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A Kronecker limit formula for totally real fields and arithmetic applications

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Abstract

We establish a Kronecker limit formula for the zeta function $\zeta_F(s, A)$ of a wide ideal class A of a totally real number field F of degree n. This formula relates the constant term in the Laurent expansion of $\zeta_F(s, A)$ at $s = 1$ to a toric integral of a $SL_n(\mathbb{Z})$-invariant function $\log G(Z)$ along a Heegner cycle in the symmetric space of $GL_n(\mathbb{R})$. We give several applications of this formula to algebraic number theory, including a relative class number formula for H/F where H is the Hilbert class field of F, and an analog of Kronecker's solution of Pell's equation for totally real multiquadratic fields. We also use a well-known conjecture from transcendence theory on algebraic independence of logarithms of algebraic numbers to study the transcendence of the toric integral of $\log G(Z)$. Explicit examples are given for each of these results.

Keywords: Maximal parabolic Eisenstein series; Heegner cycle; Kronecker limit formula; Transcendence

1 Introduction and statement of results

The celebrated Kronecker limit formula expresses the constant term in the Laurent expansion at $s = 1$ of the Dedekind zeta function $\zeta_K(s, A)$ of an ideal class A of an imaginary quadratic field K in terms of the value of $\log |\eta(z)|$ at a Heegner point τ_A in the complex upper half-plane $\mathbb{H}$ where $\eta(z)$ is the Dedekind eta function

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q := e^{2\pi i z}.$$

There are many interesting applications of this formula to algebraic number theory, including relative class number formulae and Kronecker's "solution" to Pell's equation (see e.g. [8] Chapter II). Roughly speaking, to prove the Kronecker limit formula, one computes the constant term in the Laurent expansion at $s = 1$ of the SL_2-Eisenstein series

$$E(z, s) := \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \mathrm{Im}(\gamma z)^s, \quad z \in \mathbb{H}, \quad \mathrm{Re}(s) > 1$$

then appeals to a classical identity of Dirichlet/Hecke relating $\zeta_K(s, A)$ to the value of $E(z, s)$ at the Heegner point τ_A. Note that Hecke proved a similar limit formula for real quadratic fields by relating the ideal class zeta function to an integral of $E(z, s)$ over a Heegner cycle in $\mathbb{H}$.

In the early 1980's, Bump and Goldfeld [2] proved a Kronecker limit formula for real cubic fields. This was based on an intriguing identity relating the integral of a minimal parabolic SL_3-Eisenstein series over a Heegner cycle to the Rankin/Selberg integral of a Hilbert modular Eisenstein series. Kudla [7] showed this identity was an instance of the so-called "basic identity" associated to a see-saw dual reductive pair. Efrat [4] later gave a Kronecker limit formula for non-real cubic fields by instead using the maximal parabolic SL_3-Eisenstein series. These results are just the tip of the iceberg. Indeed, as Bump and Goldfeld [2] remarked,

> "*Much work remains to be done in this direction, and one can only begin to see a whole new world of limit formulae emerging into view*".

In this paper we will prove a Kronecker limit formula for the zeta function $\zeta_F(s,A)$ of a wide ideal class A of a totally real number field F of degree $n \geq 2$, thereby extending the limit formulae of Hecke and Bump/Goldfeld (see Theorem 1). This formula relates the constant term in the Laurent expansion of $\zeta_F(s,A)$ at $s=1$ to a toric integral of a $SL_n(\mathbb{Z})$-invariant function $\log G(Z)$ along a Heegner cycle in the symmetric space of $GL_n(\mathbb{R})$. We will give some applications of the limit formula to algebraic number theory, including a relative class number formula for H/F where H is the Hilbert class field of F, and an analog of Kronecker's solution of Pell's equation for totally real multiquadratic fields (see Theorems 2 and 3). We will also use a well-known conjecture from transcendence theory on algebraic independence of logarithms of algebraic numbers to study the transcendence of the toric integral of $\log G(Z)$ (see Corollary 1). Explicit examples are given in Section 2 for each of these results.

To prove the limit formula we will generalize the method of Efrat [4]. New difficulties arise when working with the maximal parabolic SL_n-Eisenstein series for arbitrary $n \geq 2$, though many of these may be overcome by appealing to work of Friedberg [5], Goldfeld [6] and Terras [10]. A key step in the proof is an identity relating $\zeta_F(s,A)$ to a toric integral of the maximal parabolic SL_n-Eisenstein series along a Heegner cycle in the symmetric space of $GL_n(\mathbb{R})$.

In order to state our results we fix the following notation. Let $\mathcal{H}^n = GL_n(\mathbb{R})/O_n(\mathbb{R})\mathbb{R}^*$ be the symmetric space of $GL_n(\mathbb{R})$. By the Iwasawa decomposition, each coset $Z \in \mathcal{H}^n$ has a unique representative of the form

$$Z = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \dots & x_{1,n} \\ & 1 & x_{2,3} & \dots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix} \tag{1}$$

where $x_{i,j} \in \mathbb{R}$ for $1 \leq i < j \leq n$ and $y_i \in \mathbb{R}_+$ for $1 \leq i \leq n-1$. Left matrix multiplication induces an action of $GL_n(\mathbb{Z})$ on $\mathcal{H}^n$. For more details concerning these facts, see ([6] Section 1.2).

Let P be the maximal parabolic subgroup of $SL_n(\mathbb{Z})$, which consists of those matrices with bottom row $(0,\dots,0,1)$. Define the maximal parabolic Eisenstein series

$$E_n(Z,s) := \sum_{\gamma \in P\backslash SL_n(\mathbb{Z})} \operatorname{Det}(\gamma \cdot Z)^s, \quad \operatorname{Re}(s) > 1$$

where $\mathrm{Det}(\gamma \cdot Z)$ is the determinant of the unique representative of the coset $\gamma \cdot Z \in \mathcal{H}^n$ of the form (1) and $s \in \mathbb{C}$. Note that $E_n(Z,s)$ is well-defined since $\mathrm{Det}(p \cdot Z) = \mathrm{Det}(Z)$ for all $p \in P$. The completed Eisenstein series

$$E_n^*(Z,s) := \pi^{-ns/2}\Gamma(ns/2)\zeta(ns)E_n(Z,s)$$

satisfies the functional equation

$$E_n^*(Z,s) = E_n^*\left(\left(Z^T\right)^{-1}, 1-s\right)$$

and extends to a meromorphic function on $\mathbb{C}$ with simple poles at $s = 0, 1$.

Given $Z \in \mathcal{H}^n$, we may write

$$ZZ^T = \begin{pmatrix} m & \mathbf{r}^T \\ \mathbf{r} & S \end{pmatrix}$$

where

$$m = (y_1y_2 + \cdots y_{n-1})^2 + (x_{1,2}y_1y_2\cdots y_{n-2})^2 + (x_{1,3}y_1y_2\cdots y_{n-3})^2 + \cdots + x_{1,n}^2,$$

$$\mathbf{r} = Z_1 \begin{pmatrix} x_{1,2}y_1y_2\cdots y_{n-2} \\ x_{1,3}y_1y_2\cdots y_{n-3} \\ \vdots \\ x_{1,n} \end{pmatrix}$$

with

$$Z_1 = \begin{pmatrix} 1 & x_{2,3} & \dots & x_{2,n} \\ & \ddots & & \vdots \\ & & 1 & x_{n-1,n} \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1y_2\cdots y_{n-2} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix},$$

and $S = Z_1 Z_1^T$. Let

$$\mathbf{q} = S^{-1}\mathbf{r}.$$

We will prove the following result.

Proposition 1. *The Laurent expansion of $E_n^*(Z,s)$ at $s = 1$ is given by*

$$E_n^*(Z,s) = \frac{2/n}{s-1} + \gamma - \log(4\pi) - \frac{2}{n}\log\left(y_1y_2^2\cdots y_{n-1}^{n-1}\right) - 4\log g(Z) + O(|s-1|),$$

where γ is Euler's constant and

$$g(Z) := \exp\left(-\frac{\left(y_1y_2^2\cdots y_{n-1}^{n-1}\right)^{1/(n-1)} E_{n-1}^*(Z_1, n/(n-1))}{4}\right) \times \prod_{\substack{\mathbf{b}\in\mathbb{Z}^{n-1} \\ \mathbf{b}\ (\mathrm{mod}\ \pm 1) \\ \mathbf{b}\neq \mathbf{0}}} \left|1 - \exp\left(-2\pi w^{1/2}\left(\mathbf{b}^T S^{-1}\mathbf{b}\right)^{1/2} + 2\pi i \mathbf{b}^T\mathbf{q}\right)\right|.$$

Remark 1. The function $g(Z)$ is a GL_n analog of $|\eta(z)|$ which generalizes the GL_3 analog defined in ([4] p. 175).

Let F be a totally real number field of degree n and U be the group of units of F. Let A be a wide ideal class of F and define the ideal class zeta function

$$\zeta_F(s,A) := \sum_{\substack{\mathfrak{A}\in A\\ \mathfrak{A}\neq 0}} \frac{1}{N(\mathfrak{A})^s}, \quad \mathrm{Re}(s) > 1$$

where $N(\mathfrak{A})$ is the norm. The completed zeta function is defined by

$$\zeta_F^*(s,A) := \pi^{-ns/2}\Gamma(s/2)^n D_F^{s/2}\zeta_F(s,A),$$

where D_F is the discriminant of F. The function $\zeta_F^*(s,A)$ satisfies the functional equation

$$\zeta_F^*(s,A) = \zeta_F^*(1-s,A)$$

and extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s=1$. We will calculate the constant term in the Laurent expansion of $\zeta_F^*(s,A)$ at $s=1$ by relating $\zeta_F^*(s,A)$ to a toric integral of $E_n^*(Z,s)$ along a Heegner cycle in $\mathcal{H}^n$ associated to A and appealing to Proposition 1.

Fix an ideal $\mathfrak{B}\in A^{-1}$. Let $\alpha_1,\alpha_2,\ldots,\alpha_n$ be a $\mathbb{Z}$-basis for $\mathfrak{B}$ and $\alpha_1^{(i)},\alpha_2^{(i)},\ldots,\alpha_n^{(i)}$ for $i=1,2,\ldots,n$ denote their images under the real embeddings of F. Define the matrix

$$M_{\mathfrak{B}}(\mathbf{t}) := \begin{pmatrix} \alpha_1^{(1)}t_1 & \alpha_1^{(2)}t_2 & \cdots & \alpha_1^{(n-1)}t_{n-1} & \alpha_1^{(n)}(t_1t_2\cdots t_{n-1})^{-1} \\ \alpha_2^{(1)}t_1 & \alpha_2^{(2)}t_2 & \cdots & \alpha_2^{(n-1)}t_{n-1} & \alpha_2^{(n)}(t_1t_2\cdots t_{n-1})^{-1} \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_n^{(1)}t_1 & \alpha_n^{(2)}t_2 & \cdots & \alpha_n^{(n-1)}t_{n-1} & \alpha_n^{(n)}(t_1t_2\cdots t_{n-1})^{-1} \end{pmatrix}$$

where $\mathbf{t}=(t_1,t_2,\ldots,t_{n-1})\in\mathbb{R}_+^{n-1}$, and let

$$Q_{\mathfrak{B}}(\mathbf{t}) := M_{\mathfrak{B}}(\mathbf{t})M_{\mathfrak{B}}(\mathbf{t})^T.$$

The positive definite, symmetric matrix $Q_{\mathfrak{B}}(\mathbf{t})$ may be written as

$$Q_{\mathfrak{B}}(\mathbf{t}) = \mathrm{Det}(Q_{\mathfrak{B}}(\mathbf{t}))^{1/n}(y_1(\mathbf{t})^{n-1}y_2(\mathbf{t})^{n-2}\cdots y_{n-1}(\mathbf{t}))^{-2/n}\tau_{\mathfrak{B}}(\mathbf{t})\tau_{\mathfrak{B}}(\mathbf{t})^T,$$

where

$$\tau_{\mathfrak{B}}(\mathbf{t}) = \begin{pmatrix} 1 & x_{1,2}(\mathbf{t}) & x_{1,3}(\mathbf{t}) & \ldots & x_{1,n}(\mathbf{t}) \\ & 1 & x_{2,3}(\mathbf{t}) & \ldots & x_{2,n}(\mathbf{t}) \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n}(\mathbf{t}) \\ & & & & 1 \end{pmatrix}\begin{pmatrix} y_1(\mathbf{t})y_2(\mathbf{t})\cdots y_{n-1}(\mathbf{t}) & & & & \\ & y_1(\mathbf{t})y_2(\mathbf{t})\cdots y_{n-2}(\mathbf{t}) & & & \\ & & \ddots & & \\ & & & y_1(\mathbf{t}) & \\ & & & & 1 \end{pmatrix}$$

is in $\mathcal{H}^n$. Here we have suppressed the dependence of the variables $x_{i,j}(\mathbf{t})$ and $y_i(\mathbf{t})$ on $\mathfrak{B}$ and the $\mathbb{Z}$-basis $\alpha_1,\alpha_2,\ldots,\alpha_n$.

Given a unit $\varepsilon\in U$, let $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n$ denote the images of ε under the real embeddings of F. There is an action of the unit group U on $\mathbb{R}_+^{n-1}$ given by

$$\begin{aligned}\varepsilon:\quad &\mathbb{R}_+^{n-1}\longrightarrow\mathbb{R}_+^{n-1},\\ &(t_1,t_2,\ldots,t_{n-1})\longmapsto(|\varepsilon_1|t_1,|\varepsilon_2|t_2,\ldots,|\varepsilon_{n-1}|t_{n-1}).\end{aligned}$$

Let $\mathbb{R}_+^{n-1}/U$ denote a fundamental domain for this action. Then $\left\{\tau_{\mathfrak{B}}(\mathbf{t}):\ \mathbf{t}\in\mathbb{R}_+^{n-1}/U\right\}$ defines a Heegner cycle in $\mathcal{H}^n$.

We will establish the following identity.

Proposition 2. *Let F be a totally real number field of degree n and A be a wide ideal class of F. Then*

$$\zeta_F^*(s,A) = n2^{n-1} \int_{\mathbb{R}_+^{n-1}/U} \cdots \int E_n^*(\tau_{\mathfrak{B}}(\mathbf{t}),s)\frac{dt_1}{t_1}\cdots\frac{dt_{n-1}}{t_{n-1}} \tag{2}$$

where $\{\tau_{\mathfrak{B}}(\mathbf{t}) : \mathbf{t} \in \mathbb{R}_+^{n-1}/U\}$ *is the Heegner cycle associated to* $\mathfrak{B} \in A^{-1}$.

Remark 2. Although the Heegner cycle $\left\{\tau_{\mathfrak{B}}(\mathbf{t}) : \mathbf{t} \in \mathbb{R}_+^{n-1}/U\right\}$ depends on the ideal $\mathfrak{B} \in A^{-1}$ and the choice of integral basis for $\mathfrak{B}$, the integral on the right hand side of (2) depends only on the ideal class A.

Remark 3. One can identify $\mathbb{R}_+^{n-1}$ with

$$T^n = \left\{(t_1, t_2, \cdots, t_{n-1}, t_n) \in \mathbb{R}_+^n : \prod_{i=1}^n t_i = 1\right\}.$$

Using this identification, (2) can be written as an integral over T^n/U with respect to the U-invariant measure on T^n.

By combining Propositions 1 and 2, we will obtain the following Kronecker limit formula for the zeta function of a wide ideal class of a totally real field.

Theorem 1. *Let F be a totally real number field of degree n and A be a wide ideal class of F. Then*

$$\lim_{s\to 1}\left[\zeta_F^*(s,A) - \frac{2^n \mathrm{vol}\left(\mathbb{R}_+^{n-1}/U\right)}{s-1}\right] =$$
$$n2^{n-1}\gamma \mathrm{vol}\left(\mathbb{R}_+^{n-1}/U\right) - n2^{n-1}\int_{\mathbb{R}_+^{n-1}/U}\cdots\int \log G(\tau_{\mathfrak{B}}(\mathbf{t}))\frac{dt_1}{t_1}\cdots\frac{dt_{n-1}}{t_{n-1}}$$

where

$$G(Z) := 4\pi\left(y_1 y_2^2 \cdots y_{n-1}^{n-1}\right)^{2/n} g(Z)^4$$

and $\left\{\tau_{\mathfrak{B}}(\mathbf{t}) : \mathbf{t} \in \mathbb{R}_+^{n-1}/U\right\}$ *is the Heegner cycle associated to* $\mathfrak{B} \in A^{-1}$.

Proof. We may write the Laurent expansion in Proposition 1 as

$$E_n^*(Z,s) = \frac{2/n}{s-1} + \gamma - \log G(Z) + O(|s-1|) \tag{3}$$

where $G(Z)$ is defined as in the statement of the theorem. In particular, this shows that $G(Z)$ is $SL_n(\mathbb{Z})$-invariant. Inserting (3) into the integral on the right hand side of (2) immediately yields the result. □

Remark 4. By Remark 2, the integral appearing in Theorem 1 depends only on the ideal class A of F. In some of the applications which follow, it will be convenient to denote this integral by

$$\rho_n(A) := \int_{\mathbb{R}_+^{n-1}/U} \cdots \int \log G(\tau_{\mathfrak{B}}(\mathbf{t})) \frac{dt_1}{t_1} \cdots \frac{dt_{n-1}}{t_{n-1}}.$$

We now give some applications of Theorem 1 to algebraic number theory in the spirit of Siegel ([8] Chapter II). Given a number field K, let $\mathrm{CL}(K)$ be the wide ideal class group, h_K be the class number, R_K be the regulator, w_K be the number of roots of unity, and D_K be the absolute value of the discriminant. We will prove the following relative class number formula for H/F where H is the Hilbert class field of F.

Theorem 2. *Let F be a totally real number field of degree n and H be the Hilbert class field of F. Write the ideal class group of F as*

$$\mathrm{CL}(F) = \left\{ [\mathfrak{A}_1] = [\mathcal{O}_F], [\mathfrak{A}_2], \ldots, [\mathfrak{A}_{h_F}] \right\}.$$

Then

$$\frac{(-1)^{h_F-1} 2^{h_F-1}}{n^{h_F-1}} \frac{h_H}{h_F} \frac{R_H}{R_F} = \mathrm{Det} \left(\int_{\mathbb{R}_+^{n-1}/U} \cdots \int \log \left(\frac{G(\tau_{\mathfrak{A}_\ell^{-1}\mathfrak{A}_k}(\mathbf{t}))}{G\left(\tau_{\mathfrak{A}_\ell}(\mathbf{t})\right)} \right) \frac{dt_1}{t_1} \cdots \frac{dt_{n-1}}{t_{n-1}} \right)_{k,\ell}$$

where $1 \leq k, \ell \leq h_F - 1$.

We will also prove the following analog of Kronecker's solution of Pell's equation for totally real multiquadratic fields. A result of this type for real quadratic fields is given in ([8] p. 97, Proposition 13).

Theorem 3. *Let F be a totally real abelian number field with* $\mathrm{Gal}(F/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^\ell$. *Let E be an unramified real quadratic extension of F with* $\mathrm{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^{\ell+1}$ *and* $\chi_{E/F}$ *be the genus character of F associated to E/F by class field theory. Then*

$$\sum_{A \in \mathrm{CL}(F)} \chi_{E/F}(A) \rho_{2^\ell}(A) = -\frac{D_F^{1/2}}{2^{\ell-1}} \prod_{i=1}^{2^\ell} \frac{\log(\varepsilon_i) h_i}{\sqrt{\Delta_i}},$$

where $\Delta_i > 0$ *for* $1 \leq i \leq 2^\ell$ *are the discriminants of the quadratic subfields* K_i *of E which are not contained in F,* ε_i *is the fundamental unit of* K_i*, and* h_i *is the class number of* K_i.

Next, let

$$\mathcal{L} := \{\log(\alpha) : \alpha \in \overline{\mathbb{Q}}^*\}$$

be the set of logarithms of algebraic numbers. The following is a well-known conjecture on algebraic independence from transcendence theory (see e.g. [11] Conjecture 1.15).

Conjecture 1 (Algebraic Independence of Logarithms)**.** *If* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *are* $\mathbb{Q}$*-linearly independent elements of* $\mathcal{L}$*, then* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *are algebraically independent over* $\mathbb{Q}$.

Assuming Conjecture 1, we will prove the following result.

Corollary 1. *Let notation and assumptions be as in Theorem 3. If Conjecture 1 is true, then*

$$\sum_{A\in \mathrm{CL}(F)} \chi_{E/F}(A)\rho_{2^\ell}(A)$$

is transcendental.

Organization. The paper is organized as follows. In Section 2 we give some explicit examples of Theorems 2 and 3. In Sections 3 and 4 we prove Propositions 1 and 2, resp. In Sections 5, 6 and 7, we prove Theorems 2, 3 and Corollary 1, resp.

2 Examples

In this section we give some explicit examples of Theorems 2 and 3.

Example 1. Consider the tower of fields

$$\mathbb{Q} \subset F \subset F_{\mathrm{gen}} \subset H$$

where $F = \mathbb{Q}(\sqrt{5}, \sqrt{3\cdot 29})$, $F_{\mathrm{gen}} = \mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{29}, \sqrt{7+2\sqrt{5}})$ is the genus field of F, i.e., the maximal unramified extension of F which is abelian over $\mathbb{Q}$, and H is the Hilbert class field of F. The genus field F_{gen} was determined in ([12] Example 2.2 (1)). Note that because F has class number $h_F = 4$ and F_{gen}/F is an unramified abelian extension of degree $4 = h_F = [H : F]$, the genus field F_{gen} is actually the Hilbert class field H. Now, the ideal class group of F is $\mathrm{CL}(F) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. A set of representatives of the (nontrivial) ideal classes $\mathrm{CL}(F) = \{[\mathfrak{A}_1] = [\mathcal{O}_F], [\mathfrak{A}_2], [\mathfrak{A}_3], [\mathfrak{A}_4]\}$ is given by

$$\begin{aligned}
\mathfrak{A}_2 = \mathfrak{A}_2^{-1} &= \left\langle 19, \left(-\frac{9}{2}\sqrt{3\cdot 29}+3\right)\sqrt{5} - \frac{5}{2}\sqrt{3\cdot 29}+198\right\rangle \\
\mathfrak{A}_3 = \mathfrak{A}_3^{-1} &= \left\langle 31, \left(-\frac{1}{2}\sqrt{3\cdot 29}+13\right)\sqrt{5} - \frac{25}{2}\sqrt{3\cdot 29}+15\right\rangle \\
\mathfrak{A}_4 = \mathfrak{A}_4^{-1} &= \left\langle 3, \left(-\frac{1}{2}\sqrt{3\cdot 29}-\frac{3}{2}\right)\sqrt{5} + \frac{3}{2}\sqrt{3\cdot 29}+\frac{45}{2}\right\rangle.
\end{aligned}$$

Applying Theorem 2 to these particular fields yields the following formula for the class number of $H = \mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{29}, \sqrt{7+2\sqrt{5}})$,

$$h_H = -32\frac{R_F}{R_H}\mathrm{Det}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33}\end{pmatrix} \tag{4}$$

where

$$u_{k\ell} = \iiint\limits_{\mathbb{R}_+^3/U} \log\left(\frac{G(\tau_{\mathfrak{A}_\ell^{-1}\mathfrak{A}_k}(\mathbf{t}))}{G(\tau_{\mathfrak{A}_\ell}(\mathbf{t}))}\right)\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}, \quad 1 \le k, \ell \le 3.$$

In particular, if Δ denotes the determinant on the right hand side of (4), we have

$$\frac{R_F}{R_H}\Delta \in \mathbb{Q}.$$

Remark 5. A formula for the ratio of regulators R_H/R_F may be deduced from ([3] Theorem 1).

Example 2. Consider the tower of fields

$$\mathbb{Q} \subset F \subset E \subset F_{\text{gen}} = H$$

where F and $F_{\text{gen}} = H$ are as in Example 1 and $E = \mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{29})$. Then F is a real biquadratic field and E is an unramified real quadratic extension of F with $\text{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^3$. Recall the set of representatives $\text{CL}(F) = \{[\mathfrak{A}_1] = [\mathcal{O}_F], [\mathfrak{A}_2], [\mathfrak{A}_3], [\mathfrak{A}_4]\}$ for the ideal class group of F that were given in Example 1. Since the ideal $\mathfrak{A}_4$ becomes principal in E (it is generated by $\sqrt{3}$), the genus character $\chi_{E/F} : \text{CL}(F) \to \{\pm 1\}$ is given by $\chi_{E/F}([\mathfrak{A}_1]) = \chi_{E/F}([\mathfrak{A}_4]) = 1$ and $\chi_{E/F}([\mathfrak{A}_2]) = \chi_{E/F}([\mathfrak{A}_3]) = -1$. The discriminant of F is $D_F = 3027600 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 29^2$. The quadratic subfields of E not contained in F are $K_1 = \mathbb{Q}(\sqrt{3})$, $K_2 = \mathbb{Q}(\sqrt{29})$, $K_3 = \mathbb{Q}(\sqrt{3 \cdot 5})$ and $K_4 = \mathbb{Q}(\sqrt{5 \cdot 29})$. The corresponding class numbers and discriminants are $h_1 = 1$, $h_2 = 1$, $h_3 = 2$, $h_4 = 4$, and $\Delta_1 = 12$, $\Delta_2 = 29$, $\Delta_3 = 60$, $\Delta_4 = 145$, resp. The fundamental units are $\varepsilon_1 = 2 + \sqrt{3}$, $\varepsilon_2 = (5 + \sqrt{29})/2$, $\varepsilon_3 = 4 + \sqrt{15}$ and $\varepsilon_4 = 12 + \sqrt{145}$. Applying Theorem 3 to these particular fields yields the identity

$$\iiint_{\mathbb{R}_+^3/U} \log\left(\frac{G(\tau_{\mathfrak{A}_1^{-1}}(\mathbf{t}))G(\tau_{\mathfrak{A}_4^{-1}}(\mathbf{t}))}{G(\tau_{\mathfrak{A}_2^{-1}}(\mathbf{t}))G(\tau_{\mathfrak{A}_3^{-1}}(\mathbf{t}))}\right) \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3} =$$

$$-4\log(2+\sqrt{3})\log((5+\sqrt{29})/2)\log(4+\sqrt{15})\log(12+\sqrt{145}).$$

Moreover, by Corollary 1 (which assumes Conjecture 1) the number

$$\iiint_{\mathbb{R}_+^3/U} \log\left(\frac{G(\tau_{\mathfrak{A}_1^{-1}}(\mathbf{t}))G(\tau_{\mathfrak{A}_4^{-1}}(\mathbf{t}))}{G(\tau_{\mathfrak{A}_2^{-1}}(\mathbf{t}))G(\tau_{\mathfrak{A}_3^{-1}}(\mathbf{t}))}\right) \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}$$

is transcendental.

3 Maximal parabolic Eisenstein series on $SL_n(\mathbb{Z})$

In this section we compute the Laurent expansion at $s = 1$ of the maximal parabolic Eisenstein series on $SL_n(\mathbb{Z})$ and thus prove Proposition 1. We follow closely the work of Efrat [4], Friedberg [5], Goldfeld [6] and Terras [10].

For convenience, we recall the setup from Section 1. Let $\mathcal{H}^n = GL_n(\mathbb{R})/O_n(\mathbb{R})\mathbb{R}^*$ be the symmetric space of $GL_n(\mathbb{R})$. By the Iwasawa decomposition, each coset $Z \in \mathcal{H}^n$ has a unique representative of the form

$$Z = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \dots & x_{1,n} \\ & 1 & x_{2,3} & \dots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix} \tag{5}$$

where $x_{i,j} \in \mathbb{R}$ for $1 \le i < j \le n$ and $y_i \in \mathbb{R}_+$ for $1 \le i \le n-1$. Left matrix multiplication induces an action of $GL_n(\mathbb{Z})$ on $\mathcal{H}^n$.

Let P be the maximal parabolic subgroup of $SL_n(\mathbb{Z})$, which consists of those matrices with bottom row $(0, \dots, 0, 1)$. Define the maximal parabolic Eisenstein series

$$E_n(Z,s) := \sum_{\gamma\in P\backslash SL_n(\mathbb{Z})} \mathrm{Det}(\gamma\cdot Z)^s, \quad \mathrm{Re}(s) > 1$$

where $\mathrm{Det}(\gamma \cdot Z)$ is the determinant of the unique representative of the coset $\gamma \cdot Z \in \mathcal{H}^n$ of the form (5) and $s \in \mathbb{C}$. The completed Eisenstein series

$$E_n^*(Z,s) := \pi^{-ns/2}\Gamma(ns/2)\zeta(ns)E_n(Z,s)$$

satisfies the functional equation

$$E_n^*(Z,s) = E_n^*((Z^T)^{-1}, 1-s)$$

and extends to a meromorphic function on $\mathbb{C}$ with simple poles at $s = 0, 1$.

Define

$$Q = Q_Z := ZZ^T$$

and let

$$Q[\mathbf{a}] := \mathbf{a}^T Q\mathbf{a} \quad \text{for} \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Then one has the identity (see e.g. [6] p. 308-309, eq. (10.7.4))

$$\zeta(ns)E_n(Z,s) = \mathrm{Det}(Z)^s\zeta(ns/2, Q), \tag{6}$$

where

$$\zeta(s,Q) := \sum_{\substack{\mathbf{a}\in\mathbb{Z}^n\\ \mathbf{a}\neq\mathbf{0}}} Q[\mathbf{a}]^{-s}, \quad \mathrm{Re}(s) > \frac{n}{2}$$

is the Epstein zeta function of Q. In particular, since

$$\mathrm{Det}(Z) = y_1^{n-1}y_2^{n-2}\cdots y_{n-1}$$

we have

$$E_n^*(Z,s) = \pi^{-ns/2}\Gamma(ns/2)\left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right)^s \sum_{\substack{\mathbf{a}\in\mathbb{Z}^n\\ \mathbf{a}\neq\mathbf{0}}} Q[\mathbf{a}]^{-ns/2}. \tag{7}$$

We now compute the Laurent expansion of $E_n^*(Z,s)$ at $s = 1$ by splitting the sum in (7) into terms with $a_1 = 0$ and terms with $a_1 \neq 0$.

Write

$$Q = \begin{pmatrix} m & \mathbf{r}^T \\ \mathbf{r} & S \end{pmatrix}$$

where

$$m = (y_1y_2 + \cdots y_{n-1})^2 + (x_{1,2}y_1y_2\cdots y_{n-2})^2 + (x_{1,3}y_1y_2\cdots y_{n-3})^2 + \cdots + x_{1,n}^2,$$

$$\mathbf{r} = Z_1 \begin{pmatrix} x_{1,2}y_1y_2\cdots y_{n-2} \\ x_{1,3}y_1y_2\cdots y_{n-3} \\ \vdots \\ x_{1,n} \end{pmatrix}$$

with

$$Z_1 = \begin{pmatrix} 1 & x_{2,3} & \dots & x_{2,n} \\ & \ddots & & \vdots \\ & & 1 & x_{n-1,n} \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1y_2\cdots y_{n-2} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix},$$

and $S = Z_1Z_1^T$. Also, write $\mathbf{a} = \binom{a_1}{\mathbf{b}}$ where

$$\mathbf{b} = \begin{pmatrix} a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix}.$$

If $a_1 = 0$ then $Q[\mathbf{a}] = S[\mathbf{b}]$, hence the contribution of the terms with $a_1 = 0$ in (7) is

$$\pi^{-ns/2}\Gamma(ns/2)\left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right)^s \sum_{\substack{\mathbf{b}\in\mathbb{Z}^{n-1}\\ \mathbf{b}\neq\mathbf{0}}} S[\mathbf{b}]^{-ns/2}$$
$$= \left(y_1y_2^2\cdots y_{n-1}^{n-1}\right)^{s/(n-1)} E^*_{n-1}(Z_1, n/(n-1)s).$$

Let $s = 1$ to get

$$\left(y_1y_2^2\cdots y_{n-1}^{n-1}\right)^{1/(n-1)} E^*_{n-1}(Z_1, n/(n-1)). \tag{8}$$

Next, suppose that $a_1 \neq 0$. We need to analyze

$$\pi^{-ns/2}\Gamma(ns/2)\left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right)^s \sum_{\substack{\mathbf{a}\in\mathbb{Z}^n\\ a_1\neq 0}} Q[\mathbf{a}]^{-ns/2}. \tag{9}$$

Let $\mathbf{q} = S^{-1}\mathbf{r}$ and $w = m - \mathbf{q}^TS\mathbf{q}$. Then

$$Q = \begin{pmatrix} m & \mathbf{r}^T \\ \mathbf{r} & S \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{q}^T \\ & \mathrm{I}_{n-1} \end{pmatrix} \begin{pmatrix} w & \\ & S \end{pmatrix} \begin{pmatrix} 1 & \\ \mathbf{q} & \mathrm{I}_{n-1} \end{pmatrix},$$

so that

$$Q[\mathbf{a}] = w[a_1] + S[\mathbf{q}a_1 + \mathbf{b}].$$

Hence (9) may be written as

$$\pi^{-ns/2}\Gamma(ns/2)\left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right)^s \sum_{\substack{a_1\in\mathbb{Z}\\ a_1\neq 0}} \sum_{\mathbf{b}\in\mathbb{Z}^{n-1}} \left(w[a_1] + S[\mathbf{q}a_1 + \mathbf{b}]\right)^{-ns/2}. \tag{10}$$

We wish to apply the Poisson summation formula to the sum over $\mathbf{b} \in \mathbb{Z}^{n-1}$ in (10). Define

$$f(\mathbf{x}, s) := (w[a_1] + S[\mathbf{q}a_1 + \mathbf{x}])^{-ns/2}, \quad \mathbf{x} \in \mathbb{R}^{n-1}.$$

The Fourier transform is given by

$$\widehat{f}(\mathbf{y}, s) = \int_{\mathbb{R}^{n-1}} (w[a_1] + S[\mathbf{q}a_1 + \mathbf{x}])^{-ns/2} \exp\left(-2\pi i\mathbf{y}^T\mathbf{x}\right) d\mathbf{x}.$$

Write $S = W^TW$ and make the change of variables $\mathbf{u} = (w[a_1])^{-1/2}\, W(\mathbf{q}a_1 + \mathbf{x})$. Then

$$\mathbf{x} = (w[a_1])^{1/2}\, W^{-1}\mathbf{u} - \mathbf{q}a_1$$

and

$$d\mathbf{x} = (w[a_1])^{(n-1)/2}\mathrm{Det}(W)^{-1}d\mathbf{u} = (w[a_1])^{(n-1)/2}\mathrm{Det}(S)^{-1/2}d\mathbf{u},$$

so that

$$\widehat{f}(\mathbf{y},s) = (w[a_1])^{\frac{n-1}{2}-\frac{ns}{2}}\,\mathrm{Det}(S)^{-1/2}\exp\left(2\pi i\mathbf{y}^T\mathbf{q}a_1\right) I\left(2\pi(w[a_1])^{1/2}\left(\mathbf{y}^T W^{-1}\right)^T, ns/2\right)$$

where

$$I(\mathbf{y},s) := \int_{\mathbb{R}^{n-1}}\left(1+\mathbf{x}^T\mathbf{x}\right)^{-s}\exp(-i\mathbf{y}^T\mathbf{x})d\mathbf{x}, \quad \mathrm{Re}(s) > \frac{n-1}{2}.$$

We now evaluate $I(\mathbf{y},s)$. For $\mathbf{y}=\mathbf{0}$, we have (see [10] p. 480-481)

$$I(\mathbf{0},s) = \pi^{(n-1)/2}\frac{\Gamma(s-(n-1)/2)}{\Gamma(s)}. \tag{11}$$

For $\mathbf{y}\neq\mathbf{0}$, we follow ([10] p. 481). By ([9] Theorem 3.3, p. 155), we have

$$I(\mathbf{y},s) = (2\pi)^{(n-1)/2}\int_0^\infty (1+x^2)^{-s}x^{n-2}J_{\frac{n}{2}-\frac{3}{2}}(||\mathbf{y}||x)(||\mathbf{y}||x)^{\frac{3}{2}-\frac{n}{2}}dx,$$

where

$$J_\nu(x) := (1/2x)^\nu\pi^{-1/2}\Gamma(\mu+1/2)^{-1}\int_0^\pi \exp(-ix\cos(t))\sin(t)^{2\nu}dt, \quad \mathrm{Re}(\nu) > -\frac{1}{2}$$

is the J-Bessel function of the first kind and $||\mathbf{y}|| = (\mathbf{y}^T\mathbf{y})^{1/2}$. Then by ([1] p. 488, eq. (11.4.44)), we have

$$I(\mathbf{y},s) = \frac{(2\pi)^{(n-1)/2}(||\mathbf{y}||/2)^{s-(n-1)/2}}{\Gamma(s)}K_{\frac{n-1}{2}-s}(||\mathbf{y}||), \tag{12}$$

where

$$K_\nu(z) := \frac{1}{2}\int_0^\infty \exp(-z(u+\frac{1}{u})/2)u^{\nu-1}du, \quad |\arg(z)| < \frac{\pi}{2}$$

is the modified K-Bessel function.

Apply the Poisson summation formula to the sum over $\mathbf{b}\in\mathbb{Z}^{n-1}$ in (10) to get

$$\pi^{-ns/2}\Gamma(ns/2)\left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right)^s\sum_{\substack{a_1\in\mathbb{Z}\\ a_1\neq 0}}\sum_{\mathbf{b}\in\mathbb{Z}^{n-1}}\widehat{f}(\mathbf{b},s). \tag{13}$$

To analyze this expression we separate the cases $\mathbf{b}=\mathbf{0}$ and $\mathbf{b}\neq\mathbf{0}$. If $\mathbf{b}=\mathbf{0}$, then (11) yields

$$\begin{aligned}\widehat{f}(\mathbf{0},s) &= (w[a_1])^{\frac{n-1}{2}-\frac{ns}{2}}\mathrm{Det}(S)^{-1/2}I(\mathbf{0},ns/2)\\ &= \mathrm{Det}(S)^{-1/2}\pi^{(n-1)/2}\frac{\Gamma\left(\frac{ns}{2}-\frac{n-1}{2}\right)}{\Gamma(ns/2)}w^{\frac{n-1}{2}-\frac{ns}{2}}|a_1|^{(n-1)-ns}.\end{aligned}$$

Hence the contribution of $\mathbf{b}=\mathbf{0}$ to (13) is

$$2\mathrm{Det}(S)^{-1/2}\left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right)^s\pi^{-\frac{ns}{2}+\frac{(n-1)}{2}}\Gamma\left(\frac{ns}{2}-\frac{n-1}{2}\right)w^{\frac{n-1}{2}-\frac{ns}{2}}\zeta(ns-(n-1)), \tag{14}$$

where we used

$$\zeta(ns-(n-1)) = \frac{1}{2}\sum_{\substack{a_1\in\mathbb{Z}\\ a_1\neq 0}}|a_1|^{(n-1)-ns}.$$

To compute the Laurent expansion of (14) at $s = 1$, note that

$$\left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right)^s = y_1^{n-1}y_2^{n-2}\cdots y_{n-1} + \left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right)\log\left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right)(s-1) + O\left(|s-1|^2\right),$$

$$\pi^{-\frac{ns}{2}+\frac{(n-1)}{2}}\Gamma\left(\frac{ns}{2}-\frac{n-1}{2}\right) = 1 + \frac{n\gamma_0}{2}(s-1) + O\left(|s-1|^2\right)$$

where $\gamma_0 = -\gamma - \log(\pi) - 2\log(2)$ (here γ is Euler's constant),

$$w^{\frac{n-1}{2}-\frac{ns}{2}} = w^{-1/2} - \frac{n}{2}\log(w)w^{-1/2}(s-1) + O\left(|s-1|^2\right),$$

and

$$\zeta(ns-(n-1)) = \frac{1/n}{s-1} + \gamma + O(|s-1|).$$

Then multiplying terms yields the Laurent expansion

$$\begin{aligned}
&\frac{\frac{2}{n}(\mathrm{Det}(S)w)^{-1/2}\left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right)}{s-1} \\
&+ 2\gamma(\mathrm{Det}(S)w)^{-1/2}\left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right) \\
&+ \frac{2}{n}(\mathrm{Det}(S)w)^{-1/2}\left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right)\left(\log\left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right) + \frac{n\gamma_0}{2} - \frac{n}{2}\log(w)\right) \\
&+ O(|s-1|) \\
&= \frac{2/n}{s-1} + 2\gamma + \gamma_0 + \frac{2}{n}\log\left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right) - \log(w) + O(|s-1|) \\
&= \frac{2/n}{s-1} + \gamma - \log(4\pi) + \frac{2}{n}\log\left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right) - \log(w) + O(|s-1|), \qquad (15)
\end{aligned}$$

where we used

$$\mathrm{Det}(S)w = \mathrm{Det}(Q) = \mathrm{Det}(Z)^2 = \left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right)^2. \qquad (16)$$

We now calculate $w = m - \mathbf{q}^T S\mathbf{q}$. We have

$$\mathbf{q} = S^{-1}\mathbf{r} = \left(\left(Z_1^T\right)^{-1}Z_1^{-1}\right)Z_1\begin{pmatrix} x_{1,2}y_1y_2\cdots y_{n-2} \\ x_{1,3}y_1y_2\cdots y_{n-3} \\ \vdots \\ x_{1,n}\end{pmatrix} = \left(Z_1^T\right)^{-1}\begin{pmatrix} x_{1,2}y_1y_2\cdots y_{n-2} \\ x_{1,3}y_1y_2\cdots y_{n-3} \\ \vdots \\ x_{1,n}\end{pmatrix}.$$

Therefore

$$\begin{aligned}
\mathbf{q}^T S\mathbf{q} &= (x_{1,2}y_1y_2\cdots y_{n-2}, x_{1,3}y_1y_2\cdots y_{n-3}, \cdots, x_{1,n})Z_1^{-1}\left(Z_1Z_1^T\right)(Z_1^T)^{-1}\begin{pmatrix} x_{1,2}y_1y_2\cdots y_{n-2} \\ x_{1,3}y_1y_2\cdots y_{n-3} \\ \vdots \\ x_{1,n}\end{pmatrix} \\
&= (x_{1,2}y_1y_2\cdots y_{n-2})^2 + (x_{1,3}y_1y_2\cdots y_{n-3})^2 + \cdots + x_{1,n}^2,
\end{aligned}$$

so that

$$\begin{aligned}
w &= m - \mathbf{q}^T S\mathbf{q} \\
&= (y_1y_2 + \cdots y_{n-1})^2 + (x_{1,2}y_1y_2\cdots y_{n-2})^2 + (x_{1,3}y_1y_2\cdots y_{n-3})^2 + \cdots + x_{1,n}^2 \\
&\quad - \left((x_{1,2}y_1y_2\cdots y_{n-2})^2 + (x_{1,3}y_1y_2\cdots y_{n-3})^2 + \cdots + x_{1,n}^2\right) \\
&= (y_1y_2 + \cdots y_{n-1})^2.
\end{aligned}$$

Substituting this formula for w into (15) and simplifying yields the Laurent expansion

$$\frac{2/n}{s-1}+\gamma-\log(4\pi)-\frac{2}{n}\log\left(y_1y_2^2\cdots y_{n-1}^{n-1}\right)+O(|s-1|). \tag{17}$$

If $\mathbf{b}\neq\mathbf{0}$, then (12) yields

$$\begin{aligned}\widehat{f}(\mathbf{b},s)&=(w[a_1])^{\frac{n-1}{2}-\frac{ns}{2}}\mathrm{Det}(S)^{-1/2}\exp\left(2\pi i\mathbf{b}^T\mathbf{q}a_1\right)I\left(2\pi(w[a_1])^{1/2}\left(\mathbf{b}^TW^{-1}\right)^T,ns/2\right)\\&=(w[a_1])^{\frac{n-1}{2}-\frac{ns}{2}}\mathrm{Det}(S)^{-1/2}\exp\left(2\pi i\mathbf{b}^T\mathbf{q}a_1\right)\\&\quad\times\frac{(2\pi)^{(n-1)/2}\left(||2\pi(w[a_1])^{1/2}\left(\mathbf{b}^TW^{-1}\right)^T||/2\right)^{\frac{ns}{2}-\frac{n-1}{2}}}{\Gamma(ns/2)}\\&\quad\times K_{\frac{n-1}{2}-\frac{ns}{2}}\left(||2\pi(w[a_1])^{1/2}\left(\mathbf{b}^TW^{-1}\right)^T||\right)\\&=(w[a_1])^{\frac{n-1}{2}-\frac{ns}{2}}\mathrm{Det}(S)^{-1/2}\exp\left(2\pi i\mathbf{b}^T\mathbf{q}a_1\right)\\&\quad\times\frac{(2\pi)^{(n-1)/2}\left(\frac{1}{2}\cdot2\pi w^{1/2}\left(S^{-1}[\mathbf{b}]\right)^{1/2}|a_1|\right)^{\frac{ns}{2}-\frac{n-1}{2}}}{\Gamma(ns/2)}\\&\quad\times K_{\frac{n-1}{2}-\frac{ns}{2}}\left(2\pi w^{1/2}\left(S^{-1}[\mathbf{b}]\right)^{1/2}|a_1|\right),\end{aligned}$$

where we used

$$||2\pi(w[a_1])^{1/2}\left(\mathbf{b}^TW^{-1}\right)^T||=2\pi\left(w[a_1]S^{-1}[\mathbf{b}]\right)^{1/2}=2\pi w^{1/2}\left(S^{-1}[\mathbf{b}]\right)^{1/2}|a_1|.$$

Now, using the functional equation $K_{-\nu}(z)=K_\nu(z)$ and the identity

$$K_{1/2}(z)=(\pi/2z)^{1/2}e^{-z},$$

we have

$$\begin{aligned}\widehat{f}(\mathbf{b},1)&=(w[a_1])^{-1/2}\mathrm{Det}(S)^{-1/2}\exp\left(2\pi i\mathbf{b}^T\mathbf{q}a_1\right)\\&\quad\times(2\pi)^{(n-1)/2}\frac{(\pi/4)^{1/2}}{\Gamma(n/2)}\exp\left(-2\pi w^{1/2}\left(S^{-1}[\mathbf{b}]\right)^{1/2}|a_1|\right)\\&=\left(y_1^{n-1}y_2^{n-2}\cdots y_{n-1}\right)^{-1}|a_1|^{-1}\frac{\pi^{n/2}}{\Gamma(n/2)}\exp\left(2\pi i\left(\mathbf{b}^T\mathbf{q}a_1+iw^{1/2}\left(S^{-1}[\mathbf{b}]\right)^{1/2}|a_1|\right)\right),\end{aligned}$$

where we again used (16). Hence if $s=1$, the contribution of the terms with $\mathbf{b}\neq\mathbf{0}$ to (13) is

$$\begin{aligned}&\sum_{\substack{a_1\in\mathbb{Z}\\a_1\neq0}}\sum_{\substack{\mathbf{b}\in\mathbb{Z}^{n-1}\\\mathbf{b}\neq\mathbf{0}}}|a_1|^{-1}\exp\left(2\pi i\left(\mathbf{b}^T\mathbf{q}a_1+iw^{1/2}(S^{-1}[\mathbf{b}])^{1/2}|a_1|\right)\right)\\&=4\sum_{a_1=1}^{\infty}\sum_{\substack{\mathbf{b}\in\mathbb{Z}^{n-1}\\\mathbf{b}\ (\mathrm{mod}\ \pm1)\\\mathbf{b}\neq\mathbf{0}}}\frac{1}{a_1}\mathrm{Re}\left(\exp\left(\left(-2\pi w^{1/2}(S^{-1}[\mathbf{b}])^{1/2}+2\pi i\mathbf{b}^T\mathbf{q}\right)a_1\right)\right)\\&=4\sum_{\substack{\mathbf{b}\in\mathbb{Z}^{n-1}\\\mathbf{b}\ (\mathrm{mod}\ \pm1)\\\mathbf{b}\neq\mathbf{0}}}-\log\left|1-\exp\left(-2\pi w^{1/2}(S^{-1}[\mathbf{b}])^{1/2}+2\pi i\mathbf{b}^T\mathbf{q}\right)\right|\\&=-4\log\prod_{\substack{\mathbf{b}\in\mathbb{Z}^{n-1}\\\mathbf{b}\ (\mathrm{mod}\ \pm1)\\\mathbf{b}\neq\mathbf{0}}}\left|1-\exp\left(-2\pi w^{1/2}(S^{-1}[\mathbf{b}])^{1/2}+2\pi i\mathbf{b}^T\mathbf{q}\right)\right|,\end{aligned} \tag{18}$$

where we used

$$\mathrm{Re}\left(\sum_{k=1}^{\infty}\frac{z^k}{k}\right) = -\log|1-z|.$$

Finally, by combining (8), (17) and (18), we get the Laurent expansion

$$E_n^*(Z,s) = \frac{2/n}{s-1} + \gamma - \log(4\pi) - \frac{2}{n}\log\left(y_1y_2^2\cdots y_{n-1}^{n-1}\right) - 4\log g(Z) + O(|s-1|),$$

where

$$g(Z) := \exp\left(-\frac{\left(y_1y_2^2\cdots y_{n-1}^{n-1}\right)^{1/(n-1)} E_{n-1}^*(Z_1, n/(n-1))}{4}\right) \times \prod_{\substack{\mathbf{b}\in\mathbb{Z}^{n-1} \\ \mathbf{b} \pmod{\pm 1} \\ \mathbf{b}\neq\mathbf{0}}} \left|1-\exp\left(-2\pi w^{1/2}(S^{-1}[\,\mathbf{b}]\,)^{1/2} + 2\pi i\mathbf{b}^T\mathbf{q}\right)\right|.$$

This proves Proposition 1.

4 Dedekind zeta functions of totally real fields

In this section we relate the zeta function of a wide ideal class of a totally real number field of degree n to the integral of the maximal parabolic Eisenstein series $E_n^*(Z,s)$ along a Heegner cycle in $\mathcal{H}^n$ and thus prove Proposition 2.

Let F be a totally real number field of degree n and U be the group of units of F. Let A be a wide ideal class of F and fix $\mathfrak{B} \in A^{-1}$. Then the ideal class zeta function may be written as

$$\zeta_F(s,A) := \sum_{\substack{\mathfrak{A}\in A \\ \mathfrak{A}\neq 0}} \frac{1}{N(\mathfrak{A})^s} = N(\mathfrak{B})^s \sum_{\lambda\in\mathfrak{B}^*/U} \frac{1}{|N(\lambda)|^s}, \quad \mathrm{Re}(s) > 1 \tag{19}$$

where $N(\mathfrak{A})$ is the norm and $\mathfrak{B}^* = \mathfrak{B}\backslash\{0\}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the images of $\lambda \in \mathfrak{B}$ under the real embeddings of F. Then

$$|N(\lambda)|^s = |\lambda_1\lambda_2\cdots\lambda_n|^s.$$

Note that for $x > 0$,

$$x^{-s} = \frac{1}{\Gamma(s/2)}\int_0^{\infty} e^{-x^2t}t^{s/2}\frac{dt}{t}.$$

Then for $a_1, a_2, \ldots, a_n > 0$,

$$(a_1a_2\cdots a_n)^{-s}\Gamma(s/2)^n = \int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} \exp\left(-\left(a_1^2t_1 + a_2^2t_2 + \cdots a_n^2t_n\right)\right)(t_1t_2\cdots t_n)^{s/2}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\cdots\frac{dt_n}{t_n}. \tag{20}$$

Consider the change of variables

$$\begin{aligned} t_1 &= x_1^2 w \\ t_2 &= x_2^2 w \\ &\vdots \\ t_{n-1} &= x_{n-1}^2 w \\ t_n &= (x_1 x_2 \cdots x_{n-1})^{-2} w \end{aligned}$$

and the corresponding Jacobian

$$J = n 2^{n-1} (x_1 x_2 \cdots x_{n-1})^{-1} w^{n-1}.$$

Then making this change of variables in (20) yields

$$\begin{aligned} &(a_1 a_2 \cdots a_n)^{-s} \Gamma(s/2)^n = \\ &n 2^{n-1} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \left(\int_0^\infty \exp\left(- \left(\sum_{k=1}^{n-1} a_k^2 x_k^2 + a_n^2 (x_1 \cdots x_{n-1})^{-2} \right) w \right) w^{ns/2} \frac{dw}{w} \right) \\ &\quad \times \frac{dx_1}{x_1} \frac{dx_2}{x_2} \cdots \frac{dx_{n-1}}{x_{n-1}} \\ &= n 2^{n-1} \Gamma(ns/2) \int_0^\infty \int_0^\infty \cdots \int_0^\infty \left(\sum_{k=1}^{n-1} a_k^2 x_k^2 + a_n^2 (x_1 \cdots x_{n-1})^{-2} \right)^{-ns/2} \\ &\quad \times \frac{dx_1}{x_1} \frac{dx_2}{x_2} \cdots \frac{dx_{n-1}}{x_{n-1}}. \end{aligned} \tag{21}$$

We now apply the identity (21) in (19) to get

$$\begin{aligned} &\zeta_F(s,A) \Gamma(s/2)^n = n 2^{n-1} \Gamma(ns/2) N(\mathfrak{B})^s \\ &\quad \times \sum_{\lambda \in \mathfrak{B}^*/U} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \left(\sum_{k=1}^{n-1} \lambda_k^2 t_k^2 + \lambda_n^2 (t_1 \cdots t_{n-1})^{-2} \right)^{-ns/2} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \cdots \frac{dt_{n-1}}{t_{n-1}}. \end{aligned} \tag{22}$$

Given a unit $\varepsilon \in U$, let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ denote the images of ε under the real embeddings of F. There is an action of the unit group U on $\mathbb{R}_+^{n-1}$ given by

$$\begin{aligned} \varepsilon : \quad &\mathbb{R}_+^{n-1} \longrightarrow \mathbb{R}_+^{n-1}, \\ &(t_1, t_2, \ldots, t_{n-1}) \longmapsto (|\varepsilon_1| t_1, |\varepsilon_2| t_2, \ldots, |\varepsilon_{n-1}| t_{n-1}). \end{aligned}$$

Let $\mathbb{R}_+^{n-1}/U$ denote a fundamental domain for this action. Then using this action, (22) becomes

$$\begin{aligned} &\zeta_F(s,A) \Gamma(s/2)^n = n 2^{n-1} \Gamma(ns/2) N(\mathfrak{B})^s \\ &\quad \times \sum_{\lambda \in \mathfrak{B}^*} \underset{\mathbb{R}_+^{n-1}/U}{\int \cdots \int} \left(\sum_{k=1}^{n-1} \lambda_k^2 t_k^2 + \lambda_n^2 (t_1 \cdots t_{n-1})^{-2} \right)^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_{n-1}}{t_{n-1}}. \end{aligned} \tag{23}$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be a $\mathbb{Z}$-basis for $\mathfrak{B}$ and $\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_n^{(i)}$ for $i = 1, 2, \ldots, n$ denote their images under the real embeddings of F. Given $\lambda \in \mathfrak{B}$, write

$$\lambda = m_1 \alpha_1 + m_2 \alpha_2 + \cdots + m_n \alpha_n$$

where $m_1, m_2, \ldots, m_n \in \mathbb{Z}$. Then

$$\sum_{k=1}^{n-1} \lambda_k^2 t_k^2 + \lambda_n^2 (t_1 \cdots t_{n-1})^{-2} = \mathbf{m}^T \left(M_{\mathfrak{B}}(\mathbf{t}) M_{\mathfrak{B}}(\mathbf{t})^T \right) \mathbf{m}$$

where

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix}$$

and

$$M_{\mathfrak{B}}(\mathbf{t}) := \begin{pmatrix} \alpha_1^{(1)} t_1 & \alpha_1^{(2)} t_2 & \cdots & \alpha_1^{(n-1)} t_{n-1} & \alpha_1^{(n)} (t_1 t_2 \cdots t_{n-1})^{-1} \\ \alpha_2^{(1)} t_1 & \alpha_2^{(2)} t_2 & \cdots & \alpha_2^{(n-1)} t_{n-1} & \alpha_2^{(n)} (t_1 t_2 \cdots t_{n-1})^{-1} \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_n^{(1)} t_1 & \alpha_n^{(2)} t_2 & \cdots & \alpha_n^{(n-1)} t_{n-1} & \alpha_n^{(n)} (t_1 t_2 \cdots t_{n-1})^{-1} \end{pmatrix}$$

where $\mathbf{t} = (t_1, t_2, \ldots, t_{n-1}) \in \mathbb{R}_+^{n-1}$.

Define

$$Q_{\mathfrak{B}}(\mathbf{t}) := M_{\mathfrak{B}}(\mathbf{t}) M_{\mathfrak{B}}(\mathbf{t})^T,$$

and let

$$Q_{\mathfrak{B}}(\mathbf{t})[\mathbf{m}] := \mathbf{m}^T \cdot Q_{\mathfrak{B}}(\mathbf{t}) \cdot \mathbf{m}$$

be the quadratic form associated to $Q_{\mathfrak{B}}(\mathbf{t})$. Then the identity (23) becomes

$$\zeta_F(s, A) \Gamma(s/2)^n = n 2^{n-1} \Gamma(ns/2) N(\mathfrak{B})^s \int \cdots \int_{\mathbb{R}_+^{n-1}/U} \zeta(ns/2, Q_{\mathfrak{B}}(\mathbf{t})) \frac{dt_1}{t_1} \cdots \frac{dt_{n-1}}{t_{n-1}}, \quad (24)$$

where

$$\zeta(s, Q_{\mathfrak{B}}(\mathbf{t})) := \sum_{\substack{\mathbf{m} \in \mathbb{Z}^n \\ \mathbf{m} \neq \mathbf{0}}} Q_{\mathfrak{B}}(\mathbf{t})[\mathbf{m}]^{-s}, \quad \mathrm{Re}(s) > \frac{n}{2}$$

is the Epstein zeta function of $Q_{\mathfrak{B}}(\mathbf{t})$.

The positive definite, symmetric matrix $Q_{\mathfrak{B}}(\mathbf{t})$ may be written as

$$Q_{\mathfrak{B}}(\mathbf{t}) = \mathrm{Det}(Q_{\mathfrak{B}}(\mathbf{t}))^{1/n} \left(y_1^{n-1}(\mathbf{t}) y_2^{n-2}(\mathbf{t}) \cdots y_{n-1}(\mathbf{t}) \right)^{-2/n} \tau_{\mathfrak{B}}(\mathbf{t}) \tau_{\mathfrak{B}}(\mathbf{t})^T$$

where

$$\tau_{\mathfrak{B}}(\mathbf{t}) = \begin{pmatrix} 1 & x_{1,2}(\mathbf{t}) & x_{1,3}(\mathbf{t}) & \ldots & x_{1,n}(\mathbf{t}) \\ & 1 & x_{2,3}(\mathbf{t}) & \ldots & x_{2,n}(\mathbf{t}) \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n}(\mathbf{t}) \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1(\mathbf{t}) y_2(\mathbf{t}) \cdots y_{n-1}(\mathbf{t}) & & & & \\ & y_1(\mathbf{t}) y_2(\mathbf{t}) \cdots y_{n-2}(\mathbf{t}) & & & \\ & & \ddots & & \\ & & & y_1(\mathbf{t}) & \\ & & & & 1 \end{pmatrix}$$

is in $\mathcal{H}^n$. Here we have suppressed the dependence of the variables $x_{i,j}(\mathbf{t})$ and $y_i(\mathbf{t})$ on $\mathfrak{B}$ and the $\mathbb{Z}$-basis $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then $\{\tau_{\mathfrak{B}}(\mathbf{t}) : \mathbf{t} \in \mathbb{R}_+^{n-1}/U\}$ defines a Heegner cycle in $\mathcal{H}^n$.

Now, by (6) we have the identity

$$\zeta(ns/2, Q_{\mathfrak{B}}(\mathbf{t})) = \mathrm{Det}(Q_{\mathfrak{B}}(\mathbf{t}))^{-s/2} \zeta(ns) E_n(\tau_{\mathfrak{B}}(\mathbf{t}), s).$$

Moreover,

$$\begin{aligned}\mathrm{Det}(Q_{\mathfrak{B}}(\mathbf{t})) &= \mathrm{Det}\begin{pmatrix}\alpha_1^{(1)} & \alpha_1^{(2)} & \cdots & \alpha_1^{(n)}\\ \alpha_2^{(1)} & \alpha_2^{(2)} & \cdots & \alpha_2^{(n)}\\ \vdots & \vdots & & \vdots\\ \alpha_n^{(1)} & \alpha_n^{(2)} & \cdots & \alpha_n^{(n)}\end{pmatrix}^2\\ &= \mathrm{disc}(\mathfrak{B})\\ &= N(\mathfrak{B})^2 D_F\end{aligned}$$

where D_F is the discriminant of F. Then if

$$\zeta_F^*(s,A) := \pi^{-ns/2}\Gamma(s/2)^n D_F^{s/2}\zeta_F(s,A)$$

denotes the completed ideal class zeta function, (24) yields

$$\zeta_F^*(s,A) = n2^{n-1}\int_{\mathbb{R}_+^{n-1}/U}\cdots\int E_n^*(\tau_{\mathfrak{B}}(\mathbf{t}),s)\frac{dt_1}{t_1}\cdots\frac{dt_{n-1}}{t_{n-1}}.$$

This proves Proposition 2.

5 Proof of Theorem 2

Given a number field K, let $\mathrm{CL}(K)$ be the wide ideal class group, h_K be the class number, R_K be the regulator, w_K be the number of roots of unity, and D_K be the absolute value of the discriminant. Given an ideal class group character χ of K, the class group L–function is defined by

$$L_K(\chi,s) := \sum_{A\in\mathrm{CL}(K)}\chi(A)\zeta_K(s,A), \quad \mathrm{Re}(s) > 1$$

where $\zeta_K(s,A)$ denotes the ideal class zeta function of $A \in \mathrm{CL}(K)$.

If χ is trivial, then $L(\chi,s) = \zeta_K(s)$ is the Dedekind zeta function of K. The Dedekind zeta function $\zeta_K(s)$ extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$ with residue

$$\mathrm{Res}_{s=1}\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{w_K\sqrt{D_K}}, \tag{25}$$

where r_1 (resp. $2r_2$) is the number of real (resp. complex) embeddings of K.

Suppose now that F is a totally real number field of degree n and H is the Hilbert class field of F. By class field theory, one has the factorization

$$\frac{\zeta_H(s)}{\zeta_F(s)} = \prod_{\substack{\chi\in\widehat{\mathrm{CL}(F)}\\ \chi\neq 1}} L_F(\chi,s).$$

Since F is totally real of degree n and H is unramified at the infinite primes, H is totally real with $n\cdot h_F$ real embeddings. It follows from (25) that

$$\lim_{s\to 1}\frac{(s-1)}{(s-1)}\frac{\zeta_H(s)}{\zeta_F(s)} = 2^{n(h_F-1)}\frac{h_H}{h_F}\frac{R_H}{R_F}\sqrt{\frac{D_F}{D_H}}. \tag{26}$$

Here we used $w_F = w_H = 2$, since these fields are totally real and hence have only the roots of unity ± 1. On the other hand, by Theorem 1 and orthogonality, for $\chi \neq 1$ we have

$$L_F(\chi,1) = -\frac{n2^{n-1}}{D_F^{1/2}}\sum_{A\in\mathrm{CL}(F)}\chi(A)\rho_n(A) \tag{27}$$

where we recall that

$$\rho_n(A) := \int_{\mathbb{R}_+^{n-1}/U} \cdots \int \log G(\tau_{\mathfrak{B}}(\mathbf{t})) \frac{dt_1}{t_1} \cdots \frac{dt_{n-1}}{t_{n-1}}$$

and $\mathfrak{B} \in A^{-1}$. Then combining (26) and (27) yields the identity

$$\frac{(-1)^{h_F-1} 2^{h_F-1}}{n^{h_F-1}} \frac{h_H}{h_F} \frac{R_H}{R_F} \frac{D_F^{h_F/2}}{D_H^{1/2}} = \prod_{\substack{\chi \in \widehat{\mathrm{CL}(F)} \\ \chi \neq 1}} \sum_{A \in \mathrm{CL}(F)} \chi(A) \rho_n(A).$$

Since $D_H = D_F^{h_F}$, we have $D_F^{h_F/2}/D_H^{1/2} = 1$. Moreover, by a well-known result of Frobenius on group determinants (see e.g. [8] p. 78), we have

$$\prod_{\substack{\chi \in \widehat{\mathrm{CL}(F)} \\ \chi \neq 1}} \sum_{A \in \mathrm{CL}(F)} \chi(A) \rho_n(A) = \mathrm{Det}\left(\rho_n\left(A_k^{-1} A_\ell\right) - \rho_n\left(A_k^{-1}\right)\right)_{k,\ell}$$

where $1 \leq k, \ell \leq h_F - 1$. It follows that

$$\frac{(-1)^{h_F-1} 2^{h_F-1}}{n^{h_F-1}} \frac{h_H}{h_F} \frac{R_H}{R_F} = \mathrm{Det}\left(\rho_n\left(A_k^{-1} A_\ell\right) - \rho_n\left(A_k^{-1}\right)\right)_{k,\ell}.$$

Finally, if we write the ideal class group of F as

$$\mathrm{CL}(F) = \{A_1 = [\mathfrak{A}_1] = [\mathcal{O}_F], A_2 = [\mathfrak{A}_2], \ldots, A_{h_F} = [\mathfrak{A}_{h_F}]\},$$

then

$$\rho_n\left(A_k^{-1} A_\ell\right) - \rho_n\left(A_k^{-1}\right) = \int_{\mathbb{R}_+^{n-1}/U} \cdots \int \log\left(\frac{G\left(\tau_{\mathfrak{A}_\ell^{-1}\mathfrak{A}_k}(\mathbf{t})\right)}{G(\tau_{\mathfrak{A}_\ell}(\mathbf{t}))}\right) \frac{dt_1}{t_1} \cdots \frac{dt_{n-1}}{t_{n-1}}.$$

This proves Theorem 2.

6 Proof of Theorem 3

Let F be a totally real abelian number field with $\mathrm{Gal}(F/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^\ell$, and let E be an unramified real quadratic extension of F with $\mathrm{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^{\ell+1}$. Then the zeta function $\zeta_E(s)$ (resp. $\zeta_F(s)$) factors as $\zeta(s)$ times the product of the Dirichlet L–functions associated to the quadratic subfields of E (resp. F). Note that there are $2^\ell - 1$ quadratic subfields of F, $2^{\ell+1}-1$ quadratic subfields of E, and 2^ℓ quadratic subfields of E that are not contained in F. By class field theory, the unramified quadratic extension E/F gives rise to a real ideal class group character $\chi_{E/F}$ of F (a genus character) whose L-function factors as

$$L_F(\chi_{E/F}, s) = \frac{\zeta_E(s)}{\zeta_F(s)}.$$

Then by the preceding facts we obtain the factorization

$$L_F(\chi_{E/F}, s) = \prod_{i=1}^{2^\ell} L(\chi_i, s),$$

where χ_i for $1 \leq i \leq 2^\ell$ are the Kronecker symbols associated to the quadratic subfields K_i of E which are not contained in F.

By Dirichlet's class number formula, we have

$$L(\chi_i, 1) = \frac{2\log(\varepsilon_i) h_i}{\sqrt{\Delta_i}},$$

where $\Delta_i > 0$, ε_i and h_i are the discriminant, fundamental unit, and class number of K_i, resp. Therefore

$$L_F(\chi_{E/F}, 1) = 2^{2^\ell} \prod_{i=1}^{2^\ell} \frac{\log(\varepsilon_i) h_i}{\sqrt{\Delta_i}}. \tag{28}$$

Let $n = 2^\ell$ in (27) and equate this with (28) to get

$$\sum_{A \in \mathrm{CL}(F)} \chi_{E/F}(A) \rho_{2^\ell}(A) = -\frac{D_F^{1/2}}{2^{\ell-1}} \prod_{i=1}^{2^\ell} \frac{\log(\varepsilon_i) h_i}{\sqrt{\Delta_i}}.$$

This proves Theorem 3.

7 Proof of Corollary 1

Conjecture 1 asserts that if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are $\mathbb{Q}$-linearly independent elements of

$$\mathcal{L} := \{\log(\alpha) : \alpha \in \overline{\mathbb{Q}}^*\},$$

then $\lambda_1, \lambda_2, \ldots, \lambda_k$ are algebraically independent over $\mathbb{Q}$ (recall that algebraic independence over $\mathbb{Q}$ means that if $P(X_1, X_2, \ldots, X_k)$ is a nonzero polynomial with coefficients in $\mathbb{Q}$, then $P(\lambda_1, \lambda_2, \ldots, \lambda_k) \neq 0$). It follows that if $R(X_1, X_2, \ldots, X_k)$ is a nonconstant polynomial with coefficients in $\overline{\mathbb{Q}}$, then the number $R(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is transcendental.

Now, because the units ε_i, $1 \leq i \leq 2^\ell$, are multiplicatively independent, the numbers $\log(\varepsilon_i)$, $1 \leq i \leq 2^\ell$, are $\mathbb{Q}$-linearly independent. Define the (nonconstant) polynomial

$$R(X_1, X_2, \ldots, X_{2^\ell}) := \frac{D_F^{1/2}}{2^{\ell-1}} \prod_{i=1}^{2^\ell} \alpha_i X_i \in \overline{\mathbb{Q}}[X_1, X_2, \ldots, X_{2^\ell}]$$

where

$$\alpha_i := \frac{h_i}{\sqrt{\Delta_i}}.$$

Then assuming Conjecture 1, the number

$$R\left(\log(\varepsilon_1), \log(\varepsilon_2), \ldots, \log(\varepsilon_{2^\ell})\right)$$

is transcendental. However, by Theorem 3 we have

$$R\left(\log(\varepsilon_1), \log(\varepsilon_2), \ldots, \log(\varepsilon_{2^\ell})\right) = \sum_{A \in \mathrm{CL}(F)} \chi_{E/F}(A) \rho_{2^\ell}(A).$$

This proves Corollary 1.

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