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Formulae for the analysis of the flavor-tagged decay $B_s^0 \to J/\psi\phi$

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Abstract: Differential rates in the decay $B_s^0 \to J/\psi\phi$ with $\phi \to K^+K^-$ and $J/\psi \to \mu^+\mu^-$ are sensitive to the CP-violation phase $\beta_s = \arg\left((-V_{ts}V_{tb}^*)/(V_{cs}V_{cb}^*)\right)$, predicted to be very small in the standard model. The analysis of $B_s^0 \to J/\psi\phi$ decays is also suitable for measuring the B_s^0 lifetime, the decay width difference $\Delta\Gamma_s$ between the B_s^0 mass eigenstates, and the B_s^0 oscillation frequency Δm even if appreciable CP violation does not occur. In this paper we present normalized probability densities useful in maximum likelihood fits, extended to allow for S-wave K^+K^- contributions on one hand and for direct CP violation on the other. Our treatment of the S-wave contributions includes the strong variation of the S-wave/P-wave amplitude ratio with $m(K^+K^-)$ across the ϕ resonance, which was not considered in previous work. We include a scheme for re-normalizing the probability densities after detector sculpting of the angular distributions of the final state particles, and conclude with an examination of the symmetries of the rate formulae, with and without an S-wave K^+K^- contribution. All results are obtained with the use of a new compact formalism describing the differential decay rate of B_s^0 mesons into $J/\psi\phi$ final states.

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Contents

1 Introduction

The decay of $B_s^0 \to J/\psi\phi$, a transition of a pseudoscalar into two vector mesons can be thought of as six independent decays. The initial B_s^0 system consists of a heavy and a light mass eigenstate, and the $J/\psi\phi$ system to which it decays is characterized by three distinct orbital angular momentum states. A maximum amount of information about this system can be obtained from analyses which disentangle the two initial states and the three final states. The experimental technique of *flavor tagging* infers a meson's flavor at production time as B_s^0 or $\bar{B}_s^0$ and is the key to disentangling the two initial states. Flavor-tagged $B_s^0 \to J/\psi\phi$ decays are of great interest in particle physics because of their sensitivity to the CKM phases [1] and to anomalous mixing phases from physics beyond the standard model [2]. Recently the CDF and D0 collaborations have constrained the CKM phases in both untagged analyses [3, 4], and flavor-tagged analyses [5, 6]. These analyses are based on complete differential rates for the decay given in ref. [7]. They use the angular distributions of the decay products to disentangle the three final states.

In this paper we re-express the differential decay rates in ref. [7], using a new formalism that makes explicit a number of symmetries that are otherwise hidden. These formulae are then extended to the case in which the final state in the decay $B_s^0 \to J/\psi\phi$ includes decays of type $B_s^0 \to J/\psi K^+K^-$ (kaons in an S-wave state), which has been suggested [8] to be

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an important effect. After including the S-wave contribution in the theoretical description, we identify the symmetries of the modified formulae.

In addition to S-wave effects, we also investigate other aspects of the differential decay rate formulae. We include the effects of possible direct CP violation. In addition we show how interference between CP odd and CP even $J/\psi\phi$ final states effectively tags the flavor of the B_s^0 meson at decay, allowing for the possibility to observe $B_s^0 \to \bar{B}_s^0$ flavor oscillations in a flavor-tagged analysis, even in the absence of CP violation effects.

Experimentally, the differential rate formulae are used to construct likelihood functions based on normalized probability density functions (PDFs). In this paper we include normalization constants where appropriate in all expressions for transition amplitudes and PDFs. Detector angular acceptance is an important effect which must be included in these probability densities. However, the inclusion of this effect disturbs the normalization of the PDF. We present a scheme for normalizing the probability density analytically, as required for unbinned maximum likelihood fits.

2 Phenomenology of the $B_s^0 \to J/\psi\phi$ decay

We first summarize the phenomenology of the B_s^0 system and the decay $B_s^0 \to J/\psi\phi \to \mu^+\mu^- K^+K^-$. Two flavor eigenstates, $|B_s^0\rangle$ and $|\bar{B}_s^0\rangle$, mix via the weak interaction. The two mass eigenstates

$$|B_s^H\rangle = p\,|B_s^0\rangle - q\,|\bar{B}_s^0\rangle, \qquad |B_s^L\rangle = p\,|B_s^0\rangle + q\,|\bar{B}_s^0\rangle$$

are labeled "heavy" and "light". The mass and lifetime differences between the B_s^H and B_s^L states can be defined as

$$\Delta m \equiv m_H - m_L\,, \qquad \Delta\Gamma \equiv \Gamma_L - \Gamma_H\,, \qquad \Gamma = (\Gamma_H + \Gamma_L)/2\,,$$

where $m_{H,L}$ and $\Gamma_{H,L}$ denote the mass and decay width of B_s^H and B_s^L (with this definition both Δm and $\Delta\Gamma$ are expected to be positive quantities). The heavy state decays with a longer lifetime, $\tau_H = 1/\Gamma_H$, while the light state decays with the shorter lifetime $\tau_L = 1/\Gamma_L$, in analogy to the neutral kaon system. The mean lifetime is defined to be $\tau = 1/\Gamma$. Theoretical estimates predict $\Delta\Gamma/\Gamma$ to be on the order of $\sim 15\%$ [2]. Linear polarization eigenstates of the J/ψ and ϕ provide a convenient basis for the analysis of the decay [9]. The two vector mesons can have their spins transversely polarized with respect to their momentum and be either parallel or perpendicular to each other. Alternatively, they can both be longitudinally polarized. We denote these states as $|\mathcal{P}_{||}\rangle$, $|\mathcal{P}_\perp\rangle$, and $|\mathcal{P}_0\rangle$.

In the standard model, CP violation occurs through complex phases in the CKM matrix [10]. Large phases occur in the matrix elements V_{ub} and V_{td}. While these matrix elements generate large CP violation in the B^0 system, they do not appear in leading order diagrams contributing to either $B_s^0 \leftrightarrow \bar{B}_s^0$ mixing or to the decay $B_s^0 \to J/\psi\phi$. For this reason the standard model expectation of CP violation in $B_s^0 \to J/\psi\phi$ is small. In the limit of vanishing CP violation, the heavy, long-lived mass eigenstate B_s^H is CP odd and decays to the CP-odd, L=1 orbital angular momentum state $|\mathcal{P}_\perp\rangle$. The light, short-lived

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mass eigenstate B_s^L is CP even and decays to both CP-even L=0 and L=2 orbital angular momentum states, which are linear combinations of $|\mathcal{P}_0\rangle$ and $|\mathcal{P}_{||}\rangle$.

The small CP violation in $B_s^0 \to J/\psi\phi$ can be quantified in the following way: we define A_i as the decay amplitude $\langle B_s|H|\mathcal{P}_i\rangle$ and $\bar{A}_i$ as the decay amplitude $\langle \bar{B}_s|H|\mathcal{P}_i\rangle$ where i is one of $\{||, \perp, 0\}$. All CP observables in the system are characterized by three quantities $\lambda_i = \frac{q}{p}\frac{\bar{A}_i}{A_i}$. In the standard model the λ_i are given as $\lambda_i = \pm\exp(i2\beta_s)$ where the positive and negative sign applies to the CP even and odd final state, and

$$\beta_s \equiv \arg\left(-\frac{V_{ts}V_{tb}^*}{V_{cs}V_{cb}^*}\right).$$

The standard model expectation [11] is $2\beta_s = 0.037 \pm 0.002$, a very small phase which does not lead to appreciable levels of CP violation. New physics can alter the mixing phase, while leaving λ very nearly unimodular. In this paper we consider, however, also the case in which $|\lambda| \neq 1$.

3 Differential rates

The state of an initially pure B_s^0 or $\bar{B}_s^0$ meson after a proper time t has elapsed is denoted as $|B^0_{s,\text{phys}}(t)\rangle$ and $|\bar{B}^0_{s,\text{phys}}(t)\rangle$. Transitions of these states to the detectable $\mu^+\mu^-K^+K^-$ can be written as

$$\begin{aligned}
&\langle\mu^+\mu^-K^+K^-|H|B^0_{s,\text{phys}}(t)\rangle \\
&\qquad= \sum_i \langle\mu^+\mu^-K^+K^-|H|\mathcal{P}_i\rangle\,\langle\mathcal{P}_i|H|B_s^0\rangle\,\langle B_s^0|B^0_{s,\text{phys}}(t)\rangle \\
&\qquad\quad+ \sum_i \langle\mu^+\mu^-K^+K^-|H|\mathcal{P}_i\rangle\,\langle\mathcal{P}_i|H|\bar{B}_s^0\rangle\,\langle\bar{B}_s^0|B^0_{s,\text{phys}}(t)\rangle\,, \\
&\langle\mu^+\mu^-K^+K^-|H|\bar{B}^0_{s,\text{phys}}(t)\rangle \\
&\qquad= \sum_i \langle\mu^+\mu^-K^+K^-|H|\mathcal{P}_i\rangle\,\langle\mathcal{P}_i|H|B_s^0\rangle\,\langle B_s^0|\bar{B}^0_{s,\text{phys}}(t)\rangle \\
&\qquad\quad+ \sum_i \langle\mu^+\mu^-K^+K^-|H|\mathcal{P}_i\rangle\,\langle\mathcal{P}_i|H|\bar{B}_s^0\rangle\,\langle\bar{B}_s^0|\bar{B}^0_{s,\text{phys}}(t)\rangle\,,
\end{aligned} \tag{3.1}$$

where H is the weak interaction Hamiltonian. The expression can be written much more simply, by defining time-dependent amplitudes for $|B_s^0\rangle$ and $|\bar{B}_s^0\rangle$ to reach the states $|\mathcal{P}_i\rangle$ either with or without mixing:

$$\begin{aligned}
\mathcal{A}_i(t) &\equiv \langle\mathcal{P}_i|H|B_s^0\rangle\,\langle B_s^0|B^0_{s,\text{phys}}(t)\rangle + \langle\mathcal{P}_i|H|\bar{B}_s^0\rangle\,\langle\bar{B}_s^0|B^0_{s,\text{phys}}(t)\rangle\,, \\
\bar{\mathcal{A}}_i(t) &\equiv \langle\mathcal{P}_i|H|B_s^0\rangle\,\langle B_s^0|\bar{B}^0_{s,\text{phys}}(t)\rangle + \langle\mathcal{P}_i|H|\bar{B}_s^0\rangle\,\langle\bar{B}_s^0|\bar{B}^0_{s,\text{phys}}(t)\rangle\,.
\end{aligned}$$

Then:

$$\begin{aligned}
\langle\mu^+\mu^-K^+K^-|H|B^0_{s,\text{phys}}(t)\rangle &= \sum_i \mathcal{A}_i(t)e^{-imt}\,\langle\mu^+\mu^-K^+K^-|H|\mathcal{P}_i\rangle\;, \\
\langle\mu^+\mu^-K^+K^-|H|\bar{B}^0_{s,\text{phys}}(t)\rangle &= \sum_i \bar{\mathcal{A}}_i(t)e^{-imt}\,\langle\mu^+\mu^-K^+K^-|H|\mathcal{P}_i\rangle\;,
\end{aligned} \tag{3.2}$$

where the time dependence of $\mathcal{A}_i(t)$ and $\bar{\mathcal{A}}_i(t)$ is:

$$\begin{aligned}\mathcal{A}_i(t) &= \frac{e^{-\Gamma t/2}}{\sqrt{\tau_H + \tau_L \pm \cos 2\beta_s\,(\tau_L - \tau_H)}}\left[E_+(t) \pm e^{2i\beta_s} E_-(t)\right] a_i\,,\\ \bar{\mathcal{A}}_i(t) &= \frac{e^{-\Gamma t/2}}{\sqrt{\tau_H + \tau_L \pm \cos 2\beta_s\,(\tau_L - \tau_H)}}\left[\pm E_+(t) + e^{-2i\beta_s} E_-(t)\right] a_i\,,\end{aligned} \tag{3.3}$$

and where the upper sign indicates a CP even final state, the lower sign indicates a CP odd final state,

$$E_\pm(t) \equiv \frac{1}{2}\left[e^{+\left(\frac{-\Delta\Gamma}{4}+i\frac{\Delta m}{2}\right)t} \pm e^{-\left(\frac{-\Delta\Gamma}{4}+i\frac{\Delta m}{2}\right)t}\right], \tag{3.4}$$

and the a_i are complex amplitude parameters satisfying:

$$\sum_i |a_i|^2 = 1\,. \tag{3.5}$$

The final state $\mu^+\mu^-K^+K^-$ is characterized by three decay angles, described in a coordinate system[1] called the transversity basis [1]. In the J/ψ rest frame, the x-axis is taken to lie along the momentum of the ϕ and the z-axis perpendicular to the decay plane of the ϕ. The variables $(\theta,\ \varphi)$ are the polar and azimuthal angles of the μ^+ momentum in this basis. We also define the angle ψ to be the "helicity" angle in the ϕ decay, i.e. the angle between the K^+ direction and the x-axis in the ϕ rest frame. With these definitions, the muon momentum direction in the J/ψ rest frame is given by the unit vector

$$\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,. \tag{3.6}$$

Let $\mathbf{A}(t)$ and $\bar{\mathbf{A}}(t)$ be complex vector functions of time defined as

$$\begin{aligned}\mathbf{A}(t) &= \left(\mathcal{A}_0(t)\cos\psi, -\frac{\mathcal{A}_\parallel(t)\sin\psi}{\sqrt{2}}, i\frac{\mathcal{A}_\perp(t)\sin\psi}{\sqrt{2}}\right),\\ \bar{\mathbf{A}}(t) &= \left(\bar{\mathcal{A}}_0(t)\cos\psi, -\frac{\bar{\mathcal{A}}_\parallel(t)\sin\psi}{\sqrt{2}}, i\frac{\bar{\mathcal{A}}_\perp(t)\sin\psi}{\sqrt{2}}\right),\end{aligned} \tag{3.7}$$

where $\mathcal{A}_i(t)$ have now been normalized. For experimental measurements we are concerned with normalized probability density functions P_B and $P_{\bar{B}}$ for B and $\bar{B}$ mesons in the variables t, $\cos\psi$, $\cos\theta$, and φ, which can be obtained by squaring eq. (3.2). The formulae of ref. [7] are then equivalent to:

$$\begin{aligned}P_B(\theta,\varphi,\psi,t) &= \frac{9}{16\pi}|\mathbf{A}(t)\times\hat{n}|^2\,,\\ P_{\bar{B}}(\theta,\varphi,\psi,t) &= \frac{9}{16\pi}|\bar{\mathbf{A}}(\mathbf{t})\times\hat{n}|^2\,,\end{aligned} \tag{3.8}$$

which give a picture of a time-dependent polarization analyzed in the decay.[2] The factors of $9/16\pi$ are normalization constants, and are present in order that

$$\int \sum_{j=B,\bar{B}} P_j(\psi,\theta,\varphi,t)d(\cos\psi)d(\cos\theta)d\varphi dt = 1\,. \tag{3.9}$$

[1] An alternate basis called the helicity basis is discussed further in section 9.

[2] Throughout this paper, when writing the dot product of two complex vectors, we always imply complex conjugation on the second operand.

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The quantities $|a_i|^2$ give the time-integrated rate to each of the polarization states. The values of $\mathcal{A}_i(t)$ at $t = 0$ will be denoted as A_i. To translate between the a's and the A's one can use the following two sets of transformations:

$$\begin{aligned}
|A_\perp|^2 &= \frac{|a_\perp|^2 y}{1+(y-1)|a_\perp|^2} \quad & |a_\perp|^2 &= \frac{|A_\perp|^2}{y+(1-y)|A_\perp|^2} \\
|A_{||}|^2 &= \frac{|a_{||}|^2}{1+(y-1)|a_\perp|^2} \quad & |a_{||}|^2 &= \frac{|A_{||}|^2 y}{y+(1-y)|A_\perp|^2} \\
|A_0|^2 &= \frac{|a_0|^2}{1+(y-1)|a_\perp|^2} \quad & |a_0|^2 &= \frac{|A_0|^2 y}{y+(1-y)|A_\perp|^2}\,,
\end{aligned} \tag{3.10}$$

where $y \equiv (1-z)/(1+z)$ and $z \equiv \cos 2\beta_s \Delta\Gamma/(2\Gamma)$. The relation (3.5) insures that

$$\sum_i |A_i|^2 = 1\,. \tag{3.11}$$

Eq. (3.8), together with the definitions in eqs. (3.3), (3.4), and (3.6) can be used as a decay model for an event generator, and is suitable for use as a fitting function in the absence of detector effects.

4 Detector efficiency and normalization

The detector efficiency $\varepsilon(\psi, \theta, \varphi)$, when introduced into the above expression, disturbs the normalization of eq. (3.9). We restore it by dividing by a normalization factor N,

$$\begin{aligned}
P'(\psi,\theta,\varphi,t) &= \frac{1}{N} P(\psi,\theta,\varphi,t)\varepsilon(\psi,\theta,\varphi)\,, \\
N &= \int \sum_{i=B,\bar{B}} P_i(\psi,\theta,\varphi,t)\varepsilon(\psi,\theta,\varphi) d(\cos\psi) d(\cos\theta) d\varphi dt\,.
\end{aligned} \tag{4.1}$$

Suppose that the efficiency $\varepsilon(\psi, \theta, \varphi)$ can be parametrized as

$$\varepsilon(\psi,\theta,\varphi) = c^k_{lm} P_k(\cos\psi) Y_{lm}(\theta,\varphi)\,, \tag{4.2}$$

where c^k_{lm} are expansion coefficients, $P_k(\cos\psi)$ are Legendre polynomials, and $Y_{lm}(\theta, \varphi)$ are real harmonics related to the spherical harmonics through the following relations:

$$\begin{aligned}
Y_{lm} &= Y_l^m & &(m=0)\,, \\
Y_{lm} &= \frac{1}{\sqrt{2}}(Y_l^m + (-1)^m Y_l^{-m}) & &(m>0)\,, \\
Y_{lm} &= \frac{1}{i\sqrt{2}}(Y_l^{|m|} - (-1)^{|m|} Y_l^{-|m|}) & &(m<0)\,.
\end{aligned} \tag{4.3}$$

The products $P_k(\cos\psi)Y_{lm}(\theta, \varphi)$ constitute an orthonormal basis for functions of the three angles. The detector efficiency (obtained, for example, from Monte Carlo simulation) can be fit to the first few of these polynomials. A straight-forward calculation shows that:

$$
\begin{aligned}
N = \frac{3}{8\sqrt{\pi}} & \left[\frac{4c_{00}^0}{3}(|a_0|^2 + |a_\||^2 + |a_\perp|^2) \right. \\
& \left. + \frac{4c_{00}^2}{15}(2|a_0|^2 - |a_\||^2 - |a_\perp|^2) \right] \\
+ \frac{3}{8\sqrt{5\pi}} & \left[\frac{2c_{20}^0}{3}(|a_0|^2 + |a_\||^2 - 2|a_\perp|^2) \right. \\
& \left. + \frac{4c_{20}^2}{15}\left(|a_0|^2 - \frac{1}{2}|a_\||^2 + |a_\perp|^2 \right) \right] \\
- \frac{9}{16\sqrt{15\pi}} & \frac{\sin 2\beta_s(\tau_L - \tau_H)}{\sqrt{\left((\tau_L - \tau_H)\sin 2\beta_s\right)^2 + 4\tau_L\tau_H}} \\
& \times \left[(a_\|^* a_\perp + a_\| a_\perp^*) \left(\frac{4}{3}c_{2-1}^0 - \frac{4}{15}c_{2-1}^2 \right) \right] \\
+ \frac{9}{16}\frac{\sqrt{2}}{\sqrt{15\pi}} & \frac{\sin 2\beta_s(\tau_L - \tau_H)}{\sqrt{\left((\tau_L - \tau_H)\sin 2\beta_s\right)^2 + 4\tau_L\tau_H}} \\
& \times \left[(a_0^* a_\perp + a_0 a_\perp^*) \left(\frac{\pi c_{21}^1}{8} - \frac{\pi c_{21}^3}{32} + \dots \right) \right] \\
+ \frac{9}{8\sqrt{15\pi}} & \left[\frac{2c_{22}^0}{3}(-|a_0|^2 + |a_\||^2) - \frac{4c_{22}^2}{15}\left(|a_0|^2 + \frac{1}{2}|a_\||^2 \right) \right] \\
+ \frac{9}{16}\frac{\sqrt{2}}{\sqrt{15\pi}} & \left[(a_0^* a_\| + a_0 a_\|^*) \left(\frac{\pi c_{2-2}^1}{8} - \frac{\pi c_{2-2}^3}{32} + \dots \right) \right].
\end{aligned}
\tag{4.4}
$$

The numerical factors $+\pi/8$ and $-\pi/32$, appearing together with $c_{2,1}^k$ and $c_{2,-2}^k$ in the infinite series, are the integrals

$$\int P_k(\cos\psi)\cos(\psi)\sin\psi d(\cos\psi)\,. \tag{4.5}$$

While this series is infinite, the number of basis functions needed to fit detector efficiencies in a particular analysis is finite and determined chiefly by the size of the data sample. With the factors in eq. (4.5) the normalizing factor can be adapted to account for all terms used in the expansion of the efficiency. Eq. (4.4) represents an analytic normalization of the fitting function and provides an efficient way to compute the likelihood during a maximum log likelihood fit. The orthonormality of the basis functions has been used to reduce the expression to its final form.

5 Time development

The short oscillation length of the B_s^0 meson [12], $2\pi c/\Delta m \sim 106$ μm, requires us to account for resolution effects when fitting the rates of flavor-tagged decays, even using the best silicon vertex detectors, which have proper decay length resolutions on the order of 25 μm. Certain time-dependent functions arising from particle-antiparticle oscillations, particularly those expressed as the product of exponential decays and harmonic functions with frequency Δm, must be convolved with one or more Gaussian components describing detector resolution. This convolution can be carried out analytically, using the method described in ref. [13] for the evaluation of certain integrals which are equivalent to complex error functions. In this step one requires that various components of the time dependence first be separated from eq. (3.8). The time development of $\mathcal{A}_0(t)$ and $\mathcal{A}_\parallel(t)$ amplitudes are identical, but differs from that of $\mathcal{A}_\perp(t)$. We begin by decomposing

$$\mathbf{A}(t) = \mathbf{A}_+(t) + \mathbf{A}_-(t), \qquad \bar{\mathbf{A}}(t) = \bar{\mathbf{A}}_+(t) + \bar{\mathbf{A}}_-(t)\,, \tag{5.1}$$

where

$$\begin{aligned} \mathbf{A}_+(t) &= \mathbf{A}_+ f_+(t) = \left(a_0 \cos\psi, -\frac{a_\parallel \sin\psi}{\sqrt{2}}, 0\right) \cdot f_+(t)\,, \\ \bar{\mathbf{A}}_+(t) &= \bar{\mathbf{A}}_+ \bar{f}_+(t) = \left(a_0 \cos\psi, -\frac{a_\parallel \sin\psi}{\sqrt{2}}, 0\right) \cdot \bar{f}_+(t)\,, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \mathbf{A}_-(t) &= \mathbf{A}_- f_-(t) = \left(0, 0, i\frac{a_\perp \sin\psi}{\sqrt{2}}\right) \cdot f_-(t)\,, \\ \bar{\mathbf{A}}_-(t) &= \bar{\mathbf{A}}_- \bar{f}_-(t) = \left(0, 0, i\frac{a_\perp \sin\psi}{\sqrt{2}}\right) \cdot \bar{f}_-(t)\,, \end{aligned} \tag{5.3}$$

and we define

$$\begin{aligned} f_\pm(t) &= \frac{e^{-\Gamma t/2}}{\sqrt{\tau_H + \tau_L \pm \cos 2\beta_s\,(\tau_L - \tau_H)}} \left[E_+(t) \pm e^{2i\beta_s} E_-(t)\right]\,, \\ \bar{f}_\pm(t) &= \frac{e^{-\Gamma t/2}}{\sqrt{\tau_H + \tau_L \pm \cos 2\beta_s\,(\tau_L - \tau_H)}} \left[\pm E_+(t) + e^{-2i\beta_s} E_-(t)\right]\,. \end{aligned} \tag{5.4}$$

We then have in place of eq. (3.8)

$$\begin{aligned} &P_B(\theta,\psi,\varphi,t) \\ &\quad= \frac{9}{16\pi}\left\{|\mathbf{A}_+(t)\times\hat{n}|^2 + |\mathbf{A}_-(t)\times\hat{n}|^2 + 2Re((\mathbf{A}_+(t)\times\hat{n})\cdot(\mathbf{A}^*_-(t)\times\hat{n}))\right\} \\ &\quad= \frac{9}{16\pi}\left\{|\mathbf{A}_+\times\hat{n}|^2|f_+(t)|^2 + |\mathbf{A}_-\times\hat{n}|^2|f_-(t)|^2\right. \\ &\qquad\qquad \left.+\, 2Re((\mathbf{A}_+\times\hat{n})\cdot(\mathbf{A}^*_-\times\hat{n})\cdot f_+(t)\cdot f^*_-(t))\right\} \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} &P_{\bar{B}}(\theta,\psi,\varphi,t) \\ &\quad= \frac{9}{16\pi}\left\{|\bar{\mathbf{A}}_+(t)\times\hat{n}|^2 + |\bar{\mathbf{A}}_-(t)\times\hat{n}|^2 + 2Re(\bar{\mathbf{A}}_+(t)\times\hat{n})\cdot(\bar{\mathbf{A}}^*_-(t)\times\hat{n}))\right\} \\ &\quad= \frac{9}{16\pi}\left\{|\mathbf{A}_+\times\hat{n}|^2|\bar{f}_+(t)|^2 + |\mathbf{A}_-\times\hat{n}|^2|\bar{f}_-(t)|^2\right. \\ &\qquad\qquad \left.+\, 2Re((\mathbf{A}_+\times\hat{n})\cdot(\mathbf{A}^*_-\times\hat{n})\cdot \bar{f}_+(t)\cdot \bar{f}^*_-(t)\right\}\,, \end{aligned} \tag{5.6}$$

where (for $\bar{B}$) the diagonal term in eq. (5.6) is

$$|\bar{f}_\pm(t)|^2 = \frac{1}{2}\frac{(1\pm\cos 2\beta_s)e^{-\Gamma_L t} + (1\mp\cos 2\beta_s)e^{-\Gamma_H t} \pm 2\sin 2\beta_s e^{-\Gamma t}\sin\Delta m t}{\tau_L(1\pm\cos 2\beta_s) + \tau_H(1\mp\cos 2\beta_s)}\,, \tag{5.7}$$

while (for B) the diagonal term in eq. (5.5) is

$$|f_\pm(t)|^2 = \frac{1}{2}\frac{(1 \pm \cos 2\beta_s)e^{-\Gamma_L t} + (1 \mp \cos 2\beta_s)e^{-\Gamma_H t} \mp 2\sin 2\beta_s e^{-\Gamma t}\sin \Delta m t}{\tau_L(1 \pm \cos 2\beta_s) + \tau_H(1 \mp \cos 2\beta_s)} \tag{5.8}$$

and (for $\bar{B}$) the cross-term, or interference term in eq. (5.6) is

$$\bar{f_+}(t)\bar{f_-}^*(t) = \frac{-e^{-\Gamma t}\cos \Delta m t - i\cos 2\beta_s e^{-\Gamma t}\sin \Delta m t + i\sin 2\beta_s(e^{-\Gamma_L t} - e^{-\Gamma_H t})/2}{\sqrt{[(\tau_L - \tau_H)\sin 2\beta_s]^2 + 4\tau_L\tau_H}}, \tag{5.9}$$

while (for B) the interference term in eq. (5.5) is

$$f_+(t)f_-^*(t) = \frac{e^{-\Gamma t}\cos \Delta m t + i\cos 2\beta_s e^{-\Gamma t}\sin \Delta m t + i\sin 2\beta_s(e^{-\Gamma_L t} - e^{-\Gamma_H t})/2}{\sqrt{[(\tau_L - \tau_H)\sin 2\beta_s]^2 + 4\tau_L\tau_H}}. \tag{5.10}$$

This accomplishes the desired separation. In the fitting function, to accommodate the proper time resolution, one has only to replace all time-dependent functions with their smeared equivalents.

6 Sensitivity to Δm

It can be noticed that the time development of the interference term, expressions (5.9) and (5.10), contain undiluted mixing asymmetries *even in the case of no CP violation*, i.e., when $\beta_s = 0$. Let us try to better understand the mechanism by which the flavor of the B_s^0 meson is tagged at *decay* time, by first rewriting eq. (3.1) using the B_s^H and B_S^L states in the expansion rather than the B_s^0 and $\bar{B}_s^0$ states:

$$\begin{aligned} &\langle \mu^+\mu^- K^+K^-|H|B^0_{s,\mathrm{phys}}(t)\rangle \\ &\quad = \sum_i \langle \mu^+\mu^- K^+K^-|H|\mathcal{P}_i\rangle \langle \mathcal{P}_i|H|B_s^H\rangle \langle B_s^H|B^0_{s,\mathrm{phys}}(t)\rangle \\ &\qquad + \sum_i \langle \mu^+\mu^- K^+K^-|H|\mathcal{P}_i\rangle \langle \mathcal{P}_i|H|B_s^L\rangle \langle B_s^L|B^0_{s,\mathrm{phys}}(t)\rangle. \end{aligned} \tag{6.1}$$

Now, we take the limit of zero CP violation in the B_s^0 system, such that $\langle \mathcal{P}_{||}|H|B_s^H\rangle = \langle \mathcal{P}_0|H|B_s^H\rangle = \langle \mathcal{P}_\perp|H|B_s^L\rangle = 0$, and only three of the six terms in eq. (6.1) remain:

$$\begin{aligned} &\langle \mu^+\mu^- K^+K^-|H|B^0_{s,\mathrm{phys}}(t)\rangle \\ &\quad = \langle \mu^+\mu^- K^+K^-|H|\mathcal{P}_\perp\rangle \langle \mathcal{P}_\perp|H|B_s^H\rangle \langle B_s^H|B^0_{s,\mathrm{phys}}(t)\rangle \\ &\qquad + \langle \mu^+\mu^- K^+K^-|H|\mathcal{P}_0\rangle \langle \mathcal{P}_0|H|B_s^L\rangle \langle B_s^L|B^0_{s,\mathrm{phys}}(t)\rangle \\ &\qquad + \langle \mu^+\mu^- K^+K^-|H|\mathcal{P}_{||}\rangle \langle \mathcal{P}_{||}|H|B_s^L\rangle \langle B_s^L|B^0_{s,\mathrm{phys}}(t)\rangle\,. \end{aligned} \tag{6.2}$$

When the expression is squared, the interference terms are the cross terms involving both the product of a CP-even and a CP-odd amplitudes. The time dependence of these terms is contained in the factor:

$$\begin{aligned}
&\langle B_s^H|B^0_{s,\text{phys}}(t)\rangle\langle B_s^L|B^0_{s,\text{phys}}(t)\rangle \\
&\quad= \frac{1}{4}\Big[\left(\langle B_s^H|B^0_{s,\text{phys}}(t)\rangle + \langle B_s^L|B^0_{s,\text{phys}}(t)\rangle\right)^2 \\
&\qquad\quad - \left(\langle B_s^H|B^0_{s,\text{phys}}(t)\rangle - \langle B_s^L|B^0_{s,\text{phys}}(t)\rangle\right)^2\Big] \\
&\quad= \frac{1}{2}\left[\left(\frac{\langle B_s^H| + \langle B_s^L|}{\sqrt{2}}|B^0_{s,\text{phys}}(t)\rangle\right)^2 - \left(\frac{\langle B_s^H| - \langle B_s^L|}{\sqrt{2}}|B^0_{s,\text{phys}}(t)\rangle\right)^2\right] \\
&\quad= \frac{1}{2}\left[\langle B_s^0|B^0_{s,\text{phys}}(t)\rangle^2 - \langle \bar{B}_s^0|B^0_{s,\text{phys}}(t)\rangle^2\right].
\end{aligned} \tag{6.3}$$

This factor takes the value $+1/2$ when the meson is pure B_s^0, and $-1/2$ when the meson is pure $\bar{B}_s^0$, and in general oscillates between these two values. Thus the interference term effectively tags the flavor of the B_s^0 at decay. This provides a way to observe $B_s^0 \to \bar{B}_s^0$ flavor oscillations using a sample of flavor-tagged $B_s^0 \to J/\psi\phi$ decays which can be collected with a simple dimuon trigger. This may open a particularly interesting avenue for the LHC experiments to observe B_s^0 mixing using a J/ψ trigger.

7 Incorporating direct CP violation

An asymmetry either in the decay rate ($|\bar{A}_i/A_i| \neq 1$) or in the mixing ($|q/p| \neq 1$) such that $|\lambda| \neq 1$ is direct CP violation. In the case of direct CP violation λ does not lie on the unit circle in the complex plane, and we need two parameters to describe it which we will take to be $\mathcal{C} \equiv Re(\lambda)$ and $\mathcal{S} \equiv Im(\lambda)$. Experimentally, even if one sets out to extract β_s assuming the constraint $|\lambda| = 1$, it is nonetheless of interest to test that constraint, since sensitivity to $\mathcal{C}$ and $\mathcal{S}$ arise from very different features of the detector. In that case we must revisit not only the functional form of the differential decay rates, but also the normalization. The amplitudes in eq. (3.3) must now be written as:

$$\begin{aligned}
\mathcal{A}_i &= \mathcal{N}_\pm e^{-\Gamma t/2}\left[E_+(t) \pm \lambda E_-(t)\right] a_i\,, \\
\bar{\mathcal{A}}_i &= \mathcal{N}_\pm e^{-\Gamma t/2}\left[\pm E_+(t) + E_-(t)/\lambda\right] a_i\,,
\end{aligned} \tag{7.1}$$

where

$$\mathcal{N}_\pm = \left\{\frac{1}{4|\lambda|^2}\left[\left[(\tau_H+\tau_L)(1+|\lambda|^2)^2 \pm 2\mathcal{C}\cdot(\tau_L-\tau_H)(1+|\lambda|^2)\right]\right.\right. \\
\left.\left. + \frac{\tau}{1+\Delta m^2\tau^2}\cdot\left[\pm 4\mathcal{S}\cdot\left(1-|\lambda|^2\right)\Delta m\tau - 2\left(1-|\lambda|^2\right)^2\right]\right]\right\}^{-\frac{1}{2}}.$$

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These amplitudes can readily be seen to reduce to those of eq. (3.3) in the limit of $\mathcal{C}^2+\mathcal{S}^2 \equiv |\lambda|^2 \to 1$. The normalization of detector efficiency, eq. (4.4), becomes:

$$
\begin{aligned}
N = & \frac{3}{8\sqrt{\pi}}\left[\frac{4c_{00}^0}{3}\left(|a_0|^2+|a_\parallel|^2+|a_\perp|^2\right)\right. \\
& \qquad \left. + \frac{4c_{00}^2}{15}\left(2|a_0|^2-|a_\parallel|^2-|a_\perp|^2\right)\right] \\
& + \frac{3}{8\sqrt{5\pi}}\left[\frac{2c_{20}^0}{3}\left(|a_0|^2+|a_\parallel|^2-2|a_\perp|^2\right)\right. \\
& \qquad \left. + \frac{4c_{20}^2}{15}\left(|a_0|^2-\frac{1}{2}|a_\parallel|^2+|a_\perp|^2\right)\right] \\
& - \frac{9}{16\sqrt{15\pi}}\mathcal{N}_+\mathcal{N}_-\mathcal{S}\cdot(\tau_L-\tau_H) \\
& \quad \times\left[\left(a_\parallel^* a_\perp + a_\parallel a_\perp^*\right)\left(\frac{4}{3}c_{2-1}^0-\frac{4}{15}c_{2-1}^2\right)\right] \\
& + \frac{9}{16}\frac{\sqrt{2}}{\sqrt{15\pi}}\mathcal{N}_+\mathcal{N}_-\mathcal{S}\cdot(\tau_L-\tau_H) \\
& \quad \times\left[\left(a_0^* a_\perp + a_0 a_\perp^*\right)\left(\frac{\pi c_{21}^1}{8}-\frac{\pi c_{21}^3}{32}+\ldots\right)\right] \\
& + \frac{9}{8\sqrt{15\pi}}\left[\frac{2c_{22}^0}{3}\left(-|a_0|^2+|a_\parallel|^2\right)-\frac{4c_{22}^2}{15}\left(|a_0|^2+\frac{1}{2}|a_\parallel|^2\right)\right] \\
& + \frac{9}{16}\frac{\sqrt{2}}{\sqrt{15\pi}}\left[\left(a_0^* a_\parallel + a_0 a_\parallel^*\right)\left(\frac{\pi c_{2-2}^1}{8}-\frac{\pi c_{2-2}^3}{32}+\ldots\right)\right].
\end{aligned}
$$

Finally, the explicit time development, eqs. (5.7), (5.8), (5.9) and (5.10), must be replaced with the more general forms:

$$
\begin{aligned}
|\bar{f}_\pm(t)|^2 &= \frac{\mathcal{N}_\pm^2}{4|\lambda|^2}\left[\left((1+|\lambda|^2)\pm 2\mathcal{C}\right)e^{-\Gamma_L t}+\left((1+|\lambda|^2)\mp 2\mathcal{C}\right)e^{-\Gamma_H t}\right. \\
&\qquad \left. + \left(\pm 4\mathcal{S}\sin\Delta mt - 2(1-|\lambda|^2)\cos\Delta mt\right)e^{-\Gamma t}\right], \\
|f_\pm(t)|^2 &= \frac{\mathcal{N}_\pm^2}{4}\left[\left((1+|\lambda|^2)\pm 2\mathcal{C}\right)e^{-\Gamma_L t}+\left((1+|\lambda|^2)\mp 2\mathcal{C}\right)e^{-\Gamma_H t}\right. \\
&\qquad \left. - \left(\pm 4\mathcal{S}\sin\Delta mt - 2(1-|\lambda|^2)\cos\Delta mt\right)e^{-\Gamma t}\right], \\
\bar{f}_+(t)\bar{f}_-{}^*(t) &= \frac{\mathcal{N}_+\mathcal{N}_-}{4|\lambda|^2}\left[-e^{-\Gamma t}\left(2(1+|\lambda|^2)\cos\Delta mt + 4i\mathcal{C}\sin\Delta mt\right)\right. \\
&\qquad \left. + e^{-\Gamma_L t}\left((1-|\lambda|^2)+2i\mathcal{S}\right)+e^{-\Gamma_H t}\left((1-|\lambda|^2)-2i\mathcal{S}\right)\right], \\
f_+(t)f_-^*(t) &= \frac{\mathcal{N}_+\mathcal{N}_-}{4}\left[e^{-\Gamma t}\left(2(1+|\lambda|^2)\cos\Delta mt + 4i\mathcal{C}\sin\Delta mt\right)\right. \\
&\qquad \left. + e^{-\Gamma_L t}\left((1-|\lambda|^2)+2i\mathcal{S}\right)+e^{-\Gamma_H t}\left((1-|\lambda|^2)-2i\mathcal{S}\right)\right],
\end{aligned}
$$

which can be seen to reduce to expression (5.7), (5.8) and (5.9), (5.10) as $|\lambda|^2 \to 1$.

8 Incorporating a contribution from $B_s^0 \to J/\psi K^+ K^-$ (kaons in an S-wave state)

It has been suggested [8] that a contribution from S-wave K^+K^- under the ϕ peak in $B_s^0 \to J/\psi\phi$ decay may contribute up to 5-10% of the total rate. A normalized probability density for the decay $B_s^0 \to J/\psi K^+K^-$ (kaons in an S-wave state) can be worked out by considering the polarization vector of the J/ψ in the decay and proceeding as in [9]. The resulting expressions

$$\begin{aligned} Q_B(\theta,\varphi,\psi,t) &= \frac{3}{16\pi}|\mathbf{B}(t)\times\hat{n}|^2\,, \\ Q_{\bar{B}}(\theta,\varphi,\psi,t) &= \frac{3}{16\pi}|\bar{\mathbf{B}}(\mathbf{t})\times\hat{n}|^2 \end{aligned} \tag{8.1}$$

do not depend at all on the angle ψ (which is the helicity angle in the ϕ decay). In the previous expression

$$\begin{aligned} \mathbf{B}(t) &= (\mathcal{B}(t),0,0)\ , \\ \bar{\mathbf{B}}(t) &= \left(\bar{\mathcal{B}}(t),0,0\right) \end{aligned} \tag{8.2}$$

where the time-dependent amplitudes,

$$\begin{aligned} \mathcal{B}(t) &= \frac{e^{-\Gamma t/2}}{\sqrt{\tau_H+\tau_L-\cos 2\beta_s\,(\tau_L-\tau_H)}}\left[E_+(t)-e^{2i\beta_s}E_-(t)\right], \\ \bar{\mathcal{B}}(t) &= \frac{e^{-\Gamma t/2}}{\sqrt{\tau_H+\tau_L-\cos 2\beta_s\,(\tau_L-\tau_H)}}\left[-E_+(t)+e^{-2i\beta_s}E_-(t)\right] \end{aligned} \tag{8.3}$$

reflect the CP-odd nature of the $J/\psi KK$ final state.

When both P-wave and S-wave are present, the amplitudes must be summed and then squared. The P wave has a resonant structure due to the ϕ-propagator, while the S-wave amplitude is flat (but can have any phase with respect the P-wave). Suppose that in our experiment we accept events for which the reconstructed mass $m(K^+K^-) \equiv \mu$ lies within a window $\mu_{lo} < \mu < \mu_{hi}$. The normalized probability in this case is

$$\begin{aligned} \rho_B(\theta,\varphi,\psi,t,\mu) &= \frac{9}{16\pi}\left|\left[\sqrt{1-F_s}\,g(\mu)\mathbf{A}(t)+e^{i\delta_s}\sqrt{F_s}\frac{h(\mu)}{\sqrt{3}}\mathbf{B}(t)\right]\times\hat{n}\right|^2\ , \\ \rho_{\bar{B}}(\theta,\varphi,\psi,t,\mu) &= \frac{9}{16\pi}\left|\left[\sqrt{1-F_s}\,g(\mu)\bar{\mathbf{A}}(t)+e^{i\delta_s}\sqrt{F_s}\frac{h(\mu)}{\sqrt{3}}\bar{\mathbf{B}}(\mathbf{t})\right]\times\hat{n}\right|^2\ , \end{aligned} \tag{8.4}$$

where we use a nonrelativistic Breit-Wigner to model the ϕ resonance[3]

$$g(\mu) = \sqrt{\frac{\Gamma_\phi/2}{\Delta\omega}}\cdot\frac{1}{\mu-\mu_\phi+i\Gamma_\phi/2} \tag{8.5}$$

a flat model for the S-wave mass distribution

$$h(\mu) = \frac{1}{\sqrt{\Delta\mu}} \tag{8.6}$$

[3] We shall have more to say about that, later.

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and define

$$\omega_{hi} = \tan^{-1}\frac{2(\mu_{hi}-\mu_\phi)}{\Gamma_\phi} \qquad \omega_{lo} = \tan^{-1}\frac{2(\mu_{lo}-\mu_\phi)}{\Gamma_\phi} \tag{8.7}$$

and

$$\Delta\mu = \mu_{hi}-\mu_{lo} \qquad \Delta\omega = \omega_{hi}-\omega_{lo}\,. \tag{8.8}$$

In these equations, F_s is the S-wave fraction; μ_ϕ is the ϕ mass ($1019\,\mathrm{MeV/c^2}$); Γ_ϕ is the ϕ width ($4.26\,\mathrm{MeV/c^2}$), and δ_s is the phase of the S-wave component relative to the P-wave component.

In the presence of an S-wave contribution, the normalization of eq. (4.4) must be generalized; in order to do this we first define the quantities

$$\mathcal{F}(\mu) \equiv \sqrt{\frac{F_s(1-F_s)\Gamma_\phi}{2\Delta\mu\Delta\omega}}\cdot\frac{e^{-i\delta_s}}{\mu-\mu_\phi+i\Gamma_\phi/2} \tag{8.9}$$

and

$$\begin{aligned}\mathcal{I}_\mu &\equiv \int \mathcal{F}(\mu)d\mu \\ &= \sqrt{\frac{F_s(1-F_s)\Gamma_\phi}{2\Delta\mu\Delta\omega}}\cdot e^{-i\delta_s}\cdot\log\frac{\mu_{hi}-\mu_\phi+i\Gamma_\phi/2}{\mu_{lo}-\mu_\phi+i\Gamma_\phi/2}\,.\end{aligned} \tag{8.10}$$

Then the normalizing factor appropriate for eq. (8.4) is

$$\mathcal{N} = (1-F_s)\cdot N + 2Re\left[\mathcal{I}_\mu\cdot N'\right] + F_s\cdot N''\,, \tag{8.11}$$

where N is given in eq. (4.4), and

$$\begin{aligned}N' = {} & \sqrt{3}*a_0^*\left(\frac{1}{6\sqrt{\pi}}c^1_{00}+\frac{1}{12\sqrt{5\pi}}c^1_{20}-\frac{1}{4\sqrt{15\pi}}c^1_{22}\right)\\ & +\frac{3}{16}\sqrt{\frac{2}{5\pi}}a_\parallel^*\left(\frac{\pi}{2}c^0_{2-2}-\frac{\pi}{8}c^2_{2-2}+\cdots\right)\\ & +\frac{3}{16}\sqrt{\frac{2}{5\pi}}a_\perp^*\frac{\sin 2\beta_s(\tau_L-\tau_H)}{\sqrt{\left((\tau_L-\tau_H)\sin 2\beta_s\right)^2+4\tau_L\tau_H}}\left(\frac{\pi}{2}c^0_{21}-\frac{\pi}{8}c^2_{21}+\cdots\right)\end{aligned} \tag{8.12}$$

and

$$N'' = \frac{1}{2\sqrt{\pi}}c^0_{00}+\frac{1}{4\sqrt{5\pi}}c^0_{20}-\frac{3}{4\sqrt{15\pi}}c^0_{22}\,. \tag{8.13}$$

The numerical factors $+\pi/2$ and $-\pi/8$ appearing together with $c^k_{2,1}$ and $c^k_{2,-2}$ in the infinite series are the integrals

$$\int P_k(\cos\psi)\sin\psi d(\cos\psi)\,. \tag{8.14}$$

We now work out the explicit time and mass dependence of the differential rates. We will use eq. (5.5) together with the analogous equation for the pure S-wave differential rate:

$$\begin{aligned}Q_B(\theta,\psi,\varphi,t) &= \frac{3}{16\pi}|\mathbf{B(t)}\times\hat{n}|^2\\ &= \frac{3}{16\pi}|\mathbf{B}\times\hat{n}|^2|f_-(t)|^2\end{aligned} \tag{8.15}$$

and

$$
\begin{aligned}
Q_{\bar{B}}(\theta,\psi,\varphi,t) &= \frac{3}{16\pi}|\bar{\mathbf{B}}(t)\times\hat{n}|^2 \\
&= \frac{3}{16\pi}|\mathbf{B}\times\hat{n}|^2|\bar{f}_-(t)|^2\,.
\end{aligned}
\tag{8.16}
$$

where the vector $\mathbf{B}=\hat{x}=(1,0,0)$. The full probability densities, which can be used in a time-, angle-, and ϕ mass-dependent fit, are obtained by expanding eq. (8.4). We get

$$
\begin{aligned}
&\rho_B(\theta,\psi,\varphi,t,\mu) \\
&\quad = (1-F_s)\frac{\Gamma_\phi/2}{\Delta\omega}\cdot\frac{1}{(\mu-\mu_\phi)^2+\Gamma_\phi^2/4}\cdot P_B(\theta,\psi,\varphi,t) \\
&\qquad + F_s\frac{1}{\Delta\mu}Q_B(\theta,\psi,\varphi,t) \\
&\qquad + 2\frac{\sqrt{27}}{16\pi}Re\left[\mathcal{F}(\mu)\left((\mathbf{A}_-\times\hat{n})\cdot(\mathbf{B}\times\hat{n})\cdot|f_-(t)|^2\right.\right. \\
&\qquad \left.\left.+(\mathbf{A}_+\times\hat{n})\cdot(\mathbf{B}\times\hat{n})\cdot f_+(t)\cdot f_-^*(t)\right)\right]
\end{aligned}
\tag{8.17}
$$

and

$$
\begin{aligned}
&\rho_{\bar{B}}(\theta,\psi,\varphi,t,\mu) \\
&\quad = (1-F_s)\frac{\Gamma_\phi/2}{\Delta\omega}\cdot\frac{1}{(\mu-\mu_\phi)^2+\frac{1}{1+F_S}\Gamma_\phi^2/4}\cdot P_{\bar{B}}(\theta,\psi,\varphi,t) \\
&\qquad + F_s\frac{1}{\Delta\mu}Q_{\bar{B}}(\theta,\psi,\varphi,t) \\
&\qquad + 2\frac{\sqrt{27}}{16\pi}Re\left[\mathcal{F}(\mu)\left((\mathbf{A}_-\times\hat{n})\cdot(\mathbf{B}\times\hat{n})\cdot|\bar{f}_-(t)|^2\right.\right. \\
&\qquad \left.\left.+(\mathbf{A}_+\times\hat{n})\cdot(\mathbf{B}\times\hat{n})\cdot\bar{f}_+(t)\cdot\bar{f}_-^*(t)\right)\right]\,.
\end{aligned}
\tag{8.18}
$$

In case one does not want to observe the ϕ-mass variable μ, one can integrate it out. Then one obtains

$$
\begin{aligned}
&\rho_B(\theta,\psi,\varphi,t) \\
&\quad = (1-F_s)\cdot P_B(\theta,\psi,\varphi,t)+F_sQ_B(\theta,\psi,\varphi,t) \\
&\qquad + 2\frac{\sqrt{27}}{16\pi}Re\left[\mathcal{I}_\mu\left((\mathbf{A}_-\times\hat{n})\cdot(\mathbf{B}\times\hat{n})\cdot|f_-(t)|^2\right.\right. \\
&\qquad\qquad \left.\left.+(\mathbf{A}_+\times\hat{n})\cdot(\mathbf{B}\times\hat{n})\cdot f_+(t)\cdot f_-^*(t)\right)\right]\,,
\end{aligned}
\tag{8.19}
$$

$$
\begin{aligned}
&\rho_{\bar{B}}(\theta,\psi,\varphi,t) \\
&\quad = (1-F_S)\cdot P_{\bar{B}}(\theta,\psi,\varphi,t)+F_sQ_{\bar{B}}(\theta,\psi,\varphi,t) \\
&\qquad + 2\frac{\sqrt{27}}{16\pi}Re\left[\mathcal{I}_\mu\left((\mathbf{A}_-\times\hat{n})\cdot(\mathbf{B}\times\hat{n})\cdot|\bar{f}_-(t)|^2\right.\right. \\
&\qquad\qquad \left.\left.+(\mathbf{A}_+\times\hat{n})\cdot(\mathbf{B}\times\hat{n})\cdot\bar{f}_+(t)\cdot\bar{f}_-^*(t)\right)\right]\,.
\end{aligned}
\tag{8.20}
$$

9 Symmetries

In this section we examine the symmetries of our differential rate formulae, starting from the simplest case, K^+K^- in a P-wave state, eq. (3.8), but considering also the case where both P and S waves are included, eq. (8.4). In the case of pure P-wave, one can readily spot that the probability densities in eq. (3.8) are invariant to the following transformations:

- A simultaneous rotation of the vectors $\mathbf{A}(t)$ and $\hat{n}$
- An inversion of the vector $\mathbf{A}(t)$
- Complex-conjugation of the vector $\mathbf{A}(t)$

The symmetry to simultaneous rotation of the vectors $\mathbf{A}(\mathbf{t})$ and $\hat{n}$ corresponds to the well-known freedom to choose a convenient basis in which to work. An example of an alternative basis is the *helicity basis*, which derives from the *transversity basis* by a cyclic permutation of the coordinate axis: $\hat{x}_T = \hat{z}_H$, etc. One can take the angles in eq. (3.6) to be the polar and azimuthal angles in the helicity basis, but then one must transform $\mathbf{A}(\mathbf{t})$ accordingly, i.e, by permuting the elements of $\mathbf{A}(\mathbf{t})$ in the defining equation, eq. (3.7). Then, eq. (3.8) remains valid in the helicity basis. This rotational invariance implies that the choice of basis is irrelevant to the final result since the likelihood is invariant to the choice (though we do not rule out the possibility that the quality of the efficiency expansion, eq. (4.2), may depend on the choice of basis, as pointed out in [14]).

A more interesting symmetry is the symmetry that results from transforming $\mathbf{A}(t)$ to its complex conjugate. If we take, by convention, a_0 to be real and let $\delta_\parallel = \arg(a_\parallel)$, and $\delta_\perp = \arg(a_\perp)$, then as we will demonstrate below, this conjugation transformation is equivalent to the simultaneous transformation:

$$
\begin{aligned}
\beta_s &\to \pi/2 - \beta_s \\
\Delta\Gamma &\to -\Delta\Gamma \\
\delta_\perp &\to \pi - \delta_\perp \\
\delta_\parallel &\to 2\pi - \delta_\parallel \,.
\end{aligned}
\tag{9.1}
$$

That is to say that the simultaneous transformation of these four variables is a symmetry of the likelihood *because* it transforms $\mathbf{A}(\mathbf{t})$ into its complex conjugate. Since for pure P wave state the combined transformation is a well-known symmetry, this observation may appear as a curiosity; however when both P and S wave states are included, we shall see that complex conjugation teaches us how to properly extend the symmetry. First, we show how the combined transformation accomplishes the claimed complex conjugation.

1. Note from eq. (3.4) that the combined transformation transforms $E_\pm(t) \to \pm E_\pm^*(t)$.
2. Note also that the combined transformation transforms $e^{-2i\beta_s} \to -e^{+2i\beta_s}$ and $e^{+2i\beta_s} \to -e^{-2i\beta_s}$
3. Therefore, in eq. (3.3), the terms in square brackets are transformed into their complex conjugates.

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4. Note that both $\cos 2\beta_s$ and $\tau_L - \tau_H$ change sign under the transformation, so also the piece of eq. (3.3) in the denominator, under the square root sign, is invariant under the combined transformation; since that piece is real we can say that it is anyway equal to its complex conjugate.

5. The real quantity a_0 does not change under the combined transformation, but since it is real, it is anyway equal to a_0^*.

6. The combined transformation transforms $a_\parallel \to a_\parallel^*$.

7. The combined transformation transforms $ia_\perp \to -ia_\perp^*$.

8. Then looking at eq. (3.7), one sees finally that the net effect of the combined transformation has been the complex conjugation of the vector $\mathbf{A(t)}$.

Returning now to the full likelihood including both P and S wave states, eq. (8.4), we can see that, here again, complex conjugation of the term

$$\sqrt{1-F_s}\,g(\mu)\mathbf{A}(t) + e^{i\delta_s}\sqrt{F_s}\frac{h(\mu)}{\sqrt{3}}\mathbf{B}(t) \tag{9.2}$$

leaves the probability density invariant (in a parameter space now enlarged to include μ_ϕ and Γ_ϕ); however now, complex conjugation of the term $g(\mu)$, eq. (8.5), implies that the transformation $\Gamma_\phi \to -\Gamma_\phi$ should also be carried out, in addition to the transformation of β_s, $\Delta\Gamma$, $\delta_\parallel$, and $\delta_\perp$. Since negative values of Γ_ϕ are physically meaningless, this transformation is not an admissible symmetry.

However we can find a symmetry transformation that carries one set of physically meaningful parameters into another. Such a symmetry is the transformation of the terms in eq. (9.2) into their *negative complex conjugate*. This transformation is equivalent to the combined transformation already described, in addition to:

$$\begin{aligned} \delta_s &\to \pi - \delta_s \\ (\mu - \mu_\phi) &\to -(\mu - \mu_\phi)\,. \end{aligned} \tag{9.3}$$

The latter transformation carries us from a point on one side of the ϕ mass peak to another point located symmetrically on the other side. This symmetry is useful when considering likelihood functions in which the dependence on μ is integrated out. If we integrate symmetrically about the ϕ mass peak, we can consider the contribution to the integral coming from a slice in ϕ mass on one hand and the a symmetrically-located slice in ϕ mass on the other hand. While the contribution of either slice is not invariant to the transformation above, the contribution of both slices certainly is, and the combined transformation:

$$\begin{aligned} \beta_s &\to \pi/2 - \beta_s \\ \Delta\Gamma &\to -\Delta\Gamma \\ \delta_\perp &\to \pi - \delta_\perp \\ \delta_\parallel &\to 2\pi - \delta_\parallel \\ \delta_s &\to \pi - \delta_s \end{aligned} \tag{9.4}$$

is again a symmetry of the integrated likelihood. We note, however, that this symmetry requires the symmetry of the nonrelativistic ϕ-propagator, eq. (8.5), and applies to *the likelihood integrated over a finite symmetric interval of integration*.

Symmetries of the likelihood function for $B_s^0 \to J/\psi\phi$, in the presence of S-wave contribution for a fixed value of $\mu = m(K^+K^-)$ were discussed in a recent publication [15]. These formula can also used to fit for data falling within a narrow window in μ. Under those assumptions we can drop the ϕ propagator from the expression in eq. (9.2), absorb the Breit-Wigner terms into the amplitudes $\mathbf{A}(t)$, and write our model for the rates as

$$\sqrt{1-F_s}\mathbf{A}(t) + e^{i\delta_s}\sqrt{\frac{F_s}{3}}\mathbf{B}(t)\,. \tag{9.5}$$

Then one can see that the transformation in which $\delta_s \to -\delta_s$ replaces $\delta_s \to \pi - \delta_s$ in eq. (9.4) accomplishes a complex conjugation of the terms in eq. (9.5) and is a symmetry *of the likelihood at fixed* μ.

In the more general case one can notice from eqs. (8.19) and (8.20) that the probability densities integrated over μ are invariant to complex conjugation of both $\mathbf{A}(t)$ *and* the overlap integral $\mathcal{I}_{mu}$ of eq. (8.10). This can be accomplished with a more complicated adjustment of δ_s. With a nonrelativistic Breit Wigner the required transformation is

$$\delta_s \to 2\delta_{BW} - \delta_s\,,$$

where $\delta_{BW} \equiv \arg\left(\log\left((\mu_{hi} - \mu_\phi + i\Gamma_\phi/2)/(\mu_{lo} - \mu_\phi + i\Gamma_\phi/2)\right)\right)$. The phase δ_{BW} reduces to $\delta_{BW} = 0$ in the limit of an infinitesimally thin interval in μ, and to $\delta_{BW} = -\pi/2$ in the limit of a finite symmetric interval. This demonstrates real differences in the two formulations, and underscores the need for caution when applying the formulae of ref. [15] to a finite interval in $\mu = m(K^+K^-)$.

10 Conclusion

In this paper we have presented a compact formalism to easily access physical observables in the analysis of the decay $B_s^0 \to J/\psi\phi$. This formalism has practical applications, since complex vectors and their vector-algebraic operations can be easily implemented in high-level computer languages in order to model and generate such decays, but also because the symmetries of the formulae under operations such as rotation and complex conjugation are apparent and provide better physical insight into this complicated decay mode. The normalized probability densities can be used for the experimental extraction of physical parameters, in scenarios with no CP violation, with mixing-induced CP violation, or even with direct CP violation. In case of mixing induced CP violation, the effect of the S-wave contribution has also been included in the decay rate formulae.

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