_n-w^*\|^2 \\ &\quad+\frac{2(1-k)\beta_n}{1-\beta_n k}\left\{\frac{\beta_n M}{2(1-k)}+\frac{1}{1-k}\langle f(w^*)-w^*, w_{n+1}-w^*\rangle\right\} \\ &\le (1-\eta_n)\|w_n-w^*\|^2+\eta_n\delta_n,\end{aligned}$$

where $\eta_n = \frac{2(1-k)\beta_n}{1-\beta_n k}$, $\delta_n = \{\frac{\beta_n M}{2(1-k)} + \frac{1}{1-k}\langle f(w^*) - w^*, w_{n+1} - w^*\rangle\}$ and $M = \sup\{\|w_n - w^*\|^2, n \ge 0\}$.

Since $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = 0$, we have $\sum_{n=1}^{\infty} \eta_n = \infty$. Next, we will prove $\limsup_{n\to\infty} \delta_n \le 0$. Actually, $\frac{\beta_n M}{2(1-k)} \to 0$ (since $\lim_{n\to\infty} \beta_n = 0$), so we just need to prove $\limsup_{n\to\infty} \langle f(w^*) - w^*, w_n - w^*\rangle \le 0$. Take a subsequence $\{w_{n_k}\}$ in $\{w_n\}$, such that

$$\lim_{n\to\infty}\langle f(w^*)-w^*, w_{n_k}-w^*\rangle = \limsup_{n\to\infty}\langle f(w^*)-w^*, w_n-w^*\rangle.$$

Since $\{w_{n_k}\}$ is bounded, there exists a subsequence $\{w_{n_{kj}}\}$ converging weakly to ν. Suppose that $w_{n_k} \rightharpoonup \nu$ and $\lambda_{n,i} \to \lambda_i \in (0, \frac{2}{\|G^*G\|})$, according to Lemma 2.4, $\nu \in \Omega$. Since $\nu \in \Omega$ and $w^* = P_\Omega f(w^*)$,

$$\begin{aligned}\limsup_{n\to\infty}\langle f(w^*)-w^*, w_n-w^*\rangle &= \lim_{n\to\infty}\langle f(w^*)-w^*, w_{n_k}-w^*\rangle \\ &= \langle f(w^*)-w^*, \nu-w^*\rangle \le 0,\end{aligned}$$

as desired.

Therefore, $\sum_{n=1}^{\infty} \eta_n = \infty$ and $\limsup_{n\to\infty} \delta_n \le 0$ hold. All conditions of Lemma 2.5 are satisfied. Therefore $\|w_{n+1} - w^*\| \to 0$, $w_n \to w^*$.

Case 2: If $\{\|w_n - w^*\|\}$ is not a monotone sequence, we could define an integer sequence $\{\tau(n)\}$ by

$$\tau(n) = \max\{k \le n : \|w_k - w^*\| \le \|w_{k+1} - w^*\|\}.$$

It is easy to see that $\{\tau(n)\}$ is nondecreasing and when $n \to \infty$ we get $\tau(n) \to \infty$. For all $n \ge n_0$ we obtain $\|w_{\tau(n)} - w^*\| < \|w_{\tau(n)+1} - w^*\|$. Then $\{\|w_{\tau(n)} - w^*\|\}$ is a monotone sequence and according to Case 1, we have $\lim_{n\to\infty} \|w_{\tau(n)} - w^*\| = 0$ and $\lim_{n\to\infty} \|w_{\tau(n)+1} - w^*\| = 0$. Finally, from Lemma 2.6, we get

$$0 \le \|w_n - w^*\| \le \max\{\|w_n - w^*\|, \|w_{\tau(n)} - w^*\|\} \le \|w_{\tau(n)+1} - w^*\| \to 0, \quad n \to \infty.$$

Therefore, the sequence $\{w_n\}$ converges strongly to w^*.

For every $n \geq 0$, $w^* \in \Omega$ solves the GSEP if and only if w^* solves the fixed point equation $w^* = P_{S_i}(I - \lambda_{n,i} G^* G) w^*$, $i \in N$. Actually, we have proved $\lim_{n\to\infty} \|w_n - P_{S_i}(I - \lambda_{n,i} G^* G) w_n\| = 0$ and $w_n \to w^*$. Then $w^* = P_{S_i}(I - \lambda_{n,i} G^* G) w^*$, $i \in N$, that is, $w^* \in \Omega$ solves the GSEP.

Therefore, the sequence $\{w_n\}$ strongly converges to $w^* = P_\Omega f(w^*)$. This completes the proof. □

Corollary 3.3 *We define a sequence* $\{w_n\}$ *iteratively*

$$w_{n+1} = \alpha_n w_n + \sum_{i=1}^{\infty} \gamma_{n,i} P_{S_i}\bigl(I - \lambda_{n,i} G^* G\bigr) w_n, \quad n \geq 0, \tag{3.5}$$

where $\alpha_n + \sum_{i=1}^{\infty} \gamma_{n,i} \subset (0,1)$. *If* $\{\alpha_n\}$, $\{\gamma_{n,i}\}$, $\{\lambda_{n,i}\}$ *satisfy the following conditions*:

(i) $\lim_{n\to\infty}(\alpha_n + \sum_{i=1}^{\infty} \gamma_{n,i}) = 1$ *and* $\sum_{n=0}^{\infty}(1 - \alpha_n - \sum_{i=1}^{\infty} \gamma_{n,i}) = \infty$,

(ii) $\liminf_{n\to\infty} \alpha_n \gamma_{n,i} > 0$, *for each* $i \in N$,

(iii) $\{\lambda_{n,i}\} \subset (0, \frac{2}{L})$, *for each* $i \in N$, *where* $L = \rho(G^* G)$,

then $\{w_n\}$ *converges strongly to a point* w^* *which is the minimum norm solution of GSEP* (1.1).

Proof Let $f = 0$ in (3.3), then we get (3.5). We have proved $w_{n+1} \to w^* = P_\Omega f(w^*)$ in Theorem 3.2. Then,

$$\bigl\langle f\bigl(w^*\bigr) - w^*, z - w^*\bigr\rangle = \bigl\langle f\bigl(w^*\bigr) - P_\Omega f\bigl(w^*\bigr), z - P_\Omega f\bigl(w^*\bigr)\bigr\rangle \leq 0.$$

Hence, $\langle f(w^*) - w^*, z - w^*\rangle \leq 0$. Since $f = 0$, then $\langle -w^*, z - w^*\rangle \leq 0$, for all $z \in \Omega$, that is,

$$\bigl\|w^*\bigr\|^2 \leq \bigl|\bigl\langle w^*, z\bigr\rangle\bigr| \leq \bigl\|w^*\bigr\| \cdot \|z\| \quad \Rightarrow \quad \bigl\|w^*\bigr\| \leq \|z\|.$$

Thus, w^* is the minimum norm solution of GSEP (1.1). This completes the proof. □

Let $\{T_i\}_{i=1}^{\infty} : H \to H$ be a countable family of nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $T : H \to H$ be a nonexpansive mapping. Consider the variational inequality problem of finding a common fixed point of $\{T_i\}$ with respect to a nonexpansive mapping T is to

$$\text{find } x^* \in \bigcap_{i=1}^{\infty} F(T_i), \text{such that } \bigl\langle x^* - Tx^*, x^* - x\bigr\rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^{\infty} F(T_i). \tag{3.6}$$

It is easy to see that (3.6) equals the following fixed point problem:

$$\text{find } x^* \in \bigcap_{i=1}^{\infty} F(T_i), \text{such that } x^* = P_{\bigcap_{i=1}^{\infty} F(T_i)} Tx^*. \tag{3.7}$$

Letting $C_i = F(T_i)$, $Q_j = F(P_{F(T_j)} T)$, $A = I$, $B = I$, then the upper problem (3.7) is transformed into GSEP (1.1):

$$\text{find } x \in \bigcap_{i=1}^{\infty} C_i \text{ and } y \in \bigcap_{j=1}^{\infty} Q_j, \text{such that } Ax = By.$$

Therefore, GSEP (1.1) equals (3.6). Hence, we have the following result.

Theorem 3.4 *If* $\alpha_n + \beta_n + \sum_{i=1}^{\infty} \gamma_{n,i} = 1$ *and the sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{n,i}\}$, *and* $\{\lambda_{n,i}\}$ *satisfy the following conditions*:

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,
(ii) $\liminf_{n\to\infty} \alpha_n \gamma_{n,i} > 0$, *for each* $i \in N$,
(iii) $\{\lambda_{n,i}\} \subset (0, \frac{2}{L})$, *for each* $i \in N$, *where* $L = \rho(G^*G)$,

the sequence $\{w_n\}$ *defined by* (3.3) *converges strongly to a point* w^* *which solves the following variational inequality with* $w^* \in \Omega$:

$$\langle f(w^*) - w^*, z - w^* \rangle \leq 0 \quad \text{for all } z \in \Omega.$$

Proof We know from the proof of Theorem 3.2 that the sequence $\{w_n\}$ defined by (3.3) converges strongly to $w^* = P_\Omega f(w^*)$, which solves the GSEP. Also since GSEP (1.1) equals (3.6), w^* solves the variational inequality problem (3.6). Since $w^* = P_\Omega f(w^*)$, by (3.7) and (3.6), we have $\langle f(w^*) - w^*, z - w^* \rangle \leq 0$. Actually, since f is a self k-contraction mapping, $k \in (0,1)$, then f is also a nonexpansive mapping. That is to say, the condition in (3.6), that T is a nonexpansive mapping, is satisfied. Therefore, $\{w_n\}$ defined by (3.3) converges strongly to a solution of $\langle f(w^*) - w^*, z - w^* \rangle \leq 0$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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