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Counterterms for static Lovelock solutions

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Abstract In this paper, we introduce the counterterms that remove the non-logarithmic divergences of the action in third order Lovelock gravity for static spacetimes. We do this by defining the cosmological constant in such a way that the asymptotic form of the metric have the same form in Lovelock and Einstein gravities. Thus, we employ the counterterms of Einstein gravity and show that the power law divergences of the action of Lovelock gravity for static spacetimes can be removed by suitable choice of coefficients. We find that the dependence of these coefficients on the dimension in Lovelock gravity is the same as in Einstein gravity. We also introduce the finite energy-momentum tensor and employ these counterterms to calculate the finite action and mass of static black hole solutions of third order Lovelock gravity. Next, we calculate the thermodynamic quantities and show that the entropy calculated through the use of Gibbs-Duhem relation is consistent with the obtained entropy by Wald's formula. Furthermore, we find that in contrast to Einstein gravity in which there exists no uncharged extreme black hole, third order Lovelock gravity can have these kind of black holes. Finally, we investigate the stability of static charged black holes of Lovelock gravity in canonical ensemble and find that small black holes show a phase transition between very small and small black holes, while the large ones are stable.

1 Introduction

An interesting framework for studying the non-perturbative quantum field theories is through the use of anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1–3]. According to this duality, in principle, one can perform gravity calculations to find information about the field theory side or vice versa. In this context, the central charges of the dual theory (CFT) relate to coupling constants of its dual gravity. Therefore, Einstein gravity with one coupling constant restricts the dual theory to a limited class of CFT with equal central charges [4]. For extension of the duality beyond this limit, one needs to involve higher curvature terms in the gravity action. It is clear that each correction term introduces a new coupling constant and therefore one may have CFT theory with different central charges. Indeed, this procedure leads to the richness of the CFT theory [5–15]. The most natural extension of general relativity with higher curvature terms and with the assumption of Einstein – that the left hand side of the field equations is the most general symmetric conserved tensor containing no more than two-derivatives of the metric – is Lovelock theory. Lovelock [16] found the most general symmetric conserved tensor satisfying this property. The resultant tensor is nonlinear in the Riemann tensor and differs from the Einstein tensor only if the spacetime has more than 4 dimensions. Although Lovelock gravity leads to second-order field equations and it has ghost free AdS solution [19,20], it has been recently shown that quadratic and cubic gravities entail causality violation and there are stringent conditions on the coupling constants [21].

The problem with the total action of Einstein gravity is that it is divergent when evaluated on the solutions [22–24]. Due to this fact, all the other conserved quantities which is calculated through the use of this action is also divergent. One way of eliminating these divergences is through the use of background subtraction method of Brown and York [22]. In this method, the boundary surface is embedded in another (background) spacetime, and one subtracts the action evaluated on the embedded surface of the background spacetime from the total action. Such a procedure causes the resulting physical quantities to depend on the choice of reference background. Furthermore, it is not possible in general to embed the boundary surface into a background spacetime. For asymptotically AdS solutions of Einstein gravity, one may remove the non-logarithmic divergences in the action by adding a counterterm action which is a functional of the boundary curvature invariants [25–27]. Indeed, this counterterm method furnishes a means for calculating the action and

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conserved quantities intrinsically without reliance on any reference spacetime [28–31]. Although there may exist a very large number of possible invariants, only a finite number of them are non-vanishing in a given dimension on a boundary at infinity. This method has been applied to many cases such as black holes with rotation, NUT charge, various topologies, rotating black strings with zero curvature horizons and rotating higher genus black branes [32–34]. Although the counterterm method applies for the case of a specially infinite boundary, it was also employed for the computation of the conserved and thermodynamic quantities in the case of a finite boundary [35–37].

All of the works mentioned in the previous paragraph were limited to Einstein gravity. Although the counterterm of Lovelock gravity with flat horizon has been introduced [38–40], only a few works related to the counterterm method have been done for Lovelock gravity with curved horizon. This is due to the fact that even for Einstein gravity, the systematic construction that provides the form of the counterterms becomes cumbersome for high enough dimensions [25–27]. Indeed in this method, one should reconstruct the spacetime metric by solving iteratively the field equations in the Fefferman-Graham frame [41–43]:

$$ds^2 = \frac{l^2}{4\rho^2}d\rho^2 + \frac{1}{\rho}[g_{(0)ij}(x) + \rho g_{(1)ij}(x) + \rho^2 g_{(2)ij}(x) + \cdots]dx^i dx^j, \quad (1)$$

where $g_{(0)ij}(x)$ is the boundary data of an initial-value problem governed by the equations of motion. However, even for Einstein-Hilbert theory, solving the coefficients $g_{(p)}(x)$ in Eq. (1) as covariant functionals of $g_{(0)}(x)$ is only possible for low enough dimensions. Thus, it is expected that the holographic renormalization procedure would be even more complicated in Lovelock gravity because of the nonlinearity of field equations. Indeed, because of the nonlinearity of the field equations solving $g_{(p)}(x)$ in Eq. (1) as covariant functionals of $g_{(0)}(x)$ would be even more cumbersome. So, the authors in Refs. [44,45] presented an alternative construction of Kounterterms. Instead of adding counterterms to cancel the divergence at the boundary explained above, they circumvented the difficulties of the standard method by using Kounterterms which depend on the intrinsic and the extrinsic curvatures of the boundary. They selected the Kounterterms as the boundary terms which are regular on the asymptotic region. Indeed, the regularization process is encoded in the boundary terms already presented and there is no need to add further counterterms.

The exact rotating solutions of Lovelock gravity with curved horizon are not introduced till now. Indeed, only static solutions of Lovelock gravity with different matter fields are known [46–49]. So, because of the difficulties of the holographic renormalization procedure in Lovelock gravity and the nonexistance of an exact rotating solution of this theory, we limit ourselves to the case of counterterms of Lovelock gravity for static solutions. The counterterms of asymptotically AdS static solutions of Gauss-Bonnet gravity have been introduced in Refs. [50–52]. Also, the finite action and global charges of asymptotically de Sitter static solutions has been obtained in Ref. [53].

Here we like to apply the counterterm method to the case of the static solutions of the field equations of third order Lovelock gravity with curved horizon. We define the cosmological constant in such a way that the maximally symmetric AdS spacetime

$$ds^2 = -\left(k + \frac{r^2}{L^2}\right)dt^2 + \left(k + \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Sigma^2_{k,n-1} \quad (2)$$

be the vacuum solution of Lovelock gravity. In Eq. (2) $d\Sigma^2_{k,n-1}$ is the metric of an $(n-1)$-dimensional maximally symmetric space with curvature constant $(n-1)(n-2)k$ and volume $V_{k,n-1}$. Indeed, this choice of cosmological constant makes the asymptotic form of the solutions of Lovelock gravity to be exactly the same as that of Einstein gravity. Thus, we expect that the counterterm introduced for Einstein gravity in [25,26] may remove the power law divergences in the action of Lovelock gravity. Although the counterterms which should be added to Gauss-Bonnet gravity in order to remove the power law divergences of the action for static solutions are introduced in Refs. [50,51], they depend on the Gauss-Bonnet coefficient. However, because of our choice of the cosmological constant, our counterterms are the same as those of Einstein gravity and are independent of Lovelock coefficients. In order to check our counterterms, we calculate the finite action and the mass of the black hole through the use of counterterm method. Then, we use these finite quantities and the Gibbs-Duhem relation to obtain the entropy. We find that the calculated entropy of the black holes is consistent with the Wald's formula [54]. As another test of our counterterm method, we show that the mass obtained through the use of counterterm method satisfies the first law of thermodynamics. We, also, perform a stability analysis of the black hole solutions in canonical ensemble and investigate the effects of third order Lovelock term on the stability.

This paper is organized as follows. In Sect. 2, we review the well-defined action of Lovelock gravity. In Sect. 3, we introduce the counterterms for third order Lovelock gravity for static spacetimes. We also, introduce the finite stress energy tensor of this theory. Section 4 is devoted to the thermodynamics of the black hole solutions of the theory. We calculate the finite action, the total mass, the temperature, the charge and the electric potential. We calculate the entropy through the use of Gibbs-Duhem relation and Wald formula and find that they are consistent. We, also, investigate the

first law of thermodynamics. In Sect. 5, we investigate the thermal stability of the solutions in canonical ensemble. We finish our paper with some concluding remarks.

2 Action and field equations

The bulk action of Lovelock gravity in $n+1$ dimensions may be written as [16–18]

$$I_{\rm bulk}=\frac{1}{16\pi}\int_{\mathcal{M}}dx^{n+1}\sqrt{-g}\sum_{p=1}^{[n/2]}\alpha_p\left(\mathcal{L}_p-2\Lambda_p\right)+I_{\rm mat} \tag{3}$$

with $[x]$ denoting the integer part of x, α_p's ($p\geq 2$) are Lovelock coefficients,

$$\mathcal{L}_p=\frac{1}{2^p}\delta^{a_1a_2\ldots a_{2p-1}a_{2p}}_{b_1b_2\ldots b_{2p-1}b_{2p}}R^{b_1b_2}_{a_1a_2}\ldots R^{b_{2p-1}b_{2p}}_{a_{2p-1}a_{2p}}$$

is the Euler density of a $2p$-dimensional manifold, $\delta^{a_1\ldots a_{2p}}_{b_1\ldots b_{2p}}$ is the general asymmetric kronecker delta and

$$\Lambda_p=\frac{(-1)^p n(n-1)(n-2)\ldots(n-2p+1)}{2L^{2p}}. \tag{4}$$

The action (3) is written in such a way that the maximally symmetric AdS spacetime (2) is the vacuum solution of action (3). In this notation, the independent coupling constants are L and all the Lovelock coefficients.

From a geometric point of view the Lagrangian of the action (3) in $2[n/2]+1$ and $2[n/2]+2$ dimensions is the most general Lagrangian that yields second order field equations, as in the case of Einstein-Hilbert action which is the most general Lagrangian producing second order field equations in three and four dimensions. In the rest of the paper, we work in a unit system with $\alpha_1=\tilde{\alpha}_1=1$ and the dimensionless Lovelock coefficients $\tilde{\alpha}_p$ defined as

$$\tilde{\alpha}_p\equiv\frac{(n-2)\ldots(n-2p+1)}{L^{2(p-1)}}\alpha_p,\ p\geq 2. \tag{5}$$

With the definition (5), the cosmological constant for AdS spacetime is

$$\Lambda=\frac{n(n-1)}{2L^2}\sum_{p=1}^{[n/2]}(-1)^p\tilde{\alpha}_p.$$

In this paper, we consider the third order Lovelock gravity in the presence of electromagnetic field. Thus, the action of matter field is

$$I_{\rm mat}=-\frac{1}{64\pi}\int_{\mathcal{M}}dx^{n+1}\sqrt{-g}\partial_{[\mu}A_{\nu]}\partial^{[\mu}A^{\nu]},$$

where A_μ is the electromagnetic potential. The first term in Lovelock Lagrangian is the Einstein-Hilbert term R, the second term is the Gauss-Bonnet Lagrangian $\mathcal{L}_2=R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta}-4R_{\mu\nu}R^{\mu\nu}+R^2$, the third term is

$$\begin{aligned}\mathcal{L}_3=&2R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\rho\tau}R^{\rho\tau}{}_{\mu\nu}+8R^{\mu\nu}{}_{\sigma\rho}R^{\sigma\kappa}{}_{\nu\tau}R^{\rho\tau}{}_{\mu\kappa}\\&+24R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\nu\rho}R^{\rho}{}_{\mu}+3RR^{\mu\nu\sigma\kappa}R_{\sigma\kappa\mu\nu}\\&+24R^{\mu\nu\sigma\kappa}R_{\sigma\mu}R_{\kappa\nu}+16R^{\mu\nu}R_{\nu\sigma}R^{\sigma}{}_{\mu}\\&-12RR^{\mu\nu}R_{\mu\nu}+R^3,\end{aligned} \tag{6}$$

and the cosmological constant is

$$\Lambda=-\frac{n(n-1)}{2L^2}(1-\tilde{\alpha}_2+\tilde{\alpha}_3). \tag{7}$$

As in the case of Einstein-Hilbert action, the action (3) does not have a well-defined variational principle, since one encounters a total derivative that produces a surface integral involving the derivatives of $\delta g_{\mu\nu}$ normal to the boundary $\partial\mathcal{M}$. These normal derivatives of $\delta g_{\mu\nu}$ can be canceled by the variation of the surface action [38,39,55]

$$I_{\rm sur}=\frac{1}{8\pi}\int_{\partial\mathcal{M}}d^{n+1}x\sqrt{-\gamma}\sum_{p=1}^{[n/2]}\alpha_pQ_p,$$

where

$$\begin{aligned}Q_p=&p\int_0^1dt\,\delta^{[jj_1\cdots j_{2p-1}]}_{[ii_1\cdots i_{2p-1}]}K^{i_1}_{j_1}\\&\times\left(\frac{1}{2}\hat{R}^{i_2i_3}_{j_2j_3}(\gamma)-t^2K^{i_2}_{j_2}K^{i_3}_{j_3}\right)\\&\cdots\left(\frac{1}{2}\hat{R}^{i_{2p-2}i_{2p-1}}_{j_{2p-2}j_{2p-1}}(\gamma)-t^2\,K^{i_{2p-2}}_{j_{2p-2}}K^{i_{2p-1}}_{j_{2p-1}}\right).\end{aligned} \tag{8}$$

In Eq. (8) γ_{ab} and $K_{ab}=-\gamma^\mu_a\nabla_\mu n_b$ are the induced metric and extrinsic curvature of the boundary $\partial\mathcal{M}$, respectively. The explicit form of the first three terms of Eq. (8) are [38,39]:

$$\begin{aligned}I^{(1)}_{\rm sur}&=\frac{1}{8\pi}\int_{\partial\mathcal{M}}d^nx\sqrt{-\gamma}K,\\I^{(2)}_{\rm sur}&=\frac{\alpha_2}{4\pi}\int_{\partial\mathcal{M}}d^nx\sqrt{-\gamma}\left(J-2\hat{G}^{(1)}_{ab}K^{ab}\right),\\I^{(3)}_{\rm sur}&=\frac{3\alpha_3}{8\pi}\int_{\delta\mathcal{M}}d^nx\sqrt{-\gamma}\left\{P-2\hat{G}^{(2)}_{ab}K^{ab}+2\hat{R}J\right.\\&\left.-12\hat{R}_{ab}J^{ab}-4\hat{R}_{abcd}\left(2K^{ac}K^b_eK^{ed}-KK^{ac}K^{bd}\right)\right\},\end{aligned} \tag{9}$$

where J and P are the traces of

$$\begin{aligned}J_{ab}=&\frac{1}{3}\left(2KK_{ac}K^c_b+K_{cd}K^{cd}K_{ab}-2K_{ac}K^{cd}K_{db}\right.\\&\left.-K^2K_{ab}\right)\end{aligned}$$

and

$$P_{ab} = \frac{1}{5}\left\{\Big[K^4 - 6K^2K^{cd}K_{cd} + 8KK_{cd}K_e^dK^{ec} - 6K_{cd}K^{de}K_{ef}K^{fc} + 3\left(K_{cd}K^{cd}\right)^2\Big]K_{ab} - \left(4K^3 - 12KK_{ed}K^{ed} + 8K_{de}K_f^eK^{fd}\right)K_{ac}K_b^c - 24KK_{ac}K^{cd}K_{de}K_b^e + \left(12K^2 - 12K_{ef}K^{ef}\right)K_{ac}K^{cd}K_{db} + 24K_{ac}K^{cd}K_{de}K^{ef}K_{bf}\right\}, \tag{10}$$

respectively. In Eq. (9) $\hat{G}^{(1)}_{ab}$ is the Einstein tensor, $\hat{R}_{abcd}(\gamma)$ is the intrinsic curvature and $\hat{G}^{(2)}_{ab}$ is the Gauss-Bonnet tensor of the metric γ_{ab} given as

$$\hat{G}^{(2)}_{ab} = 2\left(\hat{R}_{acde}\hat{R}_b^{\ cde} - 2\hat{R}_{acbd}\hat{R}^{cd} - 2\hat{R}_{ac}\hat{R}^c_{\ b} + \hat{R}\hat{R}_{ab}\right) - \frac{1}{2}\hat{\mathcal{L}}_2\gamma_{ab}.$$

3 Counterterm method for static solutions of third order Lovelock gravity

It is well known that the action $I_{\rm bulk} + I_{\rm sur}$ is not finite for asymptotically AdS solutions. Inspired by AdS/CFT correspondence, one needs to add counterterms to the gravity action in order to get a finite action. These counterterms are made from the curvature invariants of the boundary metric with the coefficients of the higher curvature terms chosen so that power law divergences in the bulk are canceled for all possible boundary topologies permitted by the equations of motion. At any given dimension there are only a finite number of counterterms that do not vanish at infinity. This does not depend upon what the gravity theory is – i.e. whether or not it is Einstein, Gauss-Bonnet, 3rd order Lovelock, etc. Indeed, for asymptotically AdS solutions, the boundary counterterms that cancel the divergences in Einstein Gravity may also cancel the divergences in Lovelock gravity if one chooses the cosmological constant as in Eq. (4). This is due to the fact that the pth order Lovelock Lagrangian $\sqrt{\gamma}\mathcal{L}_p$ calculated for the metric (2) is independent of Lovelock coefficients. That is, the different orders of Lovelock action do not mix with each other and one may find the counterterms for different orders of Lovelock terms separately. This point makes the calculation easier. Of course, the coefficients of the various counterterms for different Lovelock terms will be different, depend only on L and will be independent of Lovelock coefficients. Thus, using the counterterms of Einstein gravity [25,26], we may write the counterterms of third order Lovelock gravity as

$$I_{\rm ct} = \sum_{p=1}^{3} I^{(p)}_{\rm ct}, \tag{11}$$

where

$$I^{(p)}_{\rm ct} = \frac{1}{8\pi}\int_{\partial M} dx^n\sqrt{-\gamma}\tilde{\alpha}_p\Big\{A_p + B_p\hat{R} + C_p\times\left(\hat{R}^{ab}\hat{R}_{ab} - \frac{n}{4(n-1)}\hat{R}^2\right) + D_p\left(\frac{3n+2}{4(n-1)}\hat{R}\hat{R}^{ab}\hat{R}_{ab} - \frac{n\,(n+2)}{16(n-1)^2}\hat{R}^3 - 2\hat{R}^{ab}\hat{R}^{cd}\hat{R}_{acbd} + \frac{n-2}{2(n-1)}\hat{R}^{ab}D_aD_b\hat{R} - \hat{R}^{ab}D^2\hat{R}_{ab} + \frac{1}{2(n-1)}\hat{R}D^2\hat{R}\right) + \cdots\Big\}. \tag{12}$$

The coefficients A_p, B_p, C_p and D_p in Eq. (12) should depend on L. These coefficients for Einstein gravity ($\tilde{\alpha}_p = 1$, $\tilde{\alpha}_2 = \tilde{\alpha}_3 = 0$) are [25,26]

$$A_1 = -\frac{(n-1)}{L},\quad B_1 = -\frac{L}{2(n-2)},\quad C_1 = -\frac{L^3}{2(n-2)^2(n-4)},\quad D_1 = \frac{L^5}{(n-2)^3(n-4)(n-6)} \tag{13}$$

One may note that the coefficients A_p, B_p, C_p and D_p are independent of Lovelock coefficients. We apply the counterterms (12) to various static solutions of Gauss-Bonnet and third order Lovelock gravity with different topolgy and find that these coefficients are

$$A_2 = -\frac{2}{3}A_1,\quad B_2 = B_1,\quad C_2 = -6C_1,\quad D_2 = -10D_1, \tag{14}$$

and

$$A_3 = \frac{6}{5}A_1,\quad B_3 = -B_1,\quad C_3 = -18C_1,\quad D_3 = 15D_1, \tag{15}$$

for Gauss-Bonnet and third order Lovelock gravity, respectively. The reason that we use exactly the counterterms of Einstein gravity is as follows. First, the boundary at r = const. for static solutions is a constant curvature hypersurface and therefore no six-derivative term will be appeared in the counterterms. In other words, all the terms of a specific order of counterterms [for example R^2 and $R_{ab}R^{ab}$ in $C_2(\hat{R}^{ab}\hat{R}_{ab} - n\hat{R}^2/[4(n-1)])$] for static solutions are proportional to r^{-4}. Also, $R_{abcd}R^{abcd}$ is proportional to r^{-4}. Therefore, in order to remove the divergences of the action which are proportional to r^{-4}, any combination of R^2, $R_{ab}R^{ab}$ and $R_{abcd}R^{abcd}$ can be used. So, we just use exactly the counterterms of Einstein gravity. Second, A_p, B_p, C_p, D_p, for $p = 2$ and 3 are proportional to those of Einstein counterterm independent of the dimensions. That is, the dimensional-

dependence of these coefficients are the same as those in Einstein gravity. Third, as we will see in the next section, the entropy calculated through the use of Gibbs-Duhem relation and the mass and action calculated by our counterterms is consistent with the entropy obtained by use of Wald's formula. Fourth, the mass calculated by our counterterms satisfies the first law of thermodynamics.

While the total action $I_{\rm bulk}+I_{\rm sur}+I_{\rm ct}$ is appropriate in grand-canonical ensemble where δA_μ is zero at the boundary, the appropriate action in the canonical ensemble where the electric charge is fixed is [56]

$$\tilde{I}=I_{\rm bulk}+I_{\rm sur}+I_{\rm ct}-\frac{1}{16\pi}\int_{\partial\mathcal{M}}d^nx\sqrt{-\gamma}n^\mu F_{\mu\nu}A^\nu. \tag{16}$$

Thus both in canonical and grand-canonical ensemble, the variation of total action about the solutions of the field equations is

$$\delta I=\delta\tilde{I}=\sum_{p=0}^{3}\left(\frac{\delta I_{\rm sur}^{(p)}}{\delta\gamma^{ab}}+\frac{\delta I_{\rm ct}^{(p)}}{\delta\gamma^{ab}}\right)\delta\gamma^{ab}. \tag{17}$$

So, the energy-momentum tensor can be written as:

$$T_{ab}=T_{ab}^{(\rm sur)}+T_{ab}^{(\rm ct)}=\frac{2}{\sqrt{-\gamma}}\sum_{p=0}^{3}\left(\frac{\delta I_{\rm sur}^{(p)}}{\delta\gamma^{ab}}+\frac{\delta I_{\rm ct}^{(p)}}{\delta\gamma^{ab}}\right). \tag{18}$$

The explicit expressions of $T_{ab}^{(\rm sur)}$ and $T_{ab}^{(\rm ct)}$ are somewhat cumbersome so we give them in the Appendix.

To compute the conserved charges of the spacetime, we choose a spacelike hypersurface $\mathcal{B}$ in $\partial\mathcal{M}$ with metric σ_{ij}, and write the boundary metric in ADM form:

$$\gamma_{ab}dx^adx^a=-N^2dt^2+\sigma_{ij}\left(d\varphi^i+V^idt\right)\left(d\varphi^j+V^jdt\right), \tag{19}$$

where the coordinates φ^i are the angular variables parameterizing the hypersurface of constant r around the origin, and N and V^i are the lapse and shift functions, respectively. When there is a Killing vector field ς on the boundary, then the quasilocal conserved quantities associated with the stress tensors of Eq. (18) can be written as

$$\mathcal{Q}(\varsigma)=\int_{\mathcal{B}}d^{n-1}\varphi\sqrt{\sigma}T_{ab}n^a\varsigma^b, \tag{20}$$

where σ is the determinant of the metric σ_{ij}, and n^a is the timelike unit normal vector to the boundary $\mathcal{B}$. In the context of counterterm method, the limit in which the boundary $\mathcal{B}$ becomes infinite ($\mathcal{B}_\infty$) is taken, and the counterterm prescription ensures that the action and conserved charges are finite. No embedding of the surface $\mathcal{B}$ into a reference of spacetime is required and the quantities which are computed are intrinsic to the spacetimes.

4 Thermodynamics of AdS charged black holes

The field equation of third order Lovelock gravity for the static metric

$$ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}+r^2d\Sigma_{k,n-1}^2$$

in the presence of electromagnetic field may be written as

$$\tilde{\alpha}_3\left(\left[\psi^3(r)-1\right]-\tilde{\alpha}_2\left[\psi^2(r)-1\right]+\left[\psi(r)-1\right]\right)+\frac{mL^2}{r^n}-\frac{2}{(n-1)(n-2)}\frac{q^2L^2}{r^{2(n-1)}}=0, \tag{21}$$

where $\psi(r)\equiv[f(r)-k]L^2/r^2$, q is the charge of the black hole and m is the integration constant which is related to the mass of the solution. The electromagnetic potential for the above metric is

$$A_\mu=-\frac{q}{(n-2)r^{n-2}}\delta_\mu^t.$$

The mass parameter m may be written in terms of horizon radius and q by using the fact that $\psi(r_+)=-kL^2/r_+^2$, and therefore

$$m=\left(1-\tilde{\alpha}_2+\tilde{\alpha}_3+k\frac{L^2}{r_+^2}\mu\right)\frac{r_+^n}{L^2}+\frac{2q^2}{(n-1)(n-2)r_+^{n-2}},$$

where

$$\mu=1+\tilde{\alpha}_2k\frac{L^2}{r_+^2}+\tilde{\alpha}_3k^2\frac{L^4}{r_+^4}. \tag{22}$$

As one expects, $\psi(r)=1$ ($f(r)=1+kr^2/L^2$) is the root of Eq. (21) for the vacuum spacetime ($m=0$ and $q=0$). The three solutions of the cubic Eq. (21) are

$$\psi_1(r)=\frac{\tilde{\alpha}_2}{3\tilde{\alpha}_3}+\delta+u\,\delta^{-1}, \tag{23}$$

$$\psi_2(r)=\frac{\tilde{\alpha}_2}{3\tilde{\alpha}_3}+-\frac{1}{2}\left(\delta+u\,\delta^{-1}\right)+i\frac{\sqrt{3}}{2}\left(\delta-u\,\delta^{-1}\right) \tag{24}$$

$$\psi_3(r)=\frac{\tilde{\alpha}_2}{3\tilde{\alpha}_3}-\frac{1}{2}\left(\delta+u\,\delta^{-1}\right)-i\frac{\sqrt{3}}{2}\left(\delta-u\,\delta^{-1}\right) \tag{25}$$

where

$$\delta=\left(v+\sqrt{v^2-u^3}\right)^{1/3},$$
$$u=\frac{\tilde{\alpha}_2^2-3\tilde{\alpha}_3}{9\tilde{\alpha}_3^2},$$

$$v = \frac{1}{2} + \frac{2\tilde{\alpha}_2^3 - 9\tilde{\alpha}_2\tilde{\alpha}_3 - 27\tilde{\alpha}_2\tilde{\alpha}_3^2}{54\tilde{\alpha}_3^3} - \frac{1}{2\tilde{\alpha}_3}\left(\frac{mL^2}{r^n} - \frac{2L^2q^2}{(n-1)(n-2)r^{2(n-1)}}\right).$$

All of the above three roots could be real in the appropriate range of $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$. The second and third solutions are real provided $u^3 > v^2$. Here, we will consider only the first solution $\psi_1(r)$ which is real provided $u^3 < v^2$ at any r.

Now, we investigate the thermodynamics of the black hole solutions. The temperature of the event horizon may be calculated through the use of analytic continuation of the metric. One obtains

$$T = \frac{1}{4\pi\eta r_+}\left(\left[1 - \tilde{\alpha}_2 + \tilde{\alpha}_3\right] n\frac{r_+^2}{L^2} + k\,(n\mu - 2\eta) - \frac{2q^2}{(n-1)r_+^{2(n-2)}}\right), \tag{26}$$

where

$$\eta = 1 + 2\tilde{\alpha}_2\frac{kL^2}{r_+^2} + 3\tilde{\alpha}_3\frac{k^2L^4}{r_+^4}. \tag{27}$$

The charge of the black holes per unit volume can be calculated by integrating the flux of the electric field as

$$Q = \frac{1}{4\pi}\int F d\Omega = \frac{1}{4\pi}q. \tag{28}$$

The electric potential Φ, measured at infinity with respect to the horizon is defined as

$$\Phi = A_\mu \chi^\mu|_{r\to\infty} - A_\mu\chi^\mu|_{r=r_+}. \tag{29}$$

where $\chi = \partial/\partial t$ is the null generator of the horizon. One obtains

$$\Phi = \frac{q}{(n-2)r_+^{n-2}}. \tag{30}$$

The finite total action in grand-canonical and canonical ensembles can be found through the use of the counterterm method introduced in the last section. It is a matter of straightforward calculations to show that the total action is finite. Since we are interested in the stability of the solutions in canonical ensemble, we calculate the finite Euclidian action per unit volume $V_{k,n-1}$ in this ensemble. One obtains

$$\tilde{I} = \frac{\beta}{16\pi}\left\{\left(\frac{\xi}{(n-1)\eta} + \frac{1}{(n-2)}\right)\frac{2q^2}{r_+^{n-2}} + (n-1)k\mu r_+^{n-2} - (1 - \tilde{\alpha}_2 + \tilde{\alpha}_3)\left[n\left(\frac{\xi}{\eta} - 1\right) + 1\right]\frac{r_+^n}{L^2} - k\,(n\mu - 2\eta)\frac{\xi}{\eta}r_+^{n-2}\right\} + I_0$$

where β is the Euclidean time period (the inverse of temperature), η is given in Eq. (27), ξ is

$$\xi = 1 + 2\tilde{\alpha}_2\frac{(n-1)kL^2}{(n-3)r_+^2} + 3\tilde{\alpha}_3\frac{(n-1)k^2L^4}{(n-5)r_+^4}, \tag{31}$$

and I_0 is

$$I_0 = \frac{\beta}{8\pi}\sum_{n/2=3,4,\ldots}(-k)^{(n/2)}\frac{((n-1)!!)^2L^{n-2}}{n!} \times\left\{1 + 2\frac{(n-1)}{(n-3)}\tilde{\alpha}_2 + 3\frac{(n-1)}{(n-5)}\tilde{\alpha}_3\right\}.$$

One should note that I_0 appears only in odd dimensions (even n).

For our static solution, there is a Killing vector field $\varsigma = \partial/\partial t$ on the boundary, and therefore the quasilocal conserved quantity associated with the stress tensors of Eq. (18) is the mass of the black hole. Using the counterterm method introduced in the last section, it is a matter of calculation to obtain the mass of black hole as

$$M = \frac{(n-1)m}{16\pi} + M_0 = \frac{(n-1)}{16\pi}\left\{\left(1 - \tilde{\alpha}_2 + \tilde{\alpha}_3 + k\frac{L^2}{r_+^2}\mu\right)\frac{r_+^n}{L^2} + \frac{16\pi^2Q^2}{(n-1)(n-2)r_+^{n-2}}\right\} + M_0, \tag{32}$$

where $M_0 = I_0/\beta$ is the Casimir energy for the vacuum AdS metric per volume $V_{k,n-1}$, which is nonzero only in odd dimensions. In order to have positive energy $M - M_0$, one should restrict the range of Lovelock coefficients in terms of Q, L and r_+ as

$$\tilde{\alpha}_3 > -\left(1 + k^3\frac{l^6}{r_+^6}\right)^{-1}\left\{1 + k\frac{l^2}{r_+^2} + \tilde{\alpha}_2\left(1 - k^2\frac{l^4}{r_+^4}\right) - \frac{2q^2L^2}{(n-1)(n-2)r_+^{2(n-1)}}\right\}. \tag{33}$$

That is, the third order Lovelock coefficient has a lower limit in terms of $\tilde{\alpha}_2$, q, L and r_+.

Using the mass and action per unit volume $V_{k,n-1}$ calculated through the use of counterterm method, one may calculate the entropy per unit volume through the use of Gibbs-Duhem relation $S = \beta M - \tilde{I}$ as

$$S = \xi\frac{r_+^{n-1}}{4}. \tag{34}$$

It is worth to note that the entropy of the black hole solution per unit volume $V_{k,n-1}$ calculated by the use of Gibbs-Duhem relation is consistent with the calculation through the use of Wald formula [54,57]

$$S = \frac{1}{4}\int d^{n-1}x\sqrt{\tilde{g}}\left(1 + kR + k^2\alpha_2\tilde{\mathcal{L}}_2\right), \tag{35}$$

where the integration is done on the $(n-1)$-dimensional spacelike hypersurface of Killing horizon, $\tilde{g}_{\mu\nu}$ is the induced metric on it, $\tilde{g}$ is the determinant of $\tilde{g}_{\mu\nu}$ and $\tilde{\mathcal{L}}_2$ is the 2nd order Lovelock Lagrangian of $\tilde{g}_{\mu\nu}$. Also, one may note that the thermodynamic quantities calculated in this section satisfy the first law of thermodynamics

$$dM = \frac{(\partial M/\partial r_+)_Q}{(\partial S/\partial r_+)}dS + \left(\frac{\partial M}{\partial Q}\right)_S dQ = TdS + \Phi dQ.$$

5 Stability in the canonical ensemble

Now, we study the thermal stability of the black hole solutions of third order Lovelock gravity in canonical ensemble. First, we investigate the conditions of having black hole solution in Lovelock gravity. The solution given by Eq. (23) presents a black hole solution provided $f(r)$ has at least one real positive root. This occurs if $r_+ \geq r_{\rm ext}$, or $q \leq q_{\rm ext}$ where $r_{\rm ext}$ and $q_{\rm ext}$ satisfy the following equation:

$$\left[1-\tilde{\alpha}_2+\tilde{\alpha}_3\right]n\frac{r_{\rm ext}^2}{L^2} + k\left[(n-2) + k(n-4)\frac{L^2}{r_{\rm ext}^2} + k^2(n-6)\frac{L^4}{r_{\rm ext}^4} - \frac{2q_{\rm ext}^2}{(n-1)r_{\rm ext}^{2(n-2)}}\right] = 0. \tag{36}$$

Also, the temperature of a physical black hole should be positive. One may note that the temperature changes its sign at the root of $\eta = 0$ ($r_{\rm crit}$). The radius $r_{\rm crit}$ depends on L and Lovelock coefficients, while $r_{\rm ext}$ depends on q too. In Fig. 1, the vertical line is $\eta = 0$ line. This figure shows that one may have only small ($\eta < 0$) and large black holes, and there is no medium black hole solution ($r_{\rm crit} < r_+ < r_{\rm ext}$). This feature does not occur for black holes of Einstein gravity or Lovelock gravity with positive Lovelock coefficients, since $\eta \neq 0$. Thus, for the case of Lovelock gravity with negative Lovelock coefficient(s) one may divide the black holes into two classes with negative and positive η.

In the canonical ensemble Q is fixed and therefore the black hole solutions are stable provided $(\partial^2 M/\partial S^2)_Q > 0$ in the range that T is positive. Using the expressions for mass and entropy given in Eqs. (32) and (34), one may calculate $(\partial^2 M/\partial S^2)_Q$:

$$\begin{aligned}\left(\frac{\partial^2 M}{\partial S^2}\right)_Q &= \left(\frac{\partial^2 M}{\partial r_+^2}\right)_Q\left(\frac{\partial S}{\partial r_+}\right)^{-2} - \left(\frac{\partial M}{\partial r_+}\right)_Q\left(\frac{\partial S}{\partial r_+}\right)^{-3}\left(\frac{\partial^2 S}{\partial r_+^2}\right)\\ &= \frac{(n-1)\sigma + 2q^2\gamma r_+^{-2n+8}}{(n-1)^2 r_+^{n+4}\eta^3}.\end{aligned} \tag{37}$$

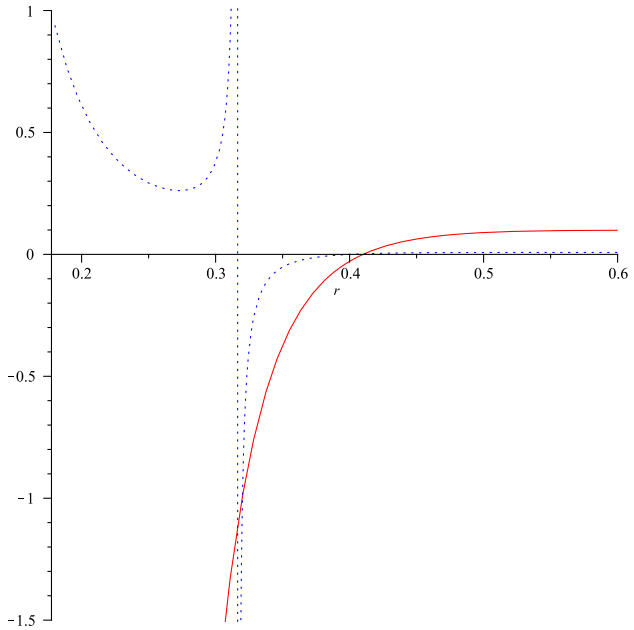

Fig. 1 $10^{-2}T$ versus r_+ for $q = 0.1$, $L = 1$, $k = 1$ and $n = 6$ in Einstein gravity (*solid*) and Lovelock gravity (*dotted*) with $\tilde{\alpha}_2 = 0.1$ and $\tilde{\alpha}_3 = -0.01$. The *vertical line* is $\eta = 0$ *line*

where

$$\begin{aligned}\gamma &= (2n-3) + 2k\tilde{\alpha}_2(2n-5)\frac{L^2}{r_+^2} + 3k^2\tilde{\alpha}_3(2n-7)\frac{L^4}{r_+^4},\\ \sigma &= nr_+^6\left(1-\tilde{\alpha}_2+\tilde{\alpha}_3\right)L^{-2}\\ &\quad + kr_+^4\left[2 + n\left(-1-6\tilde{\alpha}_2^2+6\tilde{\alpha}_2\left(1+\tilde{\alpha}_3\right)\right)\right]\\ &\quad - k^2r_+^2\left[n\left(\tilde{\alpha}_2\left(15\tilde{\alpha}_3+1\right)-15\tilde{\alpha}_3\left(1+\tilde{\alpha}_3\right)\right)-8\tilde{\alpha}_2\right]L^2\\ &\quad - 2k\left[n(\tilde{\alpha}_2^2-2\tilde{\alpha}_3)-4\tilde{\alpha}_2^2-6\tilde{\alpha}_3\right]L^4\\ &\quad - 3k^2\tilde{\alpha}_3\tilde{\alpha}_2 r_+^{-2}(n-8)L^6 - 3k\tilde{\alpha}_3^2 r_+^{-4}(n-6)L^8.\end{aligned}$$

To investigate the stability of black holes of third order Lovelock gravity in canonical ensemble, we should find the sign of $\left(\partial^2 M/\partial S^2\right)_Q$, when T is positive. In order to investigate the stability, we plot T and $\left(\partial^2 M/\partial S^2\right)_Q$ in one figure. The allowed region for investigation of $\left(\partial^2 M/\partial S^2\right)_Q$ is when T is positive. In Fig. 2, the vertical dotted-line is $\eta = 0$ line. This figure shows that the small black holes ($\eta < 0$) divided into stable and unstable black holes. That is a Hawking-Page phase transition exists for small black holes between very small and small black holes. There is no phase transition for large black holes ($\eta > 0$) as one may see in Fig. 2. For investigating the effect of third order term of Lovelock gravity on the stability of the black hole solutions, we plot T and $\left(\partial^2 M/\partial S^2\right)_Q$ versus $\tilde{\alpha}_3$ for small and large black holes. Figure 3 shows that there is no large black holes for large negative $\tilde{\alpha}_3$, while the large black holes for small negative $\tilde{\alpha}_3$ are not stable. As $\tilde{\alpha}_3$ becomes larger, large stable

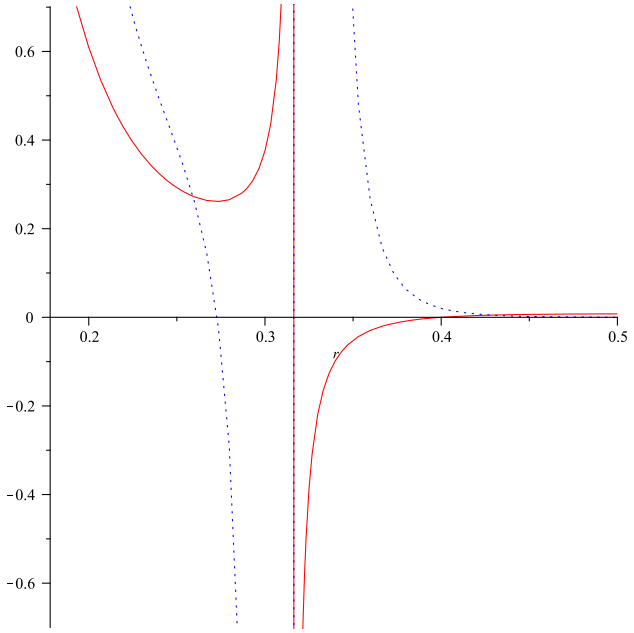

Fig. 2 $10^{-3}T$ (*dotted*) and $10^{-5}(\partial^2 M/\partial S^2)_Q$ (*solid*) versus r_+ for $\tilde{\alpha}_2 = 0.1, \tilde{\alpha}_3 = -0.01, q = 0.1, L = 1, k = 1$ and $n = 6$. The *vertical line* is $\eta = 0$ *line*

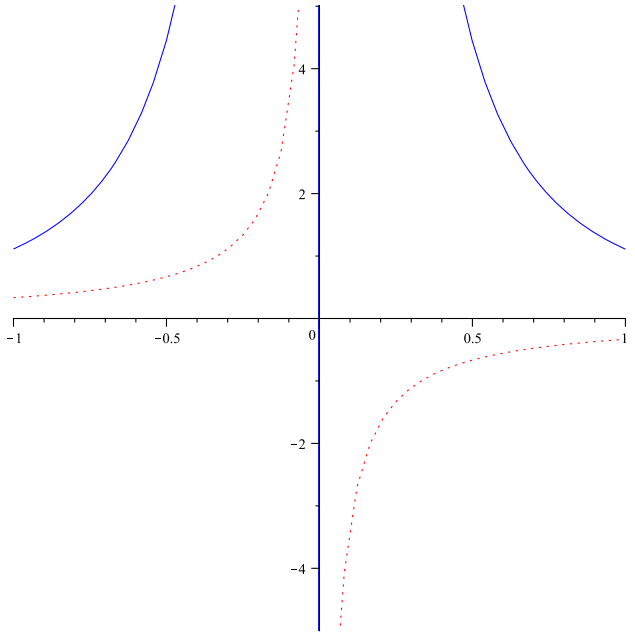

Fig. 4 $10^{-4}T$ (*dotted*) and $10^{-4}\left(\partial^2 M/\partial S^2\right)_Q$ (*solid*) versus $\tilde{\alpha}_3$ for $\tilde{\alpha}_2 = 0.1, q = 0.5, L = 1, k = 1, n = 6$ and $r_+ = 0.1$

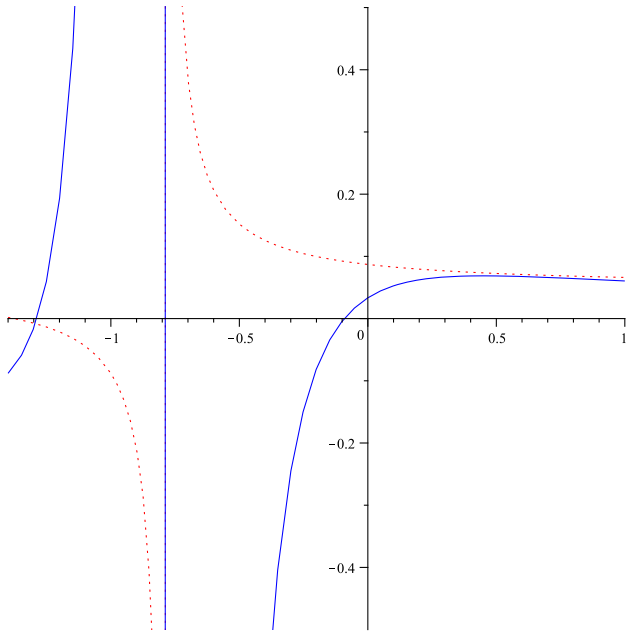

Fig. 3 $10^{-2}T$ (*dotted*) and $10^{-1}\left(\partial^2 M/\partial S^2\right)_Q$ (*solid*) versus $\tilde{\alpha}_3$ for $\tilde{\alpha}_2 = 0.1, q = 0.5, L = 1, k = 1, n = 6$ and $r_+ = 1.2$. The *vertical line* is $\eta = 0$ *line*

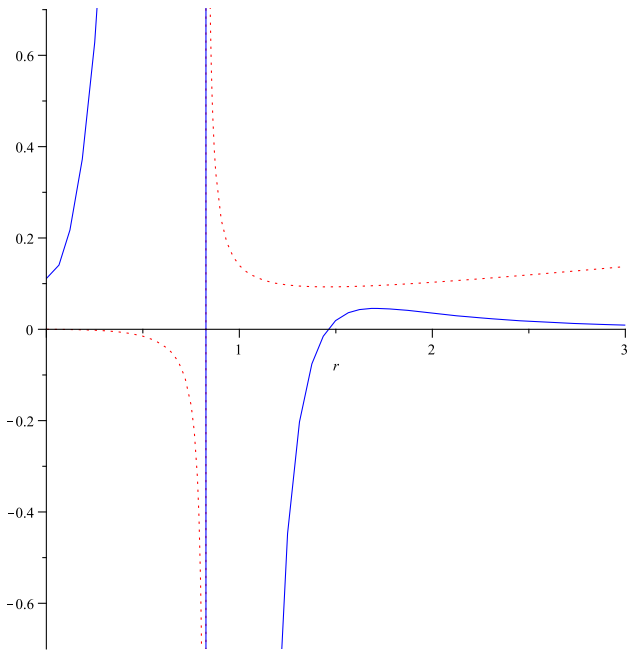

Fig. 5 $10^{-2}T$ (*dotted*) and $\left(\partial^2 M/\partial S^2\right)_Q$ (*solid*) versus r_+ for $\tilde{\alpha}_2 = 0.1, \tilde{\alpha}_3 = -0.2, q = 0, L = 1, k = 1$ and $n = 6$. The *vertical line* is $\eta = 0$ *line*

black holes may exist. Figure 4 shows that there is no small black hole solution for positive $\tilde{\alpha}_3$, while the small black hole with negative $\tilde{\alpha}_3$ are stable. Figure 5 shows that there is no negative-η (small) uncharged black hole with event horizon, while the large black holes ($\eta > 0$) show a phase transition between large and very large ones. One may see in Fig. 5 that the large black holes with positive η show a phase transition, while these black holes with negative η are not stable. Finally, Figs. 6 and 7 show the temperature and $\left(\partial^2 M/\partial S^2\right)_Q$ versus $\tilde{\alpha}_3$ for small and large uncharged black holes. Figure 6 shows

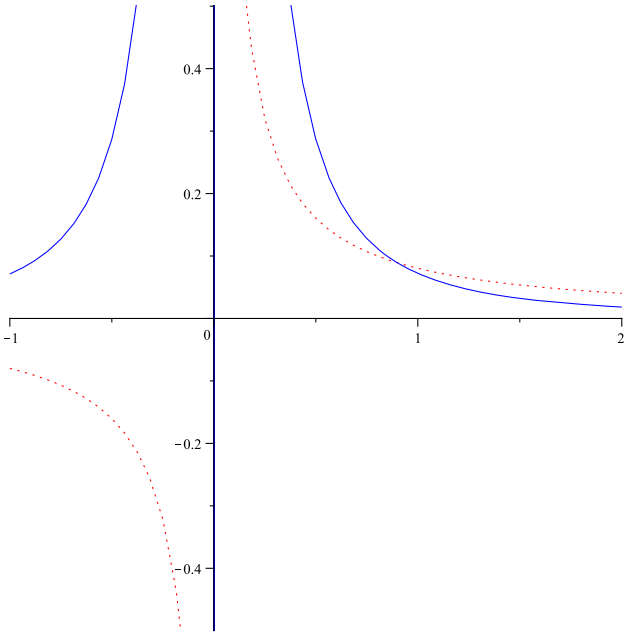

Fig. 6 $10T$ (*dotted*) and $10\left(\partial^2 M/\partial S^2\right)_Q$ (*solid*) versus $\tilde{\alpha}_3$ for $\tilde{\alpha}_2 = 0.1$, $q = 0.5$, $L = 1$, $k = 1$, $n = 6$ and $r_+ = 0.2$

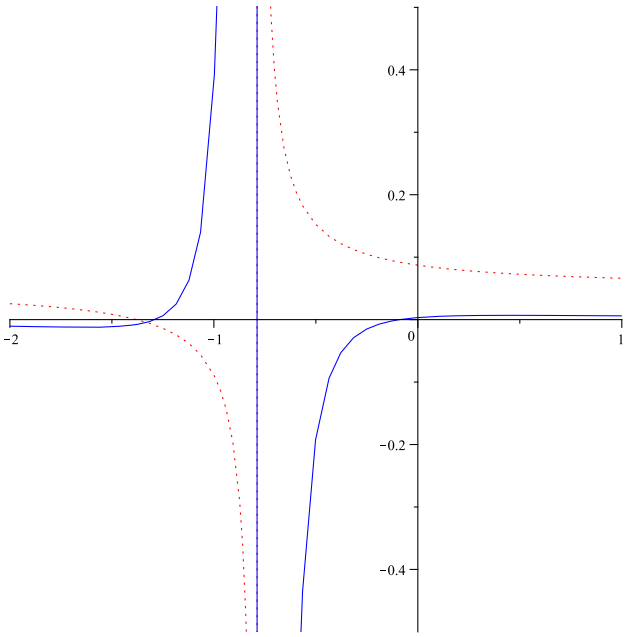

Fig. 7 $10^{-2}T$ (*dotted*) and $10^{-2}\left(\partial^2 M/\partial S^2\right)_Q$ (*solid*) versus $\tilde{\alpha}_3$ for $\tilde{\alpha}_2 = 0.1$, $q = 0.5$, $L = 1$, $k = 1$, $n = 6$ and $r_+ = 1.2$. The *vertical line* is $\eta = 0$ *line*

that there is no uncharged black hole with negative η, while the black holes with positive η are stable. On the other hand, the large uncharged black holes with positive η show a phase transition as $\tilde{\alpha}_3$ becomes larger and these solutions are not stable for very large negative $\tilde{\alpha}_3$.

6 Closing remarks

The concepts of action plays a central role in gravitation theories, but the sum of the bulk and the surface terms diverges. In this paper, we introduced the counterterms that remove the non-logarithmic divergences of static solutions of third order Lovelock gravity. We did this by defining the cosmological constant in such a way that the AdS metric (2) is the vacuum solution of Lovelock gravity. Indeed, the cosmological constant (4) makes the asymptotic form of the solutions of Lovelock gravity to be exactly the same as that of Einstein gravity. Thus, we employed the counterterms introduced for Einstein gravity in [25,26] and found that the power law divergences of static solutions in the action of Lovelock gravity can be removed by suitable choice of coefficients. We found that the counterterms are independent of Lovelock coefficients and the dimensionally dependent of them is the same as those of Einstein gravity. The main difference of our work with that of Ref. [16] is that the counterterms are exactly the same as those of Einstein gravity and do not depend on Lovelock coefficients. The only job which remains is that one needs to calculate the coefficients A_p, B_p, C_p and D_p . For example if one wants to remove the divergences of the action due to the Gauss-Bonnet term, one should calculate A_2, B_2, C_2 and D_2 that remove the divergences of $\sqrt{\gamma}\mathcal{L}_2$, without regarding the other Lovelock terms. This enables one to generalize this method to other higher curvature theories of gravity easily, including fourth order Lovelock gravity or $f(R)$ gravity. We also introduced the finite energy-momentum tensor in third order Lovelock gravity.

In addition, we employed these counterterms to calculate the finite action and mass of the static black hole solutions of third order Lovelock gravity. Calculating the temperature, the electric charge and electric potential and using the calculated finite action and mass, we showed that the entropy calculated through the use of Gibbs-Duhem relation is consistent with the calculated entropy by Wald's formula. We, also, showed that the conserved and thermodynamic quantities satisfy the first law of thermodynamics. Finally, we investigated the stability of charged black holes of Lovelock gravity in canonical ensemble. We found that the black holes with respect to the sign of η given in Eq. (27) may be divided into two classes. The negative η (small) black holes show a phase transition between very small and small black holes, while the large black holes are stable. There is no black hole solution with medium size. Of course by small black holes we mean the size of them with respect to the cosmological parameter L. We, also, investigated the effects of third order Lovelock term on the stability of the solutions. We found that there is no large black holes for large negative $\tilde{\alpha}_3$, while the black holes for small negative $\tilde{\alpha}_3$ are not stable. We also found that

as $\tilde{\alpha}_3$ becomes larger, stable black holes may exist. This shows that there is a phase transition as the third order term of Lovelock gravity becomes larger. Finally, we found that there is no small black hole solution for positive $\tilde{\alpha}_3$, while the black hole with negative $\tilde{\alpha}_3$ are stable. Finally, we considered the uncharged black holes of Lovelock gravity and found that negative η solutions are not black holes with event horizon, while the positive η black holes show a phase transition.

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7 Appendix

The first term of Eq. (18) which is the energy-momentum tensor of the surface term is

$$
\begin{aligned}
T^{(\mathrm{sur})}_{ab} = \frac{1}{8\pi}\{ & K_{ab} - K\gamma_{ab} + 2\alpha_2[3J_{ab} - J\gamma_{ab} \\
& - 2\hat{G}_{(a}{}^{c}K_{b)c} + 2\hat{R}_{ab}K - K_{ab}\hat{R} \\
& + 2K_{cd}\hat{G}^{cd}\gamma_{ab} - 2K^{cd}\hat{R}_{acbd}] \\
& + 3\alpha_3[5P_{ab} - P\gamma_{ab} + 2K\hat{G}^{(2)}_{ab} + \mathcal{L}_2(K_{ab} + K\gamma_{ab}) \\
& + 4J\hat{R}_{ab} - 24J_{(a}{}^{c}\hat{R}_{b)c} + 8K^{cd}\hat{R}_{ac}\hat{R}_{bd} \\
& - 8KK_{a}{}^{c}K_{b}{}^{d}\hat{R}_{cd} + 8KK_{ab}K^{cd}\hat{R}_{cd} \\
& - 8K^{cd}\hat{R}_{ab}\hat{R}_{cd} + 16K_{(a}{}^{c}\hat{R}_{b)}{}^{d}\hat{R}_{cd} \\
& + 16K_{(a}{}^{c}K_{b)}{}^{d}K_{d}{}^{e}\hat{R}_{ce} - 8K_{ab}K_{c}{}^{e}K^{cd}\hat{R}_{de} \\
& + 6J_{ab}\hat{R} - 8K_{(a}{}^{c}\hat{R}_{b)c}\hat{R} - 12J^{cd}\hat{R}_{acbd} \\
& - 4K^{cd}\hat{R}\hat{R}_{acbd} - 8K_{a}{}^{c}K_{bc}K^{de}\hat{R}_{de} \\
& + 16K_{c}{}^{d}K^{ef}K_{(a}{}^{c}\hat{R}_{b)edf} + 16K_{d}{}^{f}K^{de}K_{(a}{}^{c}\hat{R}_{b)ecf} \\
& + 16K_{(a}{}^{c}\hat{R}_{b)dce}\hat{R}^{de} - 16KK^{de}K_{(a}{}^{c}\hat{R}_{b)dce} \\
& - 16K^{cd}\hat{R}_{(a}{}^{e}\hat{R}_{b)cde} - 4K^{cd}\hat{R}_{ac}{}^{ef}\hat{R}_{bdef} \\
& + 8K^{cd}\hat{R}_{c}{}^{e}\hat{R}_{aebd} + 8K^{cd}\hat{R}_{c}{}^{e}\hat{R}_{adbe} \\
& - 8K^{cd}\hat{R}_{a}{}^{e}{}_{c}{}^{f}\hat{R}_{bedf} - 8K_{(a}{}^{c}\hat{R}_{b)}{}^{def}\hat{R}_{cdef} \\
& + 8K_{a}{}^{c}K_{b}{}^{d}K^{ef}\hat{R}_{cedf} - 4K_{ab}K^{cd}K^{ef}\hat{R}_{cedf} \\
& + 8K^{cd}\hat{R}_{a}{}^{e}{}_{b}{}^{f}\hat{R}_{cedf} + 2\gamma_{ab}(\hat{G}^{(2)}_{cd}K^{cd} \\
& + 6J^{cd}\hat{R}_{cd} - J\hat{R} + 2KK^{cd}K^{ef}\hat{R}_{cedf} \\
& - 4K_{c}{}^{e}K^{cd}K^{fh}\hat{R}_{dfeh})]\}
\end{aligned}
$$

and the second term corresponding to the counterterm is

$$
\begin{aligned}
T^{(\mathrm{ct})}_{ab} = & \frac{1}{8\pi}\sum_{p=1}^{3}\tilde{\alpha}_p\Big\{A_p\gamma_{ab} - 2B_p\Big(\hat{R}_{ab} - \frac{1}{2}\gamma\hat{R}\Big) \\
& - C_p\Big[-\gamma_{ab}\Big(\hat{R}^{cd}\hat{R}_{cd} - \frac{n}{4(n-1)}\hat{R}^2\Big) \\
& - \frac{n}{(n-1)}\hat{R}\hat{R}_{ab} - \frac{1}{(n-1)}\gamma_{ab}D^2\hat{R} + D^2\hat{R}_{ab} \\
& - \frac{(n-2)}{(n-1)}D_aD_b\hat{R} + 4\hat{R}_{acbd}\hat{R}^{cd}\Big] \\
& - D_p\Big[\frac{2(3n+2)}{4(n-1)}\Big[\Big(\hat{G}^{(1)}_{ab}\hat{R}^{cd}\hat{R}_{cd}\Big) \\
& - D_aD_b\Big(\hat{R}^{ef}\hat{R}_{ef}\Big) + \gamma_{ab}D^2\Big(\hat{R}^{ef}\hat{R}_{ef}\Big) + 2\hat{R}\hat{R}_{a}{}^{c}\hat{R}_{bc} \\
& + \gamma_{ab}D_cD_d\Big(\hat{R}\hat{R}^{cd}\Big) + D^2\Big(\hat{R}\hat{R}_{ab}\Big) - D^cD_b\Big(\hat{R}\hat{R}_{ac}\Big) \\
& - D^cD_a\Big(\hat{R}\hat{R}_{bc}\Big)\Big] - \frac{n(n+2)}{16(n-1)^2}\Big[-\gamma_{ab}\hat{R}^3 \\
& + 6\Big(\hat{R}^2\hat{R}_{ab} - D_aD_b\hat{R}^2 + \gamma_{ab}D^2\hat{R}^2\Big)\Big] \\
& - 2\Big[-\gamma_{ab}\hat{R}^{ef}\hat{R}^{cd}\hat{R}_{ecfd} \\
& + 3\Big(\hat{R}^{e}_{a}\hat{R}^{cd}\hat{R}_{ecbd} + \hat{R}^{e}_{b}\hat{R}^{cd}\hat{R}_{ecad}\Big) \\
& - 2D_cD_d\Big(\hat{R}_{ab}\hat{R}^{cd}\Big) + 2D_cD_d\Big(\hat{R}_{a}{}^{c}\hat{R}_{b}{}^{d}\Big) \\
& + 2\gamma_{ab}D_eD^f\Big(\hat{R}^{cd}\hat{R}^{e}_{cfd}\Big) + 2D^2\Big(\hat{R}^{cd}\hat{R}_{acbd}\Big) \\
& - 2D_eD_a\Big(\hat{R}^{cd}\hat{R}^{e}_{cbd}\Big) - 2D_eD_b\Big(\hat{R}^{cd}\hat{R}^{e}_{cad}\Big)\Big] \\
& + \frac{n-2}{2(n-1)}\Big[- D_a\hat{R}D_b\hat{R} + \hat{R}_b{}^{c}D_cD_a\hat{R} \\
& + \hat{R}_a{}^{c}D_cD_b(\hat{R}) - 2D_cD^cD_bD_a(\hat{R}) \\
& - \frac{7}{2}\hat{R}_{a}{}^{c}{}_{b}{}^{d}D_cDd(\hat{R}) + 2D_a(\hat{R}_{bc})D^c(\hat{R}) \\
& + D_b(\hat{R}_{ac})D^c(\hat{R}) + D_b\hat{R}_{ac}D^c\hat{R} - 2D_c\hat{R}_{ab}D^c\hat{R} \\
& - \hat{R}_b{}^{c}{}_{c}{}^{d}D_aD_d\hat{R} - \hat{R}_a{}^{c}{}_{c}{}^{d}D_dD_b\hat{R} \\
& - \hat{R}_a{}^{c}{}_{b}{}^{d}D_dD_c(\hat{R}) + 2\hat{R}_{ab}D_eD^e(\hat{R}) \\
& + \frac{1}{2}\gamma_{ab}(D_e\hat{R}D^e\hat{R} + D^4\hat{R})\Big] \\
& - \Big[2D_aD_b(D^2\hat{R}) - 2D^4\hat{R}_{ab} + 2D_p\hat{R}_{(a}\hat{R}^{p}_{b)} \\
& - 4\hat{R}^{pq}D_aD_b\hat{R}_{pq} + 12\,\hat{R}^{pq}D_{(b}\hat{R}_{pa)q} \\
& - 4(D^2\hat{R}^{pq})\hat{R}_{paqb} + 6D_p(\hat{R})D_{(b}\hat{R}^{p}_{a)} \\
& - 2D_a(\hat{R}^{pq})D_b(\hat{R}_{pq}) + 4D_{(a}D_q\Big(\hat{R}^{pq}\hat{R}_{b)p}\Big) \\
& + 16D^r(\hat{R}^{pq})D_{(b}\hat{R}_{|rqp|a)} - 4\,\hat{R}^{pq}\hat{R}_{pa}\hat{R}_{qb} \\
& + 8\hat{R}^{pr}\hat{R}^{q}_{r}\hat{R}_{paqb} + 4\,\hat{R}^{pq}\hat{R}^{r}_{(a}\hat{R}_{|rqp|b)}\Big] \\
& + \gamma_{ab}\Big[- D^4\hat{R} - 4D_pD_q(\hat{R})\hat{R}^{pq} + 2\hat{R}_{pq}D^2(\hat{R}^{pq}) \\
& + 4D_rD_s(\hat{R}_{pq})\hat{R}^{prqs} - D_e(\hat{R})D^e(\hat{R})
\end{aligned}
$$

$$+5\,D_r(\hat{R}_{pq})D^r(\hat{R}^{pq})-8\,D_r(\hat{R}_{pq})D^q(\hat{R}^{pr})$$
$$-4\,\hat{R}_{pq}\hat{R}^p_r\hat{R}^{qr}+4\,\hat{R}_{pq}\hat{R}_{rs}\hat{R}^{prqs}\Big]$$
$$+\frac{1}{n-1}\Big[2\,D_aD_b(D^2\hat{R})-2\,(D^2\hat{R})\hat{R}_{ab}$$
$$+D_a(\hat{R})D_b(\hat{R})+\gamma_{ab}\Big(-2\,D^4\hat{R}$$
$$-\frac{1}{2}\,D_c\hat{R}D^c\hat{R}\Big)\Big]\Big]\Big\}.$$

where A_p's, B_p's, C_p's and D_p's are given in Eqs. (13–15). The above counterterms remove the divergences of the energy-momentum tensor for $n \leq 8$. As in the case of Einstein gravity, one should add more counterterms for $n > 8$.

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