Wang et al. *Boundary Value Problems* (2016) 2016:156
DOI 10.1186/s13661-016-0664-x

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

An augmented Riesz decomposition method for sharp estimates of certain boundary value problem

Jiaofeng Wang[1], Bin Huang[2] and Nanjundan Yamini[3*]

*Correspondence:
nanjundan.yamini@gmail.com
[3]Department of Mathematics and NTIS, University of West Bohemia, Univerzitní 8, Plzeň, 306 14, Czech Republic
Full list of author information is available at the end of the article

Abstract
In this paper, by using an augmented Riesz decomposition method, we obtain sharp estimates of harmonic functions with certain boundary integral condition, which provide explicit lower bounds of functions harmonic in a cone. The results given here can be used as tools in the study of integral equations.

Keywords: Riesz decomposition method; boundary integral condition; harmonic function

1 Introduction

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space, where $n \geq 2$. Let $V = (X, y)$ be a point in $\mathbf{R}^n$, where $X = (x_1, x_2, \ldots, x_{n-1})$. Let E be a set in $\mathbf{R}^n$, the boundary and the closure of it are denoted by ∂E and $\overline{E}$, respectively.

For $P = (X, y) \in \mathbf{R}^n$, it can be re-expressed in spherical coordinates (l, Λ), $\Lambda = (\theta_1, \theta_2, \ldots, \theta_n)$ via the following transforms:

$$x_1 = l \prod_{j=1}^{n-1} \sin\theta_j \quad (n \geq 2), \qquad y = l\cos\theta_1$$

and, if $n \geq 3$,

$$x_{n-k+1} = l\cos\theta_k \prod_{j=1}^{k-1} \sin\theta_j \quad (2 \leq k \leq n-1),$$

where $0 \leq l < +\infty$, $0 \leq \theta_j \leq \pi$ $(1 \leq j \leq n-2;\, n \geq 3)$, and $-\frac{\pi}{2} \leq \theta_{n-1} \leq \frac{3\pi}{2}$ $(n \geq 2)$.

The unit sphere in $\mathbf{R}^n$ is denoted by $\mathbf{S}^{n-1}$. Let $\Gamma \subset \mathbf{S}^{n-1}$. A point $(1, \Lambda)$ on $\mathbf{S}^{n-1}$ and the set $\{\Lambda; (1, \Lambda) \in \Gamma\}$ are often identified with Λ and Γ, respectively. By $\Xi \times \Gamma$ we denote the set $\{(l, \Lambda) \in \mathbf{R}^n; l \in \Xi, (1, \Lambda) \in \Gamma\}$, where $\Xi \subset \mathbf{R}_+$. The set $\mathbf{R}_+ \times \Gamma$ is denoted by $\mathcal{T}_n(\Gamma)$, which is called a cone. We denote the sets $I \times \Gamma$ and $I \times \partial\Gamma$ by $\mathcal{T}_n(\Gamma; I)$ and $\mathcal{S}_n(\Gamma; I)$, respectively, where $I \subset \mathbf{R}$. The two sets $\mathcal{T}_n(\Gamma) \cap S_l$ and $\mathcal{S}_n(\Gamma; (0, +\infty))$ are denoted by $\mathcal{S}_n(\Gamma; l)$ and $\mathcal{S}_n(\Gamma)$, respectively.

If the Green's function on $\mathcal{T}_n(\Gamma)$ is denoted by $\mathrm{G}_\Gamma(V,W)$ $(P,Q\in\mathcal{T}_n(\Gamma))$, then the Poisson kernel on $\mathcal{T}_n(\Gamma)$ is defined by

$$c_n\,\mathrm{PI}_\Gamma(V,W)=\frac{\partial \mathrm{G}_\Gamma(V,W)}{\partial n_W},$$

where

$$c_n=\begin{cases}2\pi & \text{if } n=2,\\ (n-2)w_n & \text{if } n\geq 3,\end{cases}$$

and $\partial/\partial n_W$ denotes the differentiation at W along the inward normal into $\mathcal{T}_n(\Gamma)$.

Consider the boundary value problem (see [1])

$$\left(\Xi^*+\iota\right)\eta=0 \quad \text{on } \Gamma, \tag{1}$$

$$\eta=0 \quad \text{on } \partial\Gamma, \tag{2}$$

where Ξ^* is the spherical Laplace operator and Γ $(\subset \mathbf{S}^{n-1})$ has a twice smooth boundary. The least positive eigenvalue of (1) and (2) is denoted by ι. By $\eta(\Lambda)$ we denote the normalized eigenfunction corresponding to ι. Define

$$2\varrho^{\pm}=-n+2\pm\sqrt{(n-2)^2+4\iota},$$

$\varrho^+-\varrho^-$ will be denoted by λ.

We denote $f^+=\max\{f,0\}$ and $f^-=-\min\{f,0\}$, where f is a function defined on $\mathcal{T}_n(\Gamma)$. Throughout this paper, let A denote various constants independent of the variables in questions, which may be different from line to line. Let $\sigma(t)$ be a nondecreasing real valued function on $[1,+\infty)$ satisfying $\sigma(t)>\varrho^+$ for any $t\geq 1$.

In a recent paper, Li and Zhang (see [2], Theorem 1) solved boundary behavior problems for functions harmonic on $\mathcal{T}_n(\Gamma)$, which admit some lower bounds.

Theorem A *Let $h(V)$ be a harmonic function on $\mathcal{T}_n(\Gamma)$ and a continuous function on $\overline{\mathcal{T}_n(\Gamma)}$, where $V=(R,\Lambda)$. If*

$$h(V)\leq KR^{\sigma(R)}$$

for any $V=(R,\Lambda)\in\mathcal{T}_n(\Gamma;(1,+\infty))$ and

$$-K\leq h(V)$$

for any $V=(R,\Lambda)\in\overline{\mathcal{T}_n(\Gamma;(0,1])}$. Then we have

$$h(V)\geq -MK\left(1+\sigma(R)R^{\sigma(R)}\right)\eta^{1-n}(\Lambda),$$

where $V\in\mathcal{T}_n(\Gamma)$, K is a constant and M denotes a constant independent of R, K, and the two functions $h(V)$ and $\eta(\Lambda)$.

2 Main results

Now we state our main results in this paper.

By using a modified Carleman formula and an augmented Riesz decomposition method, we obtain sharper estimates of harmonic functions with certain boundary integral conditions. Compared with the original proof in [2], the new one is more easily applied.

Theorem 1 *Let $h(V)$ be a function harmonic on $\mathcal{T}_n(\Gamma)$ and continuous on $\overline{\mathcal{T}_n(\Gamma)}$, where $V = (R, \Lambda)$. Suppose that the two conditions* (I) *and* (II) *hold*:

(I) *For any $V = (R, \Lambda) \in \mathcal{T}_n(\Gamma; (1, \infty))$, we have*

$$\int_{\mathcal{S}_n(\Gamma;(1,R))} h^- t^{\varrho^-} \partial\eta/\partial n \, d\sigma_W \le MK\sigma(dR)R^{\sigma(dR)-\varrho^-} \tag{3}$$

and

$$\lambda \int_{\mathcal{S}_n(\Gamma;R)} h^+ R^{\varrho^- -1} \eta \, dS_R \le MKR^{\sigma(dR)-\varrho^-}. \tag{4}$$

(II) *For any $V = (R, \Lambda) \in \mathcal{T}_n(\Gamma; (0, 1])$, we have*

$$h(V) \ge -K. \tag{5}$$

Then

$$h(V) \ge -MK\bigl(1 + \sigma(dR)R^{\sigma(dR)}\bigr)\eta^{1-n}(\Lambda),$$

where $V \in \mathcal{T}_n(\Gamma)$, K is a constant, $0 < d \le 1$, and M denotes a constant independent of R, K, and the two functions $h(V)$ and $\eta(\Lambda)$.

Remark 1 By virtue of Theorem 1, we easily see that Theorem 1(I) is weaker than corresponding condition in Theorem A in the case $d \equiv 1$.

Theorem 2 *The conclusion of Theorem* 1 *remains valid if Theorem* 1(I) *is replaced by*

$$h(V) \le KR^{\sigma(dR)}, \quad V = (R, \Lambda) \in \mathcal{T}_n\bigl(\Gamma; (1, \infty)\bigr), \tag{6}$$

where $0 < d \le 1$.

Remark 2 In the case $d \equiv 1$, Theorem 2 reduces to Theorem A.

3 Lemmas

The following result is an augmented Riesz decomposition method, which was used to study the boundary behaviors of Poisson integral. For similar results for solutions of the equilibrium equations with angular velocity, we refer the reader to the paper by Wang *et al.* (see [3]).

Lemma 1 *For $W' \in \partial\mathcal{T}_n(\Gamma)$ and $\epsilon > 0$, there exist a positive number R and a neighborhood $B(W')$ of W' such that*

$$\frac{1}{c_n} \int_{S_n(\Gamma;(R,\infty))} \bigl|g(W)\bigr| \bigl|\mathrm{PI}_\Gamma(V, W)\bigr| \, d\sigma_W < \epsilon \tag{7}$$

for any $V = (r, \Lambda) \in \mathcal{T}_n(\Gamma) \cap B(W')$, *where* g *is an upper semi-continuous function. Then*

$$\limsup_{V \in \mathcal{T}_n(\Gamma), V \to W'} \mathrm{PI}_\Gamma[g](V) \le g(W'). \tag{8}$$

Proof Let $W' = (l', \Phi')$ be any point of $\partial\mathcal{T}_n(\Gamma)$ and ϵ (> 0) be any number. There exists a positive number R' satisfying

$$\frac{1}{c_n} \int_{S_n(\Gamma;(R',\infty))} \bigl|\mathrm{PI}_\Gamma(V, W)\bigr| \bigl|g(W)\bigr| \, d\sigma_W \le \frac{\epsilon}{4} \tag{9}$$

for any $V = (r, \Lambda) \in \mathcal{T}_n(\Gamma) \cap B(W')$ from (7).

Let ϕ be continuous on $\partial\mathcal{T}_n(\Gamma)$ such that $0 \le \phi \le 1$ and

$$\phi = \begin{cases} 1 & \text{on } S_n(\Gamma;(0,R']) \cup \{O\}, \\ 0 & \text{on } S_n(\Gamma;(2R',\infty)). \end{cases}$$

Let $\mathrm{G}_{\mathcal{T}_n(\Gamma;(0,j))}$ be a Green's function on $\mathcal{T}_n(\Gamma;(0,j))$, where j is a positive integer. Since $\Gamma_j(V, W) = \mathrm{G}_{\mathcal{T}_n(\Gamma)}(V, W) - \mathrm{G}_{\mathcal{T}_n(\Gamma;(0,j))}(V, W)$ on $\mathcal{T}_n(\Gamma;(0,j))$ converges monotonically to 0 as $j \to \infty$. Then we can find an integer j', $j' > 2R'$ such that

$$\frac{1}{c_n} \int_{S_n(\Gamma;(0,2R'))} \left|\frac{\partial}{\partial n_W} \Gamma_{j'}(V, W)\right| \bigl|\phi(W)g(W)\bigr| \, d\sigma_W < \frac{\epsilon}{4} \tag{10}$$

for any $V = (r, \Lambda) \in \mathcal{T}_n(\Gamma) \cap B(W')$.

Then we have from (9) and (10)

$$\begin{aligned} &\frac{1}{c_n} \int_{\partial\mathcal{T}_n(\Gamma)} \mathrm{PI}_\Gamma(V, W) g(W) \, d\sigma_W \\ &\quad\le \frac{1}{c_n} \int_{S_n(\Gamma;(0,2R'))} \frac{\partial \mathrm{G}_{\mathcal{T}_n(\Gamma;(0,j'))}(V, W)}{\partial n_W} \phi(W) g(W) \, d\sigma_W \\ &\qquad + \frac{1}{c_n} \int_{S_n(\Gamma;(0,2R'))} \bigl|\phi(W)g(W)\bigr| \left|\frac{\partial \Gamma_{j'}(V, W)}{\partial n_W}\right| d\sigma_W \\ &\qquad + \frac{2}{c_n} \int_{S_n(\Gamma;(R',\infty))} \bigl|\mathrm{PI}_\Gamma(V, W)\bigr| \bigl|g(W)\bigr| \, d\sigma_W \\ &\quad\le \frac{1}{c_n} \int_{S_n(\Gamma;(0,2R'))} \frac{\partial \mathrm{G}_{\mathcal{T}_n(\Gamma;(0,j'))}(V, W)}{\partial n_W} \phi(W) g(W) \, d\sigma_W + \frac{3}{4}\epsilon \end{aligned} \tag{11}$$

for any $V = (r, \Lambda) \in \mathcal{T}_n(\Gamma) \cap B(W')$.

Consider an upper semi-continuous function

$$\eta(W) = \begin{cases} \phi(W)g(W) & \text{on } S_n(\Gamma;(0,2R']) \cup \{O\}, \\ 0 & \text{on } \partial\mathcal{T}_n(\Gamma;(0,j')) - S_n(\Gamma;(0,2R']) - \{O\}, \end{cases}$$

on $\partial\mathcal{T}_n(\Gamma;[0,j'))$ and denote the PWB solution of the Dirichlet problem on $\mathcal{T}_n(\Gamma;(0,j'))$ by $H_\eta(P;\mathcal{T}_n(\Gamma;(0,j')))$ (see, *e.g.*, [4]); we know that

$$\frac{1}{c_n} \int_{S_n(\Gamma;(0,2R'))} \frac{\partial \mathrm{G}_{\mathcal{T}_n(\Gamma;(0,j'))}(V, W)}{\partial n_W} \phi(W) g(W) \, d\sigma_W = H_\eta\bigl(P;\mathcal{T}_n\bigl(\Gamma;\bigl(0,j'\bigr)\bigr)\bigr) \tag{12}$$

(see [5], Theorem 3). If $\mathcal{T}_n(\Gamma;(0,j'))$ is not a Lipschitz domain at O, we can prove (12) by considering a sequence of the Lipschitz domains $\mathcal{T}_n(\Gamma;(\frac{1}{m},j'))$ which converges to $\mathcal{T}_n(\Gamma;(0,j'))$ as $m\to\infty$. We also have

$$\limsup_{V\in\mathcal{T}_n(\Gamma),V\to W'} H_\eta\bigl(P;\mathcal{T}_n\bigl(\Gamma;\bigl(0,j'\bigr)\bigr)\bigr)\le \limsup_{Q\in S_n(\Gamma),Q\to W'} \eta(W)=g\bigl(W'\bigr)$$

(see, *e.g.*, [4], Lemma 8.20). Hence we know that

$$\limsup_{V\in\mathcal{T}_n(\Gamma),V\to W'} \frac{1}{c_n}\int_{S_n(\Gamma;(0,2R'))}\phi(W)\frac{\partial \mathrm{G}_{\mathcal{T}_n(\Gamma;(0,j'))}(V,W)}{\partial n_W}g(W)\,d\sigma\le g\bigl(W'\bigr).$$

With (11) this gives (8). □

The following growth properties play important roles in our discussions.

Lemma 2 (see [6]) *Let $V=(r,\Lambda)\in\mathcal{T}_n(\Gamma)$ and $W=(t,\Phi)\in S_n(\Gamma)$. Then we have*

$$\mathrm{PI}_\Gamma(V,W)\le Mr^{\varrho^-}t^{\varrho^+-1}\eta(\Lambda)\quad\left(0<\frac{t}{r}\le\frac{4}{5}\right)$$

and

$$\mathrm{PI}_\Gamma(V,W)\le Mr^{\varrho^+}t^{\varrho^--1}\eta(\Lambda)\quad\left(0<\frac{r}{t}\le\frac{4}{5}\right).$$

Let $V=(r,\Lambda)\in\mathcal{T}_n(\Gamma)$ and $W=(t,\Phi)\in S_n(\Gamma;(\frac{4}{5}r,\frac{5}{4}r))$. Then we have

$$\mathrm{PI}_\Gamma(V,W)\le M\frac{\eta(\Lambda)}{t^{n-1}}+M\frac{r\eta(\Lambda)}{|P-Q|^n}.$$

Let $\mathrm{G}_{\mathcal{T}_n(\Gamma;(t_1,t_2))}$ be the Green's function of $\mathcal{T}_n(\Gamma;(t_1,t_2))$. Then we have

$$\frac{\partial \mathrm{G}_{\mathcal{T}_n(\Gamma;(t_1,t_2))}((t_1,\Phi),(r,\Lambda))}{\partial t}\le M\left(\frac{t_1}{r}\right)^{-\varrho^-}\frac{\eta(\Phi)\eta(\Lambda)}{t_1^{n-1}}$$

and

$$-M\left(\frac{r}{t_2}\right)^{\varrho^+}\frac{\eta(\Phi)\eta(\Lambda)}{t_2^{n-1}}\le\frac{\partial \mathrm{G}_{\mathcal{T}_n(\Gamma;(t_1,t_2))}((t_2,\Phi),(r,\Lambda))}{\partial t},$$

where $0<2t_1<r<\frac{1}{2}t_2<+\infty$.

Many previous studies (see [7, 8]) focused on the following lemma with respect to the half space and its applications.

Lemma 3 (see [2], Lemma 2) *If h is a function harmonic in a domain containing $\mathcal{T}_n(\Gamma;(1,R))$, where $R>1$, then we have*

$$\lambda\int_{S_n(\Gamma;R)}h\eta R^{\varrho^--1}dS_R+\int_{S_n(\Gamma;(1,R))}h\bigl(t^{\varrho^-}-t^{\varrho^+}R^{-\lambda}\bigr)\partial\eta/\partial n\,d\sigma_W+d_1+\frac{d_2}{R^\lambda}=0,$$

where

$$d_1 = \int_{S_n(\Gamma;1)} \varrho^- h\eta - \eta(\partial h/\partial n)\, dS_1$$

and

$$d_2 = \int_{S_n(\Gamma;1)} \eta(\partial h/\partial n) - \varrho^+ h\eta\, dS_1.$$

4 Proof of Theorem 1

By Lemma 1 we have

$$-h(V) = \int_{\mathcal{S}_n(\Gamma;(0,R))} \bigl(-h(W)\bigr) \mathrm{PI}_\Gamma(V,W)\, d\sigma_W + \int_{\mathcal{S}_n(\Gamma;R)} \bigl(-h(W)\bigr) \frac{\partial \mathrm{G}_{\Gamma,R}(V,W)}{\partial R}\, dS_R \tag{13}$$

for any $V = (l,\Lambda) \in \mathcal{T}_n(\Gamma;(0,R))$.

Case 1. $V = (l,\Lambda) \in \mathcal{T}_n(\Gamma;(\frac{5}{4},\infty))$ and $R = \frac{5}{4}l$.

From (13) we know that

$$-h = \sum_{i=1}^{4} \mathfrak{U}_i, \tag{14}$$

where

$$\mathfrak{U}_1(V) = \int_{\mathcal{S}_n(\Gamma;(0,1])} \bigl(-h(W)\bigr) \mathrm{PI}_\Gamma(V,W)\, d\sigma_W,$$

$$\mathfrak{U}_2(V) = \int_{\mathrm{S}_n(\Gamma;(1,\frac{4}{5}l])} \bigl(-h(W)\bigr) \mathrm{PI}_\Gamma(V,W)\, d\sigma_W,$$

$$\mathfrak{U}_3(V) = \int_{\mathcal{S}_n(\Gamma;(\frac{4}{5}l,R))} \bigl(-h(W)\bigr) \mathrm{PI}_\Gamma(V,W)\, d\sigma_W,$$

and

$$\mathfrak{U}_4(V) = \int_{\mathcal{S}_n(\Gamma;R)} \bigl(-h(W)\bigr) \mathrm{PI}_\Gamma(V,W)\, d\sigma_W.$$

We obtain from Lemma 2

$$\mathfrak{U}_1(V) \le MK\eta(\Lambda) \tag{15}$$

and

$$\mathfrak{U}_2(V) \le MK\sigma(dR)R^{\sigma(dR)}\eta(\Lambda). \tag{16}$$

Put

$$\mathfrak{U}_3(V) \le \mathfrak{U}_{31}(V) + \mathfrak{U}_{32}(V), \tag{17}$$

where

$$\mathfrak{U}_{31}(V) = M\int_{\mathcal{S}_n(\Gamma;(\frac{4}{5}l,R))}\bigl(-h(W)\bigr)t^{1-n}\eta(\Lambda)\frac{\partial\phi(\Phi)}{\partial n_\Phi}\,d\sigma_W$$

and

$$\mathfrak{U}_{32}(V) = Mr\eta(\Lambda)\int_{\mathcal{S}_n(\Gamma;(\frac{4}{5}l,R))}\bigl(-h(W)\bigr)|V-W|^{-n}l\eta(\Lambda)\frac{\partial\phi(\Phi)}{\partial n_\Phi}\,d\sigma_W.$$

From (3) we obtain

$$\mathfrak{U}_{31}(V) \le MK\sigma(dR)R^{\sigma(dR)}\eta(\Lambda). \tag{18}$$

To estimate $\mathfrak{U}_{32}(V)$. There exists a sufficiently small number d satisfying $d > 0$ and

$$\mathcal{S}_n\left(\Gamma;\left(\frac{4}{5}l,R\right)\right) \subset B\left(V,\frac{l}{2}\right)$$

for $V = (l,\Lambda) \in \Pi(d)$, where

$$\Pi(d) = \left\{Q = (r,\Lambda) \in \mathcal{T}_n(\Gamma); \inf_{(1,z)\in\partial\Gamma}\bigl|(1,\Lambda)-(1,z)\bigr| < d, 0 < l < \infty\right\}.$$

We divide $\mathcal{T}_n(\Gamma)$ into the two sets $\Pi(d)$ and $\mathcal{T}_n(\Gamma) - \Pi(d)$.

For any $V = (l,\Lambda) \in \mathcal{T}_n(\Gamma) - \Pi(d)$, we can find a number d' satisfying $d' > 0$ and

$$d'l \le |V-W|$$

for $W \in \mathcal{S}_n(\Gamma)$, and hence

$$\mathfrak{U}_{32}(V) \le MK\sigma(dR)R^{\sigma(dR)}\eta(\Lambda). \tag{19}$$

If $V = (l,\Lambda) \in \Pi(d)$, then we have

$$H_i(V) = \left\{W \in \mathcal{S}_n\left(\Gamma;\left(\frac{4}{5}l,R\right)\right); 2^{i-1}\xi(V) \le |V-W| < 2^i\xi(V)\right\},$$

where

$$\xi(V) = \inf_{W\in\partial\mathcal{T}_n(\Gamma)}|V-W|.$$

Since $\{W \in \mathbf{R}^n : |V-W| < \xi(V)\} \cap \mathcal{S}_n(\Gamma) = \varnothing$, we get

$$\mathfrak{U}_{32}(V) = M\sum_{i=1}^{i(V)}\int_{H_i(V)}\frac{(-h(W))r\eta(\Lambda)}{|V-W|^n}\frac{\partial\eta(\Phi)}{\partial n_\Phi}\,d\sigma_W,$$

where $l(P)$ is an integer such that $2^{l(P)}\xi(V) \le r < 2^{l(P)+1}\xi(V)$.

Since

$$\eta(\Lambda) \le Ml^{-1}\xi(V),$$

where $V = (l, \Lambda) \in \mathcal{T}_n(\Gamma)$, we have

$$\int_{H_i(V)} \bigl(-h(W)\bigr)|V-W|^{-n} r\eta(\Lambda)\frac{\partial\eta(\Phi)}{\partial n_\Phi}\,d\sigma_W \le MK\sigma(dR)R^{\sigma(dR)}\eta^{1-n}(\Lambda),$$

where $l = 0, 1, 2, \ldots, l(P)$.

Thus

$$\mathfrak{U}_{32}(V) \le MK\sigma(dR)R^{\sigma(dR)}\eta^{1-n}(\Lambda). \tag{20}$$

We see that

$$\mathfrak{U}_{3}(V) \le MK\sigma(dR)R^{\sigma(dR)}\eta^{1-n}(\Lambda) \tag{21}$$

from (17), (18), (19), and (20).

On the other hand, we have from (4)

$$\mathfrak{U}_{4}(V) \le MKR^{\sigma(dR)}\eta(\Lambda). \tag{22}$$

We thus obtain (15), (16), (21), and (22) that

$$-h(V) \le MK\bigl(1+\sigma(dR)R^{\sigma(dR)}\bigr)\eta^{1-n}(\Lambda). \tag{23}$$

Case 2. $V = (l, \Lambda) \in \mathcal{T}_n(\Gamma; (\frac{4}{5}, \frac{5}{4}])$ and $R = \frac{5}{4}l$.

It follows from (13) that

$$-h = \mathfrak{U}_1 + \mathfrak{U}_5 + \mathfrak{U}_4,$$

where $\mathfrak{U}_1(V)$ and $\mathfrak{U}_4(V)$ were defined in the former case and

$$\mathfrak{U}_5(V) = \int_{\mathcal{S}_n(\Gamma;(1,R))} \bigl(-h(W)\bigr)\,\mathrm{PI}_\Gamma(V,W)\,d\sigma_W.$$

Similarly, we have

$$\mathfrak{U}_{5}(V) \le MK\sigma(dR)R^{\sigma(dR)}\eta^{1-n}(\Lambda),$$

which, together with (15) and (22), gives (23).

Case 3. $V = (l, \Lambda) \in \mathcal{T}_n(\Gamma; (0, \frac{4}{5}])$.

It is evident from (5) that we have

$$-h \le K,$$

from which one also obtains (23).

We finally have

$$h(V) \geq -KM\big(1 + \sigma(dR)R^{\sigma(dR)}\big)\eta^{1-n}(\Lambda)$$

from (23), which is required.

5 Proof of Theorem 2

By applying Lemma 3 to $h = h^+ - h^-$, we obtain

$$\begin{aligned}&\lambda \int_{\mathcal{S}_n(\Gamma;R)} h^+ R^{\varrho^- -1}\eta\, dS_R + \int_{\mathcal{S}_n(\Gamma;(1,R))} h^+\big(t^{\varrho^-} - t^{\varrho^+}R^{-\lambda}\big)\partial\eta/\partial n\, d\sigma_W + d_1 + d_2R^{-\lambda}\\ &\quad = \lambda \int_{\mathcal{S}_n(\Gamma;R)} h^- R^{\varrho^- -1}\eta\, dS_R + \int_{\mathcal{S}_n(\Gamma;(1,R))} h^-\big(t^{\varrho^-} - t^{\varrho^+}R^{-\lambda}\big)\partial\eta/\partial n\, d\sigma_W. \end{aligned} \tag{24}$$

From (6) we see that

$$\lambda \int_{\mathcal{S}_n(\Gamma;R)} h^+ R^{\varrho^- -1}\eta\, dS_R \leq MKR^{\sigma(dR)-\varrho^+} \tag{25}$$

and

$$\int_{\mathcal{S}_n(\Gamma;(1,R))} h^+\big(t^{\varrho^-} - t^{\varrho^+}R^{-\lambda}\big)\partial\eta/\partial n\, d\sigma_W \leq MKR^{\sigma(dR)-\varrho^+}. \tag{26}$$

Notice that

$$d_1 + d_2R^{-\lambda} \leq MKR^{\sigma(dR)-\varrho^+}. \tag{27}$$

We have from (24), (25), (26), and (27)

$$\int_{\mathcal{S}_n(\Gamma;(1,R))} h^-\big(t^{\varrho^-} - t^{\varrho^+}R^{-\lambda}\big)\partial\eta/\partial n\, d\sigma_W \leq MKR^{\sigma(dR)-\varrho^+}. \tag{28}$$

Hence (28) gives (6), which, together Theorem 1, gives the conclusion of Theorem 2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
NY participated in the design and theoretical analysis of the study and drafted the manuscript. JW conceived of the study and participated in its design and coordination. BH participated in the design and the revision of the study. All authors read and approved the final manuscript.

Author details
[1]Mathematical and Statistical Sciences, Vocational and Technical College, Quzhou, 324000, China. [2]Mathematical and Statistical Sciences, Quzhou University, Quzhou, 324000, China. [3]Department of Mathematics and NTIS, University of West Bohemia, Univerzitní 8, Plzeň, 306 14, Czech Republic.

Acknowledgements
This work was partially supported by a joint exchange program between the Czech Republic and Germany: by the Ministry of Education, Youth, and Sports of the Czech Republic under Grant No. 9AMB49DE002 (exchange program 'Mobility') and by the Federal Ministry of Education and Research of Germany under Grant No. 29051322 (DAAD Program 'PPP'). Meanwhile, we wish to express our genuine thanks to the anonymous referees for careful reading and excellent comments on this manuscript.

Received: 14 May 2016 Accepted: 10 August 2016 Published online: 24 August 2016

References

1. Carleman, T: Über die Approximation analytischer Funktionen durch lineare Aggregate von vorgegebenen Potenzen. Ark. Mat. Astron. Fys. **17**, 1-30 (1923)
2. Li, Z, Zhang, B: A convergence theorem for sums of dependent Hilbert space valued triangular arrays. Stat. Probab. Lett. **19**(3), 177-179 (1994)
3. Wang, J, Pu, J, Huang, B, Shi, G: Boundary value behaviors for solutions of the equilibrium equations with angular velocity. Bound. Value Probl. **2015**, 230 (2015)
4. Helms, LL: Introduction to Potential Theory. Wiley-Interscience, New York (1969)
5. Dahlberg, BEJ: Estimates of harmonic measure. Arch. Ration. Mech. Anal. **65**, 275-288 (1977)
6. Azarin, V: Generalization of a theorem of Hayman on subharmonic functions in an m-dimensional cone. Am. Math. Soc. Transl. (2) **80**, 119-138 (1969)
7. Su, B: Growth properties of harmonic functions in the upper half-space. Acta Math. Sin. **55**(6), 1095-1100 (2012)
8. Yang, P, Chen, Y, Li, Z: ADM-Padé technique for the nonlinear lattice equations. Appl. Math. Comput. **210**(2), 362-375 (2009)